Measurements and Instrumentation [SBE206A] (Fall 2018)
Tutorial 3, 4, & 5

Dr. Muhammed Rushdi   Asem Alaa

December 11, 2018

Instrument types and performance characteristics (Cont'd)

Static characteristics of instruments (Cont'd)

- The static characteristics of measuring instruments are concerned only with the steady-state reading that the instrument settles down to, such as accuracy of the reading.

Hysteresis

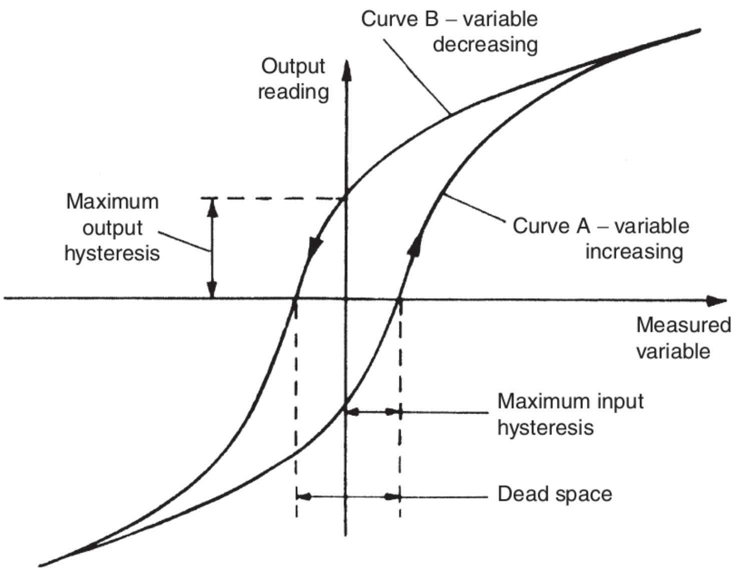

Figure 1: Instrument characteristic with hysteresis.

Dead space

- Dead space is defined as the range of different input values over which there is no change in output value.

- Any instrument that exhibits hysteresis also displays dead space, as marked on Figure .

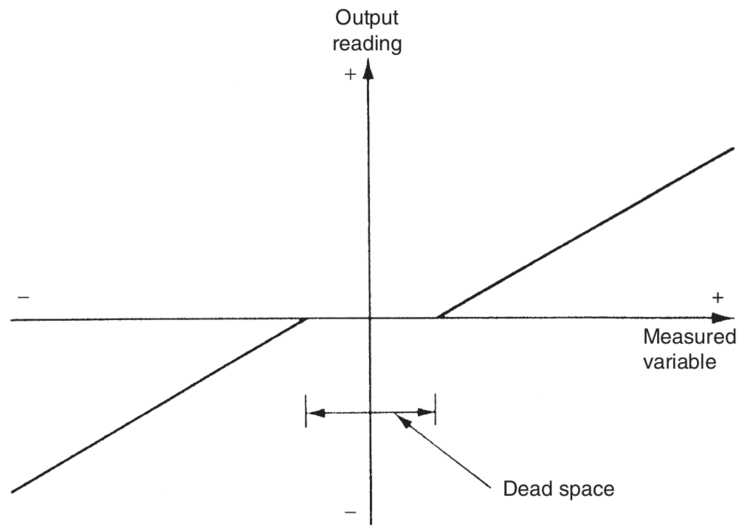

Figure 2: Instrument characteristic with dead space.

Dynamic characteristics of instruments

- The dynamic characteristics of a measuring instrument describe its behavior between the time a measured quantity changes value and the time when the instrument output attains a steady value in response.
- In any linear, time-invariant measuring system, the following general relation can be written between input and output for time $(t) > 0$:

$$a_n \frac{d^n q_o}{dt^n} + a_{n-1} \frac{d^{n-1} q_o}{dt^{n-1}} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_n \frac{d^n q_i}{dt^n} + b_{n-1} \frac{d^{n-1} q_i}{dt^{n-1}} + \cdots + b_1 \frac{dq_i}{dt} + b_0 q_i \tag{1}$$

- If we limit consideration to that of step changes in the measured quantity only, then Equation 1 reduces to:

$$a_n \frac{d^n q_o}{dt^n} + a_{n-1} \frac{d^{n-1} q_o}{dt^{n-1}} + \cdots + a_1 \frac{dq_o}{dt} + a_0 q_o = b_0 q_i \tag{2}$$

Zero-order instrument

If all the coefficients $a_1 \cdots a_n$ other than a_0 in Equation 2 are assumed zero, then:

$$a_0 q_o = b_0 q_i \quad \text{or} \quad q_o = b_0 \frac{q_i}{a_0} = K q_i \tag{3}$$

where K is a constant known as the instrument sensitivity as defined earlier. Figure shows an instrument of zero-order response.

