INTEREST RATE DERIVATIVES EXPLAINED

VOLUME 1: PRODUCTS AND MARKETS

Jörg Kienitz

FINANCIAL ENGINEERING EXPLAINED

Interest Rate Derivatives Explained

Financial Engineering Explained

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Interest Rate Derivatives Explained

Volume 1: Products and Markets

Jörg Kienitz

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To Amberley, Beatrice and Benoît

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Goals of this Book and Global Overview

I.1 Introduction and management summary

A: We have written an exposition on interest rates. This comes in two volumes.
R: Hhm!
A: The first volume is on the products, on markets, the infrastructure and curves.
R: Hhm!
A: We consider the new practice after the beginning of the financial crisis! And we do not only write down formulas but we have many examples and explain the stuff from the scratch! This includes many working Excel sheets.
R: Oh, that sounds really interesting to me since I want to get a grip on this stuff. And fast!
A: That is exactly what we provide. You get an overview of how curves are built nowadays and get a nice overview of collateralization and CSA as well as the adjustments CVA, DVA and FVA.
R: So this seems rather practical and ready to use.
A: Exactly, you can also play around with numbers, prices and risk figures since everything is illustrated on accompanying spreadsheets. So, you wish to understand a CMS Cap, just play around and study what happens when you move the curve, flatten the implied volatility, etc...
R: I see. This is why a new book on interest rates has been written!
A: Yes, we know that there are several books on the topic. Some of the most prominent figures in Quantitative Finance have contributed to this subject. But the volumes are very complete and, thus, make up several hundred pages. They have all the very details and the exact math.
R: You mean you guys are not exact?
A: No, no, this is not what I mean. But take the CMS again. If you really want to deeply understand what is going on you have to go through the whole mathematics and apply change-of-measure, calculate expectations, use de-integration formulas, and so forth. We give you the final result and we provide you with a spreadsheet with all the stuff implemented so that you can play around and study the example to get a feeling for what happens. The formulas we provide are (hopefully) exact but we do not give the derivation and explain all the underlying concepts from stochastic analysis.
R: That is nice but if I have built up the intuition and want to dig deeper into the subject...
A: Then, you can consult the reading list at the end of each chapter to find further sources of information.

R: That sounds promising! I am starting now! And this is all covered in a book of 160 pages? That is quite ambitious!

A: Honestly, we give an outline of the current market practice and you can handle a lot of products and learn about the new regulated markets. But, of course, we cannot cover every detail and every convention. But we included references for further reading. Also this volume does not cover volatility modeling or term structure models which are necessary to set up volatility surfaces and price exotic options. This is covered in the second volume of the Interest Rate Explained series. Again, this also gives an overview and you will be quite well prepared if you worked through the book but details about the numerical implementation cannot be handled in an introductory text of less than 200 pages.

I.2 Short overview

The interest rate market has undergone a significant change after the beginning of the crisis in August 2007. Figure I.1 is a road map of the crisis. Several paradigms from the markets were violated after August 2007. Expecially that a big bank cannot go bankrupt. This is reflected for instance in the money market basis spreads, the cross currency basis spread or the ECB rates corridor.

Figure I.1 shows the basis spreads reflecting the fact that standard short term deposit rate is not risk free. In fact credit and liquidity risk are immanent in the rates. The longer the tenor of the rates the higher the risks. The basis, or money market basis to be precise, is the spread of the short term money market rate with respect to the corresponding rate calculated from overnight index quotes. With regard to Figure I.1 we see that in August 2007 when BNP Paribas froze funds, the bankruptcy of Lehman in 2008 as well as the Greek crisis led to significant movements in the money market basis.

We start this book by outlining the infrastructure and the new regulations. Thus, we cover clearing mechanisms and collateral management in Chapter 1. Then, we proceed by introducing the basis notions of interest rates in Chapter 2. All this is then applied in Chapter 3 to define several relevant market structures such as bonds or repos. Risk measures and hedging is also covered to some extent. For instance we cover interest rate curve movements and we also consider what happens if the curve changes its shape. The first interest rate derivatives are introduced in Chapter 4. We cover FRA (forward rate agreements) and IRS (interest rate swaps). Here we provide an overview of the large variety of the corresponding markets. We show market quotes and explain the mechanisms providing graphs and many examples. The derivatives covered in this chapter and the basic instruments covered in the last chapter can all be priced by using yield curves. To this end we spend a whole chapter, Chapter 5, on this subject. We explain the basics of interpolation and answer the question of how to set up curves. The chapter is closed by carrying over the results for curves to higher dimensional objects. We consider surfaces and cubes which play a fundamental role in handling volatility. After being familiar with the derivatives depending only on the yield curves we cover the most relevant options: Caps/Floors and Swaptions in Chapter 6. We need a pricing model and have to provide more information; namely we take into

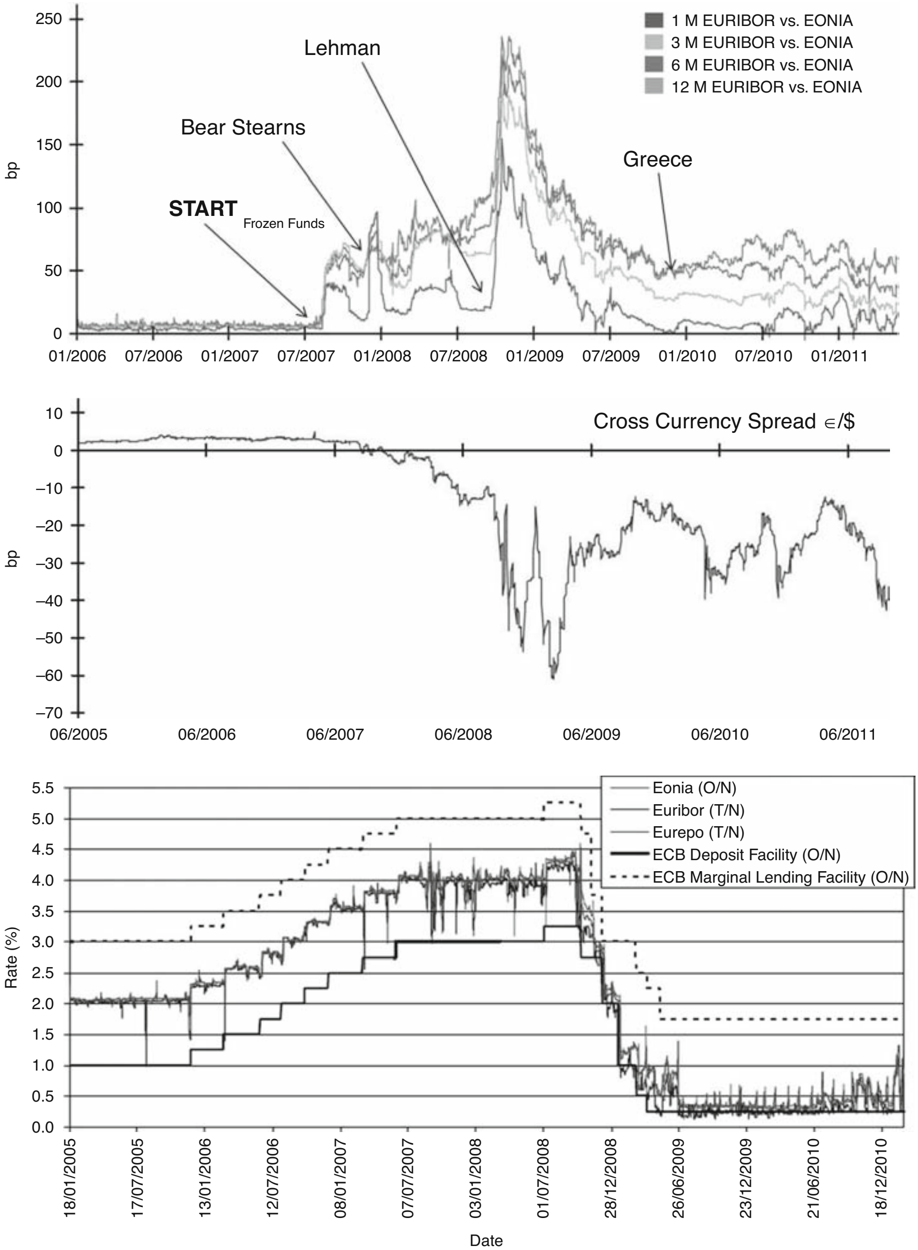


Figure I.1 Money market basis spreads (top), cross currency basis spreads (mid) and ECB rates corridor (bottom)

account an additional parameter called *volatility*. We see how such options work and how they are quoted. Again, examples can be found in the text as well as on dedicated spreadsheets. The next chapter, Chapter 7, concludes the discussion on options by tackling the CMS markets. Such products are very common when trading the shape of the curve, doing interest rate management and they are one of the most important building blocks of structured products. We discuss not only the mechanism but also give industry strength pricing methodologies. A book on the subject would not be complete without giving an overview of accounting for funding and counterparty credit risk. To this end we give many details on the usage and the calculation of adjustments in Chapter 8. The aim of this chapter is to give a thorough introduction to adjustments CVA, DVA and FVA as well as providing formulas for the calculation of the corresponding values and illustrating the concepts using examples.

After that much talking let us start with the introduction to interest rate markets and products.

I.3 Use of the book

We give a practical outline of Interest Rates and applications. Since this is the first part of two we do not cover the whole spectrum of modeling. The first part of the series covers instruments and basic option pricing. For pricing only basic structures such as yield curves and volatility surfaces assumed given are used. The second part explains how to model volatilities and use term structure models for tackling more advanced problems. This is left for the second volume of this series.

The exposition would not be complete without examples. To this end we have included many illustrations and examples in the running text but much more examples and material can be found in the accompanying spreadsheets. The sheets can also be used for a deeper analysis and the reader can play around with different numbers and see what happens. This is more interactive than static examples in a book. The sheets can be obtained via www.palgrave.com/companion/Kienitz

We always mention in the text if a spreadsheet for illustration is available.

I.4 Credits

This book could not have written without the help of many people. First of all I wish to mention my family (Beatrice my wife, Amberley my daughter and Benoit my son).

I am also indebted to my colleague Daniel Wetterau and my former colleague Manuel Wittke for helpful comments and discussions. It was a productive and prosperous time when we formed the basis of the Quantiative Analytics team at Deutsche Postbank which now sadly has ceased to exist!

Of course I would like to thank Wim Schoutens for suggesting that I write a two volume book and putting me in touch with Peter Baker from Palgrave who convinced me that the new series "Financial Engineering Explained" needs books such as "Interest Rates Explained"!

I wish you, the reader, as much fun as I had when writing this book.

Joerg Kienitz, Bonn, March 2014

I MARKETS AND LINEAR PRODUCTS

1 Clearing, Collateral, Pricing

1.1 Introduction and objectives

In this first chapter we describe the infrastructure of the interest rate market with a focus on the changes after August 2007. Especially, we stress the fact that main parts of the market moved from OTC (*over the counter*) markets to regulated markets. We think that two developments, namely *clearing* and *collateralization* are essential to be covered first. Both mechanisms represent the new regulated markets we are currently facing. We wish to outline the rationale for moving to such regulations and we discuss the process in some detail. The new setting caused banks and financial institutions to restructure the business and set up new units to cope with the changes. After addressing the new markets we wish to outline a new pricing theory taking into account the new issues of collateralization and funding. We review the exposition from Bianchetti (2013). Finally, we briefly cover the pricing paradigm that the price of a financial contract can be seen as an expected value with respect to some pricing measure. Since we work in the interest rate markets we briefly outline the *Change of Numeraire Approach*. This will prove useful when dealing with options in Chapters 6 and 7.

1.2 Netting and collateral

In this section we consider a way of reducing the counterparty credit risk in a bilateral trade. Take two parties A and B involved in trading. We consider the value of all possible cash flows until some time T. Such cash flows can be for instance payments from fixed income securities. We suppose that at each (possible) cash flow date we are allowed to sum positive and negative flows making an overall (possible) cash flow. If this amount is positive for party A it has a positive exposure against B. Otherwise it has a negative exposure against B. This notion will be made rigorous later and is called the credit exposure. This is due to the fact that cash flows might not fully be paid if the counterparty defaults.

In the sequel we consider some methods to reduce this exposure. To this end we examine *Netting* and *Collateral Management*. The latter is also often called *Margening*.

We consider the concept of Netting illustrated in Figure 1.1. Counterparties A and B have different trades with each other. The trades lead to cash flows of different sizes from A to B and vice versa. This is depicted in the top part of Figure 1.1. If both parties have a netting agreement in place they can aggregate the cash flows and calculate one cash flow called the *net cash flow* and the procedure is called *netting of cash flows*. The net cash flow then determines the counterparty credit exposure between A and B.

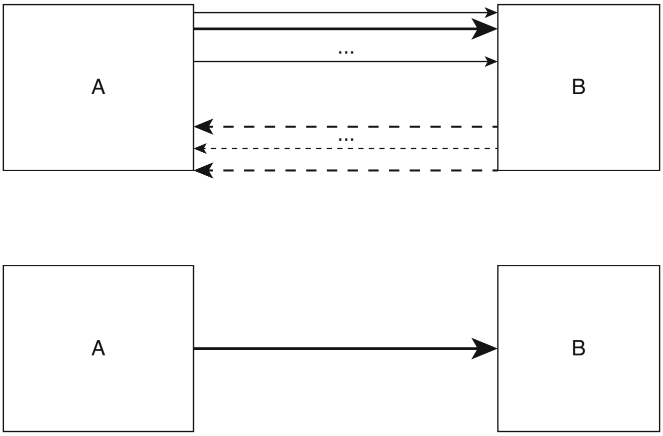


Figure 1.1 No netting applied (top) and netting applied (bottom)

Now, we can ask ourselves what can be a reasonable instrument to reduce the counterparty credit exposure. Both parties can enter into a *Collateral Agreement* and do the trades within this legal framework. In a collateral agreement it is specified how the counterparty credit exposure is reduced by posting financial instruments to the other party. Several nitty-gritty things have to be specified when setting up the collateral management process. This is the process of keeping track and controlling the collateral. We consider

- Netting agreement
  The netting agreement is at the heart of the collateral management process. In such an agreement it is specified which type of trades can be taken as a legal unit such that all trades in the unit are considered as one position. Often many different netting agreements exist. All trades falling into one such netting agreement are called the *netting set*.
- Uni- or bilateral collateral agreement
  Here it is specified whether only one or both parties take part in the collateral process. In interbank markets the collateral agreements are usually bilateral. If a firm trades with a bank often unilateral agreements are found.
- Collateral/Margening frequency
  This is the frequency at which margin calls take place. This means posting or calling collateral.
- (Minimum) Transfer amounts
  If the exposure has changed and a party should post more collateral the minimum transfer amount specifies at which level this happens. For instance let this amount be 50.000 units of currency and the exposure has changed such that 43.000 units in addition to the current amount of posted collateral would be necessary no further collateral is transferred. If the amount increases to for example 50.500 this amount has to be posted.
- Calculation agents
  A party doing the necessary calculations.

- Type of collateral
  This specifies the type of collateral and can be for instance cash in domestic or some foreign currency or bonds.
- Haircuts
  A haircut is a rate between 0% and 100% and it specifies a weight for collateral securities. Suppose we post a bond as collateral. The bond is worth 100 and we need to post 600 as collateral at a haircut of 75%. We need to post 8 bonds since $8 \cdot 100 \cdot 75\% = 600$.

All this is fixed in the *credit support annex* or *CSA* for short. There is a standard CSA from ISDA and a new standard CSA is in preparation at the time of writing. It is clear that many departments have to contribute to the collateral management process. Financial institutions establish the whole process using specific systems and policies.

The perfect CSA
Many papers and textbooks assume a *perfect CSA* to establish pricing formulas. This is a CSA where there is a zero initial margin, zero threshold and zero minimum transfer amount. The CSA is fully symmetric and the margining is continuous with instantaneous settlement. Furthermore, only cash collateral in the same currency of the trade is allowed and the collateral rate is a continuous one. There is no re-hypothecation.

In theory using this process the counterparty credit exposure is neutralized. However, this would need a continuous set-up which in practice cannot be established. Furthermore, legal issues and re-hypothecation are further issues.

Using collateral can enable a financial institution to more efficiently use credit lines and limits to do business with parties having a poor rating, reduce capital requirements and establish a more competitive pricing. After the many events leading to the crisis started in 2007 the collateral management process got much more involved. With respect to the collateral many banks had a deeper look into the process and the involved financial quantities. For instance many agreements allowed posting collateral of different types and different currencies. Thus, banks assessed the question what is the most reasonable collateral to post which means that the agreements involved embedded options. Furthermore, with the advent of the multi-curve set up outlined in the introduction of this book and considered in a rigorous way in a later chapter, the interest rate obtainable for the collateral is important for determining the correct discount factor. Thus, the collateral has impact on the correct pricing of a deal and, thus, the price of a derivative is with respect to a collateral agreement. This can lead to different prices for one and the same derivative traded with two parties with different collateral agreement. The mechanism called OIS discounting tries to overcome such issues. The risky factors of collateral management include:

- Market risk due to terms of the agreement
- Operational risk
- Liquidity and funding risk

First, the market risk depends on the specific collateral agreement. Since this agreement specifies important points such as transfer amounts and the length of the margening period it directly effects the counterparty credit exposure. Assume that some entity called A has a positive credit exposure against a party B. Now, the collateral agreement is such that collateral is posted every month. It might be the case that the market significantly moves and the exposure rises. If the interval is smaller, for instance every week the probability of big market movements in such an interval is reduced. The smaller the margening period becomes the smaller is the market risk. Mathematically, let $V(t)$ denote the value of the netting set under consideration at time t and Δ denote the margening period, then the exposure is the difference

$$max(V(T) - V(T+\Delta), 0)$$

The next relevant risk is the operational risk. First, to put collateral management in place, software has to be developed or parametrized to map the bank's work process into the software. After setting up the software it has to be maintained and properly used. Especially, the event of calling or posting collateral together with disputes about this process is very important. There might be errors due to in adequately set up systems or non-qualified personnel performing the process. This has to be taken into account and a continuous self-assessment and a loss data base due to operational risk has to be set up.

Finally, when entering into a trade we have to take care about the future development in terms of funding and collateral. Even if we enter into a trade at zero cost it might be necessary to fund the position. Typically, we have to fund future payments of cash flows and we have to be able to post collateral if the current market value turns out to be negative for us.

Some issues around collateral we consider in this book are the curve construction and the CVA/DVA/FVA calculation. All these topics are very timely and often banks fine tune and change the process since the material is complex and the developments have not reached maturity yet.

1.3 Clearing

The subprime crisis induced a credit and liquidity crisis starting in August 2007. As a reaction to this crisis governments have aimed to regulate the OTC derivatives market since they identified one of the main reasons for the crisis: the counterparty credit risk. In the wake of the crisis several strategies were proposed and published. Popular are the Dodd-Frank act and EMIR specifying and shaping the regulation of the market infrastructure. Both strategies suggest forcing the market participants into the clearing of OTC derivatives via central counterparties (CCP). This poses a full assessment of the trading practice and the IT architecture. In this section we briefly cover the idea of the CCP approach and analyze how the clearing process works.

1.3.1 What is central clearing?

First, we look into a market with participants having netting and collateral agreements in place. Figure 1.2 gives an overview of the situation. On the left side the situation is displayed as if no netting is allowed or no netting agreements of positions is in place. The right part of the figure displays bilateral netting. We face the situation that two counterparties apply the netting rules to calculate their bilateral exposure. For instance counterparty A faces the situation that it has a positive net exposure to counterparty C and D and a negative one to counterparty B. In this case appropriate collateral has to be determined and posted, respectively called from the counterparties.

Now, if we consider the same situation but instead with a central counterparty which is allowed to apply universal netting we have Figure 1.3.

In this case we see that the exposure of counterparty A has changed. It faces no exposure any more. This is due to the fact that the central counterparty can net all the outstanding cash flows from the deals of counterparties A to D.

Now, having illustrated one (maybe the most important) issue around central counterparties we start digging a little bit deeper into the basic role of CCPs. First,

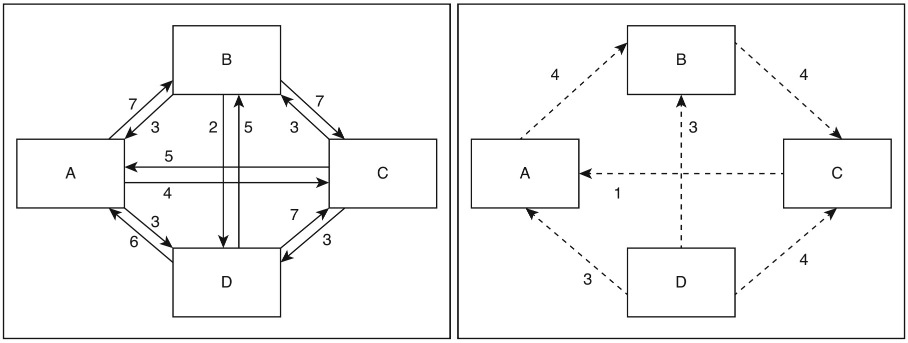


Figure 1.2 No netting (left) and bilateral netting (right)

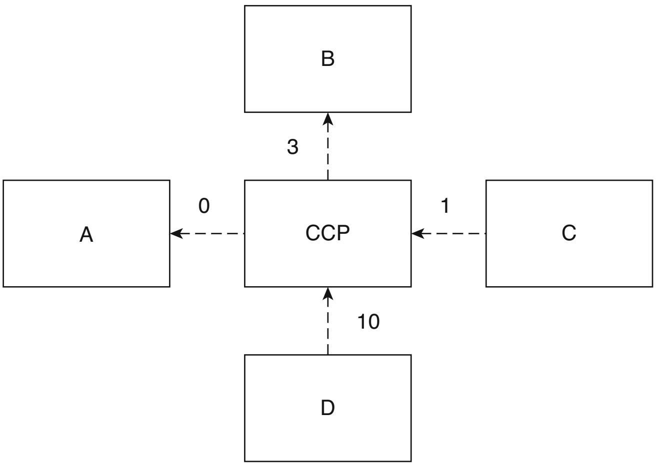


Figure 1.3 CCP and universal netting

we ask about the changes to existing deals and new deals with regards to involving a central counterparty. The legal term for that is *novation*. The involved parties transfer their rights and duties to the CCP. When a trade becomes novated the CCP has the counterparty risks that originally come from the involved parties. The CCP can now apply the universal netting illustrated in Figure 1.3. This can significantly reduce the overall exposure. Furthermore, there is a collateral agreement that applies to the credit exposure calculated on the universally netted position. Thus, such a mechanism can also significantly reduce the amount of collateral to be posted; see Figure 1.3.

Now, we examine the role of the CCP if it happens that a clearing member defaults. In the situation of only bilateral agreements in terms of netting and collateral a default can lead to significant break down of financial markets. This was observed when Lehman defaulted in 2008. In case of a default the CCP triggers an auction. In this event the CCP asks all clearing members to bid on the portfolio of the defaulted party. The portfolio can of course be split into different tranche slices. Then, each clearing member has to place a bid if they are interested in buying. Clearing members not willing to buy do not have to bid but they are fined for this. Furthermore, this set up has other drawbacks such as *moral hazard* or *adverse selection*.

Using this mechanism the CCP faces market risk. At the time of default the portfolio has a certain market value. After the default the CCP has to launch the auction and is exposed to changes in the portfolio value until the positions are assigned to other clearing members. To this end the CCP has to establish a cutting edge approach for risk management to quantify the exposure for this event. A CCP quantifies the risk in terms of a single number called the *initial margin*. Each clearing member is charged this amount and a charge for a reserve fund which then should cover the costs stemming from the described risk. Since all this has to be done to be able to handle one or more clearing members defaulting a sound risk management methodology has to be applied. This means there have to be descent methods to calculate the initial margin. Many techniques ranging from historical simulation to advanced expected shortfall models involving Non-Gaussian distributions are applied. It is outside the scope of this exposition to go into the details of such methods. An overview can be found in Table 1.1.

After all we have to remark that a CCP is meant and designed to reduce *systemic* risk of the financial markets system but it cannot remove it; neither can it remove the whole counterparty risk. What it can do is centralize the risk or transform the risk. All in all there might be risks left which are not thought of in such early stages of regulation.

Table 1.1 *Fees and margins for CCP*

Fee/Margin	Period
Initial margin	Per trade at beginning
Variation margin (Collateral)	Ongoing
Maintenance	Ongoing
Clearing fee	Per trade

1.3.2 Clearing members

Once a central counterparty is established clients wish to use this mechanism of clearing OTC derivatives. There are three different relationships of institutions to the CCP. First, market participants that are allowed to directly clear the trades with the CCP. We have

- Individual clearing members
- General clearing members

All other parties are Non-Clearing Members. Such parties can participate only through General Clearing Members. There are hard restrictions and rules for becoming a clearing member. The applicant has to demonstrate the ability to efficiently handle trades in terms of pricing and processing. To this end it has to set up an adequate IT infrastructure. To become a General Clearing Member additional effort is required since they have to set up a multi-tenant systems should be used to be able to act as a service provider for Non-Clearing Members. We include Figure 1.4 to illustrate the relationship of the three types of institutions. From a financial engineering perspective the correct pricing of instruments is most important. The procedure each applicant has to pass is the *firedrill*. In this procedure the applicant has to value a large portfolio supplied by the CCP. Thus, the corresponding data has to be processed such that the trades can be set up in a format required by the pricing system. To pass the test it is

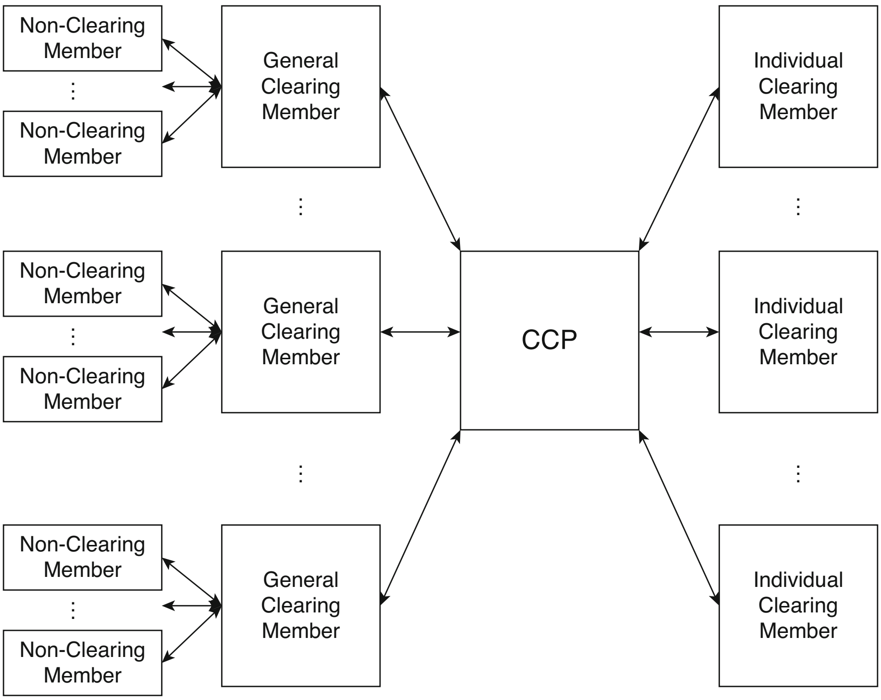


Figure 1.4 Bilateral trading and netting

necessary to value the portfolio with market data supplied by the CCP. The difference between the client valuation and that of the CCP should not be too big and is measured in terms of the Delta. If we are concerned with interest rate instruments a substitute for the Delta is the change of the present value if the yield curve moves by 1 basis point. This change is denoted as PV01, DV01 or bpv. Thus, the value has to be within $\left(NPV - \frac{PV01}{2}, NPV + \frac{PV01}{2}\right)$ of the portfolio. This test is mainly to ensure the ability for correct pricing and transferring the data into proprietary systems. The next step is than the evaluation using proprietary market data and this corresponds to a sharp evaluation of the portfolio. It must be designed in a way that the institution would buy the portfolio at this quoted level.

1.3.3 Resume

A market infrastructure with established CCPs has advantages in many ways. The most important aspect is the universal netting possibilities. Different counterparties trade but via novation the trade is done via the CCP. For the parties this offers an increasing flexibility and they do not have to worry about credit risk issues as it has to be done in bilateral trades. As described the losses are mutual ones since after a default an auction is initiated and the positions of the defaulted counterparty are transferred to the surviving clearing members. All in all the CCP ensures liquidity because of legal and operational efficiency making the transactions easy, transparent and secure using universal netting and established processes in the case of default of a clearing member. However, we should not forget that counterparty credit risk cannot fully be eliminated and there is a positive probability of a default of the CCP. Furthermore, the new infrastructure did mean and still means a new set up for market participants managing projects to prepare for the efficient use of CCPs.

On the CCPs side a well established risk management and collateral process has to be set up. This is to ensure that the initial margin and the variation margin can be calculated and processed reasonably in terms of speed and quality.

1.4 Counterparty credit risk

In this section we examine the counterparty credit risk and explain how this risk is quantified. The definition of *credit exposure* is not only important for understanding this type of risk but also serves as the basic notion when this type of risk is valued. Let us again consider two parties A and B involved in trading activities. In what follows we consider a netting set. With respect to this netting set we define the exposure. Let V^{NetSet} be the current value of the trades constituting the netting set. The *Spot Exposure* or *Current Exposure* for A with respect to B is the amount $\max\left(V^{\text{NetSet}}, 0\right)$. Thus, if we have a method in place to calculate the current value of the trades we can calculate the spot exposure by simply taking the maximum value with 0.

Put vice versa the negative exposure for A with respect to B is $\min\left(V^{\text{NetSet}}, 0\right)$. From B's perspective the positive exposure for A is the negative exposure and vice versa.

In practice and for risk management the current exposure is important but what really matters is to quantify the *future exposure*. Due to the development of the risk factors determining the value of the trades constituting the netting set the exposure varies. A positive exposure can increase or turn into a negative one. Here we see that getting a grip on the future exposure is not easy at all. First of all we need future scenarios on which we price per netting set. Not only that, after pricing we have to take into account our collateral management.

For illustration we consider only one trade that is very popular in the interest rate market. It is called a fixed against float interest rate swap. Figure 1.5 shows the present value of the instrument with respect to future scenarios of the yield curve. The spot exposure is positive. But we are interested in the exposure at a future time T. To this end we fix this time T and simulate future states of the world. For each simulation we price the swap and store the value. By repeating this exercise many times and grouping the stored values we obtain a histogram. This represents a distribution of all future outcomes. The future exposure is given by all positive values of the swap at time T. Thus, we have indicated the left part of the distribution as the part where A has an exposure against B while on the right part A has a negative exposure against B.

Figure 1.5 also reveals the nature of exposure. It is a distribution at different time horizons. Unlike calculating risk figures as VaR or Expected Shortfall we have to take into account the ageing of the trades from a given netting set and due to possible long time horizons dependencies are much more complex. Furthermore, it is interesting for exposure to be able to model it with respect to the real world measure and with respect to the risk neutral one. This makes it possible to use the exposure for analysis and planning as well as for pricing.

Furthermore, as already outlined risk mitigation has to be taken into account and simulation of applying the collateral mechanism has to be done. Now, we consider how to quantify the exposure.

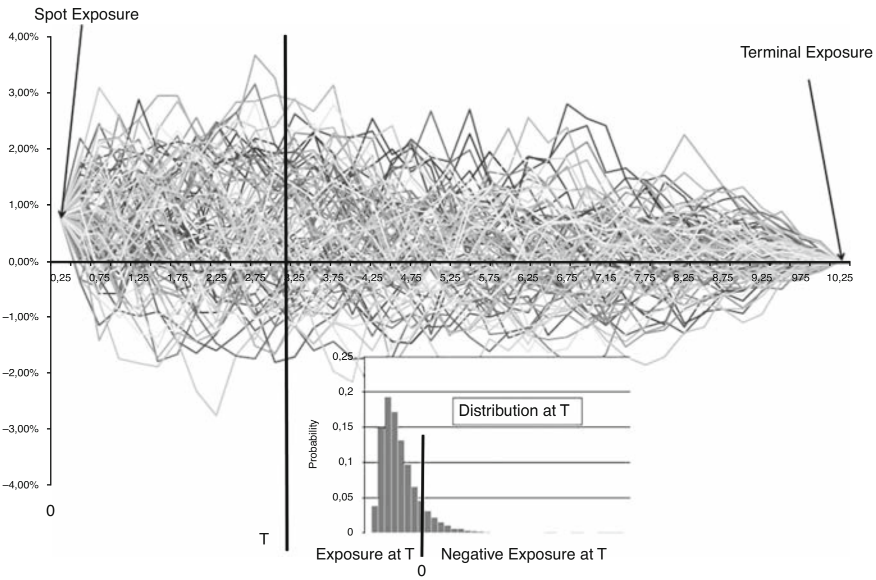


Figure 1.5 Spot and future exposure for and fixed against float interest rate swap

- Potential Future Exposure (PFE)
  This notion refers to the worst case exposure an entity faces at a future time T. The definition uses the notion of a quantile of a distribution. We fix a confidence level $p\%$, say $p\% = 99\%$. The PFE(T) then is the level of exposure that happens with a probability of not more than 1%. With respect to this definition it is quite similar to VaR but dependent on T. And we therefore have a whole path of PFE values. The PFE is shown in Figure 1.6.
- Expected Exposure (EE)
  The expected exposure is the expectation of the exposure. We have shown it in Figure 1.6. It is the expectation of the variable $\max\left(V^{\mathrm{NetSet}}, 0\right)$. Just to make sure it

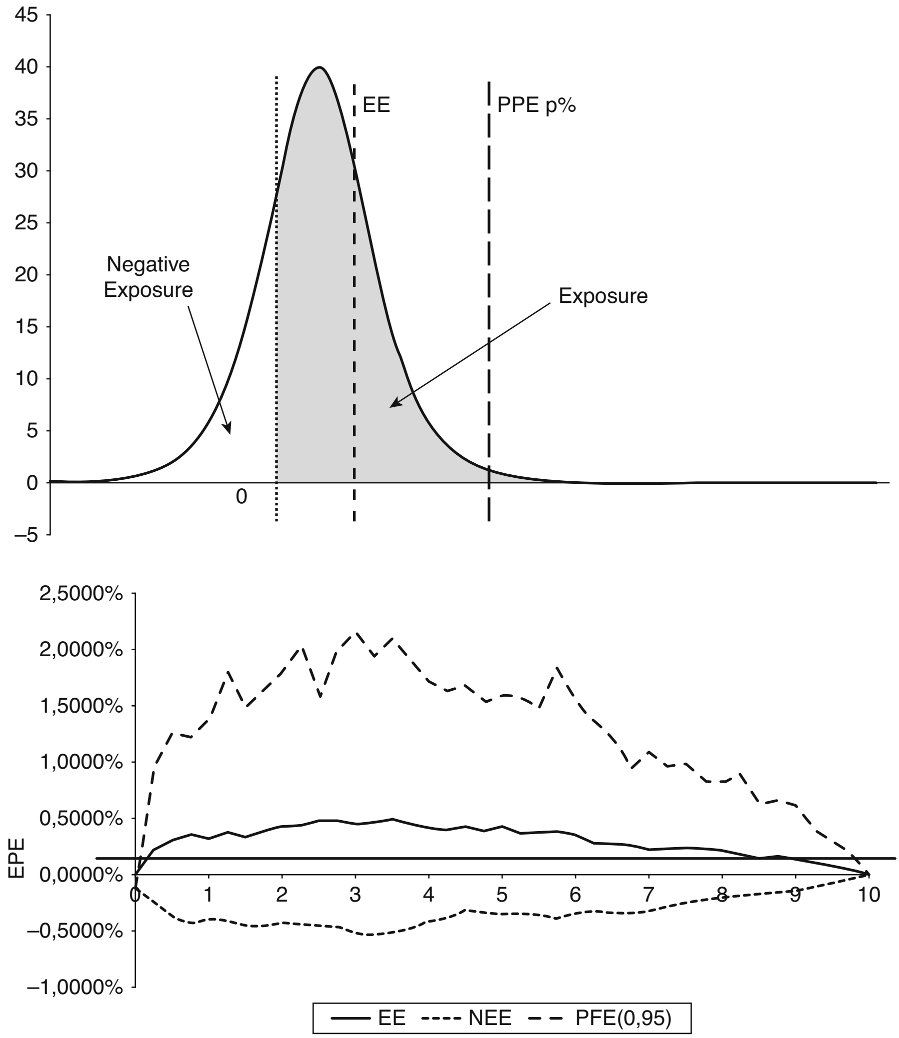


Figure 1.6 Quantifying exposure at a given time T (top) and on a time schedule (bottom)

is different to the expected value of all future outcomes which corresponds to the expectation of V^{NetSet}.

- Negative Expected Exposure (NEE)
  The negative expected exposure is the expectation of the negative exposure and, thus, the expectation of the variable $\min\left(V^{\text{NetSet}}, 0\right)$.
- Expected Positive Exposure (EPE)
  The expected positive exposure is also an expectation but now, we average over all calculated EE values over the time horizon we consider. This is illustrated in Figure 1.6 on the right. Suppose we have calculated the values $EE(T_i)$ for $i = 1, 2, \ldots, N$, then, we have

$$EPE = \frac{1}{N} \sum_{i=1}^{N} EE(T_i)$$

1.5 General pricing theory

Before we start to look into the valuation of financial instruments we need to introduce the new view on the valuation procedure. To this end we review the Black framework with funding and collateral. This has been introduced by Piterbarg (2010), Bianchetti (2013) or Ametrano and Bianchetti (2013).

The classical theory is the Black-Scholes-Merton framework. The pioneering work can be found in Black, F. and Scholes, M. (1973) and Merton, R. (1973). It was the main idea that under certain assumptions it is possible to dynamically replicate the price of an option by trading in the underlying. By doing so a pricing formula for Call and Put options was given. This is the famous Black-Scholes-Merton formula.

Let us consider the concept of replication underpinning the pricing in more detail. To this end let us assume that the market is constituted by some money market account and some assets called the basic instruments. Such assets cannot be replicated by selling or buying other assets. Suppose we have some option payoff p which is a function of one or several basic instruments. Such options are often not liquid and only traded over the counter. We wish to risk manage such instruments and determine their (fair) price. One fundamental assumption we take as an axiom is the *law of one price*, that is that there is no arbitrage. Any two financial instruments with identical payoffs should an identical price no matter what the future realizations of the market are.

There are two possible methods to deal with the risk factors determining the price of such options. We can try to neutralize risk, measured by sensitivities commonly known as *Greeks*, arising from changes in the market prices of the assets by buying or selling portions of the basic instruments, or we might try to find financial instruments such that a portfolio of these instruments mimics the payoff of the instrument under consideration. By mimicking we mean that the payoff at maturity for the option and the mimicking portfolio are the same regardless the value of the basic instruments.

The first method is known as to dynamically replicate the payoff p by buying or selling basic financial instruments. Hence it is called *dynamic replication*. On the one hand this trading strategy provides us with a recipe to rebalance the portfolio and on the other hand this trading strategy determines the price of the instrument by evaluating the strategy to replicate it dynamically.

The second method, creating a synthetic financial instrument to replicate the payoff by composing a portfolio of financial instruments and never alter it in the future, is known as *static replication*. The portfolio is the *replicating portfolio*. Applying the latter method no further trading is necessary and the price is determined by the price of the replicating portfolio.

We consider replication by describing both methods. The first one relies on several assumptions, for example the absence of transaction costs, to hold.

Let us consider dynamic replication in more detail. It depends on model assumptions, stylized facts about the market and on approximations applied due to trading restrictions like discrete trading. In the classic Black, Scholes and Merton case an idealized market has to fulfill the following issues which are not true in the real financial market:

- constant and known volatility
- constant and known carry rates
- no transaction costs
- frictionless and continuous markets
- liquid market
- complete market
- self financing trading strategy

The method of dynamic replication is based on using basic instruments only for trading. We assign some trading strategy trying to mitigate the changes arising from the options payoff. The position we hold by applying dynamic replication is linear in the basic assets and the money market account.

Once we accept the model assumptions including completeness and the no-arbitrage paradigm every option can be dynamically replicated. We focus on the main ideas and implications. In the Black, Scholes and Merton model the asset, S, is the basic security meaning it cannot be replicated by means of some other asset. The dynamic of S with respect to the risk neutral measure $\mathbb{Q}$ is given by geometric Brownian motion:

$$\begin{aligned} dS(t) &= S(t)rdt + \sigma S(t)dW(t) \\ S(0) &= s_0 \quad r,\sigma > 0 \end{aligned} \tag{1.1}$$

We consider a European Call option, that is an option paying $\max(S(T) - K, 0)$ at maturity time T. When entering into a Call option today we wish to consider the price changes if the asset price moves and the time passes. To this end we denote the Call option price by $C(S,t)$ where S is the current asset price and t the current point in time. Then,

$$\begin{aligned} C(S+\Delta S, t+\Delta t) = C(S,t) &+ \frac{\partial C}{\partial S}\Delta S \\ &+ \frac{\partial C}{\partial t}\Delta t + \frac{\partial^2 C}{\partial S^2}\frac{(\Delta S)^2}{2} + \dots \end{aligned} \tag{1.2}$$

The Black, Scholes and Merton result is that changes corresponding to S can immediately be canceled out by trading in S. The size of these trades is determined by $\frac{\partial C}{\partial S}$. The latter has to be done continuously.

If we assume that we have a long position in the option and short $\frac{\partial C}{\partial S}$ assets we are left with a profit and loss which is quadratic in the assets price change, the convexity. If we know the volatility, which is the case in the Black, Scholes and Merton model, the change in the profit and loss due to changes in the asset price is deterministic and we find that $\frac{\partial^2 C}{\partial S^2}\frac{(\Delta S)^2}{2} = \frac{\partial C}{\partial t}\Delta t$. Assuming a continuous stream of trades can be done the convexity consideration from above, Equation (1.2), holds and eventually leads to the following partial differential equation:

$$
\begin{aligned}
0 &= \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 C}{\partial S^2} - rC \qquad (1.3)\\
C(S,T) &= \max(S(T) - K, 0)\\
C(0,t) &= 0\\
C(S,t) &\to S \text{ as } S \to \infty
\end{aligned}
$$

Equation (1.3) is the Black, Scholes and Merton partial differential equation. This means in this model it is possible to dynamically hedge the option position with respect to changes in the underlying asset by shorting

$$
\Delta_C := \frac{\partial C}{\partial S} \qquad (1.4)
$$

of the asset. In reality however, continuous trading is not possible and furthermore, volatilities and carry rates are changing. The latter affect the strategy especially for options with long maturities.

