Measurements and Instrumentation [SBE206A] (Fall 2018)
Tutorial 8

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Descriptive Statistics for Measurements Errors

What is a distribution?

1. A probability distribution is a mathematical function that can be thought of as providing the probability of occurrence of different possible outcomes in an experiment wiki/Probability_distribution.
2. A distribution is a function linked to the underlying observations.
3. A distribution provides the probability that an observation has a specific value.

Visualizing distributions

- Discrete distributions are represented by bar charts, where the sum of all bar heights adds up to 1.
- Continuous distributions are visualized by a curve (probability density function).
  1. The probability of an exact value (on the x axis) is 0, and there is an unlimited amount of numbers.
  2. Probability is quantified as the area under the curve between two numbers:
  $$\mathrm{P(a < x < b)}$$
  which equals the shaded area; calculated by integration.
  3. The total area under the curve is 1.
- Both distributions can be accurately represented by a histogram, or frequency graph.
- The amount of intervals visualized in a histogram can be computed using *Sturges'* rule:
$$k = \lceil \log_2 n \rceil + 1 \tag{1}$$

Normal distribution

- Defined by the probability density function
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}.$$
- For a normal distribution, the probability of an observation falling between the mean and 1 σ away is 34.1%. Probability of falling between -2σ and $+2\sigma$ is 95.4%.

Z-score

- Percentage of observations between standard deviations: -1 to 1: 68.2%, -2 to 2: 95.4%, -3 to 3: 99.8%.
- Z-score is defined as $Z = \frac{x-\mu}{\sigma}$. It is the amount of standard deviations that a point x is away from the mean μ.
- Applied to all values, this procedure transforms the data to have $\mu = 0$ and $\sigma = 1$. A distribution with such mean and standard deviation is said to be normalized.
- Standard normal probability tables can be used to find probabilities.

Random Variables

The *probability density function* (pdf) of a continuous *Random Variable* is given by

$$\int_{x_1}^{x_2} f_X(x)dx = P(x_1 \leq X \leq x_2), \quad \forall x_1, x_2$$

And satisfies the following properties:

1. $f(x) \geq 0 \quad \forall x$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$

The *Cumulative Distribution Function*(cdf) for a continuous *Random Variable* is

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u)du, \quad \forall x$$

And satisifes the following property:

1. $\frac{\partial F_X(x)}{\partial x} = f_X(x)$ provided that F' exists and X is a continuous *Random Variable*

Sample Statistics

Mean, Median, Mode

- Sort data ascendingle when calculating these statistics, and keep duplicates if duplicates are present in the data.
- For a right-skewed distribution, the median is to the right of the mean, and the mode is to the right of the median. To remember the order: the 3 statistics are in alphabetical order for a left-skewed distribution.
- The median is usually not affected by outliers as much as the mean.

Central Tendency: sample average/mean

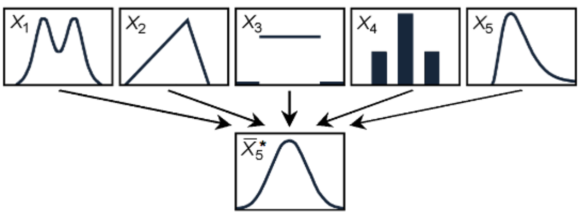
$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Scatter/dispersion: *sample variance* or *sample standard deviation*

$$\hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1};$$

$$\hat{\sigma} = S = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}}$$

Central limit theorem



- Visualize central limit theory through animation at onlinestatbook.com
- If several subsets are taken from an infinite data population with a Gaussian distribution, then, by the central limit theorem, the means of the subsets will form a Gaussian distribution about the mean of the infinite data set.
- The standard deviation of mean values of a series of finite sets of measurements relative to the true mean is defined as the standard error of the mean, $\alpha = \frac{\sigma}{\sqrt{n}}$.
- The normalized average:

$$\bar{X}_n^* = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \tag{2}$$

  n here is the count of samples, not each sample size.

- The interval $x = \bar{X}_n \pm \alpha$ includes the true mean with 68% certainity.

Interval	Confidence
$\bar{X}_n \pm \alpha$	68%
$\bar{X}_n \pm 2\alpha$	95.4%
$\bar{X}_n \pm 3\alpha$	99.7%

Estimation error from a single sample with confidence

Error	Confidence
$\pm(\sigma + \alpha)$	68%
$\pm 1.96(\sigma + \alpha)$	95.4%
$\pm 3(\sigma + \alpha)$	99.7%

Problems

1. Exercise

An integrated circuit chip contains 10000 identical transistors. Measurements are made of the current gain of each transistor. Measurements have a mean of 20.0 and a standard deviation of 1.5. The probability distribution of the measurements is Gaussian.