First-order instrument

If all the coefficients $a_2 \cdots a_n$ except for a_0 and a_1 in Equation 2 are assumed zero, then:

$$a_1 \frac{dq_o}{dt} + a_0 q_o = b_0 q_i \tag{4}$$

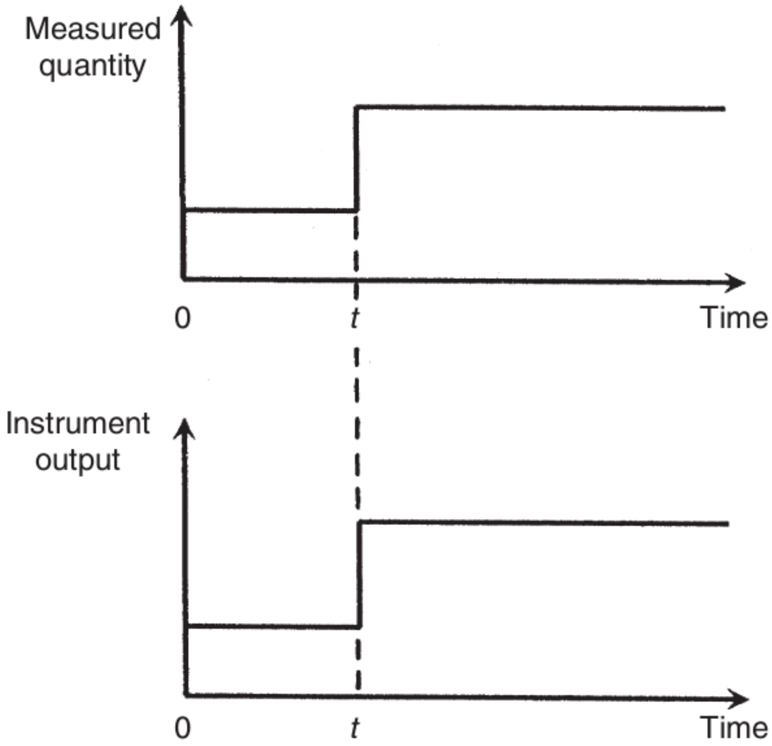

Figure 3: Zero-order instrument characteristic.

Taking the Laplace transform of Equation 4 as a method to solving the differential equation:

$$a_1\left[sQ_o(s) - q_o(0)\right] + a_0 Q_o(s) = b_0 Q_i(s) \tag{5}$$

$$(a_1 s + a_0)Q_o(s) = b_0 Q_i(s) + a_1 q_o(0) \tag{6}$$

$$Q_o(s) = \frac{b_0 Q_i(s) + a_1 q_o(0)}{a_1 s + a_0} \tag{7}$$

$$Q_o(s) = \frac{K Q_i(s) + \tau q_o(0)}{\tau s + 1} \tag{8}$$

Where $K = b_0/a_0$ as the static sensitivity and $\tau = a_1/a_0$ as the time constant of the system.

Second-order instrument

If all the coefficients $a_3 \cdots a_n$ except for a_0, a_1 and a_2 in Equation 2 are assumed zero, then:

$$a_2 \frac{d^2 q_o}{dt^2} + a_1 \frac{dq_o}{dt} + a_0 q_o = b_0 q_i \tag{9}$$

Taking the Laplace transform of Equation 9 as a method to solving the differential equation:

$$a_2\left[s^2 Q_o(s) - sq_o(0) - q_o'(0)\right] + a_1\left[sQ_o(s) - q_o(0)\right] + a_0 Q_o(s) = b_0 Q_i(s) \tag{10}$$