Example: European Call Option

For a European call option with time to maturity T Equation (1.3) can be solved and the solution is given by a closed form expression in Equation (1.6), the well known Black-Scholes-Merton formula:

$$
C(S,t) = S(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2), \qquad (1.5)
$$

$$
d_{1/2} = \frac{\ln(S(t)/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad (1.6)
$$

In Equation (1.6) we assume no dividend yields and we denote the cumulative normal distribution by $\mathcal{N}$. The corresponding quantity from Equation (1.4) for this model applied to replicate the payoff using dynamic trading is $\Delta_{BS} = \mathcal{N}(d_1)$.

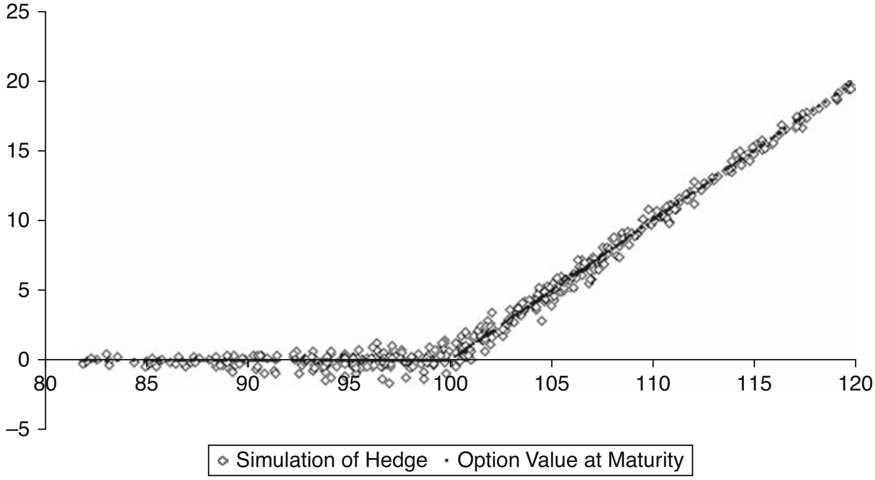


Figure 1.7 Monte Carlo simulation illustrating the performance of a weekly dynamic hedge for a European option with maturity one year. We assume a volatility of 10%, a riskless rate of 3%, spot price of 100 and a strike of 100

To illustrate the effects of discrete trading and the dependence on the parameters, volatility and carry rate we perform a Monte Carlo simulation and picture the simulated payoff using the dynamic replication strategy described above against the true payoff. Figure 1.7 shows the payoff of a European call option and a Monte Carlo simulation using dynamical hedging. We assume the maturity is one year and that trading takes place weekly.

We now consider another method to replicate the payoff p of an option. This method is completely different from dynamic replication since it does not need to adjust the position if the asset prices change.

At the heart of static replication lies the following relation:

$$p(S) = \underbrace{p(K)}_{M} + \underbrace{p'(K)(S-K)}_{A} + \underbrace{\int_K^\infty p''(K)(S-K)^+ dK}_{C} + \underbrace{\int_0^K p''(K)(K-S)^+ dK}_{P} \tag{1.7}$$

Equation (1.7) suggests that it is possible to replicate any option payoff p by taking positions in the money market account (M), the asset (A) and European call (C) and put options (P). In contrast to dynamic replication we rely on European call and put options to achieve a reasonable replication. Therefore, in contrast to dynamic replication other than the basic assets are needed. The effects of the convexity which in dynamic replication are canceled out through continuous trading are canceled out using European call and put options. The price of the replicated instrument heavily relies on the model applied to price the options involved. For example the options which are not at the money depend on how the smile or skew is modeled.

The composition of the portfolio does not change through time and therefore no dynamic rebalancing is necessary and thus only the initial transaction costs arise. Static replication was used long before the idea of dynamic hedging was introduced by Black, Scholes and Merton. Because static hedging uses options to cover the effects from convexity of the replicated instrument and not only the basic assets, it is a better hedge for delta, gamma and volatility than dynamic replication. For example static replication can be used for hedging a position in exotic options with plain vanilla options, for instance CMS Caps or Floors which can be statically replicated using European cash settled swaptions. This approach will be considered when we consider these market instruments. The method can in principle be applied to a variety of payoffs including path-dependent options. Barrier options or lookback options have been considered. Other popular examples of static replication are the *variance swap* or Digitals as Call spreads.

In practice, however, it is not always possible to hedge using static replication due to trading restrictions like liquidity or bid-ask spreads. Furthermore, the number of different options and notional amounts required can quickly become unmanageable. Therefore, a semi-static replication is used. Semi-static replication can be seen to use a manageable portfolio, around the strike for instance, and dynamically adjust the static replication if the market moves. In certain cases the transaction costs arising in dynamic replication can be reduced.

It has often been remarked that certain assumptions underlying the Black-Scholes-Merton approach are flawed and indeed there are now many more involved theories such as incomplete markets which generalize this theory. The developments following the credit and liquidity crisis also impact the general theory. In a talk at the Global Derivatives conference Bianchetti (2013) outlines an approach taking into account the new market mechanism. The classic theory for instance does not allow collateral, counterparty credit risk, dividends, repo or stochastic interest rates. The starting point is the standard theory and, then, further complexities are added. For instance collateral accounts and funding are added. Furthermore, the approach is generalized to include dividends, repo or partial collateral in multiple currencies. It is far beyond the scope of this book to even outline the approach. We, however, conclude that option pricing is still being done by computing an expected value by

$$V(t) = N(t)\mathbb{E}^{Q_N}\left[\left.\frac{V(T)}{N(T)}\right|\mathcal{F}_t\right] \tag{1.8}$$

In Equation (1.8) N is a financial asset called a scaling instrument. This scaling instrument fixes a scale to which any other financial instrument is measured to. For instance such a scaling is also done when converting a length from centimeters to inches. We could use both scales to measure a given length. In the classic theory this scale was the discount factor $\exp(-rT)$ where r was the rate offered on a save bank account until time T. This was from the good old days when a major bank could never fail and, thus, putting the money in such an account was at no risk. As in the classical theory it is possible to select different financial instruments as the scaling instruments.

1.5.1 Numeraire

A general scaling in finance is known as the *numeraire*. Especially in interest rate modeling the inspiring use of different numeraires is very important. There is a mechanism called *change of numeraire* that allows us to calculate the value stated in (1.8) with respect to another scale M. The change of numeraire mechanism says which type of scales are allowed and how the conversion should be done. Here we do not go into the details of this method but we give some hints of the application in later chapters. Popular numeraires are:

- Risk neutral measure
- T-Forward measure
- Spot measure
- Annuity measure

Numeraires and measures
Let us assume we have chosen a scale N that is a numeraire. Then, the pricing equation is

$$V(t) = N(t)\mathbb{E}^{Q_N}\left[\left.\frac{V(T)}{N(T)}\right|\mathcal{F}_t\right]$$

The change of numeraire method allows us to choose a numeraire M and modify the pricing equation to be

$$V(t) = M(t)\mathbb{E}^{Q_M}\left[\left.\frac{V(T)}{M(T)}\right|\mathcal{F}_t\right]$$

We see that not only the changes from $N(t)$ and $N(T)$ to $M(t)$ and $M(T)$ have been made but we also changed Q_N to Q_M. The latter are the corresponding measures. With respect to these measures we calculate the expected values. Thus, changing the numeraire means that we have to compute different quantities but also it means that we have to alter the measure and, thus, the distribution we use to calculate the expected values. How the actual mechanism works cannot be outlined here but it is very important to keep in mind that not only the financial instruments change but also the way we have to calculate the expectation to determine the value of a financial instrument.

1.6 Reading list

This chapter described several changes which have taken place in the financial markets. Especially, the consideration of counterparty credit risk has to be mentioned here. A text book covering the new market practice as well as a thorough compendium on the subject is Gregory (2012). The author covers the definition of counterparty credit risk in detail and gives the mathematical underpinning as well as many practical and relevant examples. Another useful source for the theory of credit risk include Brigo, Pallavicini and Torresetti (2012), Bilecki, Brigo and Patras (2011) and Brigo, Morini and Pallavicini (2013). This series of books does not only cover the aspects of counterparty credit risk as outlined in this chapter but many many more. It analyze the credit crisis and the financial instruments for trading credit as well as mathematical models for covering all aspects of credit risk modeling.

Many researchers have now considered this new market practice and proposed explanations for this new modeling perspective. Some very readable texts include Morini (2009). The more mathematically minded reader may find the papers on the theory of incorporating funding and special collateral agreements into account very useful; see Piterbarg (2010) where the effect on derivatives pricing of funding cost in a delta hedging strategy is considered, Piterbarg (2012) develops a model of an economy without a risk free rate and with all assets traded on a collateralized basis and, finally, Fujii, Shimada and Takahashi (2010a) where the impact of collateralization on derivatives pricing is considered in detail.

In Bianchetti (2013) an analogue to the Black-Scholes pricing theory is developed. This new approach takes the new market practice into account and incorporates different rates for collateralization and funding. An approach relying on partial differential equations is considered in Burgard and Kjaer (2011). An introduction to most of the concepts can be found in the book Fries (2006). The author has based recent research, see Fries (2010a) and Fries (2010), on this monograph. Finally, we recommend Andersen and Piterbarg (2010a), chapter 1 and Brigo and Mercurio (2006) for the details on numeraires and derivatives pricing.

2 Rates

2.1 Introduction and objectives

The goal of this chapter is to introduce some fundamental definitions and market conventions. As the financial markets developed participants had to agree on certain notions. For instance they had to agree on exchange schedule of payments or on the length of payment schedules. In this chapter we start with the definition of a daycount and rolling conventions. Then, we outline the basic definitions of rates which are the basic building blocks of financial instruments. We reference these definitions in the later chapters of this book. The basic definitions include *discount factors*, *zero rates*, *forward rates* and *par rates*. In real banking and trading applications such rates are inferred from *interest rate curves*. We define the notion of a curve and lay the foundations for further considerations in later chapters such as interpolation issues or curve building from given market quotes.

Before we start this chapter we consider some vocabulary on trades and dates. There is the *Settlement date* which is the date on which a trade settles and the actual cash flow is paid or asset is transferred. The settlement date is not the same as the *Value date* which is a specific date on which a fluctuating asset's price is determined. Another important date is the *Fixing date*. At this date and time a rates value is fixed. Usually this rate is a floating rate such as a LIBOR or a swap rate. Finally, we have the *Trade date* and the *Spot date*. The trade date is the day on which a trade between parties is agreed and the spot date is the settlement date seen from the trade date, respectively today.

2.2 Daycount, rolling and other conventions

In this first section we collect market conventions for counting the year fraction, the rolling and other useful conventions for handling interest rates and derivatives. Most of the material can be found in Henrad (2012b). A prototypical fixed income product consists of a set of dates and cash flows belonging to the dates. For instance we take each date of this schedule as to be specified and the cash flow has to be determined. If the rate is not fixed at the beginning as it is for a standard fixed rate bond we have to include the fixing date and the value of the rate determining the cash flow. In the above schedule the rate is not constant over time and is fixed according to the schedule given on the left column of Table 2.1. We assume today is 30.07.2013. Then, the rate in the first row is fixed on 28.07.2013 but all the other rates are not and we printed the rates and the cash flows in italic. Furthermore, we also wish to know today's value of the future cash flow schedule. To this end we also need a mechanism of projecting the cash

Table 2.1 *Example: cash flow table*

Fixing date	Start date	End date	Payment date	Coverage	Rate	Cash flow
29.07.2013	31.07.2014	31.01.2014	31.07.2014	1	3.123%	31.230
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29.01.2017	31.01.2017	31.07.2017	31.07.2017	1	4.514%	45.150

flows paid in the future to a given date t. Of utmost importance is of course $t=$today. In this section we outline how to set the dates schedule and determine the year fraction between two dates. This is important determining the cash flow. For example we earn 3% interest on 1.000 Euro over 3 months.

- Trade date
  The trade date is the day when two parties enter into a deal. Thus, the parties agree on the conditions and obligations of a transaction.
- Settlement date
  At this date the transaction is settled. Often the settlement date is some days after the trade date. This date difference is called the *spot lag*.
- Fixing date
  If the contract involves not yet fixed interest rates the fixing dates of the contract specify when the corresponding rates are fixed.
- Value date
  The value date is the day when the value of a product is determined first.
- Payment date
  At payment date the payment of the interest rate contract has to be made.

2.2.1 Daycount

Let us consider to dates $T_1 < T_2$. $T_i = d_i.m_i.y_i$, $i = 1,2$. The day count convention is used to determine the year fraction between the dates T_1 and T_2. To summarize several measures for the year fraction in the sequel let $D_2 - D_1$ be the number of days between the dates T_1 and T_2.

30/360

$$\tau^{30/360}_{T_1,T_2} := \frac{360(y_2 - y_2) + 30(m_2 - m_1 - 1) + (d_2 - d_1)}{360} \tag{2.1}$$

The 30/360 day count has many different versions. We give the 30E/360 and the 30/360US versions in the sequel.

30E/360

$$\tau^{30e/360}_{T_1,T_2} := \frac{360(y_2 - y_2) + 30(m_2 - m_1) + (\tilde{d}_2 - \tilde{d}_1)}{360} \tag{2.2}$$

with $\tilde{d}_i = d_i 1_{\{d_i=1,\dots,30\}} + 30 \cdot 1_{\{d_i=31\}}$.

30/360US

The 30/360US day count convention is also called the bond basis.

$$\tau^{30/360us}_{T_1,T_2} := \frac{360(y_2 - y_2) + 30(m_2 - m_1) + (\tilde{d}_2 - \tilde{d}_1)}{360} \tag{2.3}$$

with $\tilde{d}_1 = d_1 \cdot 1_{\{d_1 \leq 30\}} + 30 \cdot 1_{\{d_1 = 31\}}$ and $\tilde{d}_2 = d_2 \cdot 1_{\{d_2 \leq 30\}} + 30 \cdot 1_{\{d_2 = 31\}} 1_{\{d_1 = 30,31\}} + 31 \cdot 1_{\{d_2 \leq 29, d_2 = 31\}}$.

ACT/360

The actual 360 (ACT/360) day count convention is the standard money market day count convention. To this we write both dates as the number of days D_i

$$\tau^{ACT/360}_{T_1,T_2} := \frac{D_2 - D_1}{360} \tag{2.4}$$

ACT/365

Sometime the actual 365 (ACT/365) day count convention is called the Englisch money market convention. The formula is simply given by:

$$\tau^{ACT/365}_{T_1,T_2} := \frac{D_2 - D_1}{365} \tag{2.5}$$

ACT/ACT

For the actual actual (ACT/ACT) day count convention we consider $[T_1, T_2)$, thus, we include the first day T_1 but not the last date T_2.

$$\tau^{ACT/ACT}_{T_1,T_2} := \frac{D_2 - D_1}{365} 1_{\{\text{no leap year}\}} + \frac{D_2 - D_1}{366} 1_{\{\text{leap year}\}} \tag{2.6}$$

For the daycount convention involving one ACT it is necessary to determine if any year is a leap year. This can be determined by dividing it first by 4. If this is not possible without a remainder, then it is no leap year. Furthermore, if there is a remainder we divide it by 100. If this works without a remainder it is a leap year. Finally, to get all the rest of possible years we divide by 400 and if no remainder is left it is no leap year. To illustrate we take the years 1992, 2000 and 1900. The rule given above leads to the result, that the first two are leap years while 1900 is no leap year.

In the sequel we see that each market adopts its own daycount convention. We have included an Excel sheet where the reader can experiment with all the different day count conventions.

In the sequel for given time point T_j and T_{j+1} we use the notation $\tau_{j,j+1}$ instead of $\tau_{T_j,T_{j+1}}$.

Example
As an example we take the dates $T_1 = 01.02.2008$ and $T_2 = 31.05.2009$. We get Table 2.2 for the daycount conventions defined in this section

Table 2.2 *Calculated year fractions for $T_1 = 01.02.2008$ and $T_2 = 31.05.2009$ using the daycount conventions of this section. We consider the first 5 digits*

Daycount	Year fraction
30/360	1.33056
30E/360	1.33056
30/360 US	1.33333
ACT/360	1.34722
ACT/365	1.32877
ACT/ACT	1.32626

2.3 Rates

Suppose we we wish to invest a certain amount of currency at time t until time T and denote it by $D(t, T)$. We suppose the year fraction is τ_{T_1,T_2}. The amount accrues continuously over time and gives one unit of currency at time T. The following equation has to hold:

$$D(t,T) \cdot \exp(-R_1(t,T)\tau_{t,T}) = 1$$

Thus, we define the rate R_1 by setting

$$R_1(t,T) := \frac{\ln(D(t,T))}{\tau_{t,T}} \tag{2.7}$$

We call R_1 the *zero rate* or *continuously compounded rate.*

Markets use different compounding conventions. Next, we consider the rate leading to one unit of currency but solving the equation:

$$D(t,T) \cdot (1 + \tau_{t,T} R_2(t,T)) = 1$$

Thus, we define the rate R_2 by setting

$$R_2(t,T) := \frac{1 - D(t,T)}{\tau_{t,T} D(t,T)} \tag{2.8}$$

This rate is known as the *simply compounded rate.* Very important rates are simply compounded rates. For instance the XIBOR belong to this class of rates.

Now, we turn to rates solving

$$D(t,T)\cdot\left(1+\frac{R_4(t,T)}{k}\right)^{k\tau_{t,T}}=1$$

The special solution for $k=1$ is given by

$$R_3(t,T):=\frac{1}{D(t,T)^{\frac{1}{\tau_{t,T}}}}-1 \tag{2.9}$$

and the general solution is

$$R_4(t,T):=\frac{k}{D(t,T)^{\frac{1}{k\tau_{t,T}}}}-k \tag{2.10}$$

We call R_3 the *annual spot rate* and and R_4 the *k-times per year spot rate*. In the bond markets such rates are used.

Example
We illustrate rates in Table 2.3.

Table 2.3 *Examples for the rates $R_1,\dots,R_4$ and the corresponding discount factors. In this example we suppose $t=0$, $T=1,2$ and the year fraction is equal to* 1*, resp.* 2*. For the k time per year compounding we choose $k=4$*

Rate	$D(0,1)$	$D(0,2)$
$R_1=3\%$	$\exp(-3\%)\approx 0.9970045$	$\exp(-2\cdot 3\%)\approx 0.99401796$
$R_2=3\%$	$\frac{1}{1+3\%}\approx 0.97087379$	$\frac{1}{1+2\cdot 3\%}\approx 0.94339623$
$R_3=3\%$	$\frac{1}{1+3\%}\approx 0.97087379$	$\frac{1}{(1+3\%)^2}\approx 0.94259591$
$R_4=3\%$	$\left(\frac{4}{1+\frac{3\%}{4}}\right)^{-4}\approx 0.97055417$	$\left(\frac{4}{1+\frac{3\%}{4}}\right)^{-2\cdot 4}\approx 0.9419754$

2.3.1 Roll conventions – business dates

In this short subsection we consider the case of rolling out a cash flow for some financial instrument and one of the rolled out dates falls on a non-business day. There are rules on how to move this date to another one. This new date is called the adjusted date. To fix notation, we call the date on which we apply the roll convention, respectively business date adjustment T. The adjusted date is T_a.

Following
The *following* roll convention defines T_a by

$$T_a := \min_t \{t | t > T \text{ and t is a business day}\}$$

Preceding
The *preceding* roll convention does not move the date T to a date occurring after T but to one before T. We define T_a by

$$T_a := \max_t \{t | t < T \text{ and t is a business day}\}$$

Modified following
The *modified following* roll convention is a little bit more involved. It moves the date falling on a non-business date to the next available business date. But if the next business date falls into the next month we instead move T according to the preceding convention. The modified following roll convention is widely applied in interest rate modeling.

To give the formula defining T_a we need to define the function m. For a date $dd.mm.yyyy$ the function m maps the date to the month, that is $m(dd.mm.yyyy) = mm$. Then, T_a is given by

$$T_a := \begin{cases} \min_t\{t | t > T \text{ and t is a business day and}\} & \text{if } m(T) = m(T_a) \\ \max_t\{t | t < T \text{ and t is a business day}\} & \text{if } m(T) < m(T_a) \end{cases}$$

EOM
The *end of month* convention moves the date to the final business day in the month, that is we consider

$$T_a := \max\{t | t \text{ is business day and } m(t) = m(T)\}$$

LIBOR, EURIBOR, EONIA,...

There are several specific rates underlying financial contracts. We consider two rates classes, *LIBOR* and *EURIBOR*, here. Furthermore, we look into the definition of overnight (OIS) rates in some detail.

LIBOR
This is the **L**ondon **I**nterbank **O**ffered **R**ate. It was first published in 1986 and was sponsored by the British Banker Association (BBA). LIBOR rates are often referred to in ISDA standards for OTC transactions. The fixing mechanism runs as follows: The BBA polls a panel of banks for rate fixing on several maturities (1d–12M). The question to answer is for the rate the bank would be able to borrow in a reasonable market size. A weighted average is calculated by discarding the 25% highest and lowest quotes. From the remaining quotes the arithmetic average is determined and then

Table 2.4 *Currencies and conventions*

Cur	Name	Maturities	Day count	Spot
USD	LIBOR	ON - 12M	ACT/360	2
EUR	LIBOR	ON - 12M	ACT/360	2
EUR	EURIBOR	1W - 12M	ACT/360	2
GBP	LIBOR	ON - 12M	ACT/365	0
JPY	LIBOR	ON - 12M	ACT/360	2
JPY	Tibor	1W - 12M	ACT/365	2
CHF	LIBOR	ON - 12M	ACT/360	2

published. The calculation agent is Thomson Reuters. Currencies with corresponding LIBOR rates are USD, EUR, GBP, JPY, CHF, CAD, AUD, DKK, SEK, NZD. We have to remark that only one currency is used. It would not be possible to borrow in currency A and transfer to the currency under consideration, for instance to determine 3M EUR LIBOR it would not be possible to borrow in NZD and transfer this to EUR and achieve a lower quote. Per currency there are 8, 12 or 16 contributors.

If we refer to the LIBOR rate but we do not specify the currency the term *XIBOR* is used.

In July 2012 there were rumors about manipulation of the LIBOR rates by some panel banks. We do not comment on this LIBOR scandal here but in 2013 the British government decided that this rate will be calculated by the NYSE Euronext from 2014 onwards. At the time of writing there is suspicion of further rates being manipulated. Again, we do not take part in any speculation.

EURIBOR

The **E**uro **I**nterbank **O**ffered **R**ate was first published in 1998 and is sponsored by the European Banking Federation (EBF) and the Financial Markets Association (ACI). The fixing mechanism corresponds basically to LIBOR but now the poll is done for all prime banks to other prime banks for a deposit on a given time horizon for the EUR zone. In contrast to LIBOR the highest and lowest 15% are discarded and the arithmetic average over the remaining quotes is taken. All rates are for spot (today + 2 bd) and are annuliaed rates according to ACT/360. Again, the calculation agent is Thomson Reuters. A summary for different currencies is shown in Table 2.4.

OIS

OIS or overnight index rates are related to interbank lending. The lending horizon is just one day. The indexes we cover are for overnight loans and are calculated on a weighted average basis. The OIS rates per currency are displayed in Table 2.5. Here are some examples for OIS:

- EONIA
  This is the **E**uro **O**ver **N**ight **I**ndex **A**verage. To compute this rate the unsecured overnight lending in the interbank market is considered. It is a weighted average of the rates of the contributing banks calculated by the ECB. At around 19:00 CET the rate is published. We face a day count convention of ACT/360.

Table 2.5 *OIS rates for different currencies*

Currency	OIS rate	Currency	OIS rate
AUD	AONIA	HKD	HONIX
CAD	CORRA	JPY	TONAR
CHF	SARON	NZD	NZIONA
DKK	DKKOIS	SEK	SIOR
EUR	EONIA	USD	FED FUNDS
GBP	SONIA	ZAR	SAFEX O/N deposit

- USD OIS
  In USD markets the role of the OIS rate is played by the fed funds rate. Such rates are averages of trades done by major banks. The rate is calculated by the Federal Reserve Bank using revised rates by brokers. The day count convention of the rate is ACT/365.
- SONIA
  This is the **S**terling **O**ver **N**ight **I**ndex **A**verage.
- TONAR
  This is the **T**okyo **O**ver **N**ight **I**ndex **A**verage. It is the overnight rate for the JPY market. Like the other rates it is also a weighted average of unsecured overnight cash transactions between banks. The Bank of Japan calculates the rate and publishes it using a day count convention of ACT/365.
- SARON
  For the Swiss market this index is the **S**wiss **O**ver **N**ight **I**ndex **A**verage.

Comparison
The XIBOR rates are rates on unsecured funding and are expectations, respectively views of the panel banks and, thus, include credit and/or liquidity risk views. Thus, the rates are not risk free rates. The amount of credit, respectively liquidity risk to bear depends on the period of the rate. A rule of thumb is the longer the period the higher the risk. We reproduce a table from Bianchetti (2010) Let us further comment on the involved risks. To this end we take two contracts both covering a period of 6M. We first lend some amount of money for 3M and after the three months again for 3M. This strategy is called short strategy. The rate at which we lend is the 3M XIBOR rate. For the long strategy the contract is based on the 6M XIBOR. We again lend the same amount of money but now for 6M.

First, we consider the credit risk. If we lend to a bank at 6M XIBOR rate we get back the money after 6M plus the interest payment if the borrower has not defaulted. For the short contract we get our money plus interest back after 3M if the borrower has not defaulted. Then, we can decide if we again lend the money to the borrow or choose another with a better credit outlook. In this way we maximize the chance of getting our money back after another 3M. To this end the long contract carries a higher credit risk and, thus, should compensate for this. Therefore, it is expected that the 6M lending should earn more interest than the short contract solution.

Next we consider the liquidity risk. The short strategy might be preferable from a liquidity risk perspective. If after a 3M period we need liquidity to finance some of

Table 2.6 *Conventions and risk for LIBOR, EURIBOR and EONIA*

	LIBOR	EURIBOR	EONIA
Market	London Interbank	Euro Interbank	Euro Interbank
Side	Offer	Offer	Mid
Rate Quote	ACT/365, T+0	ACT/360, T+2	ACT/360, T+0
Maturities	1d-12M	1w,..,1M,...,12M	1d
Publication	12:30 CET	11:00 CET	6:45–7:00 CET
Panel Banks	8–12–16	39 (15 EU, 4 Non-EU countries)	39 (15 EU, 4 Non-EU countries)
Calc Agent	Thomson Reuters	Thomson Reuters	ECB
Transaction Based	False	False	True
Counterparty Risk	True	True	False, resp. Negligible
Liquidity Risk	True	True	False, resp. Negligible
Tenor basis	True	True	False

our business we could use at least the amount we lent plus the earned interest whereas this amount of money is available after 6M time for the long contract. However, the abdication of liquidity should somehow payoff and, thus, the 6M rate should lead to a higher return than the two 3M contracts. We summarize conventions and risk in Table 2.6.

2.3.2 Discount factors

Let us take some time in the future and call it T and consider a contract paying one unit of currency at T. The contract value at time t, $0 \leq t \leq T$ is $D(t,T)$. We have that $D(T,T) = 1$.

This is the standard definition of a *zero coupon bond* or discount factor. In the above definition we implicitly assumed that our counterparty which owes us one unit of currency at time T cannot default. That is we are guaranteed to get one unit of currency at T. Taking into account the exposition in we face the fact that even TIER I banks have a positive probability of default.

To this end the value of the zero coupon bond depends on the counterparty we do the deal with and, thus, we have to take into account the credit quality of the counterparty. On the other hand if we owe a counterparty one unit of currency in the future this party has to take into account our default probability. To this end we need either to include the probability of default into our pricing considerations or we have to invoke a certain mechanism of getting the money in case this event happens. We already covered the mechanism in Chapter 1 but other implications of counterparty credit risk are considered in Chapter 8.

2.3.3 Forward rates

We have introduced several rates at the beginning of this section, Equations (2.7)–(2.10). For given t and two different dates T_1 and T_2 we consider two such rates

$R(t,T_1)$ and $R(t,T_2)$. Now, we define a rate $\bar{R}(t,T_1,T_2)$ that satisfies the Equation (2.11):

$$R(t,T_1)\bar{R}(t,T_1,T_2) = R(t,T_2) \tag{2.11}$$

The solution is given by (2.12):

$$\bar{R}(t,T_1,T_2) = \frac{1}{\tau_{T_1,T_2}}\left(\frac{D(t,T_1)}{D(t,T_2)} - 1\right) \tag{2.12}$$

We take Equation (2.12) as a definition and call $\bar{R}(t,T_1,T_2)$ the *forward rate* between T_1 and T_2 seen at time t. The calculation is linked to the specific rates from Equations (2.7)–(2.10).

2.3.4 Other rates

We only cover the basic money market rates here. There are many more rates referring to special fixed income instruments. When we introduce the instruments in the following chapters we extend our definition of rates accordingly.

2.3.5 Interest rate curves

To fix ideas we consider two (discrete) sets of points

$$\mathcal{T} = \{t_1, t_2, \dots, t_N\}, \quad t_1 < t_2, \dots < t_N, \tag{2.13}$$
$$\mathcal{X} = \{x_1, x_2, \dots, x_N\} \tag{2.14}$$

The size of the sets $\mathcal{T}$ and $\mathcal{X}$ is the same – N in our case. Furthermore, we assume that the set Equation (2.13) is ordered since it represents the time points. The set Equation (2.14) contains the values at the specific times. We are in a position to define a (discrete) curve as the set

$$\mathcal{C}_d = \{\mathcal{T}, \mathcal{X}\} \tag{2.15}$$

In practical applications we also face the problem of determining values for time points t with $t \notin \mathcal{T}$, $t_1 \leq t \leq t_N$. To this end we need to specify how we wish to obtain such values by *interpolation.*

From the discrete curve $\mathcal{C}_d$, Equation (2.15), we wish to construct a continuous curve

$$\mathcal{C}_c : [t_1, t_N] \to \mathbb{R}, \text{ s.t. } \mathcal{C}_c(t) = \mathcal{C}_d(t), t \in \mathcal{T} \tag{2.16}$$

To this end we have to assign values for all $t \in [t_1, t_N] \backslash \mathcal{T}$. In Chapter 5 we construct a function g which is called an *interpolator* such that

$$\mathcal{C}_c(t) := g(t, \mathcal{C}_d)$$

The function is either a single function (*global interpolator*) or a structure of many different functions (*local interpolator*) appropriately glued together. For local interpolators let $m(t) := \min\{i | t_i \leq t \leq t_{i+1}, t_i, t_{i+1} \in \mathcal{T}\}$. Each local interpolating function is only used to determine values for $\{x_{m(t)}, x_{m(t)+1}\}$ or $\{x_{m(t)-1}, x_{m(t)}, x_{m(t)+1}\}$.

In our applications we use different curves for calculating discount factors, $\mathcal{C}_d$ or forward rates, $\mathcal{C}_f$.

2.4 Reading list

A very readable introduction to the subject which furthermore explains the main interest rate markets is Tuckman and Serrat (2012). This text also has many examples of special financial products and the calculations of rates. This book is very practical and, thus, ideal for all people working on the subject. Many of the new developments such as multi-curves and the new paradigm of modeling rates can be found in Henrad (2012b). This is a very detailed exposition on the subject. The paper Morini (2009) explores not only the mathematical part but explains the new setting in economical concepts and motivate the new developments. A recent book tackling the new setting and summarizing the development in a condensed form is Kenyon and Stamm (2012). For the mathematical concepts the main references are Brigo and Mercurio (2006) and Andersen and Piterbarg (2010a). Both textbooks provide many of the nitty gritty things involved in the mathematical modeling of interest rates. A rigorous modeling perspective for LIBOR rates is given in the paper Mercurio, F. (2009) and also in the book Andersen and Piterbarg (2010b), chapter 15. The model described here is a Libor Market incorporating discount and forward rates in a consistent way.

3 Markets and Products – Deposits, Bonds, Futures, Repo

3.1 Introduction and objectives

Throughout this chapter we assume that we have a continuous curve for deriving forwards and discount factors available. In this chapter we consider deposit rates, bonds and repurchasement agreements. We start by reconsidering the deposit rates introduced in Chapter 2. Especially, we give market quotes for the different rates. In Section 3.4 we introduce fixed rate bonds as well as floating rate notes. Both financial instruments play an important role in the fixed income markets. In addition we introduce well known risk measures for bonds which will be generalized to other financial instruments later in the book. The risk measures we consider are basis point value, duration and convexity. Finally, Section 3.5 concludes this chapter by introducing and explaining repurchasement agreements.

3.2 Deposits

We have already considered deposit rates to some extent in Chapter 2. We outlined how such rates are determined and we gave an overview of the corresponding market conventions.

3.2.1 Market quotes

The market quotes for EURIBOR are in terms of the bid and the ask price. Table 3.1 gives an example for market data from 01.08.2013.

3.3 Futures

Futures are contracts listed on exchanges. Typically we find short term interest rate futures or bond futures. Let us consider short term interest rate futures, *STIR*, first. Such contract are nothing but derivatives on the underlying short term interest rate.

On some settlement date a STIR future is an obligation to buy or sell a pre-defined amount of a short term interest rate at a given price. The value is settled in cash. The terms of the futures contract are standardized. For instance for EURIBOR futures the contract size is EUR 1.000.000. Futures are liquid instruments and trade as quarterly contracts each March, June, September and December.

Table 3.1 *EURIBOR bid/ask quotes as of 01.08.2013*

EURIBOR rate	Bid	Ask	Rate	Bid	Ask
EURON	0.05	0.15	EUR4M	0.16	0.26
EURTN	0.05	0.10	EUR5M	0.19	0.29
EURSN	0.05	0.10	EUR6M	0.19	0.39
EURSW	0.06	0.11	EUR7M	0.27	0.37
EUR2W	0.06	0.12	EUR8M	0.30	0.40
EUR3W	0.06	0.12	EUR9M	0.33	0.43
EUR1M	0.03	0.13	EUR10M	0.34	0.49
EUR2M	0.08	0.18	EUR11M	0.38	0.53
EUR3M	0.13	0.23	EUR1Y	0.41	0.57

Such instruments follow a simple mechanism. If rates rise the price of the future falls and vice versa. If F denotes the futures rate the quotation rule is

$$V^{\text{Future}} = 100 - (F \cdot 100) \tag{3.1}$$

Using Equation (3.1) the value of one contract is $CS \cdot (100 - \Delta(100 - F))$ with CS denoting the contract size and Δ denoting the length of the rates period. Let us consider what a change of one basis point means to a futures contract. First, we observe that the price given by Equation (3.1) changes by one basis point. It is different for the value of one contract. Assume that $CS = 10000$ and $\Delta = 0.25$, then the change would be EUR 25. The minimum price movement is often called the *tick value*. There is an interesting relation between the forward rate and the future rate. Since the future is daily marked to market and fixed by the end of each day it results in margin cash flows each day. This *margining* is not used for trading forward rates. It is only calculated at the end of a contract. A general observation is that forward rates are lower than future rates and, indeed, the rates can be related to each other and we have:

$$r_{\text{Future}} = r_{\text{Fwd}} - \text{convexity} \tag{3.2}$$

The final term in Equation (3.2) is often given in terms of volatility and we have

$$\text{convexity} = \frac{1}{2}\sigma^2 (T_2 - T_1)$$

with T_1 being the maturity of the future and T_2 the maturity of the underlying rate. The volatility has to be specified which is usually done with respect to some interest rate model. A reason why futures are popular is that they are off-balance sheet instruments. This means if one performs cash money market trades physical settlement is part of the trade. Thus, this creates on-balance sheet exposures and may demand real cash reserves for accounting for defaults. This is not the case for futures. The nominal value of the trade is not at risk at any time.

The same mechanism applies for bond futures. The underlying in this case is not a rate but a bond. Here it might be some governmental bond such as the BUND or some US Treasury bonds. We consider bonds in the next section.

3.4 Bonds

The first instruments we consider are bonds. Such instruments can be divided along many different criteria which include interest payment or issuer. We consider the first issue here. The two basic types of bonds we deal with are

- Fixed coupon bond
- Floating rate note (FRN)

A prototypical bond is shown in Figure 3.1.

To illustrate the math let us first consider the fixed coupon bond. This contract pays coupons $C_1, \dots, C_N$ at times $\mathcal{T} = \{T_1, \dots, T_N\}$. The coupons are fixed in the sense that each coupon is known at issuance of the bond. At maturity the bond pays back some amount known as the *face amount*, *face value*, *redemption* or *par value*. Often it is useful to express the coupons in terms of percentage of notional. To this end we denote the notional at the times $\mathcal{T}$ by $N_1, \dots, N_N$. If we refer to the percentage of notional we use small letters, thus, $c_1, \dots, c_N$. Therefore, both notations are linked by $c_i \cdot N_i = C_i$.

The other bond we consider is the floating rate note (FRN). Again, we fix a time schedule $\tilde{\mathcal{T}} := \{\tilde{T}_1, \dots, \tilde{T}_N\}$. The floating rate note at each time $\tilde{T}_i$ pays the floating rate fixed in $\tilde{T}_{i-1}$ for the period $[\tilde{T}_{i-1}, \tilde{T}_i)$ plus a spread s_i. Usually the spread is constant, thus, $s_i = s$. While the spread is known at the issuance of the bond the actual floating rate payments are not. As in the fixed coupon bond case at maturity the final payment is the face amount.

We need to introduce some important vocabulary for bonds. First, we denote the *settlement date* by T_S. This is the date when in a bond transaction the security changes hands and the buyer pays for the security. Second, the *ex-coupon date* is the actual date the trade at latest must be settled such that the upcoming coupon is received by the buyer. If the ex-coupon rule applies to a reference date t we denote by τ_{Ex} the ex-coupon period and only consider coupons occurring after $t + \tau_{\mathrm{Ex}}$.

The issuer consideration is closely linked to credit quality. Issuers of bonds include governments, banks or corporates. Issuers considered to certainly pay back the nominal value at maturity pay lower coupons in general. If the possibility of default increases and, thus, the risk of not paying back the full nominal amount market participants wish to be compensated for taking that risk. This is usually done by paying higher coupons.

Accreting/Amortizing bonds

For such bonds we need additional information. The information is about the increase in notional for an accreting bond, respectively about the decrease for an amortizing bond or sinking fund. Let us denote the amount of face value paid/received at time T_i by N_i.

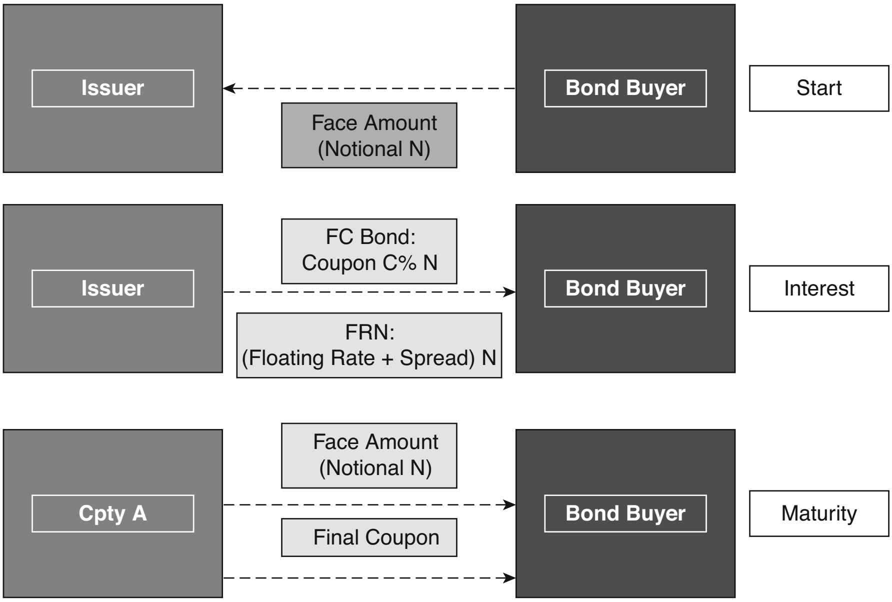


Figure 3.1 A prototypical bond: either fixed coupon bond or floating rate note

Example

Let us consider an example of an accreting bond with a fixed coupon of 3%. Suppose we buy 1.000.000 face amount and the bond yearly amortizes 100.000. Thus, we are entitled to the following stream of payments

Date	Interest	Principal
T_0	–	$-1.000.000$
T_1	$1.000.000 \cdot 3\%$	100.000
T_2	$900.000 \cdot 3\%$	100.000
$\vdots$	$\vdots$	$\vdots$
T_{10}	$100.000 \cdot 3\%$	100.000

There are several versions of such bonds, for example when dealing with mortgages we always pay an amount of 3% but the payments are split into interest payments (on a smaller notional) and prepayments on the notional.

Bullet bonds

Bullet bonds pay the notional value in one installment at the maturity.