1. Write down an expression for the number of transistors that have a current gain between 19.5 and 20.5 and show that this number is approximately 2600 transistor.
2. Calculate the number of transistors that have a current gain of 17 or more (this is the minimum current gain necessary for a transistor to be able to drive the succeeding stage of the circuit in which the chip is used).

2. Exercise

You collected 29 readings for the pressure drop (ΔP) occurring across a valve in a medical gas network, and computed a sample mean of $\Delta P = 2$ kPa, along with a sample standard deviation of $S_{\Delta P} = 0.09$ kPa. What is the 90% confidence interval (CI) you would place on the true variance of the pressure drop? (Remember the units)

3. Exercise

Consider the situation in which a large number of voltage measurements are made. From this data, the mean value of the voltage is 8.5 V and that its variance is 2.25 V^2 . The probability distribution of the measurements is Gaussian.

1. Determine the probability that a single voltage measurement will fall inside the interval between 10 V and 11.5 V.

2. Determine the probability that no voltage measurement exceeds 15 V.

4. Exercise

A pressure microsensor is tested by applying a pressure to it of 200 bar, measured by an accurate, calibrated reference pressure-measuring instrument. A set of 12 measurements are made as a reference set of measurements in order to assess the standard deviation and standard error of the mean for measurements made by the device. The measurements obtained for this reference set are:
$[199.7, 202.0, 200.9, 195.7, 200.2, 199.9, 204.4, 198.0, 203.1, 199.1, 200.5, 196.9]$

When the microsensor is subsequently used in a workplace to measure the pressure in an enclosed vessel, a reading of 184 bar is obtained. What is the likely error in this measurement, expressed to 95.0% confidence limits?

5. Exercise

You collected 22 readings for air viscosity (μ) in a medical gas network, and computed a sample mean of $\bar{\eta} = 18.27(\mu\mathrm{Pas})$, along with a sample standard deviation of $\pm 2\%$ of the sample mean. What is the 95% confidence interval (CI) you would place on the true air viscosity? (Remember the units)

6. Exercise

A sample of 50000 inductors produced in a factory is taken, and the inductance of each inductor is measured. We want to apply the χ^2 test to examine whether the data set formed by the set of 50000 inductors measurements conforms to a Gaussian distribution.

1. Using the Sturgis rule, show that the number of bins should be set to 17.
2. Assuming that the number of samples in each bin exceeds the minimum threshold, the χ^2 statistic was computed to be 7.55. Test whether the measurements follow a Gaussian distribution with significance levels α of 90% and 95%, respectively.

7. Exercise

A certain measurement gave the value of C_p, the specific heat of protein, as 1700 J/kg°C. The precision of measurement is specified by the standard deviation given by 25 J/kg°C. If the measurement is repeated what is the probability that the value is within 1700 ± 40 J/kg°C?
You may assume that the error is normally distributed.

Standard Normal Probabilities

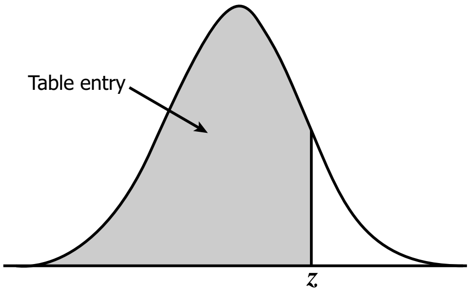
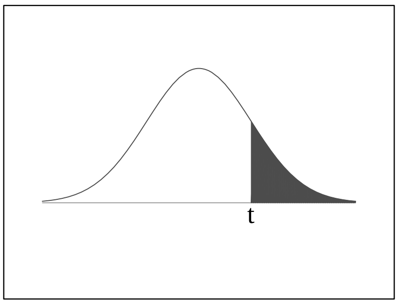


Table entry for z is the area under the standard normal curve to the left of z.

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

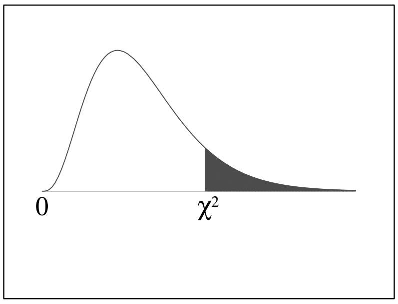
t-Distribution Table



The shaded area is equal to α for $t = t_\alpha$.

df	$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
32	1.309	1.694	2.037	2.449	2.738
34	1.307	1.691	2.032	2.441	2.728
36	1.306	1.688	2.028	2.434	2.719
38	1.304	1.686	2.024	2.429	2.712
∞	1.282	1.645	1.960	2.326	2.576

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169