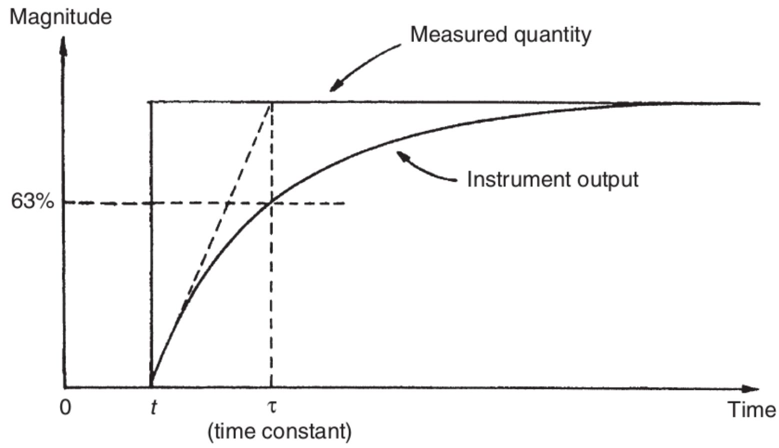

Figure 4: First-order instrument characteristic.

$$(a_2s^2 + a_1s + a_0)Q_o(s) = b_0Q_i(s) + a_2sq_o(0) + a_2q_o'(0) + a_1q_o(0) \tag{11}$$

$$Q_o(s) = \frac{b_0Q_i(s) + a_2sq_o(0) + a_2q_o'(0) + a_1q_o(0)}{a_2s^2 + a_1s + a_0} \tag{12}$$

$$Q_o(s) = \frac{K\omega_n^2Q_i(s) + q_o(0)s + q_o'(0) + 2\eta\omega_nq_o(0)}{s^2 + 2\eta\omega_ns + {\omega_n}^2} \tag{13}$$

Where:

- $K = b_0/a_0$ as the static sensitivity.
- $\omega_n = \sqrt{\frac{a_0}{a_2}}$ as the undamped natural frequency of the system.
- $\eta = \frac{a_1\omega_n}{2a_0} = \frac{a_1}{2\sqrt{a_0a_2}}$ as the *damping ratio* of the system.

Zero initial condition

To develop intuition about the solution of $q_o(t)$, let's assume that the system is driven by an input of constant quantity (i.e $q_i(t) = const.$ & $Q_i(s) = \frac{const.}{s}$), and $q_o(t)$ has zero initial value. The system is then described by the following equation:

$$Q_o(s) = \frac{M(s)}{s(s^2 + 2\eta\omega_ns + {\omega_n}^2)} \tag{14}$$

where $M(s)$ is a polynomial in s.

The solution of $q_o(t)$ is subject to computing the roots of the denumerator part $s(s^2 + 2\eta\omega_ns + {\omega_n}^2)$, where the roots of $N(s) = (s^2 + 2\eta\omega_ns + {\omega_n}^2)$ is computed as:

$$s_{1,2} = \frac{-2\eta\omega_n \pm \sqrt{4\eta^2\omega_n^2 - 4\omega_n^2}}{2} \tag{15}$$

$$= -\eta\omega_n \pm \omega_n\sqrt{\eta^2 - 1} \tag{16}$$

which has three cases:

1. $\eta > 1$: $N(s)$ has real distinct roots, then our system is called highly damped.

2. $\eta = 1$: $N(s)$ has real identical roots, then our system is called critically damped.

3. $\eta < 1$: $N(s)$ has complex roots, then our system is called lightly damped.

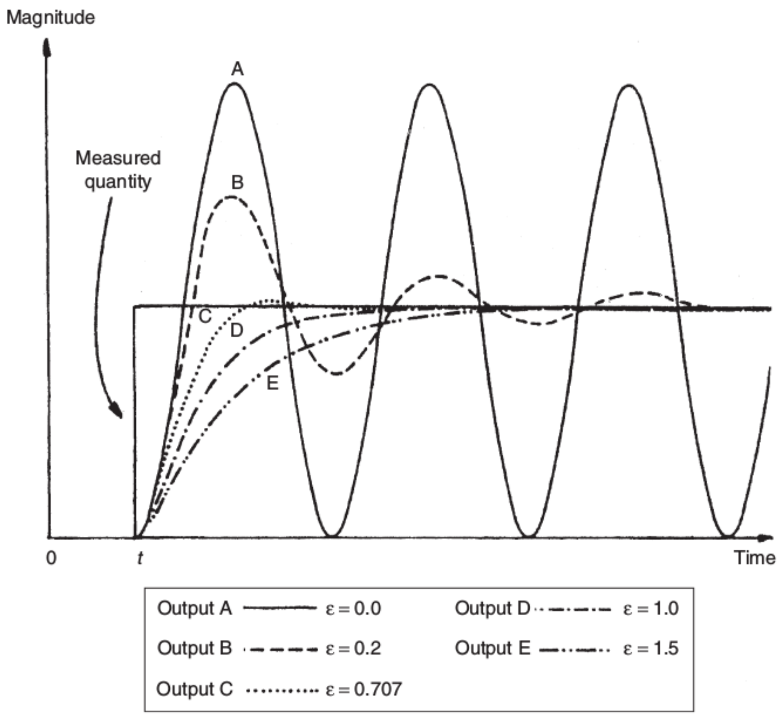

Figure 5: Second-order instrument characteristic.