3.4.1 Bond math

For the valuation of a fixed bond we need a discounting curve and for a floating rate note we additionally need a forwarding curve respectively. The discount curve is used

to project future cash flows to today and the forwarding curve is used to determine the expected values of the floating rates in the future.
Let us start with the fixed coupon bond pricing. To this end we consider the discounting curve leading to discount factors $D(0,t)$ and a discounting curve taking into account the credit quality of the issuer. Let the discount factor calculated on the latter curve be $D^C(0,t)$ and the settlement price P_S paid at the settlement date T_S. The price of a coupon bond paying coupons and notional at $T_1,\ldots,T_N$ is given by

$$\begin{aligned} V^{\text{Fixed}}(0) = &\sum_{i=1}^{N} \tau_i C_i D^C(0,T_i) + \sum_{i=1}^{N-1} (N_i - N_{i+1}) D^C(0,T_i) + N_N D^C(0,T_N) \\ &+ P_S D(0,T_S) \end{aligned} \tag{3.3}$$

For a standard bond the notional is constant and paid in one installment at the bond's last coupon date T_N. Equation (3.3) can be simplified and we have:

$$V^{\text{Fixed}}(0) = C_1 \sum_{i=1}^{N} \tau_i D^C(0,T_i) + N_1 D^C(0,T_N) + P_S D(0,T_S) \tag{3.4}$$

Equation (3.4) is the usual pricing formula. When dealing with bonds we often do not explicitly give the notional value but instead we refer to buying a given notional amount of currency of face. Thus, the number of bonds can be determined by dividing the notional value by the face amount.

Example
On 25.07.2013 we consider a bond which pays yearly coupons of 3% and maturity 30.06.2020. The face amount is 100 units of currency. We buy 1.000.000 face.
The number of bonds we hold is $1.000.000/100 = 10.000$ and we are entitled to receive the following amount of payments:

Date	Interest	Principal
30.06.2014	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2015	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2016	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2017	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2018	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2019	$3\% \cdot 1.000.000 = 30.000$	–
30.06.2014	$3\% \cdot 1.000.000 = 30.000$	1.000.000

Example (*Continued*)
Thus, assuming $T_{\text{Ex}} = 0$ we have if the zero rate is constant at 2% and the credit spread is .75%:

$$V^{\text{Fixed}}(0) = 1.000.000 \left(3\% \cdot \sum_{j=1}^{N} e^{-(2\%+.75\%)yf_j} + e^{-(2\%+.75\%)yf_N} \right) = 1.015.241,13,$$

where we denote by τ_j the corresponding year fraction using a day count convention of 30/360E.

For the valuation of a floating rate note with notional values N_i we need the forwarding curve to determine the index $L(t, \tilde{T}_{i-1}, \tilde{T}_i)$ at each coupon date. Remember that we use the $\tilde{T}$ notation for dates coming from schedules belonging to floating rates. The price is given by Equation (3.5):

$$\begin{aligned} V^{\text{FRN}}(0) = &\sum_{i=1}^{N} \tilde{\tau}_i N_i (L(0, \tilde{T}_{i-1}, \tilde{T}_i) + s_i) D^C(0, \tilde{T}_i) \\ &+ \sum_{i=1}^{N} (N_i - N_{i+1}) D^C(0, \tilde{T}_i) + N_N D^C(0, \tilde{T}_N) \end{aligned} \tag{3.5}$$

Again, assuming a constant notional being paid in one installment at the bond's maturity and having a constant spread we have Equation (3.6):

$$V^{\text{FRN}}(0) = N_1 \sum_{i=1}^{N} \tilde{\tau}_i (L(0, \tilde{T}_{i-1}, \tilde{T}_i) + s) D^C(0, \tilde{T}_i) + N D^C(0, \tilde{T}_N) \tag{3.6}$$

3.4.2 Par rate

Let us assume that we have a fixed bond with notional amount equal to 100. We assume that k times a year the bond pays a coupon of c until maturity $T = T_N$. For this contract the price is given by

$$100 \cdot c \sum_{i=1}^{kT} \tau_i D(0, T_0 + i/kY) + 100 \cdot D(0, T)$$

Now, we ask ourselves which percentage rate p leads to a bond value of 100, that means we have to solve

$$100 = 100 \cdot p \sum_{i=1}^{kT} \tau_i D(0, T_0 + i \cdot i/kY) + 100 \cdot D(0, T_N) \tag{3.7}$$

The rate p is called the *par rate*. In between coupon dates we have to take the accrued into account and, thus, we need to calculate a special first step. At maturity the bond repays its face value. This is regardless of the bond conditions such as coupon or coupon frequency.

With time to maturity becoming smaller the change in market value becomes smaller. This effect is known as *pull-to-par*.

For floating rate notes this phenomenon is also observed at the fixing dates.

3.4.3 Yield to maturity

Suppose the fixed bond pays coupons c with a frequency of k times per year until maturity T. The number of paid coupons is denoted by N and the current price of the bond is $V^{\text{Fixed}}(0)$. Then, we define the *yield to maturity* as being the solution y_{ytm} of the equation

$$V^{\text{Fixed}}(0) = 100 \cdot c \sum_{j=1}^{N} \tau_j \left(1 + \frac{y}{k}\right)^{-j} + 100 \cdot \left(1 + \frac{y}{k}\right)^{N} \tag{3.8}$$

Often the day-count convention is such that we can write the price as

$$V^{\text{Fixed}}(0) = 100 \cdot \frac{c}{k} \sum_{j=1}^{kT} \left(1 + \frac{y}{k}\right)^{-j} + 100 \cdot \left(1 + \frac{y}{k}\right)^{kT}$$

We consider the dependency of the price of the bond and the yield as well as the maturity. Figure 3.2 illustrates the relationship.

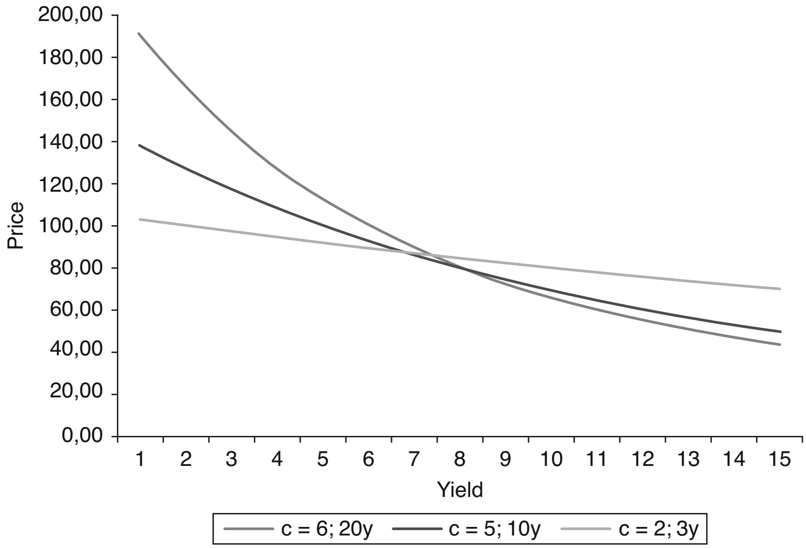


Figure 3.2 Price yield function

3.4.4 Bond risk measures

As an example we consider a fixed coupon bond and calculate what is called the *basis point value* (BPV), *duration* (D), and *convexity* (C). We assume the coupon is c which is paid every l-th month and y is the yield to maturity. For simplicity we only consider $l = 1, 2, 3, 4, 6, 12$ and denoting by P the price of the bond. Thus, denoting $k = 12/l$ we find for the risk measures:

- BPV

$$BPV = \frac{1}{10000}\frac{1}{1+y/k}\left[\frac{100c}{k}\sum_{j=1}^{kT}\frac{j}{k}\frac{1}{\left(1+y/k\right)^{j}} + T\frac{100}{\left(1+y/k\right)^{kT}}\right]$$

- Duration

$$D = \frac{1}{P}\frac{1}{1+y/k}\left[\frac{100c}{k}\sum_{j=1}^{kT}\frac{j}{k}\frac{1}{\left(1+y/k\right)^{j}} + T\frac{100}{\left(1+y/k\right)^{kT}}\right]$$

- Convexity

$$C = \frac{1}{P}\frac{1}{(1+y/k)^2}\left[\frac{100c}{k}\sum_{j=1}^{kT}\frac{j(j+1)}{k^2}\frac{1}{\left(1+y/k\right)^{j}} + T(T+1/k)\frac{100}{\left(1+y/k\right)^{kT}}\right]$$

For a zero coupon bond we find for the price

$$P = \frac{100}{\left(1+y/k\right)^{kT}}$$

and for the corresponding risk measures:

$$BPV = \frac{TP}{10000}\left(1+y/k\right), D = \frac{T}{1+y/k}, C = \frac{T(T+1/k)}{\left(1+y/k\right)^2}$$

Example:
Let us consider a bond with $k = 2$, $c = 3\%$, $T = 3$ and $y = 1\%$. Then, we get

$$\begin{aligned} P &= 100 \cdot \frac{0.03}{2}\sum_{j=1}^{2\cdot 3}\left(1+\frac{0.01}{2}\right)^{-j} + 100 \cdot \left(1+\frac{0.01}{2}\right)^{2\cdot 3} \\ &= 105.8964, \end{aligned}$$

$$
\begin{aligned}
BPV &= \frac{1}{10000}\frac{1}{1+0.01/2}\left[\frac{100\cdot 0.03}{2}\sum_{j=1}^{2\cdot 3}\frac{j}{2}\frac{1}{(1+0.01/2)^j}+3\frac{100}{(1+0.01/2)^{2\cdot 3}}\right]\\
&= 0.0305,\\
D &= \frac{1}{P}\frac{1}{1+0.01/2}\left[\frac{100\cdot 0.03}{2}\sum_{j=1}^{2\cdot 3}\frac{j}{2}\frac{1}{(1+0.01/2)^j}+3\frac{100}{(1+0.01/2)^{2\cdot 3}}\right]\\
&= 2.8806,\\
C &= \frac{1}{P}\frac{1}{(1+0.01/2)^2}\Big[\frac{100\cdot 0.03}{2}\sum_{j=1}^{2\cdot 3}\frac{j(j+1)}{2^2}\frac{1}{(1+0.01/2)^j}\\
&\quad +3(3+0.5)\frac{100}{(1+0.01/2)^{2\cdot 3}}\Big]\\
&= 9.9110
\end{aligned}
$$

Clean and dirty price

To introduce the *Clean Price* and the *Dirty Price* of a bond we first deal with the notion of *Accrued Interest* or simply *Accrued*. We consider Figure 3.3. We assume that today's date is T_S and it is between two coupon dates T_{i-1} and T_i. We assume that the purchaser of the bond holds it in T_i and, thus, is entitled to the full coupon payment at T_i. From an economical perspective it would not be fair that the purchaser gets the full coupon but the investor holding the bond up to T_S should get a fraction of the coupon with respect to the length of period $[T_{i-1}, T_S)$. We calculate this amount of money by

$$
\tau_{T_{i-1},S} N_i C_i =: Acc(T_{i-1}, T_S) \tag{3.9}
$$

The amount $Acc(T_{i-1}, T_S)$ is called the accrued interest.

The *Clean Price* of a bond is the price not including any accrued interest since the issuance or the most recent coupon payment. This means it is the price of all expected future payments minus the accrued interest discounted to today. The *Dirty Price* instead is nothing but the Clean Price plus the accrued interest. In fact Equation (3.4) is the *dirty price* of the bond and is also known as the *full price* or *invoice price*.

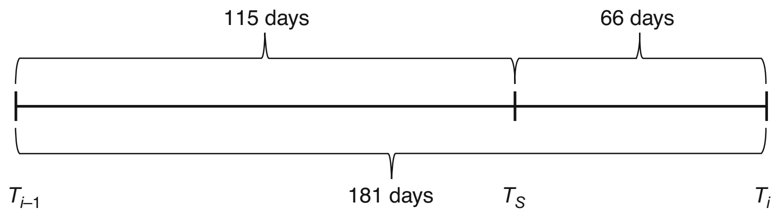


Figure 3.3 Calculation of accrued

Example
Due to Figure 3.3 we consider the dates T_{i-1}, T_i and T_S. The number of days in the intervals $[T_{i-1}, T_i)$, $[T_{i-1}, T_S)$ and $[T_S, T_i)$ is 181, 115 and 66. If we assume a notional of 1000.000 and an annual coupon of 6%, the accrued interest using Equation (3.9) would be $1000.000 \cdot 6\% \cdot 115/181 = 38121.55$.

The clean price or quoted price at time $T_{i-1} < t$ with T_{i-1} the last coupon date is given by

$$V^{\text{Bond}}_{\text{Clean}}(t) = V^{\text{Bond}}(t) - Acc(T_{i-1}, t) \tag{3.10}$$

For a Floating Rate Note the clean and the dirty price can be defined analogously. Finally, let us consider the *Moosmueller method*. Let y denote the yield and the clean price for this method is determined by

$$V^{\text{Bond}}_{\text{Clean}}(t) = \frac{C}{1+\tau_{t,T_{n+1}} y} + \sum_{j=n}^{N} \frac{C}{(1+\tau_{t,T_{n+1}} y)(1+y)^{N-j+1}} + \frac{100}{(1+\tau_{t,T_{n+1}} y)(1+y)^{N-n+1}} - Acc(T_n, t)$$

The first summand or *broken coupon* as it is called is calculated differently with respect to a bond market. In Germany for instance it is calculated using linear compounding whereas the international method, the *ISMA method*, uses exponentially compounding which results in a slightly higher result than the standard Moosmueller method.

3.5 Repos

This section deals with a contract called *repurchasement agreement* or *repo* for short. The repo is a collateralized deposit. In fact this deposit is equivalent to selling a bond and later buying exactly this bond back. The price for buying back the bond is determined at the inception of the trade. For the year fraction τ between the settlement date and the maturity we earn an interest on that transaction. This interest rate is known as the *repo rate*, R. Since the transaction is collateralized we can regard the repo rate to be risk free. Figure 3.4 shows how the repo works. Let us consider an example. Suppose the current price of a bond is 101.34. We enter into a repo deal by buying 1.000.000 face amount with a maturity of 1M. The repo rate for this deal is .19%. Using the ACT/360 market convention we find:

$$1013400\left(1 + 0.19\% \frac{31}{360}\right) = 1013565.8$$

Thus, at inception we receive 1013400 and deliver the bond to the counterparty. To unwind the repo after 1M the counterparty delivers the bond and receives a payment of 1013565.8.

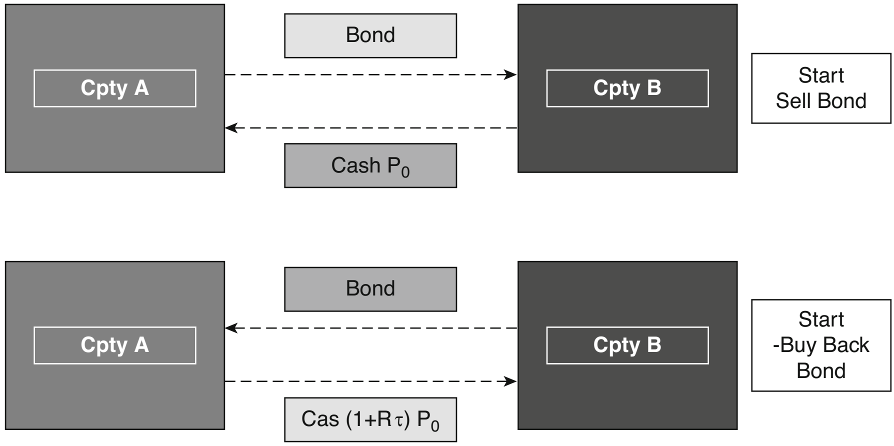


Figure 3.4 A prototypical repo

The rationale for an institution using repo deals can be manifold. It can be used for cash management, (re-)financing or liquidity management. Suppose there is a market participant holding a significant amount of cash. Since the repo is a relatively save investment it can be the deal of choice here, accepting relatively low rates but at the same time entering into a liquid and safe trade. Considering Figure 3.4 such a market participant would be Party B lending the cash and finally earing the repo rate. Since the repo is used by institutions looking for a liquid and safe investment the underlying in a repo deal are of the highest credit quality.

A repo transaction can be *overnight* or *for term*. The latter means any transaction which is different from overnight. Thus, a repo deal is a short term investment. Often the parties entering into a repo deal agree to roll the deal forward at maturity. For instance if the repo is overnight or one day and the bond owner does not need the bond, instead of buying the bond back and entering into a new repo deal the current deal is renewed or rolled forward. Sometime such deals are called *open repo*.

To round up the discussion of standard repos we have to consider *haircuts*. As stressed above repo deals are supposed to be safe deals. The cash lender has the underlying bond as collateral. On one occasion the cash lender faces a problem: if the borrower defaults and the bond declines in value. To take into account this type of event the cash lender at inception receives a higher price than the market value of the bond. The mismatch of the market price and the price received is known as the *haircut* and is set due to the creditworthiness of the borrower. Further to this securing mechanism the cash lender can call margins if the cash borrower's creditworthiness declines further.

Several banks act as borrowers of cash to refinance their businesses. For instance a way of using a repo is to refinance the buying and selling of a bond. Suppose there is a market opportunity to buy a bond and sell the bond later at a higher price. Immediately after buying the bond there might be no counterparty to which the bond can be sold. To overcome this time lag until finding a buyer the current bond holder can enter into a repo trade. This is known as *to sell the repo* or to *repo out the bond.*

Table 3.2 *Repo rates for EUR and USD from 02.08.2013*

	EUR repo		USD repo	
T	bid	ask	bid	ask
ON	0.1	0.29	0.08	0.28
TN	0.02	0.17	0.04	0.28
SN	0.02	0.17	0.04	0.28
SW	0.01	0.16	0.05	0.28
2W	0.02	0.17	0.06	0.28
3W	0.02	0.17	0.08	0.28
1M	0.03	0.18	0.1	0.28
2M	0.04	0.19	0.13	0.28
3M	0.09	0.24	0.16	0.28
4M	0.11	0.26	0.19	0.28
5M	0.14	0.29	0.21	0.28
6M	0.16	0.36	0.26	0.28
7M	0.18	0.38	0.29	0.28
8M	0.21	0.41	0.32	0.28
9M	0.25	0.45	0.35	0.28
10M	0.27	0.47	0.37	0.28
11M	0.3	0.5	0.4	0.28
12M	0.33	0.53	0.43	0.28

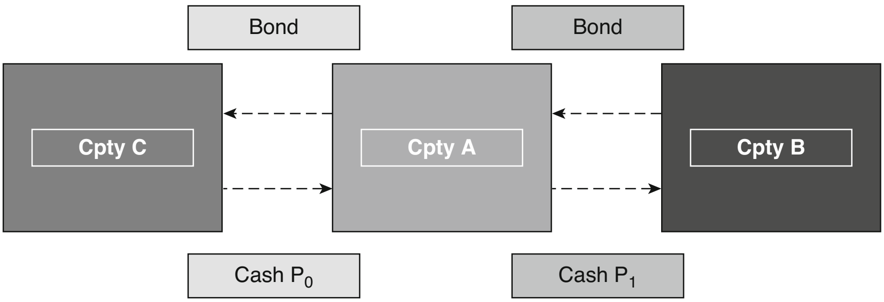


Figure 3.5 Back-to-back repo

Several variants of the standard repo deal exist. We consider *Reverse Repo* and *Back-to-Back Trades* here.

Suppose we wish to short a bond for speculating on rising rates or a relative value trade. Shorting the bond means to sell it. If we do not have the bond we need to borrow it for delivery. A standard repo is initiated for borrowing cash to lend the bond. In the current situation the bond is needed and, thus, the repo is initiated for borrowing the bond and lending cash. This kind of invoking the repo is known as *reverse repo* or *sell the repo.*

For an existing repo the deal can be rolled over or renewed. It is possible to unwind a repo and do a new repo deal with another counterparty. There might be occasions where a financial institution would prefer the latter. Unwinding and entering into a new repo deal in this way is known as a *back-to-back* repo. Figure 3.5 illustrates this situation.

3.6 Market quotes

Finally, we give market quotes for repo rates for EUR and USD, Table 3.2. The quotes are taken for 02.08.2013.

3.7 Reading list

The material presented in this chapter is also covered in Tuckman and Serrat (2012) or Fabozzi and Mann (2012). Both text books treat the fixed income markets and have special chapters on bonds and repos. The corresponding markets are explained and their specialities are pointed out. Furthermore much useful information on the conventions underlying the markets can be found. The reader might find interesting the sections describing how the usage of such instruments for funding led to the credit crisis and some banks into big losses or even bankruptcy. The interested reader will also find Henrad (2012b) a very valuable source. In this chapter the market conventions for many currencies and the instruments are described in detail. The author not only focuses on linear products but also covers options and other derivatives.

4 Markets and Products – FRAs and Swaps

4.1 Introduction and objectives

This chapter explains forward rate agreements and swaps First, in Section 4.2 we introduce the contract and explain the financial applications. We show how to value such a contract in the new multiple curves set-up. Market quotes for the EUR market conclude this section. One of the most important concepts allowing the efficient management of interest rate risk is introduced in Section 4.3. The basic interest rate swap is explained and the valuation using multiple yield curves set-up is detailed. Since the basic swap is omnipresent in fixed income business, appearing in many variants, we have included some very common variants and explain the underlying market data and the valuation. Especially, the floating against floating interest rate swaps have gained importance after mid 2007. To this end we consider two prominent examples of such swaps in Section 4.4, namely the money market basis swap and the cross currency basis swap.

All the instruments discussed serve in a later chapter as the basic instruments to be used for setting up yield curves for one and multiple currencies.

4.2 FRAs

A forward rate agreement, FRA, is a contract which at start time T_1 pays an interest rate, the fixed rate K set at trade inception minus the forward rate for the period $[T_1, T_2)$ fixed shortly before T_1. We consider simply compounded rates, Equation (2.8) and denote the rate by $L(t, T_1, T_2)$. The payoff is the value at T_2 and it is given by

$$V^{\mathrm{FRA}}(T_2, T_1, T_2) = \tau_{1,2}(K - L(T_1, T_1, T_2))$$

The value at T_1 of the FRA is

$$V^{\mathrm{FRA}}(T_1, T_1, T_2) = \tau_{1,2}\frac{(K - L(T_1, T_1, T_2))}{1 + \tau_{1,2}L(T_1, T_1, T_2)}$$

The fixing of the floating rate $L(T_1, T_1, T_2)$ takes place due to the market convention of the reference index, see Chapter 2. Figure 4.1 summarizes the setting using the trade date, the settlement date and the fixing date. The period is then determine by T_1 and T_2 in the figure.

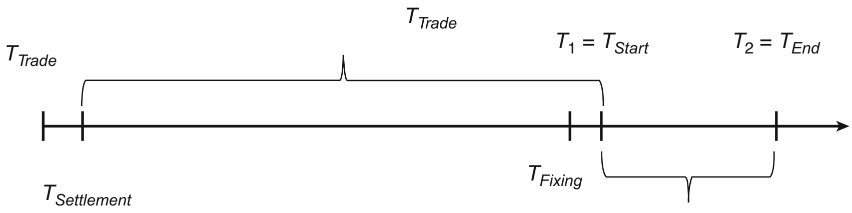


Figure 4.1 The mechanism of setting up an FRA

We outline the change in the interest rate markets in Introduction. By the definition of the rates we observed that the LIBOR rates refer to actual loans, respectively borrowing in the interbank market and thus involve risk. Since FRA contracts are quoted (traded) with respect to a CSA with cash collateral there is actually no credit risk. To this end the time T_1 value is discounted using the currencies OIS curve. The T_2 value of the FRA is discounted using the discount factor $1+\tau_{1,2}L(T_1,T_1,T_2)$ since we enter into the contract by actually lending, respectively borrowing at the FRA rate. Thus, this transaction bears risk and cannot be discounted using a risk free rate.

What is the difference between an FRA and an actual forward rate? The forward rate is the rate at which a bank agrees to borrow a certain amount of money for a given period. Thus, the forward rate involves credit risk because at maturity T_2 the borrower has to pay the notional and the interest to the bank. In contrast the FRA does not involve any lending, respectively borrowing of the notional amount.

4.2.1 FRA math

For determining the time t value of the FRA for $0 \le t_1 \le T_1$ we have to take a discount curve (with respect to the collateral agreement in place). Then, the value of the FRA is given by

$$V^{\text{FRA}}(t,T_1,T_2) = \tau_{1,2}\frac{(K-L(t,T_1,T_2))}{1+\tau_{1,2}L(t,T_1,T_2)}D(t,T_1) \tag{4.1}$$

The rate which renders this contract fair is the *FRA Rate*. It is given by

$$V^{\text{FRA}}(t,T_1,T_2) = L(t,T_1,T_2)$$

In fact the latter formula is an approximation formula. The valuation formula is given by

$$V^{\text{FRA}}(t,T_1,T_2) = \left(1-\frac{1+K\tau_{1,2}}{1+\tau_{1,2}L(T_1,T_1,T_2)}e^{C(t,T_1)}\right)D(t,T_1)$$

The factor $e^{C(t,T_1)}$ is known as the *convexity correction* or *convexity adjustment* and is model dependent. Under sound model assumptions it is shown in Mercurio, F. (2010)

Table 4.1 *FRA quotes (bid/ask) for standard FRAs (3M, 6M, 9M and 12M) for EUR as from 22.07.2013*

FRA	Rate	FRA	Rate	FRA	Rate
1×4	0.230/0.250	1×7	0.336/0.376	1×10	0.460/0.480
2×5	0.236/0.276	2×8	0.356/0.396	2×11	0.480/0.500
3×6	0.255/0.295	3×9	0.382/0.422	3×12	0.507/0.527
4×7	0.275/0.315	4×10	0.401/0.441		
5×8	0.286/0.326	5×11	0.423/0.463		
6×9	0.310/0.350	6×12	0.438/0.478		
7×10	0.320/0.360	9×15	0.504/0.544	FRA	Rate
8×11	0.338/0.378	12×18	0.555/0.595	12×24	0.790/0.810
9×12	0.370/0.390	18×24	0.670/0.710		
12×15	0.398/0.438				

that C is very small and, thus, negligible. To this end we can set $e^{C(t,T_1)} := 1$. Then, Equation (4.1) can safely be used to value an FRA.

Before the crisis only one single curve was used for discounting and forward rate calculation. Since under this assumption $1 + L(t, T_1, T_2) = D(t, T_1)/D(t, T_2)$, the time t value of the forward rate agreement could simply be written as

$$V^{\text{FRA, 1 curve}}(t, T_1, T_2) = \tau_{1,2}(K - L(t, T_1, T_2))D(t, T_2)$$

4.2.2 FRA quotes

The market quotes the rate $V^{\text{FRA}}(0, T_1, T_2)$ which renders Equation (4.1) fair for several standard tenor. For instance in the EUR market for $1M$, $3M$, $6M$ and $12M$. The quotation mechanism is given by the expression *nMxtM* where *nM* is the number of months until expiry and *tM* is the tenor. Thus, a 4*x*7 means an FRA starting in four months time which is on a three month rate. We calculate the start date of the FRA from today plus the spot lag and then we add the start period which is four months in our example. The maturity or end date is determined in the same way but we have to add a period of seven months. The period, of course, is calculated due to the index market conventions. We have summarized typical quotes for FRAs in Table 4.1.

FRA quotes depend on the shape of the given yield curve since two different points are used to calculate the value and change over time as Figure 4.2 shows. Here historical FRA quotes are plotted for a 12*x*24 FRA and the EUR market. The curves show the FRA rates calculated for one year every month. We observe that the starting point of the curves and the shape of the curves vary significantly.

4.3 Swaps

Having introduced forward rate agreements we now consider interest rate swaps or IRS for short. To this end we begin our considerations with the standard fixed against float interest rate swap.

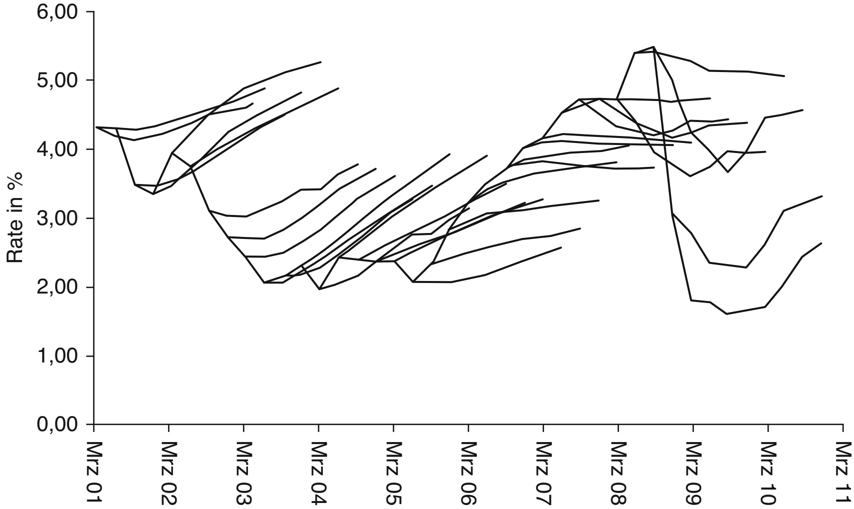


Figure 4.2 Historic FRA rates for the 12×24 FRA in EUR

4.3.1 Fixed against float interest rate swap

In a fixed against float interest rate swap two parties periodically exchange coupons on a given notional amount N. One party pays a fixed rate whereas the other pays a floating rate. The fixed rate payments are exchanged according to the dates schedule $\mathcal{T}_{\text{fix}}$ given by

$$\mathcal{T}_{\text{fix}} := \{T_m, T_{m+1}, \ldots, T_n\} \tag{4.2}$$

We include the start date T_m into the schedule despite the fact that there is no payment on this date. But the payment period is determined by two consecutive dates in the schedule. According to the market conventions of a given currency the fixed payment schedule has standard periods, for instance one year for EUR or six month for USD.

The floating rate payments are also exchanged according to a dates schedule $\mathcal{T}_{\text{float}}$. This schedule is given by

$$\mathcal{T}_{\text{float}} := \{\tilde{T}_m, \tilde{T}_{m+1}, \ldots, \tilde{T}_n\} \tag{4.3}$$

To differentiate between the time points for the fixed and the floating schedule we denote time points from the floating schedule by a tilde. For the same reason as in the fix rate case we include the start date of the swap into the schedule. The schedule $\mathcal{T}_{\text{float}}$ depends on the corresponding floating rate index. For instance the payments can be linked to a money market rate, for example the 3-month LIBOR or 3-month EURIBOR. Then, the schedule would be

$$\mathcal{T}_{\text{float}} := \left\{\tilde{T}_m, \tilde{T}_m + 3m, \ldots\right\}$$

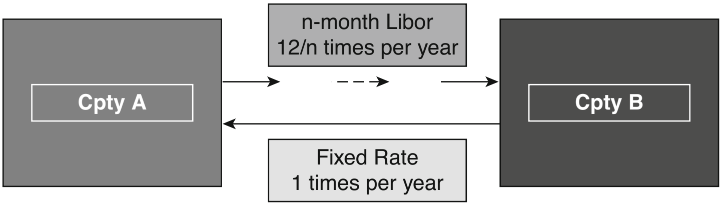


Figure 4.3 A standard interest rate swap

Finally, we have to remark that usually the final dates of the schedules coincide, that is $T_n = \tilde{T}_n$. Figure 4.3 summarizes the payments of a fixed against float interest rate swap.

We call the sum of fixed payments the *fixed leg* and the sum of the floating payments the *floating* leg. The party paying the fixed coupon payments is said to enter into a *payer swap* whereas the other party receiving the fix payments enters into a *receiver swap*.

Swap math

Let $\mathcal{C}_d$ denote the discount curve we apply for pricing the time t value of the fixed leg of a swap with swap rate K. If $\mathcal{T} = \{T_{m+1}, \ldots, T_n\}$ is the schedule of fixed cash flows the value is given by

$$V^{\text{fix}}(t) := V^{\text{fix}}_{m,n}(t) = \sum_{i=m+1}^{n} \tau_i K D(t, T_i) \tag{4.4}$$

The present value is the value at $t = 0$. The same calculation can be done for the floating leg. To this end we take a curve $\mathcal{C}_f$ for calculating the forward rates and a discount curve $\mathcal{C}_d$ for projecting the payments to time t. The time schedule for the floating cash flows is $\tilde{\mathcal{T}} = \{\tilde{T}_m, \tilde{T}_m + 1, \ldots, \tilde{T}_{\tilde{n}} = T_n\}$. Then, the time t value of the floating leg is given by

$$V^{\text{float}}(t) := V^{\text{float}}_{\tilde{m},\tilde{n}}(t) = \sum_{i=\tilde{m}+1}^{\tilde{n}} \tilde{\tau}_i L(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i) \tag{4.5}$$

Taking Equations (4.4) and (4.5) together we find for the time t value of a payer swap:

$$V^{\text{Pay}}(t) := V^{\text{float}}(t) - V^{\text{fix}}(t) = \sum_{i=m+1}^{\tilde{n}} \tilde{\tau}_i L(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i) - \sum_{i=m+1}^{n} \tau_i K D(t, T_i) \tag{4.6}$$

The rate K which makes the time t value of the swap equal to 0 is the breakeven rate and is called *Swaprate*. We define the swap rate by

$$SR_{m,n}(t) := \frac{\sum_{i=\tilde{m}+1}^{\tilde{n}} \tilde{\tau}_i L(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i)}{\sum_{i=m+1}^{n} \tau_i D(t, T_i)} \tag{4.7}$$

The denominator appearing in Equations (4.7) is called the *annuity* and is denoted by $A_{m,n}(t)$.

If the curve used to determine the forward rate and the discounting curve are the same we have the old textbook formula for the swap rate:

$$SR_{m,n}(t) = \frac{D(t, \tilde{T}_m) - D(t, \tilde{T}_n)}{\sum_{k=m+1}^{n} \tau_k D(t, T_k)}$$

In terms of the forward rate $L_i(t) := L(t, \tilde{T}_{i-1}, \tilde{T}_i)$:

$$SR_{m,n}(t) = \frac{1 - \prod_{k=\tilde{m}+1}^{\tilde{n}} \frac{1}{1+\tau_k L_k(t)}}{\sum_{k=\tilde{m}+1}^{\tilde{n}} \tau_k \prod_{l=\tilde{m}+1}^{\tilde{k}} \frac{1}{1+\tau_l L_l(t)}}$$

If we wish to specify the dependence of the tenor of the floating rate we use the notation $SR_{m,n}^{k}(t)$ with k being the floating rate tenor, for instance $k = 3M$. Finally, we have to remark that a one period forward swap and the corresponding FRA do not have the same price despite the fact that the rate is the same! This is an artifact of a multi-curve framework and is clearly not the case in a one curve setting.

4.3.2 Swap quotes

The swap rate is quoted by data providers. To actually use the data the provider has to specify which type of swap rate is quoted. We focus on the standard case of a fixed against float swap. First, we need to know against which floating rate, for instance $3M$ or $6M$ the rate is quoted. Second, we need to know when the swap starts and when it matures. There are many two different quotes around. Table 4.2 is an example for swap rate quotes in EUR. For this table we have $T_m = 0$ and T_n is given by the first column starting with 1Y and reaching out until 50Y. Another example are forward swap rates. Here Table 4.3 shows an example. In this example T_m is given by the first column and T_n can be inferred from the first column by setting $T_n = T_m + T$ where T is the corresponding number in the column. All quotes are against the $6M$ EURIBOR rate. Let us take $T_m = 5Y$ and the 6th column. Then, $T_n = 5Y + 6Y$ and, thus, the swap rate $SR_{5y,11y}(0)$ is the corresponding quote.

The rates do not stay constant over time and we face significant movements. To this end we have included Figure 4.4 for EUR and USD for illustration. We see that the overall level of the rates varies but also we face periods where long rates are higher than short term rates. For instance between 2003 and 2006 this is the case. This means

Table 4.2 *Quoted forward swap rates based on the floating rate 6M EURIBOR (left) and 3M EURIBOR (right) from 22.07.2013. The swap starts immediately, thus, $T_m = 0$ and T_n is given in the first column*

Tenor	bid 6M	ask 6M	bid 3M	ask 3M
1Y	0.384	0.415	0.286	0.316
18M	0.459	0.473	0.341	0.361
2Y	0.51	0.54	0.3801	0.4201
3Y	0.654	0.674	0.525	0.545
4Y	0.841	0.871	0.711	0.742
5Y	1.048	1.078	0.9123	0.9523
6Y	1.24	1.27	1.112	1.143
7Y	1.417	1.437	1.286	1.317
8Y	1.57	1.6	1.445	1.476
9Y	1.712	1.742	1.5884	1.6284
10Y	1.84	1.87	1.727	1.757
11Y	1.963	1.983	–	–
12Y	2.05	2.09	1.9525	1.9725
13Y	2.153	2.16	–	–
14Y	2.223	2.23	–	–
15Y	2.27	2.29	2.176	2.196
20Y	2.405	2.413	2.306	2.346
25Y	2.4127	2.4527	2.355	2.362
30Y	2.4117	2.4517	2.36	2.368
40Y	2.4567	2.4967	2.409	2.429
50Y	2.5027	2.5427	2.459	2.479

the yield curve is steep. Periods where the different rates are very close indicate that the curve is flat.

To further illustrate the stochastic nature of the rates and, thus, of the curve we have plotted the curve starting with the 6M rate up to 10Y for EUR and USD. Figure 4.5 shows how the curve varied in a 16 month period between March 2007 and July 2008. Not only the overall level of the rates but also the slope and the curvature varied significantly.

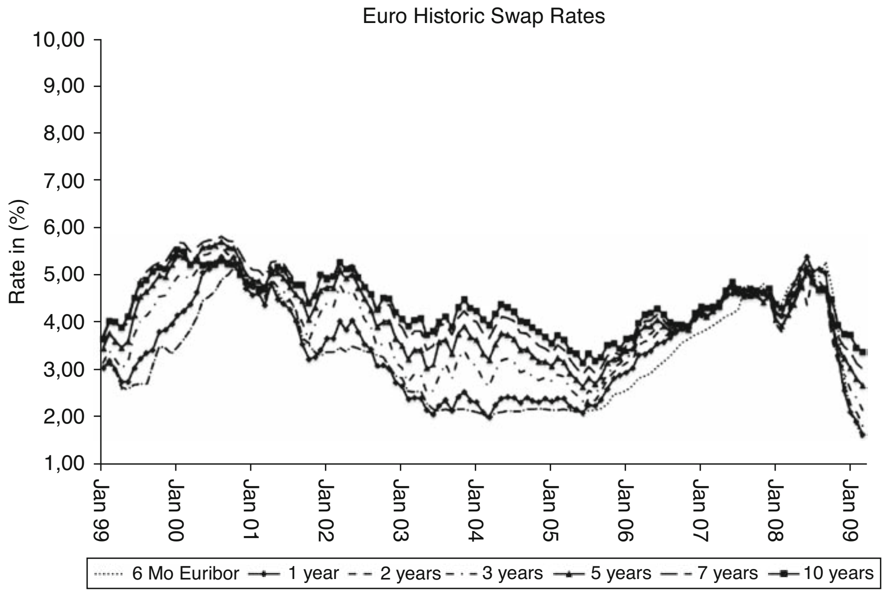
In the market there are a variety of variants of the standard fix against float interest rate swap. Some of the variants are considered in the sequel.

4.3.3 In arrears swaps

Usually the swap rate of a fixed against float swap on the nM index is calculated using Equations (4.7) with the curves $\mathcal{C}_{nM}$ to calculate the forward rates and $\mathcal{C}_d$ for the discount factors. Sometimes swaps can be *in arrears* which means that the forward rate which is fixed is immediately paid. We consider the valuation later since it cannot be done using yield curves only.

Table 4.3 *Quoted forward swap rates for EUR from 22.07.2013. The left column denotes the time at which the swap starts, T_m, and the top row is the corresponding tenor T meaning the swap ends at $T_m + T$. The quotes are against 6M EURIBOR*

F/T	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	…	15Y	20Y	25Y	30Y
1Y	0.642	0.795	1.010	1.232	1.430	1.603	1.759	1.900	2.028	2.140	…	2.471	2.540	2.538	2.526
2Y	0.949	1.196	1.432	1.632	1.803	1.953	2.090	2.214	2.321	2.412	…	2.643	2.661	2.636	2.615
3Y	1.447	1.680	1.867	2.025	2.165	2.293	2.408	2.508	2.591	2.659	…	2.792	2.768	2.720	2.694
4Y	1.915	2.081	2.224	2.352	2.472	2.580	2.672	2.749	2.810	2.855	…	2.901	2.844	2.782	2.754
5Y	2.250	2.383	2.504	2.619	2.722	2.809	2.880	2.935	2.973	2.995	…	2.972	2.891	2.820	2.794
6Y	2.519	2.634	2.747	2.848	2.930	2.995	3.043	3.074	3.090	3.094	…	3.015	2.916	2.843	2.820
7Y	2.752	2.866	2.963	3.039	3.098	3.139	3.163	3.171	3.167	3.153	…	3.033	2.923	2.853	2.835
8Y	2.984	3.074	3.142	3.191	3.224	3.239	3.239	3.227	3.205	3.177	…	3.032	2.914	2.851	2.839
9Y	3.167	3.224	3.265	3.289	3.295	3.286	3.266	3.237	3.202	3.167	…	3.008	2.891	2.838	2.832
10Y	3.282	3.316	3.332	3.329	3.312	3.284	3.248	3.207	3.166	3.127	…	2.965	2.855	2.816	2.817
11Y	3.350	3.358	3.346	3.320	3.285	3.242	3.195	3.150	3.107	3.068	…	2.909	2.812	2.788	2.796
12Y	3.366	3.344	3.309	3.267	3.218	3.167	3.118	3.072	3.031	2.993	…	2.846	2.766	2.756	2.771
13Y	3.321	3.279	3.232	3.178	3.123	3.072	3.025	2.983	2.945	2.912	…	2.778	2.721	2.725	2.745
14Y	3.236	3.185	3.127	3.069	3.017	2.970	2.929	2.891	2.860	2.830	…	2.713	2.679	2.696	2.721
15Y	3.133	3.071	3.011	2.959	2.912	2.873	2.836	2.807	2.779	2.751	…	2.655	2.644	2.672	2.700
20Y	2.660	2.628	2.612	2.592	2.568	2.547	2.531	2.515	2.507	2.500	…	2.530	2.591	2.639	2.666
25Y	2.432	2.432	2.419	2.422	2.423	2.435	2.450	2.468	2.487	2.507	…	2.600	2.663	2.695	2.712
30Y	2.499	2.523	2.549	2.574	2.601	2.625	2.647	2.669	2.689	2.707	…	2.765	2.787	2.797	2.821



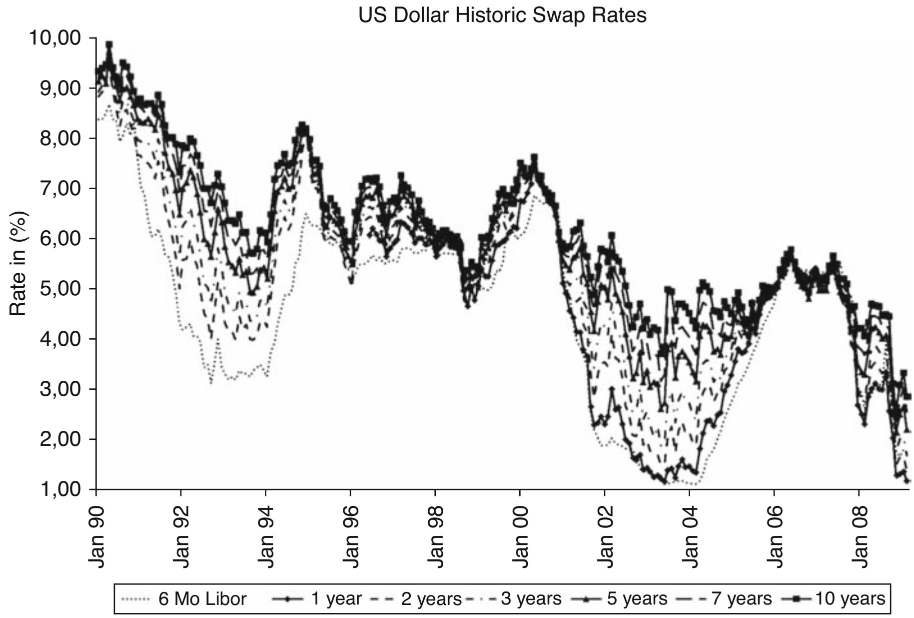


Figure 4.4 Historic rates for EUR (top) and USD (bottom) 6M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y

OIS swaps

Overnight Index Swaps (OIS) are fixed against float interest rate swaps but the floating schedule is linked to an overnight rate and, thus, pays the daily compounded coupons over the given coupon period. To this end let $\tilde{\mathcal{T}}$ and $\mathcal{T}$ be the floating and the fixed period time schedule, N_i being the number of days between $\tilde{T}_m$ and $\tilde{T}_i$. Then, the value

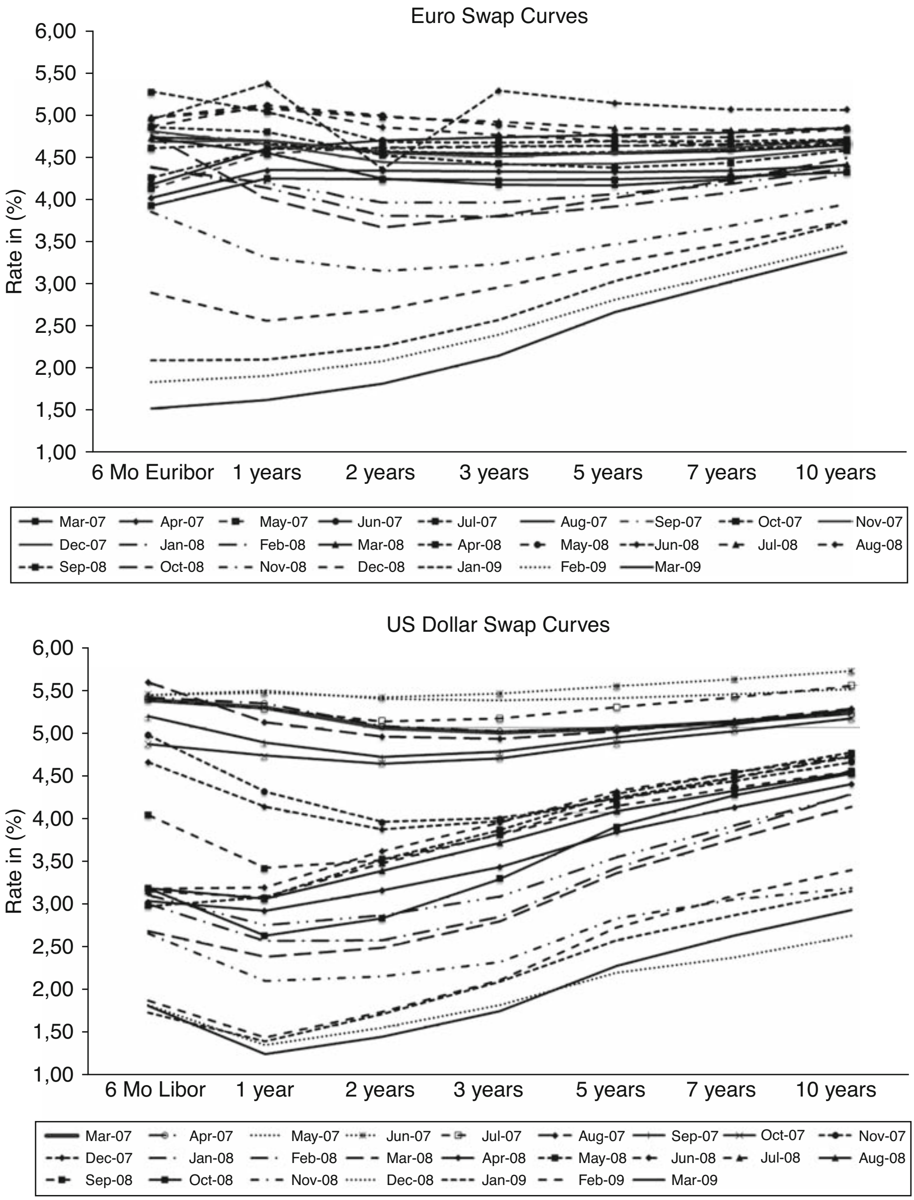


Figure 4.5 Historic swap curves for EUR (top) and USD (bottom) plotted monthly from March 2007 until July 2008

of the floating leg in this time period is given by

$$V^{\text{float}}(t, \tilde{T}_i) = \frac{D(t, \tilde{T}_i)}{\tilde{\tau}_{i,m}} \left(\prod_{j=k+1}^{N_i} \left(1 + L(t, \tilde{T}_{j-1}, \tilde{T}_j)\tilde{\tau}_{j-1,j}\right) - 1 \right) \tag{4.8}$$

$$= \frac{D(t,\tilde{T}_i)}{\tilde{\tau}_{i,m}} \left(\prod_{k=m+1}^{N_i} \left(1 + \left(\frac{D(t,\tilde{T}_{j-1})}{D(t,\tilde{T}_j)} - 1 \right) \right) - 1 \right)$$

$$= D(t,\tilde{T}_i) \frac{1}{\tilde{\tau}_{i,m}} \left(\frac{D(t,\tilde{T}_m)}{D(t,\tilde{T}_i)} - 1 \right) = D(t,\tilde{T}_i) L^{\mathrm{OIS}}(t,\tilde{T}_m,\tilde{T}_i)$$

For the fixed leg and a fixed rate K we have

$$V^{\mathrm{fix}}(t,T_i) = K\tau_{i-1,i} D(t,T_i) \tag{4.9}$$

Thus, determine the value of an OIS payer swap by

$$V^{\mathrm{Pay}}(t) = \sum_{j=m+1}^{n} V^{\mathrm{float}}(t,\tilde{T}_j) - V^{\mathrm{fix}}(0,T_j) \tag{4.10}$$

$$= \frac{D(t,\tilde{T}_m) - D(t,\tilde{T}_n)}{A(t,T_n)}. \tag{4.11}$$

The market convention is that a fixed interest rate is exchanged for the overnight rate which is compounded and paid at maturity. On both legs there is a single payment for maturity up to 1Y, yearly payments with short stub for longer maturities.

Accreting and amortizing swaps

For many purposes the coupon notional is not the same for each swaplet. If the coupon notional decreases we call the swap an *amortizing swap*. On the other hand if the notional increases the swap is called and *accreting swap*. Such type of swaps are for instance popular when dealing with mortgage structures where prepayment happens due to a prescribed schedule. To hedge the interest rate risk for each period the notional decreases. Thus, using an amortizing swap would be the instrument of choice.

Zero coupon swaps

The standard fixed against floating interest rate swap pays a floating rate against a fixed rate at prescribed dates. Now, we consider a contract based on a floating rate and a fixed rate but with (a) the fixed leg payment made at maturity or (b) the fixed leg and floating leg payments made at maturity. Such swaps, called *zero coupon swaps*, are used by corporates to hedge a loan with the interest being capitalized and paid at maturity. Banks use such instruments for hedging customers, flows.

For valuation take a floating schedule $\tilde{\mathcal{T}}$ and a fixed schedule $\mathcal{T}$ of a standard interest rate swap. On the floating leg we pay the standard coupons for contract (a) or compounded floating rates for (b):

$$V^{\mathrm{float,a}}(t) = \sum_{i=m+1}^{\tilde{N}} \tilde{\tau}_i L(t,\tilde{T}_{i-1},\tilde{T}_i) D(t,\tilde{T}_i) \tag{4.12}$$

$$V^{\text{float,b}}(t) = D(t, \tilde{T}_{\tilde{N}}) \prod_{i=m+1}^{\tilde{N}} \tilde{\tau}_i L(t, \tilde{T}_{i-1}, \tilde{T}_i) \tag{4.13}$$

For the fixed leg we pay all the coupons at maturity. Thus, we have

$$V^{\text{fix}}(t) = \tau_{T_{m+1},T_N} KD(t, T_N)$$

There are also variants of the zero coupon swap which pay the fixed leg at inception. This contract is known as a *reverse zero coupon swap*.

Stub rates

Sometimes it is necessary to trade swaps for time periods which do not exactly fit into the floating rate schedule. In doing so we face the problem that we cannot divide the swap period into periods of the underlying floating rate. This can lead to a period shorter than that of the floating rate at the beginning, respectively at the end or a longer period at the beginning or at the end. These periods are known as *stub periods*.

Short front stub:
The first interest rate period is shorter than the floating rate period for the swap. For instance a swap starts at 04.01.2014 and ends at 01.05.2017. The first interest period is from 04.01.04 to 01.05.04 and then yearly. The parties trading the swap have to agree on the curve used for determining the fixing for the first period. Often the curve determining the rates for the floating rate is chosen to be the forward curve closest to the short period.

Long front stub:
Instead of having a first short period and then having regular payments according to the floating rates period the parties can agree to pay a floating rate for the period constituted by the short stub plus one floating rate period. This is known as a long stub period. Again, the parties have to agree upon the curve from which the first floating rate for this period is calculated.

Short end stub:
In this case the final floating rate period is shorter than the period of the floating rate. Here we need to fix the forward curve from which this period is calculated.

Long end stub:
The last interest rate period is longer than the period of the floating rate.

Cross currency and quanto swaps

We consider two versions of an interest rate swap involving foreign currencies. The variants we consider are

- Cross currency swap
- Quanto swap

Let us fix two schedules $\mathcal{T}_{\text{float}d}$ and $\mathcal{T}_{\text{float}f}$. One schedule corresponds to an index, usually the 3M LIBOR, of the domestic currency and the other schedule corresponds

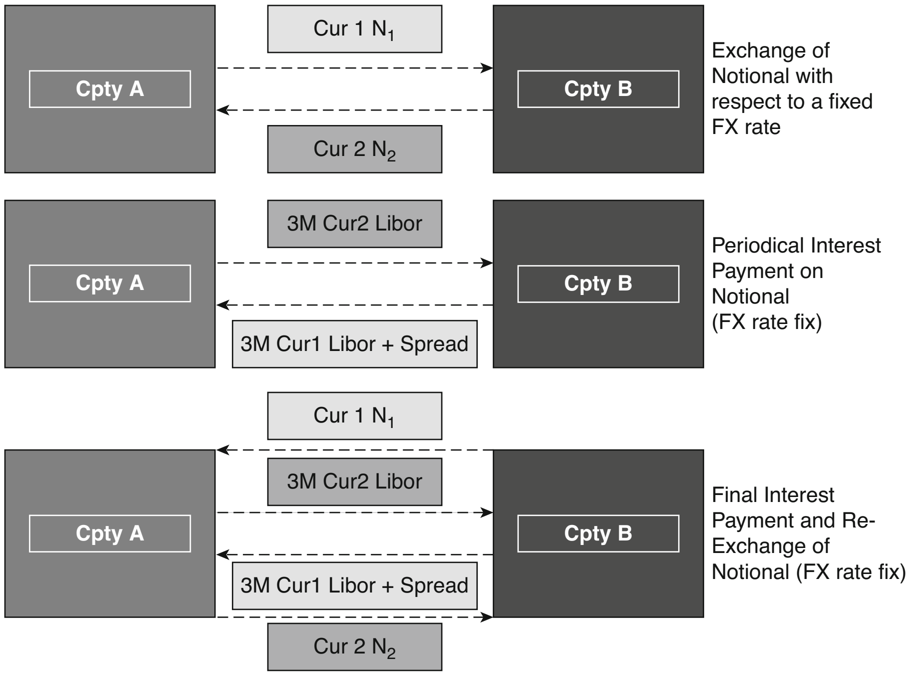


Figure 4.6 Mechanics of a cross currency swap

to an index, again usually the 3M LIBOR, in the foreign currency. At inception of the cross currency swap the notional is exchanged. The conversion of the notional is done according to the current spot exchange rate fx_0. Then, periodical payments are made for each floating leg in the floating leg's currency. At maturity the notional is again exchanged with respect to the rate fx_0.

The market quotes for cross currency swaps are given in terms of the spread appearing in Figure 4.6. The spreads are quoted with respect to a given collateral currency. This currency is usually USD. Some quotes are given in Table 4.4.

The valuation of the cross currency swap is not that easy. We need several curves. First, we need the forwarding curves in the domestic and in the foreign currency. Using these curves we determine the forward rates for the floating rates. For discounting we need to have information on the CSA and in particular which currency is the collateral currency. Suppose the foreign currency is the collateral currency. Then, we use the OIS curve in the foreign currency to discount all foreign cash flows and the domestic yield curve due to the foreign currency as the collateral currency to discount all domestic currency cash flows. This discount factor is denoted by D_{CSA}. Thus, the value of the domestic floating leg is

$$V^{\text{float, dom, nM}}(t) = -N_{\text{dom}} \sum_{i=m+1}^{N_{nM}} \tilde{\tau}_{\text{dom,i}} L_{nM,dom}(t, \tilde{T}_{i-1}, \tilde{T}_i) D_{\text{CSA}}(t, \tilde{T}_i)$$
$$- N_{\text{dom}} D_{\text{CSA}}(t, \tilde{T}_m) + N_{\text{dom}} D_{\text{CSA}}(t, \tilde{T}_{N_{nM}})$$

Table 4.4 *Example of cross currency basis swap spreads – all currencies vs. 3M USD LIBOR as from 01.08.2013*

	REC/PAY EUR	REC/PAY GBP	REC/PAY JPY	REC/PAY CHF	REC/PAY SEK
1Y	−13.75/-18.75	+02.00/−08.00	−15.50/−23.50	−11.00/−21.00	−10.50/−16.50
2Y	−20.75/−25.75	−03.75/−08.75	−32.00/−40.00	−22.00/−28.00	−09.50/−15.50
3Y	−23.25/−28.25	−05.25/−10.25	−44.50/−52.50	−30.00/−36.00	−07.00/−13.00
4Y	−23.75/−28.75	−06.50/−11.50	−53.75/−61.75	−36.00/−42.00	−03.50/−09.50
5Y	−23.25/−28.25	−07.25/−12.25	−60.75/−68.75	−41.75/−47.75	−00.50/−06.50
7Y	−22.00/−27.00	−09.25/−14.25	−68.00/−76.00	−47.50/−53.50	+02.25/−03.75
10Y	−20.00/−25.00	−11.00/−16.00	−66.50/−74.50	−48.00/−54.00	+05.00/−01.00
15Y	−17.75/−22.75	−14.00/−19.00	−51.25/−59.25	−48.50/−54.50	+12.75/+06.75
20Y	−16.25/−21.25	−14.50/−19.50	−35.50/−43.50	−48.00/−54.00	+19.25/+13.25
30Y	−15.00/−20.00	−10.75/−15.75	−19.25/−27.25	−43.75/−49.75	+24.25/+18.25
40Y	−14.25/−19.25	−06.50/−11.50			
50Y	−13.50/−18.50	−02.50/−07.50			

and the value of the foreign floating leg is

$$V^{\text{float, for, nM}}(t) = \sum_{i=m+1}^{N_{nM}} \tilde{\tau}_{\text{for,i}} L_{nM,for}(t,\tilde{T}_{i-1},\tilde{T}_i) D_{\text{OIS for}}(t,\tilde{T}_i)$$
$$-N_{\text{for}} D_{\text{CSA}}(t,\tilde{T}_m) + N_{\text{for}} D_{\text{CSA}}(t,\tilde{T}_{N_{nM}})$$

N_{for} is given by converting the notional in domestic currency with a fixed exchange rate, thus, $N_{\text{for}} = N_{\text{dom}} \cdot fx(0)$.

A quanto swap is in principle a standard fix against floating interest rate swap on some given notional N. The only thing which differs is that the floating rate is a floating rate from another currency. For instance the swap pays in EUR but the rate determining the payments is the USD 3M LIBOR rate. The payments corresponding to this floating rate are paid in EUR with respect to an exchange rate agreed upon at the inception of a deal. This is usually the foreign exchange rate prevailing in the market at inception. Let the exchange rate be $fx(0)$, then, we have for the floating and the fixed leg:

$$V^{\text{fix}}(t) = \sum_{i=m+1}^{n} \tau_i K fx(0) D(t,T_i),$$
$$V^{\text{float}}(t) = \sum_{i=m+1}^{N} \tilde{\tau}_i L_{for}(t,\tilde{T}_{i-1},\tilde{T}_i) fx(0) D(t,\tilde{T}_i)$$

The rates $L_{for}(t,\tilde{T}_{i-1},\tilde{T}_i)$ have to be adjusted by the risks stemming from the varying exchange rate over the lifetime of the quanto swap. By only using $fx(0)$ there is a guarantee that we can keep the value of the exchange rate constant. The corresponding costs for achieving this is valued by adjustment of the rate $L_{for}(t, \tilde{T}_{i-1}, \tilde{T}_i)$ accordingly. To this end a model is necessary to model the stochastic evolution of the exchange rate.

4.4 Float against float interest rate swaps

4.4.1 Money market basis swaps

In a money market basis swap or basis swap for short we consider two floating schedules $\mathcal{T}^{mM}_{\text{float}}$ and $\mathcal{T}^{nM}_{\text{float}}$. The schedules belong to two different indices with a period of m months and n months. Figure 4.7 illustrates the construction of a money market basis swap.

For the pricing of such a swap starting at $\tilde{T}_k$ we need to calculate two present values of floating legs:

$$V^{\text{float, mM}}(t) = \sum_{i=k+1}^{N_{mM}} \tilde{\tau}_{i,mM} L_{mM}(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i), \tag{4.14}$$

$$V^{\text{float, nM}}(t) = \sum_{i=k+1}^{N_{nM}} \tilde{\tau}_{i,nM} L_{nM}(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i) \tag{4.15}$$

Thus, the present value of the swap is

$$V^{\text{MMbasis}}(t) = \sum_{i=k+1}^{N_{mM}} \tilde{\tau}_{i,mM} L_{mM}(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i) - \sum_{i=k+1}^{N_{nM}} \tilde{\tau}_{i,nM} L_{nM}(t, \tilde{T}_{i-1}, \tilde{T}_i) D(t, \tilde{T}_i) \tag{4.16}$$

Even in the single curve framework where forward rates and discount rates are calculated using the same curve this present value would not be exactly 0 but very

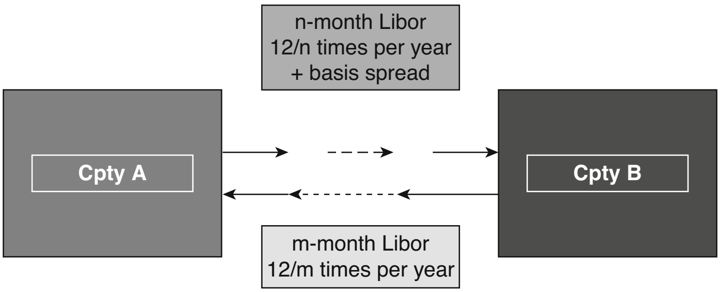


Figure 4.7 Mechanics of a money market basis swap

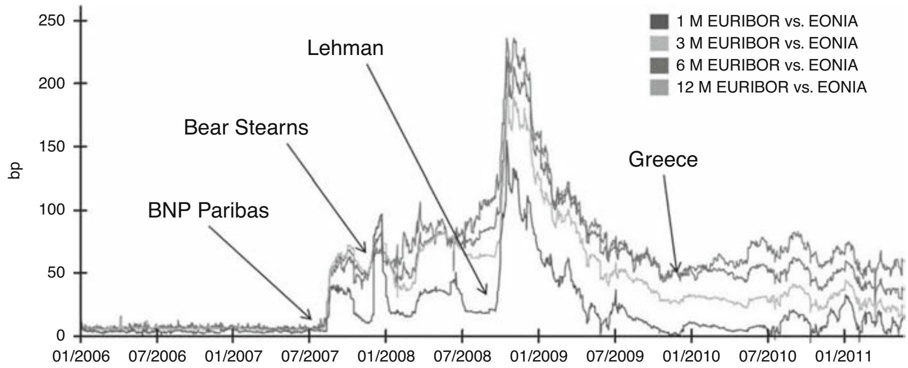


Figure 4.8 Basis swap spread – EUR

very close. The only difference would arise from paying the annuity on a different schedule. In practice the market would have charged some fractions of a basis point to adjust for liquidity or credit risk. In the current markets, however, the plain difference given by Equation (4.17) is significant as Figure 4.8 suggests. In fact we have to adjust the present value of the index with the shorter period with a spread. This spread is known as the *money market basis spread.* The present value of the corresponding swap becomes:

$$V^{\text{Basis}}(0) = \sum_{i=k+1}^{N_{mM}} \tilde{\tau}_{mM,i}(L_{mM}(0,\tilde{T}_{i-1},\tilde{T}_i) + s_{\text{mM/nM}})D(0,\tilde{T}_i) - \sum_{i=k+1}^{N_{nM}} \tilde{\tau}_{nM,i}L_{nM}(0,\tilde{T}_{i-1},\tilde{T}_i)D(0,\tilde{T}_i), \tag{4.17}$$

where $s_{\text{mM/nM}}$ is chosen such that $V^{\text{Basis}}(0)$ is equal to 0.

The quotes for the basis are available in two ways. First, the basis is quoted as if we would enter into a money market basis swap. This is the type of swap we described above. Second, it would be possible to synthesize the position using two swaps, one swap being a payer swap on the m-month index and the other being a receiver swap on the n-month index. Then, the basis is chosen such that the sum of the present values of the swaps is equal to 0. The two possible quotations are very close but not exactly the same. Table 4.5 displays the basis quoted as one swap and the analogue by using two swaps which is the number in brackets.

This case was the standard quoting mechanism until the money market basis widened significantly. In this case we have two fixed schedules $\mathcal{T}_1, \mathcal{T}_2$ with N payments and two float schedules $\tilde{\mathcal{T}}_1$, $\tilde{\mathcal{T}}_2$. We denote the number of LIBOR payments in the corresponding swap by N_s and N_l, $s < l$. The value of the money market basis swap is

Table 4.5 *Basis spread quoted for EUR expressed using 2 swaps and 1 swap in brackets. Quotes as of 01.08.2013*

	3M vs 6M	1M vs 3M	1M vs 6M	6M vs 12M	3M vs 12M
1Y	11.10 (10.9)	11.6 (11.4)	22.7 (22.4)	14.0 (13.7)	25.1 (24.6)
2Y	12.65 (12.5)	12.8 (12.6)	25.5 (25.1)	14.7 (14.4)	27.3 (27.1)
3Y	12.95 (12.8)	13.9 (13.7)	26.8 (26.4)	14.6 (14.4)	27.5 (26.7)
4Y	12.85 (12.6)	14.2 (14.0)	27.1 (26.6)	14.3 (14.1)	27.2 (26.9)
5Y	12.60 (12.4)	14.2 (13.9)	26.8 (26.3)	14.0 (13.8)	26.6 (26.1)
6Y	12.25 (12.0)	14.0 (13.7)	26.3 (25.7)	13.8 (13.5)	26.0 (25.5)
7Y	11.90 (11.7)	13.7 (13.4)	25.6 (25.1)	13.5 (13.3)	25.4 (24.9)
8Y	11.45 (11.2)	13.4 (13.1)	24.9 (24.3)	13.2 (12.9)	24.6 (24.1)
9Y	11.05 (10.8)	13.1 (12.8)	24.1 (23.6)	12.8 (12.6)	23.9 (23.4)
10Y	10.65 (10.4)	12.7 (12.4)	23.4 (22.8)	12.5 (12.2)	23.1 (22.6)
11Y	10.25 (10.0)	12.3 (12.0)	22.6 (22.0)	12.1 (11.8)	22.3 (22.8)
12Y	9.80 (9.6)	12.0 (11.7)	21.8 (21.3)	11.7 (11.5)	21.5 (21.0)
15Y	8.75 (8.6)	10.8 (10.5)	19.6 (19.1)	10.6 (10.3)	19.3 (18.9)
20Y	7.40 (7.2)	9.5 (9.3)	16.9 (16.5)	9.4 (9.2)	16.8 (16.4)
25Y	6.50 (6.3)	8.6 (8.4)	15.1 (14.7)	8.6 (8.3)	15.0 (14.7)
30Y	5.85 (5.7)	8.0 (7.8)	13.8 (13.5)	7.9 (7.8)	13.8 (13.5)
40Y	5.10 (5.0)	7.0 (6.8)	12.1 (11.8)	7.1 (6.9)	12.1 (11.8)
50Y	4.65 (4.5)	6.5 (6.3)	11.1 (10.9)	6.5 (6.3)	11.1 (10.8)
60Y	4.35 (4.2)	6.1 (5.9)	10.4 (10.2)	6.1 (6.0)	10.5 (10.3)

then given by

$$
\begin{aligned}
V^{\text{Basis}}(t) = &\sum_{i=k+1}^{N_l} \tilde{\tau}_i L(t,\tilde{T}_{i-1},\tilde{T}_i) D(t,\tilde{T}_i) - \sum_{i=k+1}^{N_s} \tilde{\tau}_i L(t,\tilde{T}_{i-1},\tilde{T}_i) D(t,\tilde{T}_i) \\
&- \sum_{i=k+1}^{N_s} \tilde{\tau}_i s D(t,\tilde{T}_i)
\end{aligned}
$$

This means if we have a portfolio of two swaps the basis spread is quoted with respect to the fixed time schedule. Table 4.5 shows the corresponding quotes.

After the spreads have widened the market now quotes in terms of one float against float swap. To this end we only need to consider two floating schedules $\tilde{\mathcal{T}}_1$ and $\tilde{\mathcal{T}}_2$ and the corresponding present value is given by Equation (4.17). The spread is quoted against the time schedule of the shorter floating period.

We also introduce different deposit rates for the EUR market, namely EURIBOR and EUR LIBOR rates. Both rates do not coincide and it is actually possible to trade both indices in a money market basis swap. The market quotes for a swap where one party pays the 3M EURIBOR and receives the 3M EUR LIBOR are given in Table 4.6.

Table 4.6 *Spreads for a swap paying 3M EURIBOR against 3M Eur LIBOR for different maturities. Quotes as of 01.08.2013*

Tenor	Bid	Ask
3M	−12.75	−6.75
6M	−14.50	−8.50
9M	−16.75	−10.75
1Y	−18.00	−14.00
2Y	−25.25	−21.25
3Y	−27.75	−23.75
4Y	−28.25	−24.25
5Y	−27.75	−23.75
6Y	−27.5	−23.50
7Y	−26.5	−22.50
8Y	−26.00	−22.00
9Y	−25.25	−21.25
10Y	−24.50	−20.50
11Y	−24.25	−20.25
12Y	−23.75	−19.75
13Y	−23.25	−19.25

4.4.2 Constant maturity swaps (CMS)

Another well known example of a float against float swap is the *constant maturity swap* or *CMS* for short. In a prototypical constant maturity swap we exchange an nY swap rate against a money market mM floating rate. This type of swap cannot be priced using the prevailing yield curves. Furthermore, options on swap rates have to be taken into account. Therefore, we do not discuss the valuation of this type of swap in detail here but in a later chapter of the book we give all the necessary details for the pricing and risk management of such contracts.

CMS quotes

Due to the structure of a constant maturity swap the leg paying the short period floating rate is priced at a lower level than the floating leg paying the swap rate. To this end the market quotes a spread on the floating leg paying the short period rate. In EUR this is either the 3M or the 6M rate. Table 4.7 shows such quotes as observed on the 01.08.2013.

4.4.3 Other type of swaps

There are in fact many more types of swap contracts and we cannot list all the varieties here. Especially, we can combine features of the swaps we have already examined. In the sequel we consider two more types of swaps which we think are important to know but do not belong to any class discussed so far. We begin with an asset swap.

Table 4.7 *CMS quotes with EONIA discounting and against the 3M EURIBOR (top) and 6M EURIBOR (bottom). The left column is the maturity and the row gives the tenor of the underlying swap rate. The quotes are observed on 01.08.2013.*

	2Y index	5Y index	10Y index	20Y index	30Y index
5Y	49.0/55.0	102.1/111.1	160.8/170.8	178.7/198.7	173.0/198.0
10Y	43.7/49.7	84.4/93.4	121.3/131.3	122.0/142.0	120.8/145.8
15Y	36.2/42.2	64.3/73.3	92.4/102.4	87.3/107.3	91.2/116.2
20Y	30.7/36.7	53.6/62.6	72.9/92.9	75.7/95.7	78.5/108.5
	2Y Index	5Y Index	10Y Index	20Y Index	30Y Index
5Y	36.7/42.7	89.9/98.9	148.6/158.6	166.5/186.5	160.9/185.9
10Y	33.4/39.4	74.2/83.2	111.2/121.2	111.9/131.9	110.6/135.6
15Y	27.8/33.8	56.0/65.0	84.1/94.1	79.0/99.0	82.9/107.9
20Y	23.6/29.6	46.5/55.5	65.9/85.9	68.7/88.7	71.5/101.5

Asset swaps

The asset swap consists of two parts. This financial product is a cash bond with a fixed against float interest rate swap. The aim of this product is to transform the interest rate coupon of the underlying bond to either another basis or from fixed rate to floating rate. In a standard asset swap transaction as depicted in Figure 4.9 the investor receives the bond and at the same time enters into a fixed against float interest rate swap. The fixed leg pays the bond's coupon rate and the floating leg pays the n-month floating rate plus a spread. This spread is necessary to reflect the bond's coupon. This coupon is usually higher than the market prevailing swap rate since the bond coupon reflects the credit risk of the issuer of the bond. The transaction is often at par but they can also be at the bond's market price. The latter means that the asset swap value is determined by the difference between par and the bond's market value.

The price of an asset swap is calculated by valuing an interest rate swap and an upfront payment reflecting the purchase asset in return for par. Thus, denoting by $V^{\text{Bond}}(t)$ the bond's price at time t we have:

$$1 = V^{\text{Bond}}(t) + C\sum_{i=1}^{N} \tau_i D(t, T_i) - \sum_{i=1}^{\tilde{N}} \tilde{\tau}_i(L(\tilde{T}_{i-1}, \tilde{T}_i) + s)D(t, \tilde{T}_i) \tag{4.18}$$

Using the pricing formula Equation (4.18) the asset swap spread is determined by Equation (4.19):

$$s = \frac{1 - V^{\text{Bond}}(t) + V^{\text{IRS}}_{T_N}(t)}{\sum_{i=1}^{\tilde{N}} \tilde{\tau}_i D(t, \tilde{T}_i)} \tag{4.19}$$

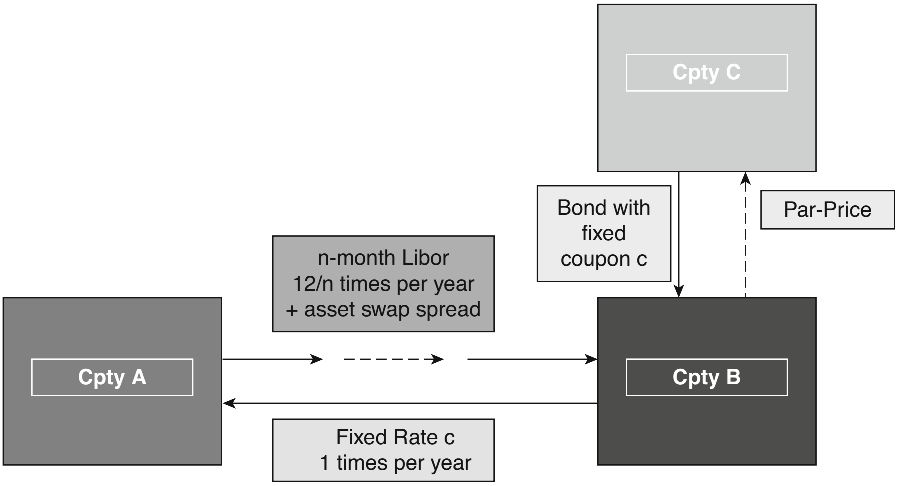


Figure 4.9 Prototypical structure of an asset swap

Example

Let us consider an example of an asset swap. We take a bond paying a coupon of 7.625% semi annually until 21.11.2022 with clean price 99.9250. We wish to par asset swap this bond with respect to the 3M USD LIBOR rate on a 1.000.000 notional. Using the USD OIS curve for discounting we find:

Date	Coupon	LIBOR	Spread	Discount	PV LIBOR	PV spread	PV coupon
21.08.2013	0	0,095	1,206	1	13003,52	946,307	12183,635
21.11.2013	−38125	0,095	1,206	1	−25108,92	950,63	12180,212
21.02.2014	0	0,104	1,206	0,999	13083,63	1034,574	12175,387
21.05.2014	−38125	0,076	1,206	0,999	−25278,86	762,031	12171,488
21.08.2014	0	0,131	1,206	0,999	13347,07	1307,071	12166,249
21.11.2014	−38125	0,148	1,206	0,998	−24536,65	1476,305	12158,439
21.02.2015	0	0,17	1,206	0,997	13721,96	1699,086	12148,936
21.05.2015	−38125	0,154	1,206	0,996	−24431,12	1532,467	12135,363
21.08.2015	0	0,237	1,206	0,995	14348,14	2355,707	12118,171
21.11.2015	−38125	0,287	1,206	0,992	−23014,15	2851,173	12089,429
21.02.2016	0	0,348	1,206	0,99	15372,38	3441,182	12056,301
21.05.2016	−38125	0,381	1,206	0,987	−21957,77	3758,426	12019,725
21.08.2016	0	0,484	1,206	0,983	16615,48	4762,594	11977,167
21.11.2016	−38125	0,56	1,206	0,979	−20026,29	5481,886	11922,497
21.02.2017	0	0,622	1,206	0,974	17791,79	6052,28	11862,601
21.05.2017	−38125	0,609	1,206	0,968	−19346,44	5899,307	11799,842
21.08.2017	0	0,719	1,206	0,963	18533,77	6925,598	11729,894

Example (*Continued*)

21.11.2017	−38125	0,778	1,206	0,956	−17491,02	7440,827	11653,123
21.02.2018	0	0,826	1,206	0,95	19292,48	7840,923	11571,637
21.05.2018	−38125	0,803	1,206	0,943	−17003,76	7575,913	11488,518
21.08.2018	0	0,912	1,206	0,936	19814,46	8533,114	11399,636
21.11.2018	−38125	0,932	1,206	0,929	−15558,35	8652,84	11316,286
21.02.2019	0	0,972	1,206	0,922	20074,72	8961,471	11229,78
21.05.2019	−38125	0,939	1,206	0,915	−15255,92	8585,036	11143,238
21.08.2019	0	1,042	1,206	0,907	20389,29	9452,939	11051,018
21.11.2019	−38125	1,038	1,206	0,899	−14109,39	9331,263	10956,141
21.02.2020	0	1,071	1,206	0,891	20288,51	9542,391	10858,804
21.05.2020	−38125	1,052	1,206	0,883	−13732,82	9291,088	10761,357
21.08.2020	0	1,124	1,206	0,875	20385,32	9836,3	10659,633
21.11.2020	−38125	1,091	1,206	0,866	−13134,7	9449,752	10555,948
21.02.2021	0	1,116	1,206	0,858	19911,95	9569,9	10450,484
21.05.2021	−38125	1,063	1,206	0,849	−13111,92	9025,296	10346,921
21.08.2021	0	1,156	1,206	0,84	19844,76	9712,585	10238,423
21.11.2021	−38125	1,137	1,206	0,831	−12218,17	9451,983	10128,609
21.02.2022	0	1,157	1,206	0,822	19424,76	9511,047	10017,662
21.05.2022	−38125	1,098	1,206	0,813	−12272,78	8928,312	9909,371
21.08.2022	0	1,188	1,206	0,804	19249,65	9554,709	9796,609
21.11.2022	−1038125	1,176	1,206	0,795	−11367,77	9349,253	9683,116

From the above cash flow table we can calculate the present value from the LIBOR leg and the coupon leg. The spread is now determined such that the value of both legs is the same. We find $s = 487.3571156$. Using this spread we have PV of the floating leg is 240833.5651 + the spread leg 430111.6493 making together 670945.2144. The coupon leg is -670945.2144. This is exactly the same value but of opposite sign. The initial payment to account for the bond price is $99.925 - 100 = -0.075$ times the notional.

Structured swaps

There are no restrictions on how to structure the coupon of a swap. The coupon can be linked to indices such as constant maturity swap rates including options or even to the performance of an equity. The general structure is given in Figure 4.10.

The valuation methodology is always the same. We have to evaluate the structured swap leg and the floating leg. At inception the swap should have a value of 0. To this end

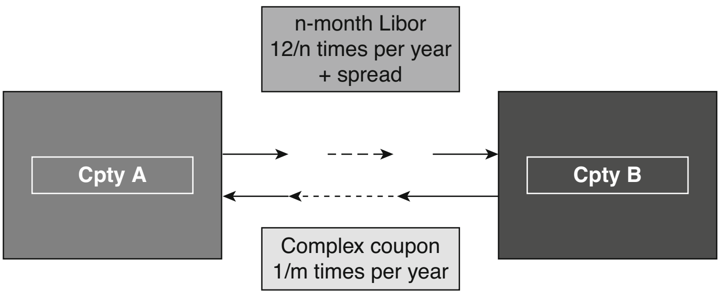


Figure 4.10 General structured swap

it might be necessary to put a spread on the floating rate. This spread is determined by the difference between the structured leg and the floating leg. The spread is an annual spread.

4.5 Reading list

Since swaps are one of the main fixed income instruments they are covered in the already cited reference Tuckman and Serrat (2012). This text covers the conventions for swap trading in different markets. Furthermore, the linkage of the swap and the bond markets is described. Many different types of swap contracts are considered and the authors treat the market conventions and the practical details of swaps. Furthermore, the valuation methodologies and the task of risk management are considered. The text is very practitioner minded. The mathematical theory for swaps on the other hand can be found in Andersen and Piterbarg (2010a) or Brigo and Mercurio (2006). The mathematical definitions of curves and how the corresponding swap rates can be calculated are considered there. The changes after the credit crisis can be found in the papers Morini (2009), Henrad (2007) and Henrad (2009).

5 Using Curves

5.1 Introduction and objectives

Up until now for valuing a product we have assumed that the necessary yield curves are given and the necessary rates can be interpolated. For instance when considering a swap we assumed the index curve to retrieve the forward rates and the discounting curve for calculating the discount factors to be given. In this chapter we consider several aspects of yield curves themselves. First, we outline methodologies for interpolation and, second, we wish to find ways to set up yield curves for practical purposes. Section 5.2 shows how to set up an interpolator starting from a discrete curve. To this end we consider different methodologies for interpolating discrete data. Section 5.3 is meant to describe the process of setting up curves. Here we need to specify the market data we wish to use as input and the methodology for using the input data. As outlined with respect to the current market practice care has to be taken when building yield curves. The good old days when a single curve per currency could be used are long gone. A hierarchy of curves has to be set up to consistently price future cash flows. The curves depend on the purpose of application and on the prevailing collateral agreement. We consider all aspects of curve construction. We determine input data, bootstrap and calibration methodologies for single and multi currency set-ups. A subsection on validating the curves is also given. We close this chapter by deriving risk measures associated with curve and rates movement in Section 5.4. Standard all-purpose measures such as *BPV*, *Duration* and *Convexity* are introduced but we also give a short introduction to *principal component analysis* which is very useful for understanding the parametric methods which are applied by central banks to set up their curves.

Finally, we show how the interpolation methods can be extended to higher dimensions. We consider surfaces and cubes on which we wish to apply interpolation. This is very important for handling options. For options practitioners consider a number called volatility. This number is not only time-dependent such as a yield but also depends on other parameters. This makes it necessary to consider higher dimensional structures. We show some examples where the underlying financial instruments will be introduced in the next chapter.