Figure shows a system of second-order reponse with different η values.

Case 1: highly damped system ($\eta > 1$)

The Equation 14 can be factorized as following:

$$\begin{aligned} Q_o(s)|_{\eta>1} &= \frac{M(s)}{s(s+(\alpha+\beta))(s+(\alpha-\beta)} \\ &= \frac{A}{s} + \frac{B}{s+(\alpha+\beta)} + \frac{C}{s+(\alpha-\beta)} \end{aligned}$$

where $\beta = \omega_n\sqrt{\eta^2-1}$.
and the general solution of $q_o(t)|_{\eta>1}$ becomes:

$$q_o(t)|_{\eta>1} = A + Be^{-(\alpha+\beta)t} + Ce^{-(\alpha-\beta)t} \tag{17}$$

Case 2: critically damped system ($\eta = 1$)

The Equation 14 can be factorized as following:

$$\begin{aligned} Q_o(s)|_{\eta=1} &= \frac{M(s)}{s(s+\alpha)^2} \\ &= \frac{A}{s} + \frac{B}{s+\alpha} + \frac{C}{(s+\alpha)^2} \end{aligned}$$

and the general solution of $q_o(t)|_{\eta=1}$ becomes:

$$q_o(t)|_{\eta=1} = A + Be^{-\alpha t} + Cte^{-\alpha t} \tag{18}$$

Case 3: lightly damped system ($\eta < 1$)

The Equation 14 can be expressed as following:

$$\begin{aligned} Q_o(s)|_{\eta<1} &= \frac{A}{s} + \frac{Bs+C}{s^2 + 2\eta\omega_n s + {\omega_n}^2} \\ &= \frac{A}{s} + \frac{Bs+C}{(s+\eta\omega_n)^2 + {\omega_n}^2 - \alpha^2} \\ &= \frac{A}{s} + \frac{Bs+C}{(s+\alpha)^2 + \omega_d^2} \end{aligned}$$

where:

- $\alpha = \eta\omega_n$, is called the attenuation coeffecent.
- $\omega_d = {\omega_n}^2 - \alpha^2 = \omega_n\sqrt{1-\eta^2}$, is called the damped natural frequency, which is responsible of the oscillation frequency of the system response.

and the general solution of $q_o(t)|_{\eta<1}$ becomes:

$$q_o(t)|_{\eta<1} = A + Be^{-\alpha t}\sin(\omega_d t - \phi) \tag{19}$$

Problems

Example on 1_{st}-order device **1. Exercise**

A balloon is equipped with temperature and altitude measuring instruments and has radio equipment that can transmit the output readings of these instruments back to ground. The balloon is initially anchored to the ground with the instrument output readings in steady state. The altitude-measuring instrument is approximately zero order and the temperature transducer first order with a time constant of 15 seconds. The temperature on the ground, T_0 , is 10 °C and the temperature T_x at an altitude of x metres is given by the relation: $T_x = T_0 - 0.01x$

1. If the balloon is released at time zero, and thereafter rises upwards at a velocity of 5 metres/second, draw a table showing the temperature and altitude measurements reported at intervals of 10 seconds over the first 50 seconds of travel. Show also in the table the error in each temperature reading.
2. What temperature does the balloon report at an altitude of 5000 metres?

1. Solution

Example on 1_{st}-order device

2. Exercise

An unmanned submarine is equipped with temperature and depth measuring instruments and has radio equipment that can transmit the output readings of these instruments back to the surface. The submarine is initially floating on the surface of the sea with the instrument output readings in steady state. The depthmeasuring instrument is approximately zero order and the temperature transducer first order with a time constant of 50 seconds. The water temperature on the sea surface, T_0 , is 20 °C and the temperature T_x at a depth of x metres is given by the relation: $T_x = T_0 - 0.01x$

1. If the submarine starts diving at time zero, and thereafter goes down at a velocity of 0.5 metres/second, draw a table showing the temperature and depth measurements reported at intervals of 100 seconds over the first 500 seconds of travel. Show also in the table the error in each temperature reading.