5.2 Interpolation methods

We consider the setting of 2.3.5. Thus, we have two sets of points $\mathcal{T}$ and $\mathcal{X}$ given by

$$\mathcal{T} = \{t_1, t_2, \ldots, t_N\}, \quad t_1 < t_2, \ldots < t_N$$
$$\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$$

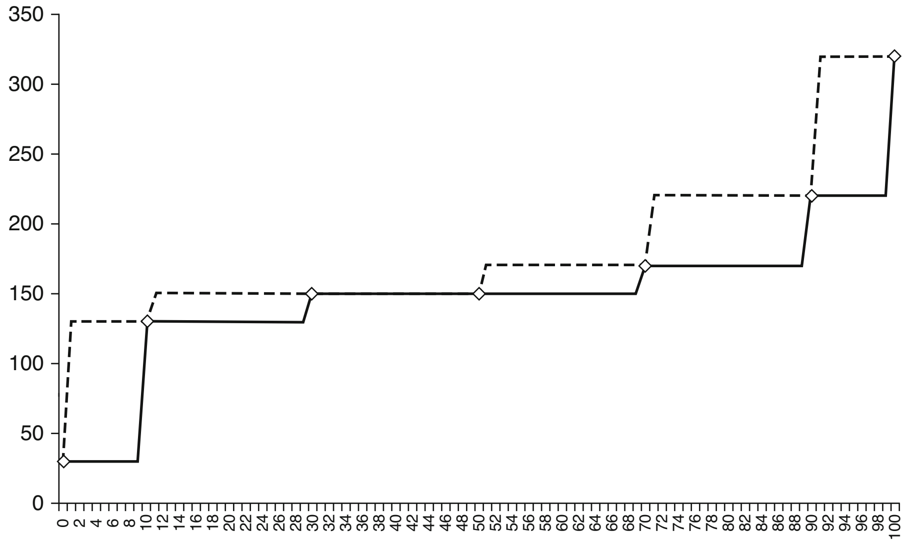


Figure 5.1 Piecewise constant interpolation from the left (solid line) and from the right (dashed line)

and denote the (discrete) curve by $\mathcal{C}_d = \{\mathcal{T}, \mathcal{X}\}$, see (2.15). Now, we wish to determine possible functions g to determine a continuous curve $\mathcal{C}_c$ as defined in Equation (2.16). In the sequel we consider local interpolation methods. Such methods specify the interpolation values locally. We implicitly assume that local changes only have local effects. Later we also consider methods which use a parametric function where this assumption is abandoned.

5.2.1 Constant interpolation

First, we consider a very simple interpolation method. The constant interpolation from the left, respectively from the right is given by considering the functions:

$$\mathcal{C}_c(t) := x_{m(t)}, \quad \mathcal{C}_c(t) := x_{m(t)+1}$$

The value $m(t)$ is defined as $m(t) := \max\{i | t \leq t_i \in \mathcal{T}\}$. Despite the fact that the interpolator is simple and easy to code we observe that it is not continuous and often this is not a reasonable approach for practical problems. Figure 5.1 shows the resulting curve $\mathcal{C}_c$.

5.2.2 Linear interpolation

We consider a more sophisticated method called *linear interpolation*. For $t_{i-1} \leq t \leq t_i$ it is given by

$$\mathcal{C}_c(t) := \frac{t - t_{i-1}}{t_i - t_{i-1}} x_i + \frac{t_i - t}{t_i - t_{i-1}} x_{i-1}$$

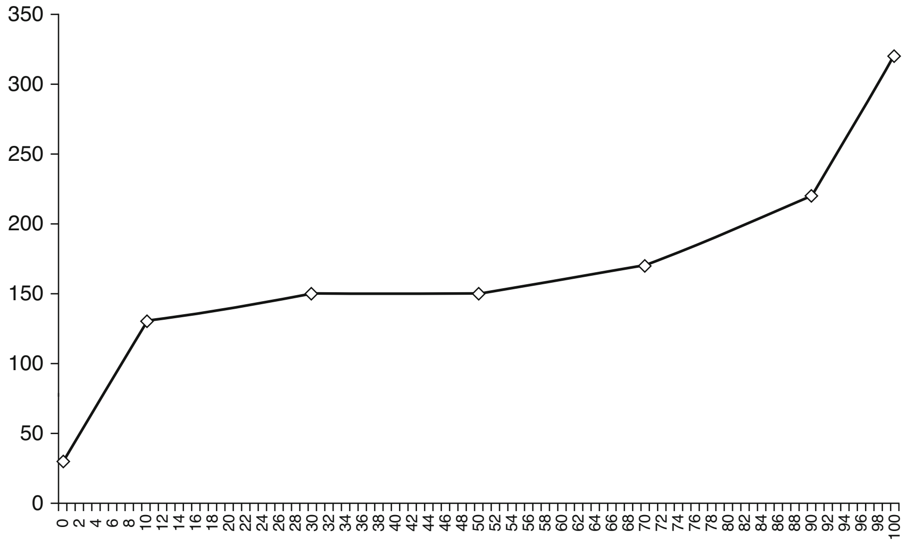


Figure 5.2 Piecewise constant interpolation (from the left)

We observe that the resulting curve $\mathcal{C}_c$ is continuous but not necessarily differentiable at the points $\mathcal{C}_d$, see Figure 5.2.

5.2.3 Cubic spline interpolation

If we consider the interpolation methods we specify some functional dependence. For the *spline interpolator* and $t_{i-1} \le t \le t_i$ the function is

$$\mathcal{C}_c(t) := a_i(t-t_i)^3 + b_i(t-t_i)^2 + c_i(t-t_i) + d_i$$

This gives us $4(N-1)$ unknown coefficients. Using the fact that we wish that the initial points coincide on $\mathcal{C}_d$, the resulting curve is continuous and differentiable. We have $3N-4$ constraints ($N-1$ values at left interval bounds, $N-1$ values at right interval bounds and $N-2$ constraints from differentiability). Together with these values we often define constraints on the derivatives $\left.\frac{\partial C_c(t_{i-1})}{\partial t}\right|_{t=t_i} = s_{i-1}$ and $\left.\frac{\partial C_c(t_i)}{\partial t}\right|_{t=t_i} = s_i$ where s_i are free parameters and can be determined by the modeler. The choice of the letter s suggests that the derivative determines the slope. Thus, setting the values s_i poses constraints on the slope of the interpolating function.

To fully determine the problem we need another N constraints. For instance we consider

(i) $\mathcal{C}_c(t_i) = x_i$
Differentiability $\rightarrow N-2$ constraints and $\frac{\partial^2 \mathcal{C}_c(t)}{\partial t^2}(t_1) = \frac{\partial^2 \mathcal{C}_c(t)}{\partial t^2}(t_N) = 0$ (Natural Cubic Spline).

(ii) $\mathcal{C}_c(t_i) = x_i$
Differentiability $\to N-2$ constraints and linear interpolation to the very left and constant to the very right.

(iii) $\mathcal{C}_c(t_i) = x_i t_i$
Differentiability $\to N-2$ constraints and linear on the very right and quadratic on the very left. This is outlined in McCulloch, J. H. and Kochin (2000).

(iv) $\mathcal{C}_c(t_i) = x_i$ and $\mathcal{C}_c(t_i) = x_i t_i$
Differentiability $\to N-2$ constraints and choosing the values for c_i. They are the slope at t_i of the quadratic that passes through (t_j, x_j), $j = i-1, i, i+1$, for c_1, $j = 1,2,3$ and for c_N, $j = N-2, N-1, N$, deBoor (1978).

(v) $\mathcal{C}_c(t_i) = x_i$
Differentiability $\to N-2$ constraints (differentiability) and uses Hyman, J. M. (1983) where values for c_i are specified.

For instance for method (i) we get:

$$\begin{aligned}
a_i(t_{i+1}-t_i)^3 + b_i(t_{i+1}-t_i)^2 + c_i(t_{i+1}-t_i) &= x_{i+1}-x_i \\
3a_{i-1}(t_i-t_{i-1})^2 + 2b_{i-1}(t_i-t_{i-1}) + c_{i-1} - c_i &= 0 \\
6a_{i-1}(t_i-t_{i-1}) + 2b_{i-1} &= 0 \\
b_1 &= 0 \\
6a_{N-1}(t_N-t_{N-1}) + 2b_{N-1} &= 0
\end{aligned}$$

Figure 5.3 shows the resulting interpolation.

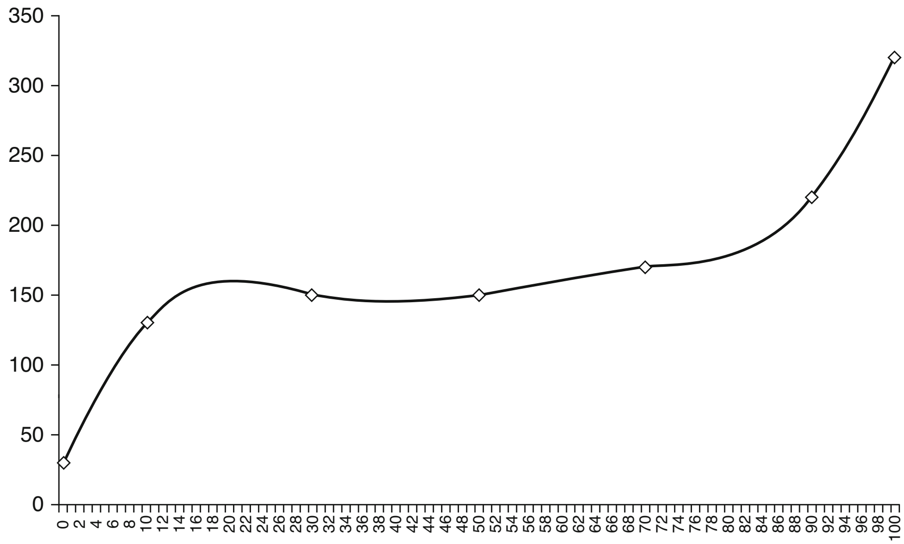


Figure 5.3 Piecewise cubic spline interpolation

We have to remark that problems can arise by simply applying one of the methods:

- Oscillatory behavior:
  Some of the spline methods have shown to lead to oscillatory behavior.
- Monotonicity:
  The interpolators do not necessarily lead to monotone interpolation. Suppose we have the same values x at T_{i-1} and T_i a general spline interpolation mechanism does not guarantee a value of x for all $t \in [T_{i-1}, T_i]$.
- Non-local methods:
  Since we use points from more than one interval for deriving the coefficients of the interpolator the method is not local but spreads around different intervals.

To overcome the problems several ideas have been put forward. These include

- Hermite splines:

$$d_{i-1} = x_{i-1}, \quad c_{i-1} = s_{i-1}, \quad b_{i-1} = \frac{3\frac{x_i - x_{i-1}}{t_i - t_{i-1}} - s_i - 2s_{i-1}}{t_i - t_{i-1}},$$

$$a_{i-1} = \frac{-2\frac{x_i - x_{i-1}}{t_i - t_{i-1}} - s_i - s_{i-1}}{(t_i - t_{i-1})^2}$$

  The values s_i are chosen such that the interpolation is as local as possible.

- Bessel splines:

$$s_{i-1} = \frac{(t_{i-1} - t_{i-1})\frac{x_i - x[i-1}{t_i - t_{i-1}}) + (t_i - t_{i-1})\frac{x_{i-1} - x_{i-2}}{t_{i-1} - t_{i-2}}}{t_i - t_{i-1}}$$

  The original method can be improved by considering filtering. This means we have to set the values for s_{i-1} to some different value. For instance we can take
  – Hyman83 filter for monotonicity, see Duffy and Germani (2013).
  – Hyman89 filter, from LeFloch and Kennedy (2013) for monotonicity is much more complicated and we do not reproduce it here. It can be found in Dougherty and Edelman and Hyman (1989) and also Duffy and Germani (2013).

- Harmonic splines:

$$\frac{1}{s_{i-1}} = \frac{t_{i-1} - t_{i-2} + 2(t_i - t_{i-1})}{3(t_i - t_{i-2})}\frac{t_{i-1} - t_{i-2}}{x_{i-1} - x_{i-2}} + \frac{2(t_{i-1} - t_i) + t_i - t_{i-1}}{3(t_i - t_{i-2})}\frac{t_i - t_{i-1}}{x_i - x_{i-1}}$$

We have used the algorithms described in Duffy and Germani (2013) to produce Figure 5.4 to show the effect on the forwards using the above described interpolation methodologies.

In the sequel we consider further adjustments to splines: *Kruger splines*, *Tension splines* and the *Monotone convex method*.

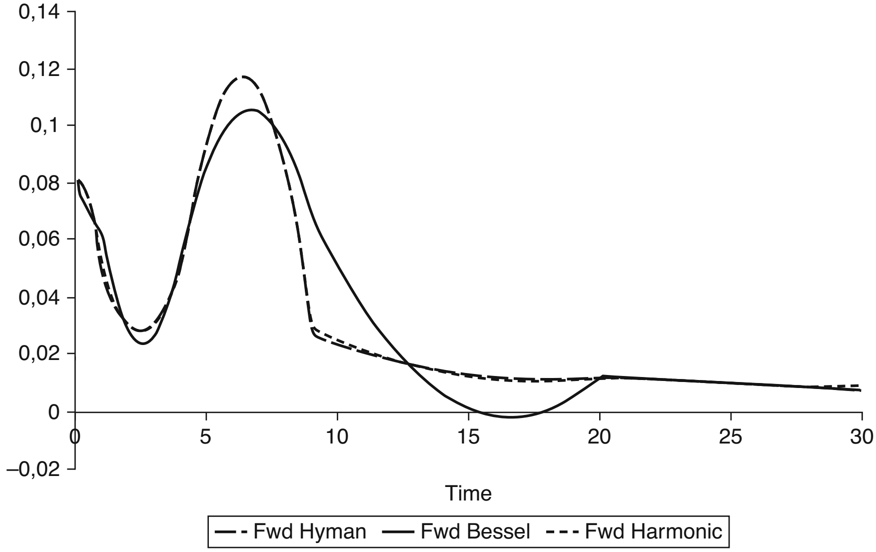


Figure 5.4 Forward curves for Bessel, Bessel with Hyman filter and Harmonic spline interpolation

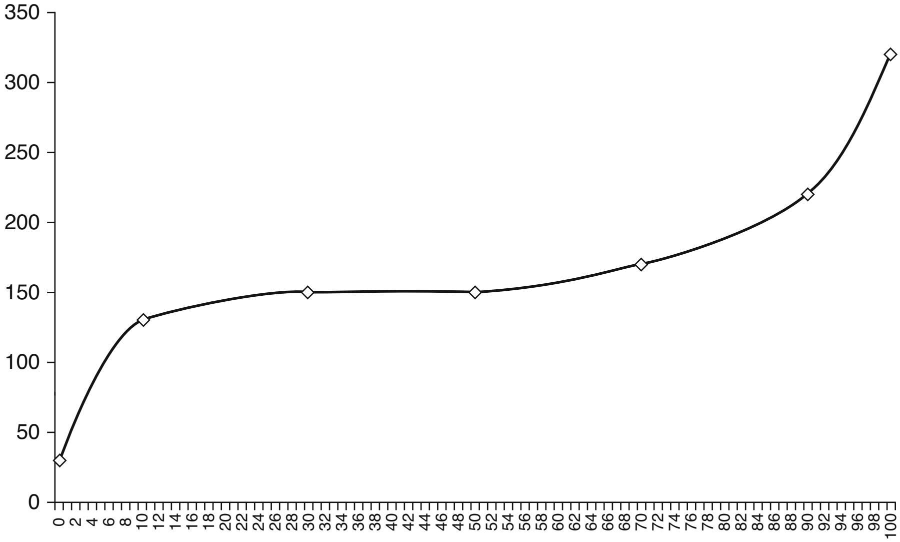


Figure 5.5 Piecewise Kruger spline interpolation

Kruger splines

The interpolation using Cubic splines is in general not monotonic. We consider a version which preserves monotonicity. This version is given in Kruger (2003).

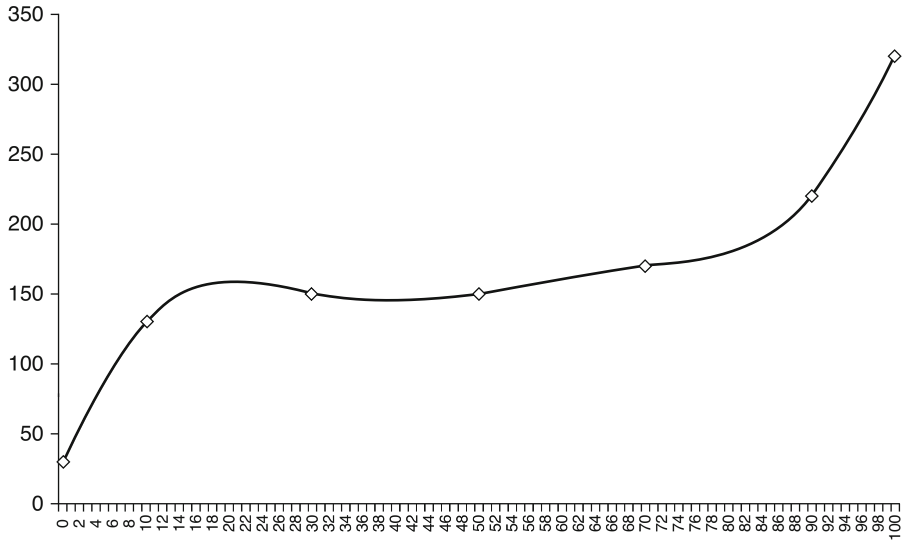


Figure 5.6 Piecewise tension spline interpolation

Fix the first order derivative at each point t_i by setting

$$\frac{\partial \mathcal{C}_c}{\partial t}(t_i) = \frac{\partial \mathcal{C}_{i,i+1}}{\partial t}(t_i) = \begin{cases} 0 & , \text{ slope changes sign at } t_i \\ 2\left(\frac{t_{i+1}-t_i}{x_{i+1}-x_i} - \frac{t_i-t_{i-1}}{x_i-x_{i+1}}\right)^{-1} & , \text{ else} \end{cases}$$

and

$$\frac{\partial \mathcal{C}_c}{\partial t}(t_1) = \frac{3}{2}\frac{x_2-x_1}{t_2-t_1} - \frac{\mathcal{C}_c(t_2)}{2}, \quad \frac{\partial \mathcal{C}_c}{\partial t}(t_N) = \frac{3}{2}\frac{x_N-x_{N-1}}{t_N-t_{N-1}} - \frac{\mathcal{C}_c(t_{N-1})}{2}$$

The resulting interpolator is displayed in Figure 5.5

Tension splines

The interpolation using splines is in general not local. By local we mean that a change affects not only points in the neighborhood but other or even all points on the curve. We consider a version which introduces a parameter which is used to localize the spline. This parameter is called *tension*. The corresponding splines are called *tension splines*.

The idea is to replace the constraints condition on the second derivative by:

$$\begin{aligned} \frac{\partial^2 \mathcal{C}_c}{\partial t^2}(t) - \sigma^2 \mathcal{C}_c(t) = {} & \frac{t_{i+1}-t}{t_{i+1}-t_i}\left(\frac{\partial^2 \mathcal{C}_c}{\partial t^2}(t_i) - \sigma^2 \mathcal{C}_c(t_i)\right) \\ & + \frac{t-t_i}{t_{i+1}-t_i}\left(\frac{\partial^2 \mathcal{C}_c}{\partial t^2}(t_{i+1}) - \sigma^2 \mathcal{C}_c(t_{i+1})\right) \end{aligned} \tag{5.1}$$

This Equation (5.1), can be solved by integration if σ is known. Requiring continuity of the first derivative we obtain the values for the second derivative at the boundaries. The parameter σ is called the *tension* of the spline. The resulting interpolator is shown in Figure 5.6. We have used different tensions, that is different values for σ to generate Figure 5.7. Here we have to remark that using different tensions might lead to unstable behavior of the corresponding yield curves. Thus, the tension has to be used with care.

Finally, we show all the interpolation methods for one example in Figure 5.8. Clearly, changing the interpolation leads to different values of a financial instrument such as an FRA or a swap.

5.2.4 Which method to use and how?

We have outlined several interpolation methods. Now, we wish to clarify how interpolation on a yield curve should be done. First of all, the curve should not include arbitrage. To this end the forwards have to be positive. The calculated forwards must be as continuous as possible and as local as possible. The latter means that if we change a rate the effect should only be observable at at time close to this rate. The latter is very important for computing risk factors such as sensitivities with respect to rate shifts since such exposure has to be hedged with instruments having a specific maturity. As a rule of thumb we take: the more local the shift effects on the curve the more local are the hedges. The final restriction is with regard to the forwards. Do the forwards behave in a stable way? To this end it is a common procedure to consider a rates shift of the input rates and quantify the effect on the calculated forwards.

In Chapter 2 we outlined different rates using different compounding methodologies. Thus, it is possible to transform the rates to different compounding and day count conventions. The most important method is to transform spot, zero, forward or discount rates into one another. Using such transforms we can use the curves interchangeably. Furthermore, given some monotone function such as the logarithm to transform rates we are also able to transform back using the inverse function. Such transforms are sometimes useful when applying interpolation.

Usually, the interpolation is done on the logarithm of the discount factors using one or the other interpolation method. But in practice many different ways of applying interpolation is found. The modeler has to decide which method leads to the most stable and adequate results.

Let us quickly look to some interpolation methods. The linear interpolator fulfills many of the requirements but the forwards are not continuous. One standard method, the natural cubic spline interpolation, does not produce positive forwards since it is not monotonic and since it uses information from three time points it is also not local. Some of the correction methods also cannot keep the non-localness of the interpolator.

5.3 Curve construction

After outlining different methodologies for interpolation given a discrete curve $\mathcal{C}_d$ we wish to outline how to set up curves for discounting and forwarding which are in line with given market quotes for different financial instruments. In general we divide the curve construction into two types of methods:

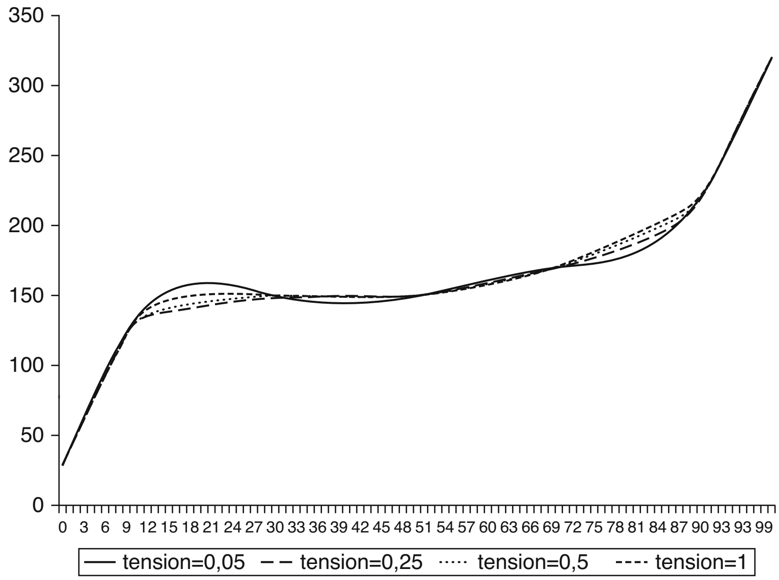


Figure 5.7 Different tension settings and the consequences for interpolation

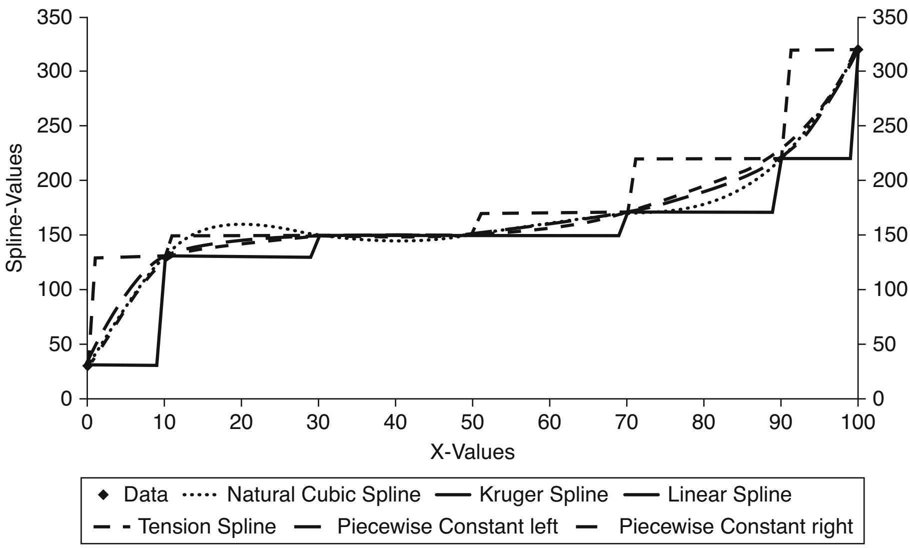


Figure 5.8 Comparison of all interpolation methods

- Bootstrapping
- Optimization

To apply the *Bootstrapping* methodology we sort the underlying market rates with respect to maturity. Then, we start with the shortest maturity and build up the corresponding curve step by step taking into account the instrument with the next maturity. If additional data is necessary to construct the curve we apply one of the above introduced interpolation methodologies.

The second method assumes that we have market data, interpolation methodology and a pricing algorithm available as well as an optimization methodology. Each pricing algorithm needs input parameters from one or more yield curves, for instance discounting or forward rate curves. By specifying a set of market quoted instruments and a measure which evaluates the current fit to market rates that can be minimized by the optimization methodology. Again we can run into difficulties if we need more market data to determine all unknown values for the curves. If this is the case, we can again rely on interpolation methods to create market data synthetically.

For both methods we choose interest bearing instruments together with market conventions and interpolation methods. We wish to find curves such that given market quotes are priced (nearly) correct. We start with the example determining a single curve for discounting and forwarding. The multi-curve case is also examined.

5.3.1 Bootstrapping

First, we consider the bootstrapping methodology by giving an example. Our aim is to set up a curve such that quoted instruments can be priced on this curve consistently. We wish to determine the discount factors given the market quotes for

- Deposit rates up to 1Y
- Swap rates with respect to $6M$ floating rate from 1Y onwards

Since a cash flow paid today does not have to be discounted we choose $D(0,0) = 1$. Then, we consider the deposit rate y_T with maturity T. Since we know that a deposit rate is simply compounded and linked to the discount factor by Equation (2.8) we set $D(0,T) = (1 + \tau(0,T)y_T)$. This procedure is used up to $1Y$. Precisely speaking, deposit rates are quoted for the period from settlement to maturity and, thus, the actual discount factor is $D(nd,T)$. For the EUR market for instance we have $n = 2$ and we need to include the ON and SN rates to get the exact discount factor, that is $D(0,T) = (1 + \tau(0,1d)y_{ON}(1 + \tau_{1d,2d})y_{SN}(1 + \tau_{2d,T})y_T)$. From one year onwards only swap rates are quoted on a yearly basis. The first problem we face here is that for $1Y$ we have two possible rates which we can apply for setting up the curve. First, we have the $1Y$ deposit rate and, second, we have the $1Y$ swap rate. We choose the rate which determines the $1Y$ value. Then, we consider the second available swap rate. In the sequel we consider two type of swap rates. We assume that the quotes are available every year and the fixed rate is paid (a) half-yearly and (b) yearly. Let us start with (a) by taking the USD market. The swap rate is given by

$$SR_{0,2Y}(0) = \frac{1 - D(0,2Y)}{\sum_{j=1}^{4} \tau_j D(0, j \cdot 6M)} \tag{5.2}$$

Here we face a problem. To determine the discount factors $D(0,1Y6M)$ and $D(0,2Y)$ we only have one piece of information, namely the $2Y$ swap rate. To this end we need to synthetically create a $1Y6M$ swap rate. Then, using

$$SR_{0,1Y6M}(0) = \frac{1 - D(0,1Y6M)}{\sum_{j=1}^{3} \tau_j D(0, j \cdot 6M)}$$

we can determine the discount factor $D(0,1Y6M)$ as follows:

$$D(0,1Y6M) = \frac{1 - A(0,1Y) \cdot SR_{0,1Y6M}(0)}{1 + SR_{0,1Y6M}(0) \cdot \tau_{1Y,1Y6M}} \tag{5.3}$$

With this result, Equation (5.3), the discount factor $D(0,2Y)$ can uniquely be determined from (5.2). The synthetic swap rate $SR_{0,1Y6M}(0)$ depends on the rates $SR_{0,1Y}(0)$ and $SR_{0,2}(Y)0$ and the applied interpolation algorithm.

The general recipe for setting up the curve is, thus, synthetically determine the missing rates by using interpolation. Using the synthetic rates it is possible to uniquely determine the missing discount factors.

Suppose we have calculated the discount factors up to nY years and we are given the $(n+1)Y$ swap rate. We synthetically determine the swap rate $SR_{0,nY+0.5}(0)$. With this swap rate we use

$$D(0, nY + 0.5) = \frac{1 - A(0, nY) \cdot SR_{0,nY+0.5}(0)}{1 + SR_{0,nY+0.5}(0) \cdot \tau_{nY,nY+0.5}}$$

This illustrates essentially the idea of bootstrapping. Now, we consider case (b), for instance the EUR market. Suppose we are interested in deriving the discount factor for $T = 3.5Y$. The bootstrapping becomes more simple since for the $2Y$ swap rate we have

$$SR_{0,2Y}(0) = \frac{1 - D(0,2Y)}{\sum_{j=1}^{2} \tau_j D(0, j \cdot 1Y)} \tag{5.4}$$

This means we can directly solve Equation (5.4) for the discount factor $D(0,2Y)$ without using a synthetic rate as in case (a). We proceed by solving for the discount factors up to $4Y$. Now, to get the discount factor $D(0,3.5Y)$ we simply apply interpolation to the derived discount curve. However, there might exist a time from which onwards the rates are quoted for different intervals than $1Y$. In this case we have to apply interpolation as described in (a) to get the yearly rates before we are able to derive the discount factors.

After this introductory example we turn to setting up yield curves in the current markets involving discounting and forward curves. Before the new market paradigm had been established it was common sense to create one single curve for calculating the discount factors and the forward rates. This curve was called the *swap curve*. The standard rate curve was built using:

- ON, TN (for curve defined from today)
- Spot: SN, SW, 1M, 2M, etc. (at least up to the first IMM date)
- Futures (8 contracts, maybe one serial)
- Swaps (2Y, 3Y, .., 30Y and beyond)

This is not the way the markets work anymore!

OIS curves

The above outline has already highlighted the essence of bootstrapping. To implement the bootstrapping procedure for the new multiple curve setting we always start with a discount curve. In the sequel we take the OIS market quotes, see for instance section 4.3.3 and the pricing formula (4.11) for OIS swaps for bootstrapping. Let us first recall the standard ISDA for the OIS rate during a given accrual period. To this end we denote by N the number of business days in the period, τ_i the year fraction between the consecutive business days and, finally, r_i is the OIS rate published for the specific date. Then, the interest over the period is given by

$$\prod_{i=1}^{N}(1+\tau_i r_i) - 1$$

For the OIS rates we observe that there are some specialities due to the nature of this rate. Especially the short end of the curve depends on the rates announcement of the corresponding central bank. This announcement takes place on meeting dates. We observe that between meeting dates there is a fluctuation around an average value. Furthermore, there are often seasonality effects observable at each quarter or end of year. This can be considered in the bootstrapping of the OIS curve. Thus, for setting up reasonable OIS curves it might become necessary to tweak the curve a little bit. For one example see Section 5.3.1.

First, we assume that seasonality is incorporated in the rates r_i by adding a spread s_i. Let us start with the short and the mid part of the curve. The constituting times of the short part of the curve are given by the dates

$$\mathcal{T}_{\text{short}} = \{0 = t_0, t_1, \ldots, t_n\}$$

This time schedule may incorporate rates at meeting dates as well as regular tenor OIS rates. The corresponding quoted rates at these periods are

$$\tilde{r}_{\mathcal{T}} = \{\tilde{r}_1, \ldots, \tilde{r}_n\}$$

Our aim is to determine the rates r_i. Since these rates are daily rates there is no chance to uniquely determine these rates from much fewer quoted rates. For example the relation of the rates $\tilde{r}_1$ and the rates r_i is

$$\underbrace{\tilde{r}_1 \tau_{t_0,t_1}}_{=:\text{fixed}} = \underbrace{\prod_{i=1}^{N_{t_1}}(1+\tau_{t_i}(r_i + s_i))}_{=:\text{floating}}$$

with N_{t_1} the number of business days in the interval from today to t_1. With the assumption of constant rates r_i the equation can be simplified and we consider the rates $r_{i,i+1}$ between the given time points to get

$$\tilde{r}_1 \tau_{t_0,t_1} = \prod_{i=1}^{N_{t_1}} (1 + \tau_i (r_{t_0,t_1} + s_i)$$

If the seasonality effects are determined by some model and are known we can solve for r_{t_0,t_1}. This can be done for instance by assuming some parametric form for s_i or simply taking $s_i = c$ with some constant c.

Using the rate r_{0,t_1} it is then possible to calculate the rate r_{0,t_2} by assuming constant rates between t_1 and t_2 and solving

$$\tilde{r}_{t_2} \tau_{t_0,t_2} = \prod_{i=1}^{N_{t_1}} \tau_i (r_{t_0,t_1} + s_i) \cdot \prod_{i=t_1+1}^{N_{t_1,t_2}} \tau_i (r_{t_1,t_2} + s_i) - 1$$

Here N_{t_1,t_2} denotes the number of business days between t_1 and t_2. For any period which is added a new factor enters into the equation but assuming that we aim to determine the rates for the full period we only have one single unknown number and can solve for this quantity. For the next period from t_2 to t_3 we, thus, get

$$\tilde{r}_{t_3} \tau_{t_0,t_3} = \prod_{i=1}^{N_{t_1}} \tau_i (r_{t_0,t_1} + s_i) \cdot \prod_{i=t_1+1}^{N_{t_1,t_2}} \tau_i (r_{t_1,t_2} + s_i) \cdot \prod_{i=t_2+1}^{N_{t_2,t_3}} \tau_i (r_{t_2,t_3} + s_i) - 1$$

Finally, we consider the long period of the OIS curve. Using Equations (4.8) and (4.9) leading to Equation (4.10) we take the OIS swap rates which are calculated by

$$SR_{0,N}(0) = \frac{1 - D(0, T_N)}{A(0, T_N)}$$

We can then use the techniques for single curve bootstrapping. Furthermore, based on the observation that a risk free floating rate note trades at par another method based on matrix inversion can be applied. That means if we denote the forward rate calculated on the overnight curve by FRA^{OIS} we have

$$\begin{aligned} 100 &= \sum_{i=1}^{N} FRA^{\text{OIS}}(0, T_{i-1}, T_i) \tau_i D(0, T_i) + 100 \cdot D(0, T_N) \\ &= \sum_{i=1}^{N} \tau_i SR_{0,T_N} D(0, T_i) + 100 \cdot D(0, T_N) \end{aligned} \tag{5.5}$$

The latter also implies that the fixed rate bond paying the rate SR_{0,T_N} is at par. This is due to the fact that the floating leg's present value is the same as the fixed leg's for a fair swap.

Denoting by $SR_{0,T}(0)$ the OIS Swap rate to get a bond worth par which is 100 we, by Equation (5.5), must have:

$$\begin{pmatrix} SR_{0,T_1}(0)+100 & 0 & 0 & \cdots & 0 \\ SR_{0,T_2}(0) & SR_{0,T_2}(0)+100 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ SR_{0,T_N}(0) & SR_{0,T_N}(0) & \cdots & \cdots & SR_{0,T_N}(0)+100 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ \vdots \\ 100 \end{pmatrix}$$

This matrix equation can be solved for the discount factors d_i by matrix inversion and matrix vector multiplication.

Example

We take $N = 4$ and consider the calculation on a yearly basis. To simplify notation we set $\tau_i = 1$. The equation for the given rates due to (5.5) is given by:

$$\begin{pmatrix} 0.009 \\ 0.013 \\ 0.017 \\ 0.019 \end{pmatrix} \times \begin{pmatrix} 100.9 & 0 & 0 & 0 \\ 1.3 & 101.3 & 0 & 0 \\ 1.7 & 1.7 & 101.7 & 0 \\ 1.9 & 1.9 & 1.9 & 101.9 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$

Inverting the matrix and multiplying by the par vector gives:

$$\begin{pmatrix} 0.009910803 & 0 & 0 & 0 \\ -0.000127187 & 0.009871668 & 0 & 0 \\ -0.000163541 & -0.000165013 & 0.009832842 & 0 \\ -0.000179373 & -0.000180988 & -0.000183341 & 0.009813543 \end{pmatrix} \times \begin{pmatrix} 100 \\ 100 \\ 100 \\ 100 \end{pmatrix}$$
$$= \begin{pmatrix} 0.991080278 \\ 0.974448131 \\ 0.950428729 \\ 0.926984116 \end{pmatrix}$$

For the whole bootstrapping we use the quotes displayed in Table 5.1.

In the next two subsections we consider a typical seasonality effect and expand on synthetic rates.

Turn of Year (TOY)

The *Turn of Year* effect is the name of an effect that is observed in market data quotation and time series data. This effect is observed as a jump in rates spanning an interval with technical dates. Typically – and hence the name – this is the end of the year but can be other dates as well. In Figure 5.9 we consider the time series for the $1M$ EURIBOR floating rate. We observe jumps of different heights in December. Usually,

Table 5.1 *Instrument selection for setting up the EUR OIS curve (EONIA curve). Quotes are as of 02.08.2013*

Description	Bloomberg	Reuters	Quote
	EONIA Curncy	EONIA	0.093
ON	EUSWE1D Curncy	EUREONON	0.1908
1W	EUSWE1Z Curncy	EUREON1W	0.097
2W	EUSWE2Z Curncy	EUREON2W	0.097
3W	EUSWE3Z Curncy	EUREON3W	0.097
1M	EUSWEA Curncy	EUREON1M	0.101
2M	EUSWEB Curncy	EUREON2M	0.1055
3M	EUSWEC Curncy	EUREON3M	0.107
4M	EUSWED Curncy	EUREON4M	0.107
5M	EUSWEE Curncy	EUREON5M	0.1123
6M	EUSWEF Curncy	EUREON6M	0.1148
7M	EUSWEG Curncy	EUREON7M	0.1173
8M	EUSWEH Curncy	EUREON8M	0.123
9M	EUSWEI Curncy	EUREON9M	0.1305
10M	EUSWEJ Curncy	EUREON10M	0.15
11M	EUSWEK Curncy	EUREON11M	0.1478
1Y	EUSWE1	EUREON1Y	0.156
13M	EUSWE1A	EUREON13M	0.1653
14M	EUSWE1B	EUREON14M	0.1733
15M	EUSWE1C	EUREON15M	0.1817
16M	EUSWE1D	EUREON16M	0.1908
17M	EUSWE1E	EUREON17M	0.1993
18M	EUSWE1F	EUREON18M	0.2082
19M	EUSWE1G	EUREON19M	0.2172
20M	EUSWE1H	EUREON20M	0.228
21M	EUSWE1I	EUREON21M	0.237
22M	EUSWE1J	EUREON22M	0.2488
23M	EUSWE1K	EUREON23M	0.2588
2Y	EUSWE2	EUREON2Y	0.262
30M	EUSWE2F	EUREON30M	0.35
3Y	EUSWE3	EUREON3Y	0.4433
4Y	EUSWE4	EUREON4Y	0.666
5Y	EUSWE5	EUREON5Y	0.894
6Y	EUSWE6	EUREON6Y	1.096
7Y	EUSWE7	EUREON7Y	1.274
8Y	EUSWE8	EUREON8Y	1.437
9Y	EUSWE9	EUREON9Y	1.586
10Y	EUSWE10	EUREON10Y	1.7199
11Y	EUSWE11	EUREON11Y	1.838
12Y	EUSWE12	EUREON12Y	1.941
13Y	EUSWE13	EUREON13Y	2.032
14Y	EUSWE14	EUREON14Y	2.104
15Y	EUSWE15	EUREON15Y	2.161
16Y	EUSWE16	EUREON16Y	2.177
17Y	EUSWE17	EUREON17Y	2.212

Table 5.1 *(Continued)*

Description	Bloomberg	Reuters	Quote
18Y	EUSWE18	EUREON18Y	2.239
19Y	EUSWE19	EUREON19Y	2.257
20Y	EUSWE20	EUREON20Y	2.303
25Y	EUSWE25	EUREON25Y	2.337
30Y	EUSWE30	EUREON30Y	2.346
35Y	EUSWE35	EUREON35Y	2.333
40Y	EUSWE40	EUREON40Y	2.358
50Y	EUSWE50	EUREON50Y	2.405

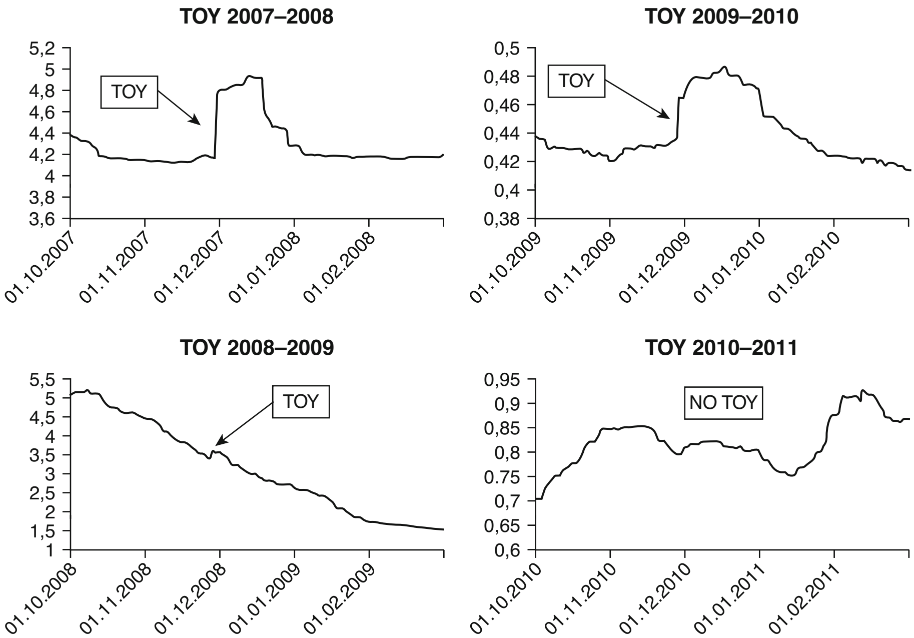


Figure 5.9 TOY effects for different years

the TOY is observed in short tenor rates and is not or is less significant in long tenor rates. For instance for the 5Y swap rate there is no observable jump observable.

Knowing that this effect is observable we might wish to include it in our curve construction procedure using bootstrapping or calibration. If the market quotes include a TOY the calibration procedure takes this into account. The bootstrapping can be sensitive to this effect and it is a good idea to exclude it from the data and after the bootstrapping procedure model the effect on top of the bootstrapped curve.

To this end we have to outline a method to estimate the TOY effect and, first, exclude it from the quotes and, second, include it again. Ametrano and Bianchetti (2013) consider the effect and propose to beg out the TOY by considering

- Futures 3M strip
- FRA 6M strip
- IRS 1M strip

We refer to Ametrano and Bianchetti (2013) for the details.

Synthetic rates and assumptions

For building yield curves from market instruments we face the problem that often not enough information is available to exactly determine the values for all tenors. For instance consider a fixed against floating interest rate swap on a six month money market rate. The swap rates for this contract are quoted yearly. But if we wish to set up a consistent yield curve for the six month money market rate we might need these rates on a half yearly basis as outlined in example (a). To this end we use interpolation to obtain such rates.

In fact this is the first example of a synthetic rate. This rate is somewhat arbitrary since it depends on the interpolation mechanism we use. Thus, synthetic rates are not something evil but something practical if handled with care.

Another example for a synthetic rate but not as easy to obtain as interpolated rates are extrapolated data points. Here, we have less information. We only know the left or the right boundary value instead of having two anchor points. In this case arguing in favor of extrapolation is more cumbersome and often there are disputes on applying this methodology. However, in practice this method is also applied and is an example of a synthetic rate.

Using the multi-curve set-up for a single currency we create a forwarding curve for each available floating rate tenor. The corresponding building blocks used to determine the discrete curve are only those referencing the tenor under consideration. To this end we would only use FRAs with a period of six months and swap rates based on $6M$ LIBOR rates to set up the 6M forwarding curve. This implies that the first entry is on the six month tenor. If we wish to compute the zero curve from a given forwarding curve there is a lot of freedom for the period from today up to six months. To this end practitioners make assumptions to be able to synthesize a six month rate for one month, two months and so forth. This is somewhat arbitrary but may be justified because the corresponding discounting curves can be determined more easily.

In fact we also used a synthetic rate when we used the six month deposit rate as the first data point for the forwarding curve. In fact a deposit rate is different from an FRA rate. And using the six month short term rate would be an approximation for the 0x6 FRA rate. Thus, another example of a synthetic rate.

Summarizing we see that determining synthetic rates is practical in many circumstances but has to be handled with care and there should be sound reasoning for applying this method.

Forward curves

Now, we assume that a discount curve with respect to the domestic collateral is available. The standard case would be the OIS curve which is the EONIA curve for the EUR market. To take into account the new pricing framework including multiple curves we have to set up appropriate forward curves. To construct the forward curves by applying a bootstrapping methodology we consider homogeneous rates. Such rates only involve instruments which refer to the same tenor of rates. For instance if we set up a 3M curve we consider swap rates and other instruments referring to floating rates of 3M period only such as FRA rates on 3M FRAs and money market basis swap spreads where the 3M rate is involved.

The basic set-up in a single currency should involve the tenors 1M, 3M, 6M and 12M. Sometimes it is necessary to consider other rates lying in between the standard tenors. This is not a standard case and we do not give a detailed solution here. We may wish to calculate a 2M rate by interpolating between the corresponding rates on 1M and 3M curve rates. Here we need further interpolation techniques which are considered later and illustrated for volatility surfaces.

Now we start by gathering market quotes which can be applied to the curve construction. We start with the most liquid segment which is for the EUR the 6M curve. Assuming we wish to consider future cash flows which are secured by domestic collateral we need only one discount curve since with respect to the collateral two identical future cashflows must have the same present value. Traditionally bootstrapping was used for both forwarding and discounting. This is not the way it is done today. We keep a discounting curve and only bootstrap the forwarding curve assuming the discount curve given. The swap rates we use as input are rates tradable between collateralized parties and, thus, residual credit risk is negligible. Opposed to that un-collateralized trades have residual credit risk. Such trades should be discounted with the capital market's funding rate of the bank but not with the curve we use here.

To start we use the instruments specified in Table 5.2. First, we know from the market quotes the prices for FRAs and Swaps. The valuation formulas are given by Equations (4.1) and (4.6). Since the floating leg has the same value as the fixed leg we can determine the forward rates by solving:

$$\begin{pmatrix} \tilde{\tau}_1 D(\tilde{T}_1) & 0 & 0 & \cdots & 0 \\ \tilde{\tau}_1 D(\tilde{T}_1) & \tilde{\tau}_2 D(\tilde{T}_2) & 0 & \cdots & 0 \\ \tilde{\tau}_1 D(\tilde{T}_1) & \tilde{\tau}_2 D(\tilde{T}_2) & \tilde{\tau}_3 D(\tilde{T}_3) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\tau}_1 D(\tilde{T}_1) & \tilde{\tau}_2 D(\tilde{T}_2) & \tilde{\tau}_3 D(\tilde{T}_3) & \cdots & \tilde{\tau}_N D(\tilde{T}_N) \end{pmatrix} \times \begin{pmatrix} L(0,\tilde{T}_0,\tilde{T}_1) \\ L(0,\tilde{T}_1,\tilde{T}_2) \\ L(0,\tilde{T}_2,\tilde{T}_3) \\ \vdots \\ L(0,\tilde{T}_{N-1},\tilde{T}_N) \end{pmatrix} = \begin{pmatrix} V^{\text{fixed}}_{0,T_1}(0) \\ V^{\text{fixed}}_{0,T_2}(0) \\ V^{\text{fixed}}_{0,T_3}(0) \\ \vdots \\ V^{\text{fixed}}_{0,T_N}(0) \end{pmatrix} \tag{5.6}$$

The solution to Equation (5.6) can be obtained using matrix inversion. The missing values to fully specify all necessary information to solve this equation are replaced by synthetic values obtained by interpolation. At the end of this subsection we give a calculated example.

Table 5.2 *Instrument selection for seting up the 6M EUR curve. Quotes are as of 02.08.2013*

Description	Bloomberg	Reuters	*Quote*
FRA 0x6	EUR006M Index	EURIBOR6MD	0.34
FRA 1x7	EUFR0AG Curncy	EUR1X7F	0.366
FRA 2x8	EUFR0BH Curncy	EUR2X8F	0.389
FRA 3x9	EUFR0CI Curncy	EUR3X9F	0.415
FRA 4x10	EUFR0DJ Curncy	EUR4X10F	0.442
FRA 5x11	EUFR0EK Curncy	EUR5X11F	0.471
FRA 6x12	EUFR0F1 Curncy	–	0.499
FRA 7x13	EUFR0AG Curncy	–	0.366
FRA 8x14	EUFR0G1A Curncy	–	0.531
FRA 9x15	EUFR0H1B Curncy	–	0.556
FRA 10x16	EUFR0I1C Curncy	–	0.579
FRA 11x17	EUFR0J1D Curncy	–	0.603
FRA 12x18	EUFR011F Curncy	–	0.656
SwapRate1Y	EUSA1	EURAB6E1Y	0.428
SwapRate2Y	EUSA2	EURAB6E2Y	0.585
SwapRate3Y	EUSA3	EURAB6E3Y	0.78
SwapRate4Y	EUSA4	EURAB6E4Y	1.0087
SwapRate5Y	EUSA5	EURAB6E5Y	1.236
SwapRate6Y	EUSA6	EURAB6E6Y	1.436
SwapRate7Y	EUSA7	EURAB6E7Y	1.612
SwapRate8Y	EUSA8	EURAB6E8Y	1.767
SwapRate9Y	EUSA9	EURAB6E9Y	1.9115
SwapRate10Y	EUSA10	EURAB6E10Y	2.039
SwapRate11Y	EUSA11	EURAB6E11Y	2.149
SwapRate12Y	EUSA12	EURAB6E12Y	2.242
SwapRate13Y	EUSA13	EURAB6E13Y	2.3225
SwapRate14Y	EUSA14	EURAB6E14Y	2.387
SwapRate15Y	EUSA15	EURAB6E15Y	2.438
SwapRate16Y	EUSA16	EURAB6E16Y	2.477
SwapRate17Y	EUSA17	EURAB6E17Y	2.5052
SwapRate18Y	EUSA18	EURAB6E18Y	2.5238
SwapRate19Y	EUSA19	EURAB6E19Y	2.538
SwapRate20Y	EUSA20	EURAB6E20Y	2.546
SwapRate21Y	EUSA21	EURAB6E21Y	2.5525
SwapRate22Y	EUSA22	EURAB6E22Y	2.5555
SwapRate23Y	EUSA23	EURAB6E23Y	2.5577
SwapRate24Y	EUSA24	EURAB6E24Y	2.559
SwapRate25Y	EUSA25	EURAB6E25Y	2.557
SwapRate26Y	EUSA26	EURAB6E26Y	2.5555
SwapRate27Y	EUSA27	EURAB6E27Y	2.554
SwapRate28Y	EUSA28	EURAB6E28Y	2.5525
SwapRate29Y	EUSA29	EURAB6E29Y	2.5495
SwapRate30Y	EUSA30	EURAB6E30Y	2.549
SwapRate31Y	EUSA31	–	2.549
SwapRate32Y	EUSA32	–	2.549
SwapRate33Y	EUSA33	–	2.551
SwapRate34Y	EUSA34	–	2.552

Table 5.2 *(Continued)*

Description	Bloomberg	Reuters	*Quote*
SwapRate35Y	EUSA35	EURAB6E35Y	2.5548
SwapRate36Y	EUSA36	–	2.558
SwapRate37Y	EUSA37	–	2.562
SwapRate38Y	EUSA38	–	2.566
SwapRate39Y	EUSA39	–	2.57
SwapRate40Y	EUSA40	EURAB6E40Y	2.574
SwapRate41Y	EUSA41	–	2.578
SwapRate42Y	EUSA42	–	2.5651
SwapRate43Y	EUSA43	–	2.5611
SwapRate44Y	EUSA44	–	2.589
SwapRate45Y	EUSA45	–	2.592
SwapRate46Y	EUSA46	–	2.596
SwapRate47Y	EUSA47	–	2.5754
SwapRate48Y	EUSA48	–	2.602
SwapRate49Y	EUSA49	–	2.606
SwapRate50Y	EUSA50	EURAB6E50Y	2.6095
SwapRate55Y	EUSA55	–	2.625
SwapRate60Y	EUSA60	EURAB6E60Y	2.64

Next, we consider input instruments based on the 3M floating rate. Such instruments include FRAs on the 3M rate, swap rates calculated against the 3M EURIBOR. Eventually such rates have to be calculated using money market basis swap quotes against the 6M swap rates. For stability we might wish to include further instruments not directly observable. We consider such methods later. Table 5.3 summarize our input data.

After stripping the 6M and the 3M curves we consider the 1M and the 12M curves. We use the quotes from Table 5.4. For the 1M floating rate basis swaps against other floating rates as well as swap rates are quoted. To this end we can decide which types of quotes to use. There are no swap rate quotes for the 12M floating rate available. Thus, we can rely on money market basis swap quotes with respect to the 3M or 6M curve. Ametrano and Bianchetti (2013) propose including synthetic deposits and synthetic FRA rates into the bootstrapping procedure. This means that artificial rates are included into the bootstrapping. This can be seen as posing constraints of the yield curve and might be useful for stabilizing the curve construction.

Finally we consider the forward curves for different interpolation schemes. Figure 5.10 shows the effect on the discount, respectively forward curves. We observe that for the discount curve we see virtually no difference between the schemes but it is significant for the forward curve. The piecewise constant method and the linear methods have to be applied with care. In our example on the monotone preserving Kruger spline interpolator leads to reasonable results. Especially, we were able to create smooth forwards in this case.

Let us give a calculated example using the same methodology as for the OIS curve construction. To this end we assume that the OIS curve has been set up and we have quoted swap rates $SR_{0,T_i}(0)$. Using the OIS curve for discounting and denoting the

Table 5.3 *Instrument selection for setting up the 3M EUR curve. Quotes are as of 02.08.2013*

Description	Bloomberg	Reuters	Quote
FRA 0x3	EUR003M Index	EURIBOR3MD	0.228
FRA 1x3	EUFR0AD Curncy	EUR1X4F	0.242
FRA 2x5	EUFR0BE Curncy	EUR2X5F	0.2595
FRA 3x6	EUFR0CF Curncy	EUR3X6F	0.28
FRA 4x7	EUFR0DG Curncy	EUR4X7F	0.296
FRA 5x8	EUFR0EH Curncy	EUR5X8F	0.316
FRA 6x9	EUFR0FI Curncy	EUR6X9F	0.342
FRA 7x10	EUFR0GJ Curncy	EUR7X10F	0.369
FRA 8x11	EUFR0HK Curncy	EUR8X11F	0.395
FRA 9x12	EUFR0I1C Curncy	–	0.579
FRA 10x13	EUFR0J1D Curncy	–	0.603
FRA 11x14	EUFR0K1E Curncy	–	0.628
FRA 12x15	EUFR011C Curncy	–	0.478
FRA 15x18	EUFR1C1F2 Curncy	–	0.48
FRA 18x21	EUFR1F1I Curncy	–	0.542
FRA 21x24	EUFR1I2 Curncy	–	0.554
SwapRate 6M	EUSWFV3 Curncy	–	0.258
SwapRate 9M	EUSWIVC Curncy	–	0.286
SwapRate1Y	EUSW1V3 Curncy	EURAB3E1Y	0.319
SwapRate 18M	EUSWV31F Curncy	–	0.387
SwapRate2Y	EUSW2V3 Curncy	EURAB3E2Y	0.4592
SwapRate3Y	EUSW3V3 Curncy	EURAB3E3Y	0.6485
SwapRate4Y	EUSW4V3 Curncy	EURAB3E4Y	0.8795
SwapRate5Y	EUSW5V3 Curncy	EURAB3E5Y	1.1095
SwapRate6Y	EUSW6V3 Curncy	EURAB3E6Y	1.3125
SwapRate7Y	EUSW7V3 Curncy	EURAB3E7Y	1.491
SwapRate8Y	EUSW8V3 Curncy	EURAB3E8Y	1.6535
SwapRate9Y	EUSW9V3 Curncy	EURAB3E9Y	1.8002
SwapRate10Y	EUSW10V3 Curncy	EURAB3E10Y	1.932
SwapRate11Y	EUSW11V3 Curncy	EURAB3E11Y	2.046
SwapRate12Y	EUSW12V3 Curncy	EURAB3E12Y	2.1435
SwapRate13Y	EUSW13V3 Curncy	EURAB3E13Y	2.2261
SwapRate14Y	EUSW14V3 Curncy	EURAB3E14Y	2.2946
SwapRate15Y	EUSW15V3 Curncy	EURAB3E15Y	2.3497
SwapRate16Y	EUSW16V3 Curncy	EURAB3E16Y	2.3925
SwapRate17Y	EUSW17V3 Curncy	EURAB3E17Y	2.4233
SwapRate18Y	EUSW18V3 Curncy	EURAB3E18Y	2.4459
SwapRate19Y	EUSW19V3 Curncy	EURAB3E19Y	2.4616
SwapRate20Y	EUSW20V3 Curncy	EURAB3E20Y	2.4727
SwapRate21Y	EUSW21V3 Curncy	EURAB3E21Y	2.481
SwapRate22Y	EUSW22V3 Curncy	EURAB3E22Y	2.4868
SwapRate23Y	EUSW23V3 Curncy	EURAB3E23Y	2.4906
SwapRate24Y	EUSW24V3 Curncy	EURAB3E24Y	2.4927

Table 5.3 *(Continued)*

Description	Bloomberg	Reuters	Quote
SwapRate25Y	EUSW25V3 Curncy	EURAB3E25Y	2.4928
SwapRate26Y	EUSW26V3 Curncy	EURAB3E26Y	2.4931
SwapRate27Y	EUSW27V3 Curncy	EURAB3E27Y	2.4922
SwapRate28Y	EUSW28V3 Curncy	EURAB3E28Y	2.4911
SwapRate29Y	EUSW29V3 Curncy	EURAB3E29Y	2.4903
SwapRate30Y	EUSW30V3 Curncy	EURAB3E30Y	2.487
SwapRate35Y	EUSW35V3 Curncy	EURAB3E31Y	2.499
SwapRate40Y	EUSW40V3 Curncy	EURAB3E32Y	2.525
SwapRate50Y	EUSW50V3 Curncy	EURAB3E33Y	2.564

discount factors $d_1,\ldots,d_N$ it is possible to calculate the present values for the fixed legs. We denote these values by $P_1,\ldots,P_N$. This can be done by the following set of equations:

$$\begin{pmatrix} SR_{0,T_1}(0)+100 & 0 & 0 & \cdots & 0 \\ SR_{0,T_2}(0) & SR_{0,T_2}(0)+100 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ SR_{0,T_N}(0) & SR_{0,T_N}(0) & \cdots & \cdots & SR_{0,T_N}(0)+100 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix}$$

Using the present values of the fixed legs we now consider the payments of the floating leg with respect to a single curve. This curve is used for discounting the floating payments. We get the following set of equations with $\tilde{d}_j$ being the discount factor on this new curve.

$$\begin{pmatrix} 100d_1 & 0 & 0 & \cdots & 0 \\ 100d_1 & 100d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 100d_1 0 & 100d_2 & \cdots & \cdots & 100d_n \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} = \begin{pmatrix} P_1-100d_1 \\ P_2-100d_2 \\ \vdots \\ P_N-100d_N \end{pmatrix} \tag{5.7}$$

Solving for $\tilde{d}_j$ gives the solution by setting

$$\tilde{d}_j := \prod_{i=1}^{j} \frac{1}{1+r_i}, \quad j=1,\ldots,N$$

Let us give a specific example for four rates and check if the corresponding rates really reproduce the assumed market rates. We take for the discount factors ($d_{2d} = 0.999990411$, $d_{1Y} = 0.994629003$, $d_{2Y} = 0.984758715$, $d_{3Y} = 0.971085364$, $d_{4Y} = 0.951999586$, $d_{5Y} = 0.929930159$) and swap rates (0.8, 0.95, 1.14, 1.48, 1.76)which

Table 5.4 *Instrument selection for setting up the 1M EUR curve (top) and 12M EUR curve (bottom). Quotes are as of 02.08.2013*

Description	Bloomberg	Reuters	Quote
FRA 0 × 1	EUR001M Index	EUIBOR1MD	0.13
ShortSwap 2M	EUSWBV1 Curncy	EUR2X1S	0.1335
ShortSwap 3M	EUSWCV1 Curncy	EUR3X1S	0.1375
ShortSwap 4M	EUSWDV1 Curncy	EUR4X1S	0.137
ShortSwap 5M	EUSWEV1 Curncy	EUR5X1S	0.142
ShortSwap 6M	EUSWFV1 Curncy	EUR6X1S	0.146
ShortSwap 7M	EUSWGV1 Curncy	EUR7X1S	0.154
ShortSwap 8M	EUSWHV1 Curncy	EUR8X1S	0.152
ShortSwap 9M	EUSWIV1 Curncy	EUR9X1S	0.17
ShortSwap 10M	EUSWJV1 Curncy	EUR10X1S	0.175
ShortSwap 11M	EUSWKV1 Curncy	EUR11X1S	0.184
SwapRate 1Y	EUSW1V1 Curncy	EURAB1E1Y	0.199
SwapRate 2Y	EUSW2V1 Curncy	EURAB1E2Y	0.325
SwapRate 3Y	EUSW3V1 Curncy	EURAB1E3Y	0.51
SwapRate 4Y	EUSW4V1 Curncy	EURAB1E4Y	0.736
SwapRate 5Y	EUSW5V1 Curncy	EURAB1E5Y	0.966
SwapRate 6Y	EUSW6V1 Curncy	EURAB1E6Y	1.171
SwapRate 7Y	EUSW7V1 Curncy	EURAB1E7Y	1.355
SwapRate 8Y	EUSW8V1 Curncy	EURAB1E8Y	1.519
SwapRate 9Y	EUSW9V1 Curncy	EURAB1E9Y	1.67
SwapRate 10Y	EUSW10V1 Curncy	EURAB1E10Y	1.806
SwapRate 11Y	EUSW11V1 Curncy	EURAB1E11Y	1.923
SwapRate 12Y	EUSW12V1 Curncy	EURAB1E12Y	2.024
SwapRate 15Y	EUSW15V1 Curncy	EURAB1E15Y	2.242
SwapRate 20Y	EUSW20V1 Curncy	EURAB1E20Y	2.377
SwapRate 25Y	EUSW25V1 Curncy	EURAB1E25Y	2.407
SwapRate 30Y	EUSW30V1 Curncy	EURAB1E30Y	2.411
FRA 0x12	EUR012M Index	EURIBOR12MD	0.531
FRA 12x24	EUFR012 Curncy	–	0.897
Basis 6M12M	EUBSST2 Curncy	EUR6E12E2Y	13.5
Basis 6M12M	EUBSST3 Curncy	EUR6E12E3Y	13.9
Basis 6M12M	EUBSST4 Curncy	EUR6E12E4Y	13.8
Basis 6M12M	EUBSST5 Curncy	EUR6E12E5Y	13.6
Basis 6M12M	EUBSST6 Curncy	EUR6E12E6Y	13.5
Basis 6M12M	EUBSST7 Curncy	EUR6E12E7Y	13.3
Basis 6M12M	EUBSST8 Curncy	EUR6E12E8Y	13
Basis 6M12M	EUBSST9 Curncy	EUR6E12E9Y	12.6
Basis 6M12M	EUBSST10 Curncy	EUR6E12E10Y	12.2
Basis 6M12M	EUBSST11 Curncy	EUR6E12E11Y	11.9
Basis 6M12M	EUBSST12 Curncy	EUR6E12E12Y	11.4
Basis 6M12M	EUBSST15 Curncy	EUR6E12E15Y	10.35
Basis 6M12M	EUBSST20 Curncy	EUR6E12E20Y	9.35
Basis 6M12M	EUBSST25 Curncy	EUR6E12E25Y	8.5
Basis 6M12M	EUBSST30 Curncy	EUR6E12E30Y	7.8

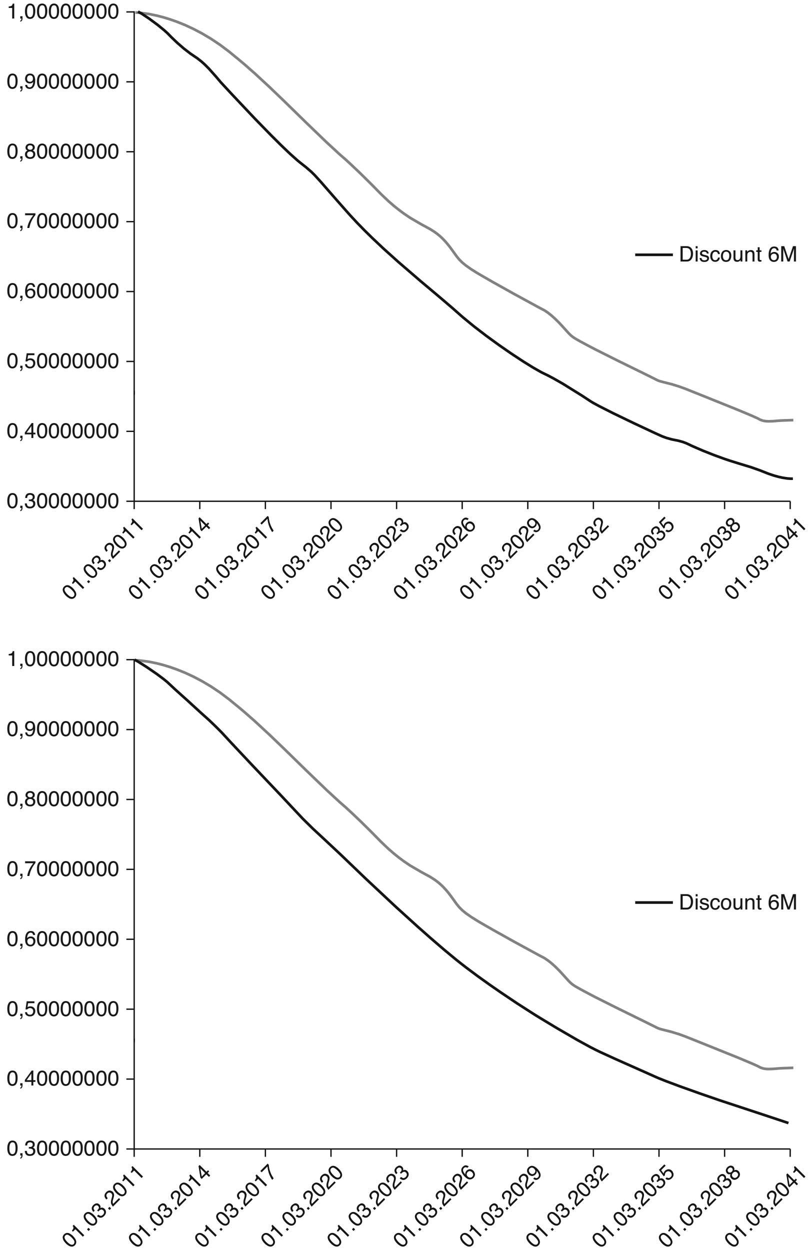
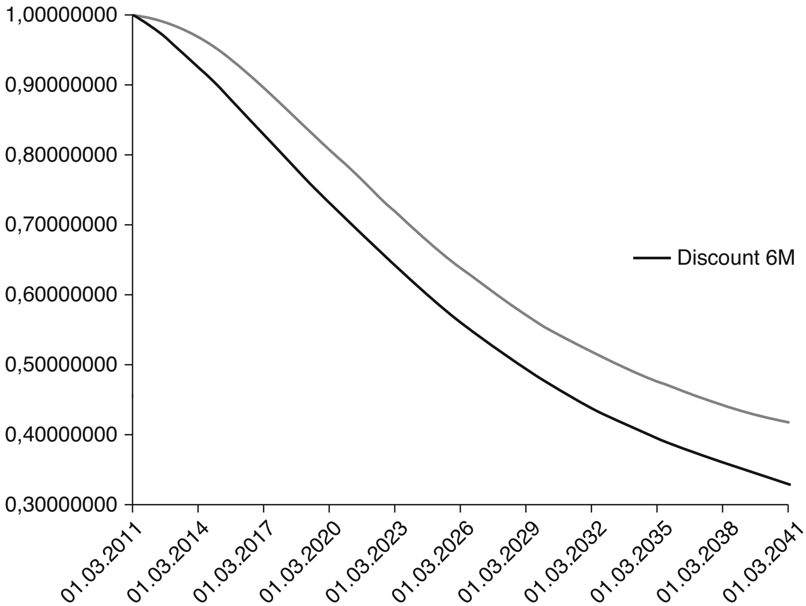


Figure 5.10 Different interpolation methods and the effect on curve calculation. Piecewise constant (top), linear (mid) and Kruger (bottom). On the top we have discount and on the bottom forward curves



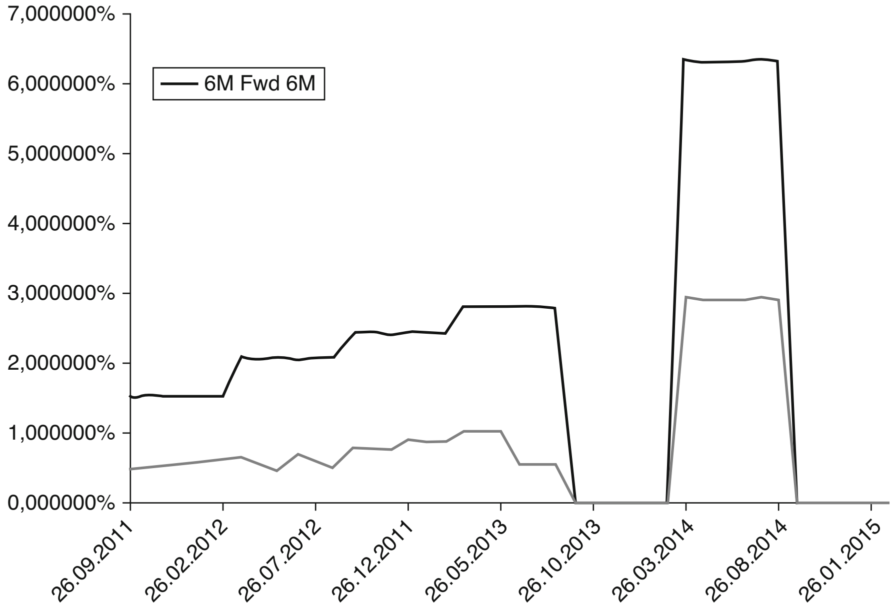
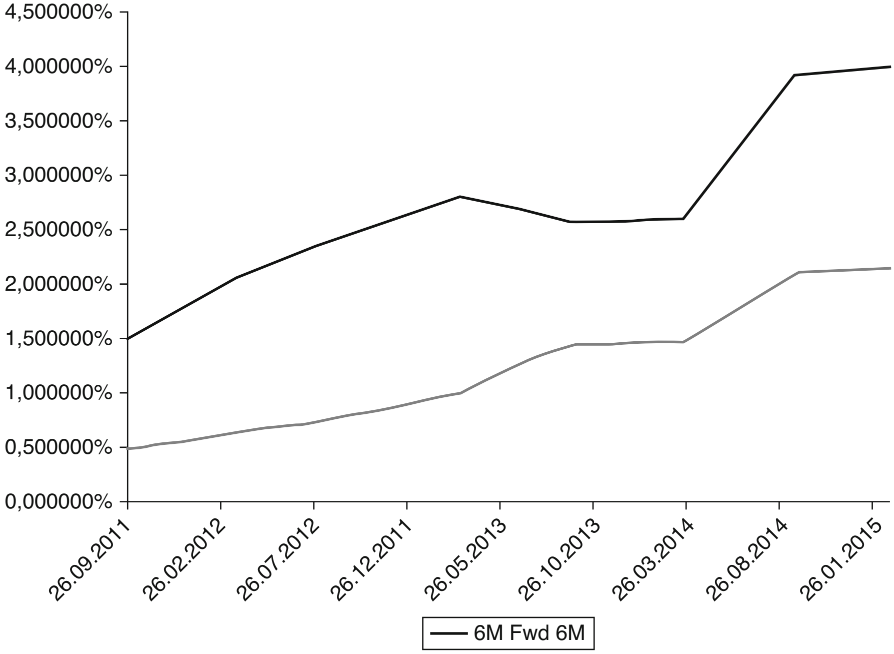


Figure 5.10 (*Continued*)



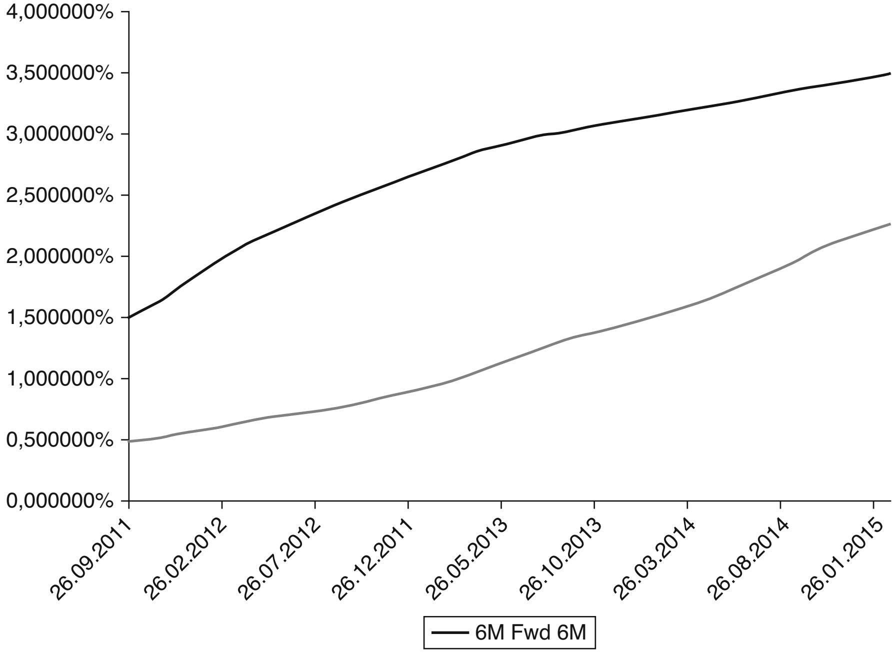


Figure 5.10 (*Continued*)

leads to prices for the fixed leg

$$\begin{pmatrix} 100.8 & 0 & 0 & 0 & 0 \\ 0.95 & 100.95 & 0 & 0 & 0 \\ 1.14 & 1.14 & 101.14 & 0 & 0 \\ 1.48 & 1.48 & 1.48 & 101.48 & 0 \\ 1.76 & 1.76 & 1.76 & 1.76 & 101.76 \end{pmatrix} \times \begin{pmatrix} 0.994629003 \\ 0.984758715 \\ 0.971085364 \\ 0.951999586 \\ 0.929930159 \end{pmatrix} = \begin{pmatrix} 100.2586035 \\ 100.3562898 \\ 100.4720757 \\ 100.9756182 \\ 101.4980449 \end{pmatrix}$$

Due to the calculation the present values are given by $(100.2586035, 100.3562898, 100.4720757, 100.9756182, 101.4980449)$. This leads to consider the equation derived from Equation (5.7):

$$\begin{pmatrix} 99.46290034 & 0 & 0 & 0 & 0 \\ 99.46290034 & 98.47587146 & 0 & 0 & 0 \\ 99.46290034 & 98.47587146 & 97.10853638 & 0 & 0 \\ 99.46290034 & 98.47587146 & 97.10853638 & 95.19995862 & 0 \\ 99.46290034 & 98.47587146 & 97.10853638 & 95.19995862 & 92.9930159 \end{pmatrix}$$
$$\times \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} 0.795703203 \\ 1.880418332 \\ 3.363539313 \\ 5.775659549 \\ 8.505028975 \end{pmatrix}$$

The values $(99.46290034, 98.47587146, 97.10853638, 95.1999586, 92.9930159)$ are obtained by subtracting the values $100 \cdot d_i$ from the vector of the fixed leg present values. Inverting the matrix and multiplying the latter results in:

$$\begin{pmatrix} 0.010054 & 0 & 0 & 0 & 0 \\ -0.010154772 & 0.010154772 & 0 & 0 & 0 \\ 0 & -0.010297756 & 0.010297756 & 0 & 0 \\ 0 & 0 & -0.010504206 & 0.010504206 & 0 \\ 0 & 0 & 0 & -0.010753496 & 0.010753496 \end{pmatrix}$$
$$\times \begin{pmatrix} 0.795703203 \\ 1.880418332 \\ 3.363539313 \\ 5.775659549 \\ 8.505028975 \end{pmatrix} = \begin{pmatrix} 0.008 \\ 0.011015035 \\ 0.015272818 \\ 0.025337408 \\ 0.029350262 \end{pmatrix}$$

The last vector is the rates vector $(r_1, \ldots, r_5)$ and we get the LIBOR discount factors by calculating

$$\tilde{d}_j = \prod_{i=1}^{j} \frac{1}{1+r_i}, \quad j = 1, \ldots, 5$$

This yields the vector $(0.992063492, 0.981254935, 0.966493851, 0.942610542, 0.915733523)$. The reader can check that this is inline with the initial market quotes.

Monotone convex – Hagan/West approach

In the sequel we describe a sophisticated method for bootstrapping often applied by practitioners, since for doing the next step in a bootstrapping procedure it might be necessary to use interpolated values. For instance if we wish to calculate the discount factor at $T + 6M$ using the discount curve already built up to time T but the next available market quote is given at $T + 1Y$. Thus, the bootstrapping and the interpolation are closely tied together. Using different interpolation schemes leads to a different bootstrapped curve. Furthermore, the interpolation used for the bootstrapping should also be used for retrieving values from the curve later on.

The method we describe is especially designed for building interest rate curves and has been introduced in Hagan and West (2006a). If the set $\mathcal{X}$ is constituted of rates we transform the rates to forwards. This can be done by considering the set

$$\mathcal{Y} := \{y_0, y_1, \ldots, y_N\}$$

and set

$$y_0 := 0, \quad y_i := \frac{x_i t_i - x_{i-1} t_{i-1}}{t_i - t_{i-1}}$$

We should have from no-arbitrage reasons that $y_i > 0$ for $i = 1, \ldots, N$. From a financial viewpoint by construction y_i can be seen as the discrete rate belonging to the interval $[t_{i-1}, t_i]$.

From the rates $\mathcal{Y}$ we construct rates $\tilde{\mathcal{X}}$ such that each $\tilde{x} \in \tilde{\mathcal{X}}$ is the midpoint of the rates in $\mathcal{Y}$, i.e.

$$\tilde{x}_i = \frac{t_i - t_{i-1}}{t_{i+1} - t_i} y_{i+1} + \frac{t_{i+1} - t_i}{t_{i+1} - t_i} y_i$$

$$\tilde{x}_0 = y_1 - \frac{\tilde{x}_1 - y_1}{2}$$

$$\tilde{x}_N = y_N - \frac{\tilde{x}_{N-1} - y_N}{2}$$

We have that $\tilde{x}_i \geq 0$ for $i = 0, \ldots, N$ if the input already was positive.

We wish to determine some function f

$$f : [t_0, t_N] \to \mathbb{R}$$

such that f recovers $\tilde{\mathcal{X}}$ and satisfies:

(i) $\frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(t)dt = y_i$
(ii) f is positive
(iii) f is continuous
(iv) If $y_{i-1} < y_i < y_{i+1}$ then $f(t)$ is increasing on $[t_{i-1}, t_i]$ and if $y_{i-1} > y_i > y_{i+1}$ then $f(t)$ is decreasing on $[t_{i-1}, t_i]$

Normalization leads us to seek functions g_i (denoted as simply g), $i = 1, \ldots, N$ such that

$$g : [0,1] \to \mathbb{R}, \quad g(x) = f(t_{i-1} + (t_i - t_{i-1})x) - y_i$$

such that (i–iv) for f are satisfied and we have

$$f(t) = g\left(\frac{t - t_{i-1}}{t_i - t_{i-1}}\right) + y_i$$

We know that $g(0) = \tilde{x}_{i-1} - y_i$, $g(1) = \tilde{x}_i - y_i$ and $\int_0^1 g(x)dx = 0$. To cope with the problem of finding g we assume that g is of some functional form. To keep things simple and computable we choose

$$g(x) = K + Lx + Mx^2$$

Then, we have three constraints and three unknown variables K, L and M. This can be solved!

We obtain

$$g(x) = g(0)(1 - 4x + 3x^2) + g(1)(3x^2 - 2x)$$

We take this as a basis and modify the function to meet all the constraints. To meet all the constraints we have to consider four cases (1–4), see Figure 5.11.

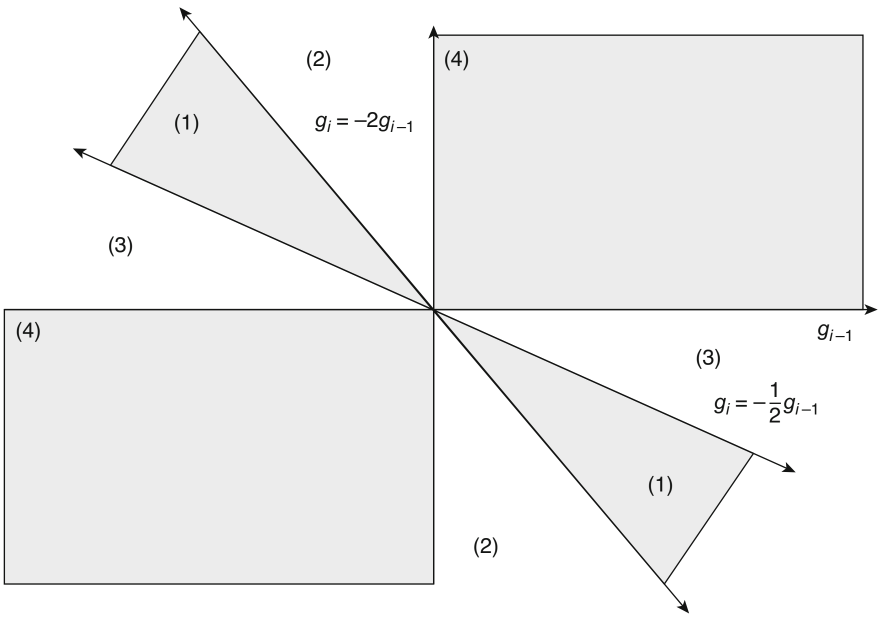


Figure 5.11 Cases which can arise for monotone convex method

(1) The function g_i does not need any modification

(2) One solution is to insert a flat curve segment

$$g(x) = \begin{cases} g(0) & 0 \leq x \leq \eta \\ g(0) + (g(1) - g(0)) \left(\frac{x-\eta}{1-\eta}\right)^2 & \eta < x \leq 1 \end{cases}$$

with

$$\eta = \frac{g(1) + 2g(0)}{g(1) - g(0)}$$

(3) Again the solution is to use flat curve segements

$$g(x) = \begin{cases} g(1) + (g(0) - g(1)) \left(\frac{\eta-x}{\eta}\right)^2 & 0 \leq x < \eta \\ g(1) & \eta \leq x \leq 1 \end{cases}$$

with $\eta = \frac{3g(1)}{g(1)-g(0)}$

(4)To cope with the last case we take the modification given by

$$g(x) = \begin{cases} A + (g(0) - A) \left(\frac{\eta-x}{\eta}\right)^2 & 0 < x < \eta \\ A + (g(1) - A) \left(\frac{x-\eta}{1-\eta}\right)^2 & \eta < x < 1 \end{cases}$$

with $\eta = \frac{g(1)}{g(1)+g(0)}$ and $A = -\frac{g(0)g(1)}{g(0)+g(1)}$

We finally ask if the above construction leads to positivity of the interpolator f. If we follow the recipe:

- Determine $\mathcal{Y}$ as outlined
- Determine $\tilde{\mathcal{X}}$ as outlined
- For f being positive we propose to collar f_0, f_i and f_N between 0 and $2y_1$, $2\min(y_i, y_{i+1})$ and $2y_N$. Otherwise we do not collar!
- Construct the functions g_i as shown
- Determine the function f using the functions g_i and $\mathcal{Y}$

we reach an interpolator with satisfies (i–iv).