2. What temperature does the submarine report at a depth of 1000 metres?

2. Solution

3. Exercise

Write down the general differential equation describing the dynamic response of a second order measuring instrument and state the expressions relating the static sensitivity, undamped natural frequency and damping ratio to the parameters in this differential equation. Sketch the instrument response for the cases of heavy damping, critical damping and light damping, and state which of these is the usual target when a second order instrument is being designed.

3. Solution

Midterm 2014

4. Exercise

The dynamic error in a temperature measurement using a thermometer is 70% at 3 seconds after an input step change in temperature. Determine:

1. the magnitude ratio at 3 seconds,
2. the thermometers time constant (in seconds), and
3. the magnitude ratio at 1 second.

4. Solution

Midterm 2017

5. Exercise

A thermocouple is immersed in a liquid to monitor its temperature fluctuations. Assume the thermocouple acts as a first-order system.

1. In a planned experiment, the thermocouple is to be exposed to a step change in temperature. The response characteristics of the thermocouple must be such that the thermocouples output reaches 95% of the final temperature within 5 seconds. Assume that the thermocouples bead (its sensing element) is spherical with a density $\rho = 8000\ \mathrm{Kg/m^3}$, a specific heat at constant volume $\mathrm{C_v} = 380\ \mathrm{J/(Kg.K)}$ and a convective heat transfer coefficient $h = 210\ W/(m^2K)$. The time constant of the thermocouple is related to those parameters by $\tau = \frac{\rho \mathrm{d} \mathrm{C_v}}{6h}$, where d is the diameter of the thermocouple's bead. Determine the *maximum* diameter that the thermocouple can have and still meet the desired response characteristics.

2. In another experiment on the same thermocouple, the temperature fluctuations (in $^\circ C$) vary in time as $T(t) = 50 + 25\cos(4t)$. The output of the thermocouple transducer system $E(t)$ (in mV) is linearly proportional to temperature and has a static sensitivity of $2\ mV/^\circ C$. Find the output $E(t)$ (in mV).

5. Solution

Midterm 2017

6. Exercise

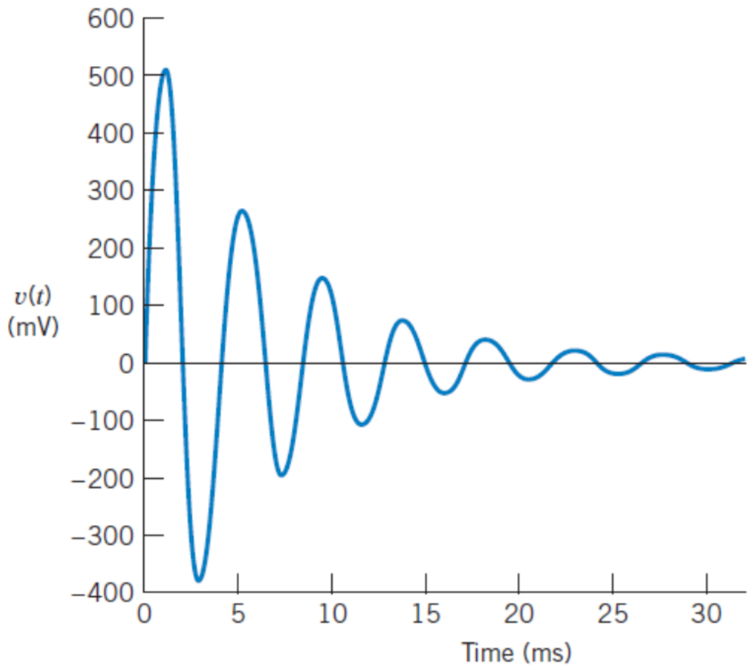

Figure 6: Second-order response of the system.

An instrument is modeled as a parallel RLC circuit whose differential equation is

$$\frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = 0$$

where $v(t)$ is the capacitor voltage. The natural response of the instrument is measured and plotted in the Figure 6. Using this chart, determine:

1. the damped natural frequency,
2. the damping ratio (using the log-decrement method),
3. the undamped natural frequency,
4. an expression for $v(t)$, and
5. the R, L, and C values.

6. Solution