Finally, Hagan and West (2006a) and Hagan and West (2006b) describe a method to smooth the curves. This method is called *amelioration*. But since they suggest not to use it for bootstrapping yield curve we do not explain it here in detail.

5.3.2 Curve construction – optimization

Now, we turn to the other method for determining the yield curves used to value different financial instruments. For this approach we assume that we wish to price a set of financial instruments $\mathcal{I} = \{I_1,\ldots,I_{N_I}\}$ with given market prices V_j. For each of the instruments we have to consider the set of curves we need for the valuation. For instance to value a domestic fixed against float interest rate swap with domestic EUR cash collateral we need the discounting and the forwarding curves. For other instruments such as cross currency swaps we need even more curves to price such instruments correctly. We consider the curves $\mathcal{C}_1,\ldots,\mathcal{C}_{N_C}$ with the standard tenors and some pre-defined interpolation method.

The sets $\mathcal{X}_k$, $k = 1,\ldots,N_C$ have to be determined. To do so we first have to translate this problem into some optimization problem. To this end we specify some function, called *error measure*, which is then minimized. A standard error measure is the *least squares* distance which in our case is given by

$$\frac{1}{N_I}\sum_{j=1}^{N_I}\left(V_j - \tilde{V}_j\right)^2 \tag{5.8}$$

The prices $\tilde{V}_j$ are the prices of the instruments from the set $\mathcal{I}$ obtained by pricing using the curves $\mathcal{C}_1,\ldots,\mathcal{C}_{N_C}$. The initial values of the curves $\mathcal{X}_1,\ldots,\mathcal{X}_{N_C}$ are chosen appropriately. For instance we could choose these values to be the values obtained from the last calibration or the plain quoted market values for the specific tenors.

Finally, we choose an optimization algorithm and some constraints on the free parameters. The optimization algorithm can be some local optimizer such as the well known Levenberg-Marquardt algorithm, some LBFGS method or another Newton solver or we might wish to choose some global optimizer such as differential evolution or simulated annealing. A discussion of the different methods is outside the scope of this book but we included references to relevant literature in the last section of this chapter.

For the market quotes used to determine the error function, Equation (5.8), we take the corresponding quotes from Tables 5.1–5.4. Now, we have all the necessary tools available. We use our – hopefully fast and stable – pricing mechanism, the curves and the optimizer. As an output we get the discrete curves $\mathcal{X}_1,\ldots,\mathcal{X}_{N_C}$ and the final least squares error. This error is a measure of how well the market quotes could fit.

Multiple currency curves

Using the described methodology we are able to determine curves for pricing linear instruments in a given currency if the collateral is also posted in this currency. The different collateral agreements and market practices make it necessary to consider linear instrument valuation with respect to another currency for collateral or even multiple possible currencies.

This setting cannot be handled by the methods introduced so far. To challenge this setting we have to consider quotes from the cross currency market. As outlined earlier the market quotes are given in terms of USD collateral. To this end we have to set up *cross currency basis curves.*

- Discounting cash flows where the collateral currency is different from the deal currency.
- The discounting curve is changed due to the collateral currency.

- The forward curves are not changed. The curves for the deal currency are used.
- Benchmark quotes are the liquid market quotes.

Let us outline the approach by considering the EUR/USD case. We consider the 3M EURIBOR and the 3M USD LIBOR rates. The collateral currency is USD. We calculate the floating rates on the EUR 3M curve for EURIBOR and on the USD 3M curve for USD LIBOR. The discounting, since we have USD collateral, is done with respect to the OIS USD curve for USD payments. For the EUR payments, also collateralized in USD, we have to set up a new curve which is consistent with the market quotes. Having set up such a curve we are then able to discount cash flows in EUR which are collateralized in USD.

This partly solves the problem of collateral agreements in different currencies. However, some collateral agreements allow posting of the collateral in different currencies. Thus, we could ask ourselves which might be the currencies which is best suited to post as collateral. Since we have the right to choose any of the collateral currency this is some option we own. And this option has to be valued and the outcome of this valuation procedure is a discount curve which accounts for that option. There are several approaches for setting up such curves which we do not discuss here. One practical and simple solution would be to determine the discount curve for each currency and take the maximum discount factor with respect to each currency at each time point. Finally, smooth the curve. This curve can then be used to account for multiple collateral.

Mirrors and triangles

For a general setting $\mathcal{C}_{d,X,Y}$ is the discount curve on a currency X with collateral posted in currency Y. Thus, the discount curve used in the latter example is $\mathcal{C}_{d,EUR/USD}$. If we assume that the cross currency spreads and the collateral rate are not correlated we can find a corresponding spread $s_{X/Y}$ upon the X discount curve, which in the EUR case would be the EONIA curve, see Fujii, Shimada and Takahashi (2010a) or Macey, G. (2011). The corresponding formula is

$$D_{X/Y}(t) = D_X(t)\exp\left(-\int_0^t s_{X/Y}(u)du\right)$$

To collateralize a Y cash flow in the currency X the spread $s_{Y/X}$ is set to be the *mirror spread* $-s_{X/Y}$ and we have

$$D_{X/Y}(t) = D_Y \exp\left(-\int_0^t s_{Y/X}(u)du\right) = D_Y \exp\left(\int_0^t s_{X/Y}(u)du\right)$$

If another currency Z is involved we apply a triangle relation and use the equation

$$s_{X/Z} = s_{X/Y} - s_{Y/Z}$$

Variants

Sometimes people apply the global optimization technique together with bootstrapping methods. For instance we might wish to determine the discounting curve by the bootstrapping method and then apply an optimization method to get the corresponding forwarding curves for the different tenors 1M, 3M, 6M and 12M.

Sometimes it might be necessary to use further assumptions when setting up curves. Especially, the problem is under-determined if there are fewer equations than unknown variables. To this end it is often necessary to synthetically create quotes by using interpolation. As explained the quotes are almost all for swaps with USD collateral. Thus, there are no market quotes available for such swaps with other currencies as collateral. Now, to value a cross currency swap with GBP collateral for instance we cannot derive the corresponding curve and value the corresponding cross currency swap. To be able to value such instruments it is necessary to assume

- The market values are independent of the collateral and we do get the same quotes if we collateralize in GBP or USD.
- The forward values are independent of the collateral and we do get the same forwards if we collateralize in GBP or USD.

Once we have agreed to the one or the other assumption we can beg out the discount curve for the collateralization with GBP.

Validation

On the other hand it might also be the case that if the modeler includes all observable quotes into the calibration the problem becomes over-determined. This means there are more equations than unknown variables. A bootstrapping method cannot be applied here but we can still use the optimization method. This is for instance the case when working with money market basis swaps, swap rates and cross currency swaps. Before we use the constructed yield curves for tackling financial problems such as financial instrument pricing we should do some validation. The first validation methodology we could think of is to test whether the built curves correctly reprice the market instruments which served as an input to our methodology. If this is not the case there might be something wrong with the bootstrapping procedure, with the optimizer or with the valuation formulas applied to price the instruments.

The next test could be on market data which was not used as input but which were available from data providers or brokers. For instance we can calculate forward swap rates using the derived curves and we can then compare the data to quoted data. This is a good indicator to see if our framework is consistent with the market or not.

We could also pose further restrictions on the output curves. For instance we could determine some other market rates which should be recovered exactly since the trading desk especially works with such rates. Another requirement might be that the forward rates generated by the output curves are sufficiently smooth.

For the final test we take the annuity $A_{t,T}(0)$ corresponding to some swap rate $SR_{t,T}(0)$. We have

$$0 = V^{\text{Swap}}(0,T)) = SR_{0,t}(0)A_{0,t}(0) + SR_{t,T}(0)A_{t,T}(0) - V^{\text{Float}}_{0,t}(0) - V^{\text{Float}}_{t,T}(0)$$

This implies

$$SR_{0,T}(0)A_{0,T}(0) = SR_{0,t}(0)A_{0,t}(0) + SR_{t,T}(0)A_{t,T}(0)$$

This quantity can be used to test the OIS discounting framework. The resulting error should be equal or very close to 0.

5.4 Risk measures

This section is meant to illustrate how the yield curves can be used to derive risk measures used in practice to manage interest rate risk. The measures we introduce only take into account how a given contract or portfolio of interest related instruments behave if the curve changes its overall level, the slope or the curvature. The considered measures generalize the yield based measures we introduced when dealing with bonds in Chapter 3.

5.4.1 Risk measures 1D

First, we consider the risk arising from a shift in one rate only. To this end we take a given curve and shift one rate on the curve. Here, we have two possibilities. We could shift the market rate and build the pricing curves with the shifted rate or we take the pricing curves and directly shift the corresponding zero rate.

To calculate the sensitivity of a given financial instrument or portfolio of instruments we price with the current curve and with the shifted curve. The difference is the sensitivity against the shift. If the shift is 1 bp the corresponding risk measure is called *BPV* or *DV01*. Another measure which is closely related to the BPV is the duration, respectively modified duration. This risk measure is the local approximation of a given yield curve by a linear function and it is, thus, related to the slope of the curve in the corresponding time point.

BPV

We consider the *basis point value*, BPV. To this end let P be the price function of a fixed income security in terms of the yield y. The BPV is defined by

$$BPV := -\frac{\Delta P}{10000\Delta y}$$

Thus, this quantity denotes the absolute change in value of one basis point in the given currency if yield changes. Practitioners use BPV to hedge fixed income positions. Suppose we have a futures option position of $1.000.000$ and we would like to hedge with the underlying future. The market rates are given in Table 5.5.

Then, the hedge we would apply is

$$N_{\text{NotionalFuture}} = N_{\text{NotionalOption}} \frac{BPV_{\text{Future}}}{BPV_{\text{Option}}}$$

Table 5.5 *Example for basis point value*

Rate	Futures price	BPV	Option price	BPV
2.76%	119.9780		1.9014	
2.77%	119.2061	0,4221	1.6363	0,20125
2.78%	119.1338		1.4989	

Duration

As outlined in the introduction of this Subsection we consider the *duration*, D. Again, let P be the price function of a fixed income instrument in terms of the yield y. Then, duration is defined by:

$$D := -\frac{1}{P}\frac{\Delta P}{\Delta y}$$

Thus, duration measures the value change of an instrument when yield changes. It is a percentage change. We consider the same example as for the *BPV* hedge.

$$N_{\text{NotionalFuture}} = N_{\text{NotionalOption}} \frac{D_{\text{Future}} D_{\text{Option}}}{P_{\text{Future}} P_{\text{Option}}} = 47.678.275,29$$

As we might expect the corresponding hedge is equal to the hedge calculated using the BPV.

Convexity

Next, we wish to take the curvature of the yield curve into account. This corresponds to the second derivative in a given point on the curve. The risk measure is called *convexity*, C. Let P be the price function of a fixed income instrument in terms of yield y. The convexity is defined by:

$$C := \frac{1}{P}\frac{d^2 P}{dy^2}$$

It measures the change in duration if yields change. Let us consider a Taylor approximation of the price of an instrument in terms of the yield:

$$P(y+\Delta y) \approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2 P}{dy^2}(\Delta y)^2$$

- Price changes

$$\Delta P \approx \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2 P}{dy^2}(\Delta y)^2$$

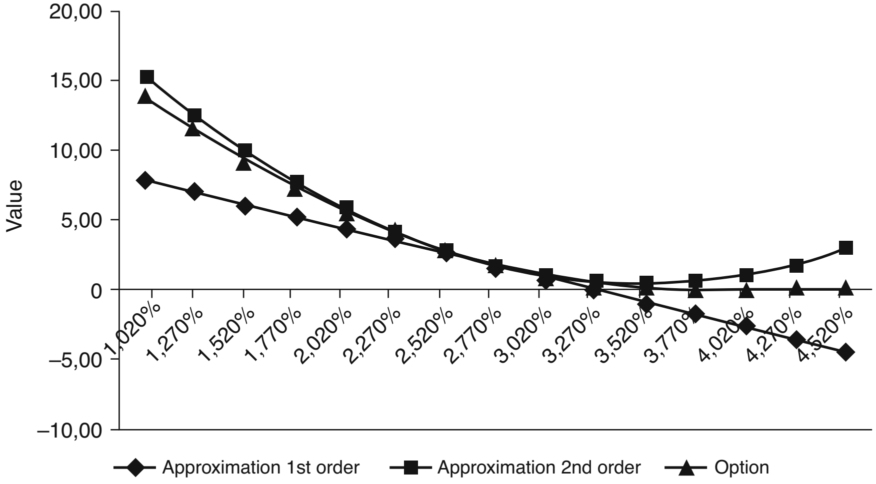


Figure 5.12 Price rate curve with first (duration) and second (convexity) order approximation

- Percentage price changes

$$\frac{\Delta P}{P} \approx \frac{1}{P}\frac{dP}{dy}\Delta y + \frac{1}{2P}\frac{d^2P}{dy^2}(\Delta y)^2$$

- In terms of duration and convexity

$$\frac{\Delta P}{P} \approx -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

We consider a portfolio $P = \sum_k P_k$ of financial instruments, then, we have

$$BPV = \sum_k BPV_k, \quad D = \sum_k \frac{P_k}{P} D_k, \quad C = \sum_k \frac{P_k}{P} C_k$$

5.4.2 Risk measures nD

Often it does not suffice to consider the change in price if a single rate moves. We wish to consider more complex risk measures. To this end it is popular among practitioners to shift several rates at a time or even consider different shift sizes among rates. Using the latter approach it is possible to find out how a given instrument, respectively portfolio moves when the curve changes its shape.

Key rate shifts, key date duration

In contrast to the risk measures introduced before, we wish to account for the risk distributed along the whole curve.

To this end we assign some rates (e.g. 6m, 1Y, 2Y, 5Y, 10Y, 20Y, 30Y) from which we think they capture many risks involved in the curve.

For each time point we consider the price change of a fixed income instrument if one of the rates moves.

Let P again be the price function in terms of the key rates $y_1,\dots,y_N$. The *key rate*, BPV^k, is defined by

$$BPV^k = -\frac{1}{10000}\frac{\partial P}{\partial y_k}$$

According to this definition we can also consider the key rate duration, D^k defined by

$$D^k = -\frac{1}{P}\frac{\partial P}{\partial y_k}$$

5.4.3 Forward buckets

While key rates provide access to different part of the curves there is yet a better way to access this topic.

We consider the forwards for certain terms. Then, we shift the corresponding forward rate by a given value and consider the changes in price of the fixed income instrument under consideration. As market data we take the ones given by Table 5.6. As a final application we wish to illustrate how sensitivities calculated on a curve can be used to hedge a given financial instrument using basic instruments. To this end we take the same example given in Tuckman and Serrat (2012). We consider an option, in this case a 5x10 Payer Swaption, and consider three different hedge deals. First, we take a 10y swap payer swap, second, a 5x10 payer swap and a combination of a 15y and a 5y payer swap. Table 5.7 shows the sensitivities for the option and for the hedging portfolios. Figure 5.13 graphically illustrates the hedging application.

Principal components

Finally, we consider the key drivers of the yield curve by analyzing the *principal components* of the yield curve. We wish to attribute the changes of an underlying instrument to the height, slope and curvature of the yield curve and, furthermore, derive a percentage of the power of the explanatory factor. In doing so we first have to define what a principal component is. To round up this subsection we give two

Table 5.6 *Market rates for hedging analysis*

Instrument	Rate	Fwd buckets				
		0–2	2-5	5–10	10–15	Sum
5x10 Payers	4.04%	0.001	0.0016	−0.0218	−0.0188	−0.038
5y IRS	2.12%	0.0196	0.0276	0	0	0.0472
10y IRS	2.94%	0.0194	0.0269	0.0394	0	0.0857
15y IRS	3.29%	0.0194	0.0265	0.0383	0.0323	0.1165
5x10 IRS	4.04%	0	0	0.0449	0.0366	0.0815

Table 5.7 *Example: hedging a payer swap using different instruments*

Instrument	Hedge ratio	Sensi 1	Sensi 2	Sensi 3	Sensi 4	Sensi net
5x10 payers		0.001	0.0016	−0.0218	−0.0188	−0.038
Hedge 1						
L 44,34% 10y IRS	44.34%	0.0086	0.011928	0.017470		
		0.0096	0.013528	−0.004329	−0.0188	6.93889E-18
Hedge 2						
L 46,66% 5x10 IRS	−46.63%	0	0	−0.020934	−0.017065	−0.038
		0.001	0.0016	−0.042734	−0.035865	−0.076
Hedge 3						
L 56,92% 15y IRS	56.92%	0.011042	0.015083	0.0218	0.018384	0.066310705
S 59,98% 5y IRS	−59.98%	−0.01175	−0.016554	0	−0.028310	
		−0.000713	−0.001471	0.0218	0.018384	0.038
		0.000286	0.000129	3.42434E-15	−0.000415	−1.75127E-13

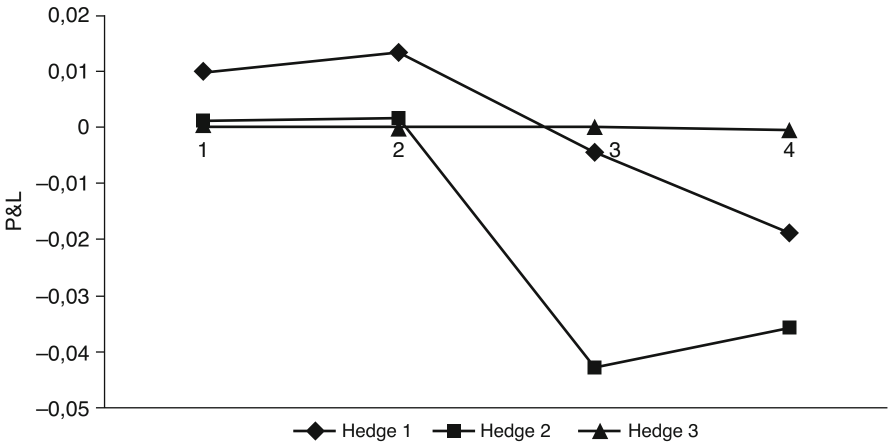


Figure 5.13 Hedge performance for 3 hedges

parametric methods for constructing yield curves which are frequently used by central banks. These methods are called *Nelson-Siegel* and *Svensson*. There are some question to raise when principal components are applied:

- Is PCA an all purpose method to reduce the risk factors?
- Are there situations when PCA might lead to wrong results?
- Are there situations when PCA leads to very good results?

The first question can be answered with “No”. To the following two questions a “Yes” can be assigned. We point to West (2005) for corresponding results and illustrations on several data sets. Let us consider why the PCA is not an all purpose method. Of course working in a low dimensional environment has some advantages. For instance we can assign meaning to the variables. In terms of the yield curve the first three principal components are the height, the slope and the curvature of the curve. For statistical purposes it might be of benefit to do testing on only a few variables than on many. Finally, to use the technique of scenario generation it is reasonable to assign a distribution to the principal variables and simulate them. This enables the user to better interpret the results and keep the computational time for simulation small.

But one has to be aware that the usage of principal components also has disadvantages. First of all if you apply this technique to a yield curve the result would actually be a factor model and not a yield curve model. Furthermore, the approach based on the reduced number of factors might be of limited freedom to correctly model the market observed correlation structure and does not reprice all market instruments correctly.

Principal component math

To illustrate the methodology and its applicability to yield curves we start with yield curve data R_p^i. The index p represents the tenor of a curve rate, for instance $p = 3M$ or $p = 5Y$. The index $i = 1, \ldots, N$ is the number of observations. In general we consider a time-series of rates given by

$$\begin{matrix} R_1^1 & R_2^1 & R_3^1 & \ldots & R_d^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^N & R_2^N & R_3^N & \ldots & R_d^N \end{matrix}$$

We are interested in the variance of the time series. Before we start with the actual calculation we have to standardize the variables by setting $\tilde{R}_i = \frac{R_i - \mu_i}{\sigma_i}$. In the latter equation $\mu_i = \frac{1}{N}\sum_{j=1}^N R_i^j$ and $\sigma_i^2 = \frac{1}{N-1}\sum_{j=1}^N (R_i - \mu_i)^2$. From the transformed rates we calculate the correlation matrix ρ. A real correlation matrix is a symmetric matrix and if we solve the equation $\rho x = \lambda x$ we find that there exist N tuples (λ_i, e_i), $i = 1, \ldots, N$ such that $\rho e_i = \lambda_i e_i$. The values λ_i are called *eigenvalues* and e_i are the corresponding *eigenvectors*. For a correlation matrix the eigenvalues are real and bigger than zero.

REMARK

In practice, however, the calculated correlation matrix is often not a true correlation matrix. There are methods available to transform the calculated matrix into a true correlation matrix. The generated correlation matrix is close to the calculated matrix in a certain mathematical sense. The discussion of such methods lies outside the scope of this book.

Due to a mathematical theorem the correlation matrix can be written as

$$\rho = ODO^\top$$

The entries of the matrix $= D^{1/2}$ are called *factor loadings*. Furthermore, we can order the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ and we find for the matrix $A_m := \sum_{i=1}^m \lambda_i e_i e_i^\top$:

$$\|(A - A_m)x\|_2^2 \leq \lambda_{m+1}^2 \|x\|$$

If the values λ_i get small very quickly A_m is a good approximation for A. Since the approximation is in the least squares sense we have shown that most of the variance is explained by the matrix A_m. In fact the *Lidskii* theorem yields $\lambda_1 + \ldots + \lambda_N = N$ and thus the fraction $\sum_{i=1}^m \lambda_i / N$ is called the *explanatory power*.

Parsimonious models used by many central banks and governments apply the Nelson-Siegel method, Nelson and Siegel (1987) or the Svensson extension, Svensson (1994). In general we have N rates, $y^M(T_i)$ on the curve, $i = 1M, 2M, 3M, \ldots, 11M, 1Y, 2Y, \ldots, 30Y$. Therefore, to apply the Nelson-Siegel method we have to solve for β_0, β_1 and β_2:

$$\begin{pmatrix} 1 & \frac{1-\exp(-\lambda T_1)}{\lambda T_1} & \frac{1-\exp(-\lambda T_1)}{\lambda T_1} - \exp(-\lambda T_1) \\ 1 & \frac{1-\exp(-\lambda T_2)}{\lambda T_2} & \frac{1-\exp(-\lambda T_2)}{\lambda T_2} - \exp(-\lambda T_2) \\ \vdots & \ddots & \vdots \\ 1 & \frac{1-\exp(-\lambda T_N)}{\lambda T_N} & \frac{1-\exp(-\lambda T_N)}{\lambda T_N} - \exp(-\lambda T_N) \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} y^M(T_1) \\ y^M(T_2) \\ \vdots \\ y^M(T_N) \end{pmatrix}$$

This is a regression problem and in fact the parameters are chosen such that the first three factors explain most of the percentage of the curve! For further considerations on the parametric method we refer to Diebold and Li (2006). For illustration we give the parametric forms for Nelson-Siegel Equation (5.9) and Svensson Equation (5.10) using the extension in terms of the short rate by Diebold and Li (2006).

$$r_{\text{ns}}(0,T) = \beta_0 + \beta_1 \frac{1-\exp(-\lambda_1 T)}{\lambda_1 T} + \beta_2 \left(\frac{1-\exp(-\lambda_1 T)}{\lambda_1 T} - \exp(-\lambda_1 T) \right) \tag{5.9}$$

$$\begin{aligned} r_{\text{nss}}(0,T) = \beta_0 + \beta_1 \frac{1-\exp(-\lambda_1 T)}{\lambda_1 T} + \beta_2 \left(\frac{1-\exp(-\lambda_1 T)}{\lambda_1 T} - \exp(-\lambda_1 T) \right) \\ + \beta_3 \left(\frac{1-\exp(\lambda_2 T)}{\lambda_2 T} - \exp(-\lambda_2 T) \right) \end{aligned} \tag{5.10}$$

In this way the parameters β_0, β_1 and β_2 can be interpreted in terms of level, slope and convexity. The parameters λ_1 and λ_2 are decay parameters.

Example

We take $\beta_0 = 0.05$, $\beta_1 = 0.02$, $\beta_2 = -0.05$, $\beta_3 = 0.05$, $\lambda_1 = 9$ and $\lambda_2 = 10$. Then, for the first 5 years we get the following results:

T	1	2	3	4	5
r_{ns}	0.046673249	0.048333334	0.048888889	0.049166667	0.049333333
r_{nss}	0.051670752	0.050833334	0.050555556	0.050416667	0.050333333

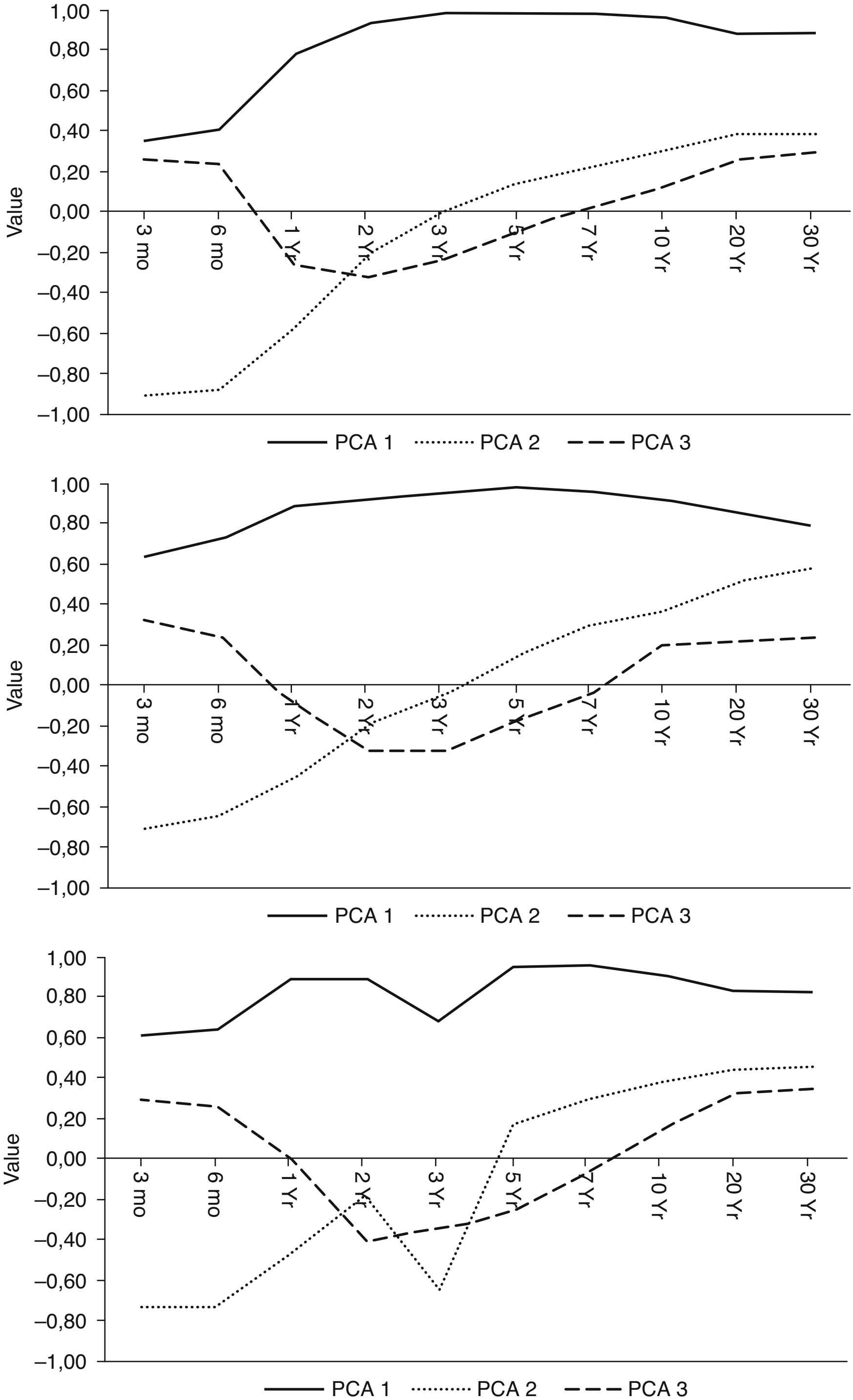


Figure 5.14 PCA for USD (top), EUR (2nd), GBP (3rd) and JPY (bottom) columns swap rates – 15.6.2005 to 15.6.2010

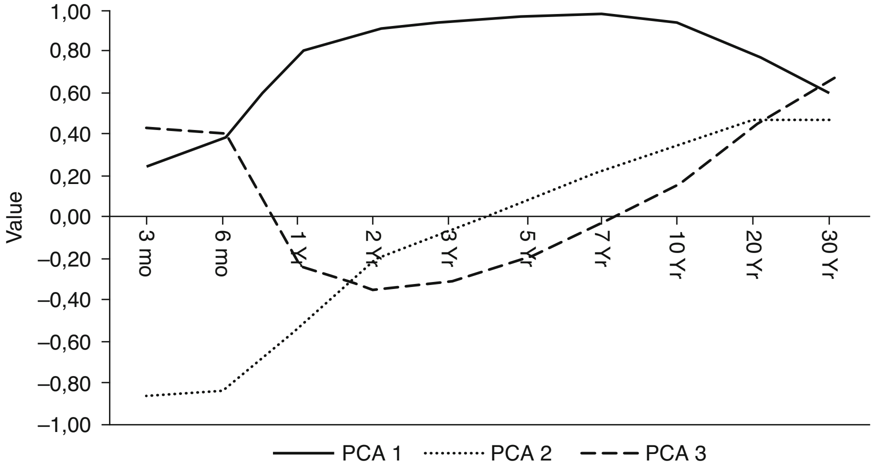


Figure 5.14 (*Continued*)

We apply the methodology of principal component analysis to four different currencies and curves. To this end we take monthly dates for the rates from 15.6.2005 to 15.6.2010. The rates below to the following tenors: $3m, 6m, 1y, 2y, 3y, 5y, 7y, 10y, 20y$ and $30y$. Figure 5.14 shows the outcome of the analysis. The corresponding data is summarized in Table 5.8. The partial sum of the eigenvalues is shown in the left column. Thus, the first entry is the explanatory power of the first, the second entry of the second principal component. Thus, our analysis shows that the height, the slope and the curvature explain more than 95% of the variance of the whole yield curve. More information and illustration can be found on the spreadsheet on principal component analysis.

As a final application we show a hedging strategy using principal component analysis. Suppose we have rates $r_1, \ldots, r_N$ and corresponding $BPV_1, \ldots, BPV_N$. Furthermore, we assume that we have done a PCA analysis using the rates $r_1, \ldots, r_N$ with levels $l_1, \ldots, l_N$ and slopes $s_1, \ldots, s_N$.

We wish to determine amounts N_n and N_m for rates r_n and r_m in a hedge for rate 100 of r_k. The following equations have to be solved:

$$N_n \frac{BPV_n}{100} \cdot l_n + N_m \frac{BPV_m}{100} \cdot l_m + 100 \frac{BPV_k}{100} \cdot l_k = 0$$
$$N_n \frac{BPV_n}{100} \cdot s_n + N_m \frac{BPV_m}{100} \cdot s_m + 100 \frac{BPV_k}{100} \cdot s_k = 0$$

Solving for N_n and N_m gives the result.

Example

We assume that the sensitivities are given by Table 5.9. We have to solve for N_n and N_m. This is possible since we have two equations and two unknown variables. The solution is

$$N_m = 23.1489936, \quad N_n = 106.7503947.$$

Table 5.8 *Explanatory power and eigenvalues for USD, EUR, GBP and JPY*

USD		EUR		GBP		JPY	
69.487%	2.6360	70.876%	2.6623	69.458%	2.6355	59.906%	2.4476
92.945%	1.5316	90.164%	1.3888	88.799%	1.3907	82.415%	1.5003
98.048%	0.7144	95.644%	0.7403	96.996%	0.9054	94.433%	1.0963
98.902%	0.2922	97.253%	0.4011	98.411%	0.3762	98.110%	0.6064
99.383%	0.2193	98.775%	0.3901	99.351%	0.3066	98.996%	0.2977
99.760%	0.1942	99.340%	0.2377	99.727%	0.1939	99.607%	0.2472
99.942%	0.1349	99.627%	0.1694	99.882%	0.1245	99.920%	0.1769
99.978%	0.0600	99.801%	0.1319	99.950%	0.0825	99.979%	0.0768
99.994%	0.0400	99.907%	0.1030	99.993%	0.0656	99.990%	0.0332
100.000%	0.0245	100.000%	0.0964	100.000%	0.0265	100.000%	0.0316

Table 5.9 *Market data for hedge parameter calculation*

BPV_n	0.003	l_n	0.5	s_1	0.05
BPV_m	0.0031	l_m	0.55	s_2	−0.03
BPV_k	0.00305	l_k	0.525	s_3	0.02

5.5 Interpolation in two and three dimensions

We wish to extend the methods to higher dimensions since it will become necessary to interpolate from surfaces and cubes when we consider the valuation of options. Relying on the methods we already established for the one dimensional case we build up a methodology for extending the interpolation algorithms to multiple dimensions. To this end we break down the problem into lower dimensional problems and solve these problems.

To fix ideas we consider a surface which is nothing but a selection of three sets of points

$$
\begin{aligned}
\mathcal{T} &= \{t_1, t_2, \ldots, t_{N_T}\}, \quad t_1 < t_2, \ldots < t_{N_T}, \\
\mathcal{K} &= \{k_1, k_2, \ldots, k_{N_K}\}, \quad k_1 < k_2, \ldots < t_{N_K}, \\
\mathcal{V} &= \{y_{1,1}, \ldots, y_{1,N_K}, y_{2,1}, \ldots, y_{2,N_K}, \ldots, y_{N_T,1}, \ldots, y_{N_T,N_K}\}
\end{aligned}
$$

The size of $\mathcal{T}$ is N_T and $\mathcal{K}$ is N_K. The set

$$
\mathcal{C}_d^2 = \{\mathcal{T}, \mathcal{K}, \mathcal{Y}\}
$$

is a (discrete) surface. We are interested in values which correspond to $(t,k) \notin \mathcal{T} \times \mathcal{K} \times \mathcal{S}$. This is useful for the interpolation of volatility surfaces. We observe such structures when we consider swaption smiles, i.e. implied volatility for a given swap tenor and different times and strikes.

$$
\begin{aligned}
\mathcal{T} &= \{t_1, t_2, \ldots, t_{N_T}\}, \quad t_1 < t_2, \ldots < t_{N_T}, \\
\mathcal{K} &= \{k_1, k_2, \ldots, k_{N_K}\}, \quad k_1 < k_2, \ldots < t_{N_K}, \\
\mathcal{S} &= \{s_1, s_2, \ldots, s_{N_S}\}, \quad s_1 < s_2, \ldots < t_{N_S}, \\
\mathcal{V} &= \{y_{1,1,1}, \ldots, y_{1,1,N_S}, \ldots, y_{N_T-1,N_K,N_S}, \ldots, y_{N_T,N_K,N_S}\}
\end{aligned}
$$

The size of $\mathcal{T}$ is N_T, $\mathcal{K}$ is N_K and $\mathcal{S}$ is N_S. The set

$$\mathcal{C}_d^3 = \{\mathcal{T}, \mathcal{K}, \mathcal{S}, \mathcal{Y}\}$$

is a (discrete) cube. We are interested in values which correspond to $(t, k, s) \notin \mathcal{T} \times \mathcal{K}$. This is useful for the interpolation of volatility cubes. We observe such structures when we consider the swaption smile for different times, strikes and swap tenors.

For the examples we consider the two dimensional case, namely interpolating a surface of volatilities. For a given tenor there are some strike values $\mathcal{K}$ for each of these values a number called the volatility is given. But in practice we do not have only one but many different tenors denoted by $\mathcal{T}$. Now, we have to think about evaluating a number for data (t, k) with $t \notin \mathcal{T}$ and $k \notin \mathcal{K}$. We first create a new curve $\tilde{\mathcal{K}}$ which consist of all interpolated values with respect to k. This new curve is one dimensional and consists of values for a single strike k but different tenor values $\mathcal{T}$. Now, to get the value at t we can use this curve for interpolation.

5.6 Example: cap and floor volatilities

We take the interpolation of the cap and floor volatilities. To this end we have collected market quotes for different tenors and different strikes. Such quotes are available, see for instance Tables 6.3, 6.4 or 6.5. We apply different interpolator functions. For our strike and time grid we get Figure 5.15. Thus, the volatilities and therefore the prices for the corresponding contracts depend on the used interpolation method.

Especially, to point out that significant differences can arise for the interpolated volatilities we have included Figure 5.7.

The difference not only affects the cap prices but also the stripping algorithm to calculate caplet prices from cap prices will lead to different caplet volatilities. To this end we have to be careful when choosing the interpolation algorithm.

5.7 Reading list

We suggest Andersen and Piterbarg (2010a), Chapter 6, as a guide for the yield curve building problem. All the necessary math can be found and the analysis of local and global interpolator functions is very detailed. This is then applied to the study of risk measures such as BPV, Duration and Convexity. A thorough analysis of the different

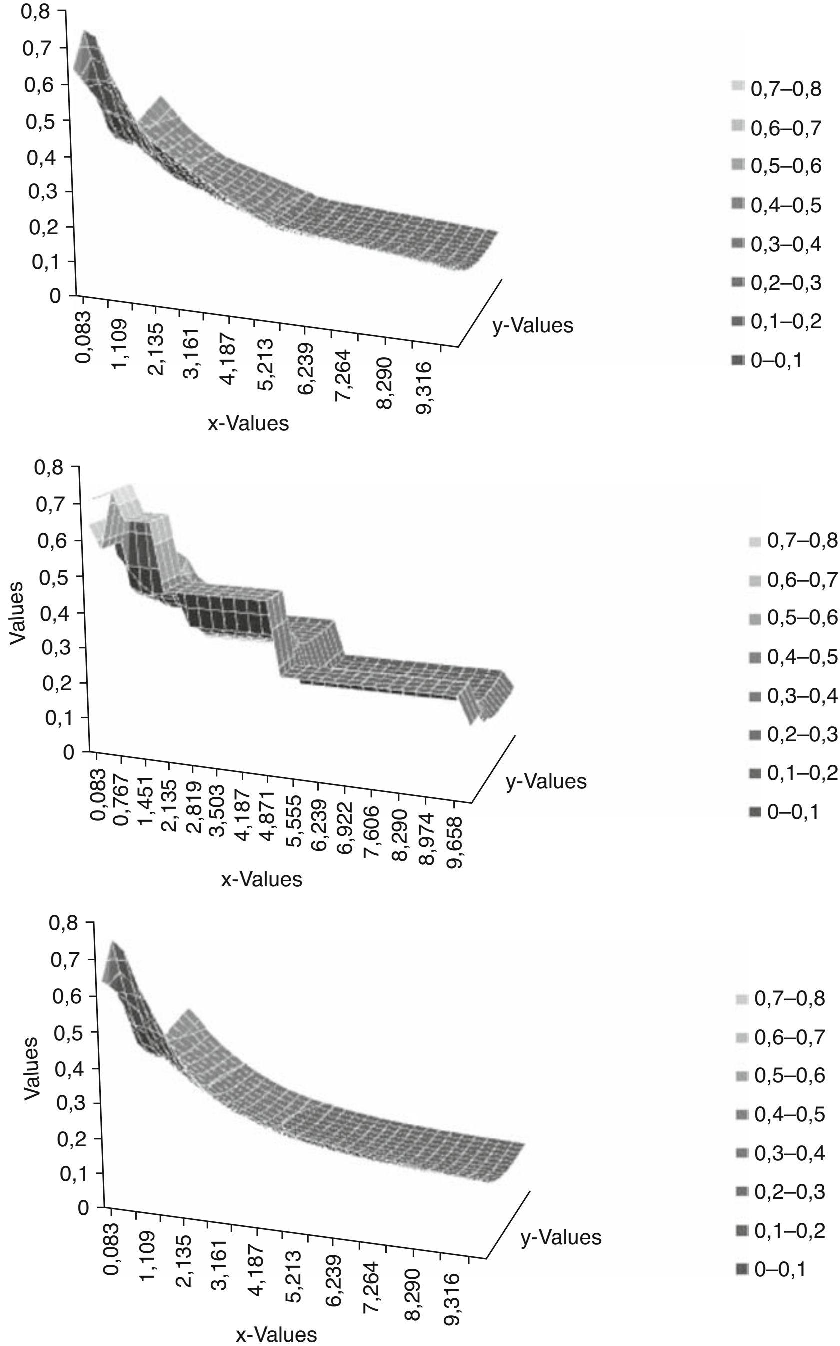


Figure 5.15 Different interpolation methods for two dimensional interpolation. Bilinear (top), piecewise constant (2nd), cubic (3rd) and Kruger (bottom)

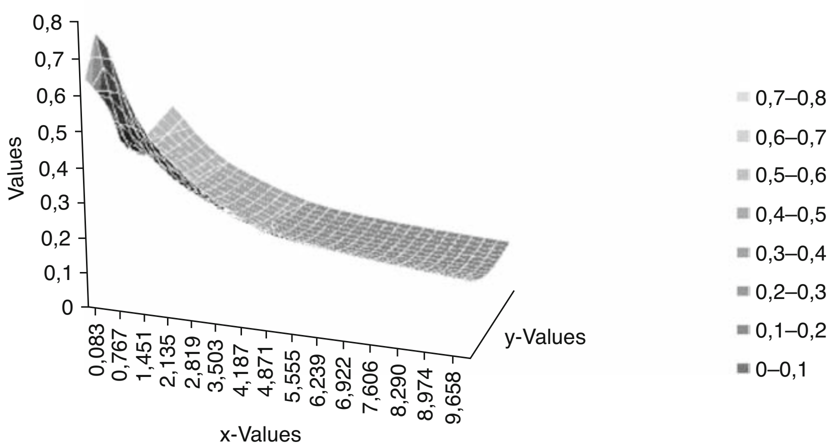


Figure 5.15 (*Continued*)

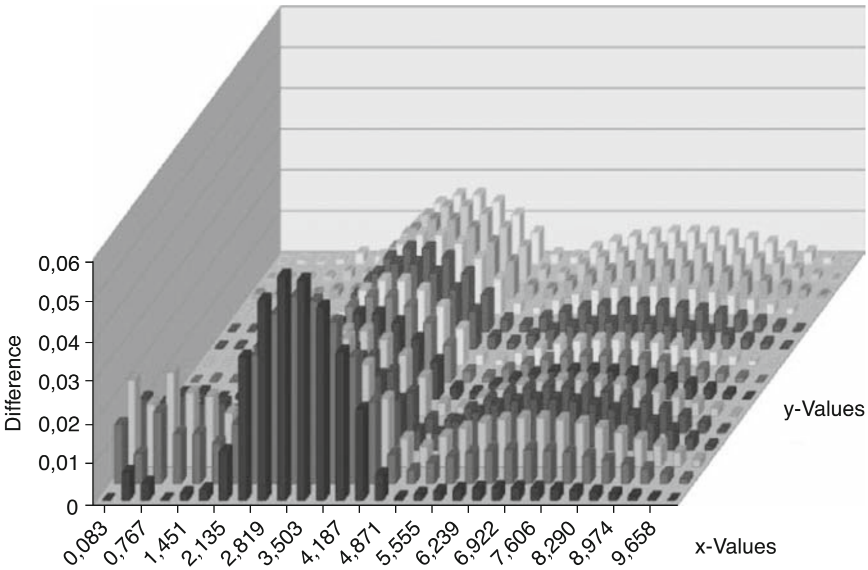


Figure 5.16 Difference between bilinear and cubic interpolation

interpolation schemes and the effect on building yield curves is analyzed in detail. Another book which focuses on the implementation of the concepts is Duffy and Germani (2013). They provide an introduction to interpolation and curve building by describing the whole process using C# classes. They provide numerous examples and describe the practical setting of developing the necessary algorithms. This includes the monotone convex approach to yield curve building published in Hagan and West (2006a) which we have reviewed in this chapter.

The application of principal component analysis is wide-spread. One reference which is very readable is West (2005). Parametric approaches for building yield curves relying on principal component analysis include the pioneering work Nelson and Siegel (1987) and Svensson (1994). Many central banks and treasury departments used such parametric approaches for modeling yield curves or their future evolution.

The papers Fujii, Shimada and Takahashi (2010a) and Fujii and Shimada and Takahashi (2010b) are among the first to consider the multi-curve approach and the relation to collateral. These papers are very mathematical and treat the curve building in a rigorous way. The bootstrapping methodology can be found in Andersen and Piterbarg (2010a) and for the multi-curve approach in Ametrano and Bianchetti (2009) and Ametrano and Bianchetti (2013). The latter reference covers many aspects of the new modern curve building methodologies including the choice of proper rates and hints on the implementation. It also has recipes for using different types of rates necessary to set up the new yield curve universe for pricing and risk management.

The calibration methodology which applies optimizing a score function is detailed in Fries (2012). The author details all the instruments necessary to construct the yield curves. Furthermore, Java classes for implementing this approach are available from the author's website and the software architecture is described there. Using this framework it is possible to tackle many of the problems of curve building and very satisfying results have been prodcued.

II MARKETS AND NON-LINEAR PRODUCTS

6 Options I

6.1 Introduction and objectives

The key objective of this chapter is to introduce financial products which to be priced need further information than the current yield curve. In fact we introduce options on the basic rates and products we have introduced so far.

6.2 Market conventions

In this section we cover the new market conventions for quoting option prices. First, let us start with the standard methodology for option price quotes, the *Black Implied Volatility*.

As already outlined in the previous chapters of this book the interest rate markets faced several changes. These changes also affected the way options on interest rates are handled. Not only is OIS discounting used for quoting prices but for instance the yield curves for several currencies including EUR and CHF led to the problem of negative forward rates. Obviously the standard approach using implied Black volatilities cannot be applied in this setting. We describe two approaches, the *Normal model* and the *Displaced Diffusion model*. The latter is also sometimes called the *Shifted Black model*.

For risk management purposes *sensitivities* or *Greeks* are calculated. Within the models we consider such sensitivies correspond to derivatives of the pricing formula with respect model parameters. Well known Greeks are

- Delta, $\Delta = \frac{\partial P}{\partial S}$
- Gamma, $\Gamma = \frac{\partial^2 P}{\partial S^2}$
- Vega, $V = \frac{\partial P}{\partial \sigma}$

For all the models we give the corresponding Greeks.

6.2.1 Black model

The current market conditions have to take into account negative rates and very large values for implied volatility. To this end data providers have started to contribute other volatilities than the standard Black implied volatilities.

These volatilities are either *normal volatilities* or *displaced diffusion volatilities* which we cover in this section.

We have based our considerations above on equivalent Black volatilities but using results from Hagan et al. (2002) we can also rely on equivalent normal volatilities. We briefly consider the pricing approaches which are detailed in Kienitz and Wetterau

(2012). First, to define Black implied volatilities we re-consider the Black formula:

$$C_{LN}(S(0), K, T, \sigma) = S(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2) \tag{6.1}$$

$$P_{LN}(S(0), K, T, \sigma) = K\mathcal{N}(-d_2) - S(0)\mathcal{N}(-d_1) \tag{6.2}$$

Here we denote $d_1 = \frac{\log\left(\frac{S(0)}{K}\right) + \frac{\Sigma^2}{2}}{\Sigma}$ and $d_2 = d_1 - \Sigma$ and $\Sigma = \sigma\sqrt{T}$.

Suppose now that a European Call/Put has price V_C/V_P. Then, the Black implied volatility is the number σ_{impl} such that $V_C = C_{LN}(S(0), K, T, \sigma_{\text{impl}})/V_P = P_{LN}(S(0), K, T, \sigma_{\text{impl}})$.

In general if we have some pricing formula depending on a volatility input we can define the implied volatility in this way. In the sequel we consider the *Normal Model* and the *Displaced Diffusion Model* since these models are currently used to quote implied volatilities by data vendors.

> **Example**
> Let us take $S(0) = 0.03$, $K = 0.032$, $T = 2$, $r = 0$ and $\sigma = 0.2$. We obtain $d_1 = -0.086756773$ and $d_2 = -0.369599486$ which leads to a Call option price of 0.002576078.

6.2.2 Normal model

For the Normal model which is also sometimes also called Gaussian or Bachelier model the dynamic of an asset S is given by

$$dS(t) = \sigma_{\text{N}} dW(t), \quad S(0) = s_0 \tag{6.3}$$

For a European Call, respectively Put option price V_C/V_P with strike K and maturity T and dynamic given by Equation (6.3) we have

$$C_N(S(0), K, T, \sigma) = \big(S(0) - K\exp(-rT)\big)\mathcal{N}(d(\sigma_{\text{N}})) + \sigma\sqrt{T}n(d(\sigma_{\text{N}})) \tag{6.4}$$

$$P_N(S(0), K, T, \sigma) = (K - S(0))\mathcal{N}(-d(\sigma_{\text{N}})) + \sigma\sqrt{T}n(d(\sigma_{\text{N}})) \tag{6.5}$$

with $d(\sigma) = \frac{S(0)-K}{\sigma_{\text{N}}\sqrt{T}}$. Since $S(T)$ is distributed with respect to a Gaussian distribution the values of $S(T)$ can become negative. The volatilities σ_{impl} matching a given price, that is $V_C = C(S(0), K, T, \sigma_{\text{impl}})$, are called Normal implied volatilities or bp volatilities and are currently used to quote prices for interest rate options. The fact that $S(T)$ can become negative is very important here since using this model it is possible to quote values for strikes $K \leq 0$.

> **Example**
> Let us take $S(0) = 0.03$, $K = 0.032$, $T = 2$, $r = 0$ and $\sigma = 0.2$. We obtain $d = -0.01767767$ which leads to a Call option price of 0.110355547.

6.2.3 Displaced diffusion model – shifted black model

Another approach to take negative rates into account is to consider the following dynamics:

$$dS(t) = (S+b)\sigma\, dW(t), \quad S(0) = s_0 \tag{6.6}$$

Together with the volatility σ there is a *displacement* parameter b. The pricing equations for European Call and Put options for dynamic Equation (6.6) are:

$$C_{DD}(S(0), K, T, \sigma) = (S(0)+b)\mathcal{N}(d_1) - (K+b)\mathcal{N}(d_2) \tag{6.7}$$

$$P_{DD}(S(0), K, T, \sigma) = (K+b)\mathcal{N}(-d_2) - (S(0)+b)\mathcal{N}(-d_1) \tag{6.8}$$

with $d_1 = \frac{\log\left(\frac{S+b}{K+b}\right) + \frac{\Sigma_{\text{dd}}^2}{2}}{\Sigma_{\text{dd}}}$ and $d_2 = d_1 - \Sigma_{\text{dd}}$, $\Sigma_{\text{dd}} = \sigma\sqrt{T}$. Data providers have started to quote σ together with the displacement parameter b. Then, the option prices can be obtained by Equation (6.7). We observe that the pricing within the displaced diffusion model is very close to the Black model. It is just to replace the input data and adjust the data by the displacement coefficient. For a given displacement parameter there is only one input parameter – the volatility – which can uniquely be used to determine the price for European Call and Put options. Using the same methodology as for the Black and the Normal models we define the Displaced Diffusion implied volatility.

Example

Let us take $S(0) = 0.03$, $K = 0.032$, $T = 2$, $r = 0$, $\sigma = 0.2$ and $b = 0.005$. We obtain $d_1 = -0.33068341$ and $d_2 = -0.613526122$ which leads to a Call option price of 0.001671505.

6.3 Caps and floors

In this section we outline the pricing of Caps and Floors on LIBOR rates.

Pricing in *T*-Fwd measure

The price of an option with payoff $V(T)$ at maturity T can be written in terms of an expected value with respect to the risk neutral measure $\mathbb{Q}$:

$$V(t) = \mathbb{E}_{\mathbb{Q}}[V(T)D(t,T)|\mathcal{F}_t]$$

Now, we take another measure. This measure is chosen such that the zero coupon bond maturity at T is the numeraire. Then, using the change of numeraire technique we find:

$$V(t) = \mathbb{E}_{\mathbb{Q}}[V(T)D(t,T)|\mathcal{F}_t] = D(t,T)\mathbb{E}_T[V(T)|\mathcal{F}_t],$$

since $D(T,T) = 1$. $\mathbb{E}_T[\,\cdot\,]$ is the expectation with respect to the T-forward measure.

6.3.1 Cap/Floor math

We consider a time grid $\tilde{\mathcal{T}} = \left\{\tilde{T}_0, \tilde{T}_1, \ldots\right\}$ with corresponding floating rates $L_{nM}(t, \tilde{T}_{k-1}, \tilde{T}_k)$. For a given strike value K the payoff of a Caplet is given by

$$P^{\text{Caplet}} = \tilde{\tau}_k \left(L_{nM}(\tilde{T}_{k-1}, \tilde{T}_{k-1}, \tilde{T}_k) - K\right)^+, \; P^{\text{Floorlet}} = \tilde{\tau}_k \left(K - L_{nM}(\tilde{T}_{k-1}, \tilde{T}_{k-1}, \tilde{T}_k)\right)^+ \tag{6.9}$$

Taking the expectation with respect to the $\tilde{T}_k$-Forward measures on the discount curve leads to the pricing formula for a Caplet/Floorlet:

$$V^{\text{Caplet}}(t, \tilde{T}_k, K) = \tilde{\tau}_k \mathbb{E}_{\tilde{T}_k}\left[\left(L_{nM}(t, \tilde{T}_{k-1}, \tilde{T}_k) - K\right)^+ \middle| \mathcal{F}_t\right] D(t, \tilde{T}_k) \tag{6.10}$$

The value of the expectation in Equations (6.10) and (6.11) depends on the pricing model. For the corresponding value of the Floorlet $V^{\text{Floorlet}}(t, \tilde{T}_k, K)$ we have to calculate the expectation of $\left(K - L_{nM}(\tilde{T}_{k-1}, \tilde{T}_k)\right)^+$. In explicit form we obtain for $i = LN, N, DD$:

$$V^{\text{Caplet}}(t, \tilde{T}_k, K) = \tilde{\tau}_k D(t, \tilde{T}_k) C_i \left(L_{nM}(t, \tilde{T}_{k-1}, \tilde{T}_k), K, \tilde{T}_{k-1}, \sigma_k \sqrt{\tilde{T}_{k-1} - t}\right)$$
$$V^{\text{Floorlet}}(t, \tilde{T}_k, K) = \tilde{\tau}_k D(t, \tilde{T}_k) P_i \left(L_{nM}(t, \tilde{T}_{k-1}, \tilde{T}_k), K, \tilde{T}_{k-1}, \sigma_k \sqrt{\tilde{T}_{k-1} - t}\right)$$

Thus, since the payoff of Cap/Floor is the sum of the Caplets/Floorlets we find the price of a Cap/Floor to be:

$$V^{\text{Cap}}(t) = \sum_k = V^{\text{Caplet}}(t, \tilde{T}_k, K), \text{ resp. } V^{\text{Floor}}(t) = \sum_k = V^{\text{Floorlet}}(t, \tilde{T}_k, K) \tag{6.11}$$

Using the *Put-Call-Parity* we find an interesting relation for the ATM strike of a Cap/Floor. The ATM strike is nothing but the forward Swap Rate on a fixed against floating interest rate swap.

Example
For the Cap pricing example we take the forward rates from Table 6.1. We consider the pricing models from the beginning of this chapter, namely Black, Displaced Diffusion and Normal. For the constituting Caplets we get the prices displayed in Table 6.2.

The interesting thing here is that the prices do not differ much if we have to input different volatilities to the models.

Table 6.1 *Term structure of forward rates*

Period	Forward
$0->3$	0.20%
$3->6$	0.23%
$6->9$	0.23%
$9->12$	0.23%
$12->15$	0.23%
$15->18$	0.23%
$15->21$	0.23%
$21->24$	0.23%

Table 6.2 *Caplet prices with respect to different models and volatilities*

Model strike	Black 0.23%	Displaced 0.23%	Normal 0.23%
σ	65%	11.9251%	0.1466%
	0.029728%	0.029288%	0.029272%
	0.041844%	0.041400%	0.041384%
	0.051018%	0.050689%	0.050677%
	0.058650%	0.058517%	0.058512%
	0.065285%	0.065410%	0.065414%
	0.071205%	0.071639%	0.071655%
	0.076578%	0.077365%	0.077393%

6.3.2 Cap/Floor quotes

Finally, we give examples for market quotes of Caps and Floors. Figures 6.3–6.5 display Caps/Floors Black implied volatilities, Normal implied volatilities and Displaced Diffusion implied volatilities for the EUR market from 22.07.2013. These are the volatilities that have to be plugged into the pricing Equations (6.1), (6.2), (6.4), (6.5), (6.7) and (6.8) to retrieve the prices quoted in Table 6.6.

Caplet stripping

The market quotes shown in this subsection, Tables 6.3–6.6, are not available for all different LIBOR tenors. For instance taking EUR it is market convention to quote Caps against the $3M$ EURIBOR rate up to 2 years and against the $6M$ EURIBOR rate for all larger maturities. To this end to use the market quotes to price Caplets on a given floating rate we have to convert the quoted volatilities to the correct tenor. This will be covered in a later section. After the conversion has been done we have to convert the Cap/Floor volatilities into Caplet/Floorlet volatilities using a bootstrapping method which we briefly outline here.

Table 6.3 *Caps/Floors Black volatilities for EUR as of 22.07.2013*

T	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
1Y	96.39	95.64	95.08	94.59	94.16	93.78	93.46	93.16	92.90	92.67	92.46	92.26
2Y	96.50	94.49	93.32	92.41	91.64	90.99	90.43	89.94	89.50	89.11	88.75	88.43
3Y	71.64	68.81	68.09	67.73	67.48	67.29	67.12	66.97	66.84	66.72	66.62	66.52
4Y	67.48	61.19	59.23	58.39	57.92	57.63	57.41	57.25	57.11	57.00	56.90	56.82
5Y	63.67	55.08	52.00	50.60	49.86	49.39	49.08	48.85	48.67	48.53	48.41	48.30
6Y	60.46	50.67	47.03	45.41	44.57	44.08	43.76	43.54	43.38	43.25	43.15	43.06
7Y	57.73	47.32	43.40	41.69	40.83	40.36	40.07	39.87	39.74	39.64	39.56	39.50
8Y	55.35	44.59	40.47	38.68	37.82	37.36	37.09	36.93	36.82	36.75	36.69	36.65
9Y	53.11	42.15	37.86	36.01	35.15	34.72	34.49	34.36	34.28	34.24	34.21	34.20
10Y	51.08	40.03	35.62	33.72	32.87	32.47	32.28	32.18	32.14	32.13	32.13	32.14
12Y	48.08	37.03	32.42	30.37	29.45	29.04	28.85	28.77	28.75	28.76	28.79	28.82
15Y	45.38	34.41	29.76	27.64	26.70	26.28	26.10	26.05	26.04	26.07	26.11	26.16
20Y	43.68	32.65	28.09	26.06	25.17	24.80	24.65	24.62	24.63	24.67	24.73	24.79
25Y	43.26	31.98	27.42	25.39	24.48	24.08	23.91	23.85	23.85	23.87	23.91	23.95
30Y	43.26	31.68	27.10	25.05	24.12	23.70	23.51	23.43	23.41	23.41	23.43	23.46

Table 6.4 *Caps/Floors normal volatilities for EUR as of 22.07.2013*

T	STK	ATM	0.00	0.25	0.50	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	5.00	10.0
1Y	0.32	34.10	26.6	31.4	40.8	58.7	73.0	86.0	92.3	98.6	111	123	136	161	292
18M	0.37	38.80	30.5	34.7	43.3	61.2	76.3	90.2	96.8	103	116	128	141	166	290
2Y	0.42	43.90	34.9	39.0	46.3	64.0	79.8	94.3	101	108	121	133	146	170	288
3Y	0.72	54.30	38.7	42.3	47.7	63.4	79.0	93.5	100	107	120	132	144	167	274
4Y	0.92	63.50	41.4	45.3	51.0	66.0	80.2	93.7	100	106	118	130	141	163	259
5Y	1.13	70.50	43.1	47.0	52.7	67.0	80.1	92.2	98.0	103	114	124	135	154	241
6Y	1.32	75.30	44.9	48.7	54.2	67.5	79.4	90.4	95.5	100	110	119	128	146	224
7Y	1.49	78.40	47.7	50.9	55.9	68.0	78.7	88.3	92.8	97.2	105	114	122	137	209
8Y	1.64	80.30	49.9	52.7	57.3	68.3	77.8	86.3	90.4	94.3	101	109	116	130	197
9Y	1.78	81.20	51.4	53.9	58.1	68.2	76.8	84.5	88.0	91.5	98.4	105	111	124	188
10Y	1.91	81.50	52.5	54.8	58.6	68.0	75.8	82.7	86.0	89.2	95.4	101	107	120	180
12Y	2.11	81.30	53.9	55.9	59.3	67.4	74.1	80.0	82.8	85.5	90.8	96.1	101	112	166
15Y	2.31	79.50	54.8	56.5	59.4	66.1	71.7	76.6	79.0	81.2	85.8	90.4	95.1	104	153
20Y	2.42	75.60	52.9	54.8	57.4	63.0	67.8	72.1	74.2	76.2	80.4	84.4	88.6	97.0	139
25Y	2.44	72.10	49.4	51.7	54.3	59.7	64.3	68.5	70.5	72.6	76.7	80.4	84.2	91.8	129
30Y	2.43	69.50	46.9	49.3	51.9	57.2	61.7	65.8	67.8	70.1	74.0	77.5	81.1	88.1	121

Let us consider a given tenor and strike value denoted by nM and K and start with the first quote. We assume that the Cap quotes are given for the time grid $\tilde{\mathcal{T}} = \{\tilde{T}_{2nM}, \tilde{T}_{3nM}, \ldots, \tilde{T}_{NnM}\}$. We do not consider $\tilde{T}_{nM}$ since the floating rate for $[0, \tilde{T}_{nM}$ is already fixed and there is no uncertainty about this rate and, thus, no time value to the Caplet price. The T_{knM} Cap consists of $k-1$ Caplets and the price is

Table 6.5 *Caps/Floors shifted black volatilities for EUR with a shift of 1% as of 22.07.2013*

T	STK	ATM	0.00	0.25	0.50	1.00	1.50	2.00	2.25	2.50	3.00	3.50	4.00	5.00	10.0
1Y	0.32	25.78	22.9	24.4	28.9	35.6	39.2	41.6	42.7	43.7	45.6	47.3	49.0	52.0	63.9
18M	0.37	28.21	25.7	26.4	30.0	36.4	40.2	42.9	44.0	45.0	46.9	48.5	50.0	52.6	62.7
2Y	0.42	30.77	28.9	29.1	31.5	37.4	41.3	44.1	45.2	46.2	48.0	49.5	50.8	53.1	61.5
3Y	0.72	31.35	28.9	28.7	29.6	33.7	37.2	39.8	40.8	41.8	43.4	44.8	46.0	47.9	53.9
4Y	0.91	32.64	29.5	29.3	30.1	33.1	35.4	37.3	38.1	38.8	40.1	41.1	42.0	43.5	47.8
5Y	1.12	32.35	29.3	29.1	29.7	31.9	33.5	34.7	35.2	35.7	36.5	37.1	37.8	38.8	42.0
6Y	1.32	31.48	29.3	29.0	29.3	30.8	31.8	32.5	32.8	33.1	33.6	34.0	34.4	35.1	37.5
7Y	1.49	30.40	29.9	29.1	29.2	30.0	30.4	30.7	30.8	30.9	31.2	31.4	31.6	32.1	34.0
8Y	1.64	29.22	30.1	29.1	28.9	29.2	29.2	29.2	29.2	29.1	29.2	29.2	29.3	29.6	31.2
9Y	1.78	27.98	30.1	28.9	28.5	28.5	28.2	27.9	27.7	27.6	27.5	27.4	27.5	27.6	29.1
10Y	1.91	26.81	29.8	28.6	28.1	27.7	27.2	26.7	26.5	26.4	26.1	26.0	26.0	26.1	27.4
12Y	2.11	24.92	29.2	27.9	27.3	26.6	25.7	25.1	24.8	24.5	24.1	23.9	23.8	23.7	24.6
15Y	2.31	22.98	28.3	27.0	26.2	25.2	24.2	23.4	23.1	22.8	22.3	22.0	21.8	21.7	22.4
20Y	2.42	21.42	26.9	25.7	24.9	23.8	22.7	21.9	21.6	21.3	20.9	20.6	20.4	20.2	20.6
25Y	2.44	20.62	25.5	24.5	23.7	22.7	21.8	21.1	20.8	20.6	20.2	19.9	19.7	19.4	19.5
30Y	2.43	20.15	24.6	23.6	23.0	22.0	21.1	20.5	20.2	20.1	19.8	19.5	19.2	18.9	18.7

given by

$$V^{\text{Cap } T_{knM}}(t) = \sum_{j=nM}^{knM} V^{\text{Caplet}}(t, \tilde{T}_j, K)$$

Using this relationship we find for the Caplet spanning the period $[\tilde{T}_{(k-1)nM}, \tilde{T}_{knM})$ that it is the difference of two Caps:

$$V^{\text{Cap } T_{(k+1)nM}}(t) - V^{\text{Cap } T_{knM}}(t) = V^{\text{Caplet}}(t, \tilde{T}_{(k+1)nM}, K)$$

From the latter equation it is possible to infer the implied Caplet Volatility using one of the introduced models – Black, Normal, Displaced Diffusion.

The bootstrapping methodology only works if enough Cap quotes are available. If this is not the case we rely on inter-/extrapolation. A second method to determine the Caplet volatilities is to consider the problem as an optimization problem given some function for the Caplet volatility depending on parameters which control the shape of this function. The values taken from this functional form and subtracting the market quotes for the implied Caplet volatilities calculated from market quotes should be minimized. Often quants do not take the sum of the absolute differences but the sum of the squared differences. If we use a different parametric form for the intervals between time points of quoted market values we need to specify what happens at the end points of the interval. Usually, we need the implied Caplet volatilities to be continuous meaning that the values using two different parametric forms coincide on the common boundary. Often a smoothness restriction is also applied. In practice we

Table 6.6 *Cap (top) and Floor (bottom) premiums for EUR as of 22.07.2013*

Cap premiums												
T	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
1Y	1	0	0	0	0	0	0	0	0	0	0	0
2Y	13	5	3	2	1	1	1	0	0	0	0	0
3Y	48	21	11	7	5	3	2	2	1	1	1	1
4Y	123	61	36	24	17	12	9	7	6	4	4	3
5Y	236	128	78	51	36	26	20	15	12	9	8	6
6Y	376	215	133	88	62	45	34	26	21	17	13	11
7Y	536	317	199	133	94	69	52	41	32	26	21	18
8Y	714	434	276	186	132	97	74	58	46	37	31	25
9Y	904	561	359	242	172	127	97	76	61	50	41	34
10Y	1102	694	447	302	214	159	122	96	77	63	52	44
12Y	1507	974	632	426	301	222	169	133	106	87	72	60
15Y	2078	1375	903	609	430	317	242	190	153	125	104	87
20Y	2839	1908	1269	866	619	462	357	284	231	192	161	137
25Y	3447	2338	1568	1080	778	585	456	365	299	250	211	181
30Y	3980	2723	1845	1285	935	711	560	453	375	315	269	232
Floor premiums												
T	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
1Y	52	127	203	279	354	430	506	582	657	733	809	885
2Y	116	285	459	635	811	988	1164	1341	1518	1694	1871	2048
3Y	120	345	588	836	1086	1336	1588	1840	2092	2344	2596	2848
4Y	152	443	769	1108	1453	1800	2149	2498	2848	3199	3550	3901
5Y	179	520	920	1343	1778	2218	2661	3106	3553	4000	4448	4897
6Y	203	587	1051	1553	2072	2602	3137	3675	4216	4758	5301	5845
7Y	224	645	1168	1743	2344	2960	3584	4213	4846	5480	6116	6753
8Y	244	697	1272	1915	2594	3293	4003	4720	5442	6167	6893	7621
9Y	260	740	1361	2067	2819	3597	4390	5192	5999	6810	7624	8440
10Y	275	777	1439	2204	3026	3880	4753	5636	6527	7422	8321	9222
12Y	306	849	1584	2454	3405	4402	5426	6466	7516	8572	9634	10698
15Y	360	964	1799	2813	3941	5135	6367	7623	8893	10172	11458	12749
20Y	486	1205	2215	3462	4864	6357	7902	9478	11075	12685	14304	15930
25Y	640	1482	2662	4124	5772	7530	9351	11211	13095	14996	16908	18828
30Y	804	1764	3103	4760	6627	8620	10685	12795	14934	17091	19262	21442

certainly do need at least to allow different parameters for the short, the mid and the long part of the implied Caplet volatility curve.

We give the algorithm for the non-parametric Caplet stripping in the sequel and apply it to an example. As input variables we take:

- curve – dates and rates
- n_1 – number of Caplets
- CapVola – term structure for the Cap volatilities

The following variables are used during the calculation:

- 2-dimensional arrays
  $CapletVola(n_1+1,2)$, $vc(n_1,2)$, $vf(n_1,2)$, $pc(n_1,2)$, $pf(n_1,2)$

- 1-dimensional arrays
  dateval(n_1), *timefromtoday*(n_1), *discountf*(n_1), *cpv*(n_2)
- Variables
  Indices: *i*, *j*, *k*; Numbers: *s*, *x*, *v*, *T*, *today*
- Functions *DiscountFactor* – calculates the discounts; *BlackPrice* – calculates the Black price; *ImpliedBlackVola* – calculates the Black implied volatility

The dates and the market quotes are stored into the 2 dimensional array *curve*.

Algorithm

```
today = curve(1, 1)

for i = 1 to n1 step 1
        dateval(i) = CapVola(i + 1, 1)
        timefromtoday(i) = today + dateval(i)/yearlength
        discountf(i) = DiscountFactor(timefromtoday(i), curve)
next i

for j = 1 to 2 step 1
        cpv(j) = CapVola(1, 2)
next j

for j = 1 to 3 step 1
        for i = 1 To n1 + 1 step 1
            CapletVola(i – 1, j – 1) = CapVola(i, j)
        next i
next j

for j = 1 to 2
        pc(1, j) = 0
        for i = 2 to n1 step 1
            pc(i, j) = 0
            for k = 2 to i step 1
                s = discountf(k – 1) – discountf(k)
                x = cpv(j)· (dateval(k) – dateval(k – 1)) ·yrf· discountf(k)
                v = CapVola(i + 1, j + 1)
                T = dateval(k – 1)
                pc(i, j) = pc(i, j) + BlackPrice(s, x, v, T, 1)
            Next k
        next i
next j

for j = 1 to 2 step 1
        for i = 2 to n1 step 1
            pf(i, j) = (pc(i, j) – pc(i – 1, j))
```

Algorithm (*Continued*)

$s = \text{discountf}(i-1) - \text{discountf}(i)$

$x = cpv(j) \cdot (\text{dateval}(i) - \text{dateval}(i-1)) \cdot yrf \cdot \text{discountf}(i)$

$v = \text{CapVola}(i+1, j+1)$

$T = \text{dateval}(i-1)$

$\text{CapletVola}(i, j) = \text{ImpliedBlackVol}(s, x, v, T, \text{pf}(i, j))$

next i

next j

This algorithm stores the results into the array *CapletVola.*

Example

We consider a set of given Cap volatilities. From this set we wish to infer the Caplet volatilities. We can use a parametric form for the Caplet volatilities. To this end we set up a function consisting of some parameters and try to find the parameters for which the observed prices for Caps are matched or appropriately approximated. We can also use a non-parametric bootstrapping method to find the Caplet prices in line with the quoted Caps. Figure 6.1 illustrates the setting.

We applied the stripping algorithm and the data from Table 6.7 to get Figure 6.1.

6.4 Swaptions

In this section we consider options on swap rates. For swaptions we have to take the expectation with respect to the annuity measure. Let the annuity be given by the

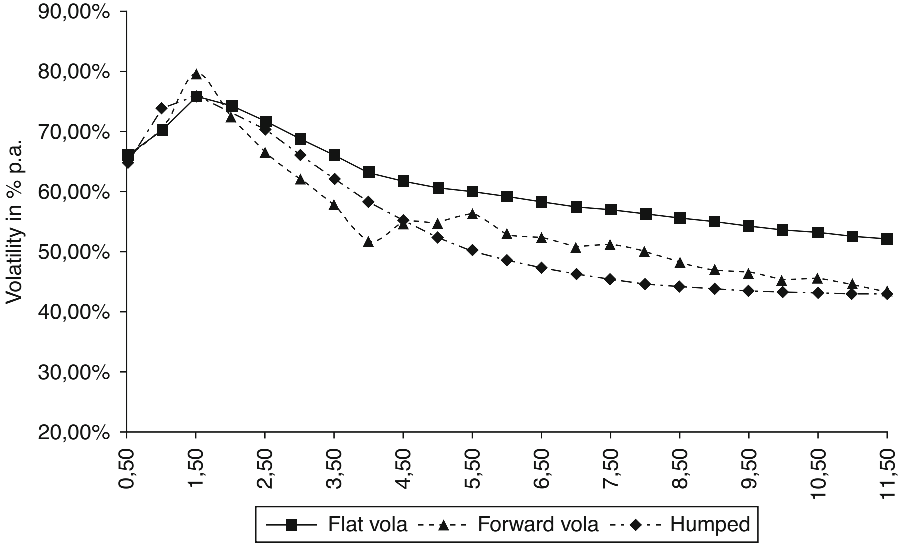


Figure 6.1 Stripped caplet volatilities using a parametric and a non-parameteric approach

Table 6.7 *Term structure of cap and caplet volatilities. The caplet volatilities are obtained using a non-parametric and a parametric form for the volatilities*

Time	Capvol	Capletvol parametric	Capletvol non-parametric
0.5	0.66290696	0.66290696	0.647732742938319
1	0.7032204	0.7032204	0.738982835071111
1.5	0.758708	0.797082540092787	0.757983322002763
2	0.7441956	0.725416551799863	0.739205948113948
2.5	0.7170008	0.665382730744613	0.703134163182852
3	0.689806	0.621692217986084	0.661498093186444
3.5	0.6630116	0.5783908567088	0.620643519679744
4	0.6332172	0.518050659556524	0.583678681794835
4.5	0.6182904	0.549501370218173	0.551825703538306
5	0.6073636	0.548739093567452	0.525258315048529
5.5	0.6013044	0.56424474714752	0.503610795888354
6	0.5922452	0.529688851795764	0.486278826600076
6.5	0.5843184	0.523468276868518	0.472590455465163
7	0.5763916	0.509273584011926	0.461897435052621
7.5	0.5701634	0.512568601600661	0.45361889323686
8	0.5639352	0.501334118053717	0.447257414465625
8.5	0.5571408	0.483312112800537	0.442399939096367
9	0.5503464	0.470918589729268	0.438710989308439
9.5	0.5441182	0.46612164639362	0.435922635673091
10	0.53789	0.454583991223151	0.43382369212796
10.5	0.5325111	0.455962877387001	0.432249450198941
11	0.5271322	0.445855548643373	0.431072562180552
11.5	0.5217533	0.435673981737238	0.430195279954259

denominator of Equation (4.7):

$$A(t) = \sum_{i=m+1}^{n} \tau_i D(t, T_i) \tag{6.12}$$

Pricing with respect to the annuity measure

A swaption can be priced by taking the expected value with respect to the risk neutral measure $\mathbb{Q}$. Indeed we take a payoff $V(T)$ at maturity and calculate the expected value. As we did for calculating an expectation for a Caplet, respectively Floorlet we apply a change of measure technique here. To simplify the calculation we consider the measure $\mathbb{A}$ for which the swap rate maturing at time T is a martingale. The corresponding quantity we have to consider here is (6.12). The change of numeraire technique leads to

$$V(t) = \mathbb{E}_{\mathbb{A}}[V(T)D(t,T)|\mathcal{F}_t] = A(t,T)\mathbb{E}_{A}[V(T)|\mathcal{F}_t]$$

$\mathbb{E}_{\mathbb{A}}$ is the expectation with respect to the annuity measure associated to the annuity, (6.12) is called the *Swap measure*. Let us fix the following notation:

$$
\begin{aligned}
\mathcal{T} &:= \{T_m, T_{m+1}, T_{m+2}, \ldots, T_{N_1} = T_n\} \text{ fixed rate time schedule} \\
\mathcal{T} &:= \{\tilde{T}_m, \tilde{T}_{m+1}, \tilde{T}_{m+2}, \ldots, \tilde{T}_{N_2} = T_n\} \text{ float rate time schedule} \\
SR_{m,n}(T) &= \text{swap rate at } T \text{ for a swap starting in } T_m \text{ ending in} T_n \\
D(t,T) &= \text{zero coupon bond at } t \text{ maturing in } T \\
\mathbb{E}_{\mathbb{P}} &= \text{expectation operator with respect to the measure } \mathbb{P} \\
\mathbb{V}_{\mathbb{P}} &= \text{variance operator with respect to the measure } \mathbb{P} \\
\mathbb{C}_{\mathbb{P}} &= \text{covariance operator with respect to the measure } \mathbb{P}
\end{aligned}
$$

$V^{\text{Receiver}}(0) = \sum_{i=1}^{N_1} K\tau_i D(0,T_i) - \sum_{i=1}^{N_2} FRA(\tilde{T}_{i-1}, \tilde{T}_{i-1}, \tilde{T}_i)\tilde{\tau}_i D(0,\tilde{T}_i)$. The present value is known, Equation (4.7), and we have (with respect to the annuity measure $\mathbb{A}$):

$$
\begin{aligned}
\mathbb{E}_{\mathbb{A}}[V] &= \mathbb{E}_{\mathbb{A}}\left[\sum_{i=1}^{N_1} K\tau_i D(0,T_i) - \sum_{i=1}^{N_2} L(\tilde{T}_{i-1}, \tilde{T}_{i-1}, \tilde{T}_i)\tau_i D(0,\tilde{T}_i)\right] \\
&= \sum_{i=1}^{N_1} K\tau_i D(0,T_i) - \sum_{i=1}^{N_2} \mathbb{E}_{\mathbb{A}}\left[L(\tilde{T}_{i-1}, \tilde{T}_{i-1}, \tilde{T}_i)\tilde{\tau}_i D(0,\tilde{T}_i)\right] \\
&= \sum_{i=1}^{N_1} K\tau_i D(0,T_i) - D(0,T_i)\sum_{i=1}^{N_2} \tilde{\tau}_i \mathbb{E}_{\tilde{T}_i}\left[L(\tilde{T}_{i-1}, \tilde{T}_{i-1}, \tilde{T}_i)\right] \\
&= \sum_{i=1}^{N_1} K\tau_i D(0,T_i) - \sum_{i=1}^{N_2} FRA(0, \tilde{T}_{i-1}, \tilde{T}_i)\tilde{\tau}_i D(0,\tilde{T}_i)
\end{aligned}
$$

From the latter equation we obtained the definition of the swap rate, $SR_{m,n}(t) = \frac{\sum_{i=m+1}^{N_1} \tilde{\tau}_i D(t,\tilde{T}_i) FRA(t,\tilde{T}_{i-1},\tilde{T}_i)}{\sum_{i=m+1}^{N_2} \tau_i D(t,T_i)}$, where we used $T_n = \tilde{T}_{N_2} = T_{N_1}$

6.4.1 Swaption math

Let us consider the nY forward swap rate $SR_{m,m+nY}(t)$. A swaption is an option giving the holder the right to enter at time T into a fixed against float interest rate swap. This swap could either be a payer or a receiver swap. We take the period of the floating rate to be p:

$$
V^{\text{P/R}}(t,T,K) = \sum_{j=m+1}^{n/p} \tau_j \mathbb{E}_A\left[\left(\pm SR_{m,m+nY}(t) \mp K\right)^+ \middle| \mathcal{F}_t\right] D(t,T_k) \tag{6.13}
$$

For swaptions banks often classify these options into *Gamma Swaptions* and *Vega Swaptions*. This is due to the fact that swaptions with longer maturities are much more exposed to volatility than for shorter maturities. The change in prices for the latter options is mainly affected by changes in the underlying swap rate and not so much by changes in the volatility. The maturities up to 1 year are classified as Gamma Swaptions and the ones with maturities longer than 1 year are Vega swaptions. Some banks use a different value here, for instance 2 years.

Physical/Cash settled swaptions

Two types of swaptions are traded. Especially in the European and the Great Britain markets most swaptions are *cash settled Swaptions*. The difference to standard or *physical settled Swaptions* considered so far is that the annuity term is replaced by another quantity. It is assumed that there is a constant discounting rate over the lifetime of the underlying swap. Thus, instead for a swap based on the nM LIBOR rate leading to a payment of $m = 12/n$ with floating payments per year and N floating payments alltogether, the annuity is replaced by

$$\frac{1}{m}\sum_{j=1}^{N}\frac{1}{(1+\frac{1}{m}SR)^j} \tag{6.14}$$

Example

Let us consider $m = 4$ and a swap rate $SR = 0.015$. The following table gives the values for $Denom = \left(1+\frac{1}{m}SR\right)$, $Denom^j = \left(1+\frac{1}{m}SR\right)^j$ and the corresponding annuity values $Ann_N = \frac{1}{m}\sum_{j=1}^{N}\frac{1}{(1+\frac{1}{m}SR)^j}$ for $N = 1,\ldots,5$.

j	1	2	3	4	5
$Denom^{-1}$	0.996264	0.992542	0.988834	0.985140	0.981459
$Denom^{-j}$	0.996264	1.988806	2.977640	3.962779	4.944239
Ann_N	0.249066	0.497201	0.744410	0.990695	1.236060

The advantage of this approach is that the payoff does not depend on the discount curve and, thus, is not dependent on the used bootstrapping, respectively calibration methodology for the curve. The disadvantage is that the established pricing formulas cannot be applied. Instead if we replace the annuity by the factor from Equation (6.14) we in fact use a (hopefully) good approximation to the true price. The approximation is just the replacement of the expectation with respect to the annuity measure A to an expectation with the Cash-Settled measure $\mathbb{C}$:

$$D(0,T)C(S)\mathbb{E}_{\mathbb{C}}[(K-S)^+] \approx D(0,T)C(S)\mathbb{E}_A[(K-S)^+]$$

Henrad M. (2012a) investigates the difference with respect to several interest rate models and Mercurio (2008) shows that the standard pricing formulas with standard smile descriptions lead to arbitrage.

6.4.2 Swaption quotes

The market quotes swaption volatilities as well as option premiums. Before we consider the quotes, we briefly summarize which types of volatilites are considered in the market:

- **Implied volatilities**
  Implied Volatilities are inferred from market prices. The implied volatilities are calculated with respect to one of the pricing Equations (6.1)–(6.7). We call the corresponding implied volatilities *Bachelier*, *Black* or *Displaced Diffusion* implied volatilities. For the Bachelier implied volatilities people often use *Normal* implied volatilities since they are calculated with respect to the model based on Gauss, respectively Normal distribution.

- **Realized volatilites**
  The realized volatilities are calculated from historical data

$$\log(S_i/S_{i-1})$$

- **bp volatilities**
  The implied and the realized volatilities are based on considerations of relative changes in the rates and not on absolute values. We consider the volatility on a absolute level. For instance take the $6M5Y$ swap rate. Assuming this forward rate for two years time is at a level of 4.56% and has an implied volatility of 11.22% we calculate a bp volatility of 32.23 bp. The bp volatility is closely related to the Bachelier volatility. Actually, it is the one day Bachelier volatility and we can approximate it by

$$\frac{\sigma_N}{\sqrt{252}} \approx \frac{\sigma_B S}{\sqrt{252}}$$

  This formula is only a rough proxy. For more accurate formulas we refer to second volume of this series.

Payer and receiver swaptions can be seen as Call and Put options on a swap rate. For the quotes price the premium of an option strategy called a *straddle* is considered. First, the straddle is simply the sum of a Call and a Put option. Both have the same strike. Second, this straddle can be seen as an option with given strike and maturity. To get the implied volatility we could assume that we have a price paid at spot and convert this price into an implied volatility or we can see it as a forward premium for which the implied volatility is calculated and then discounted back to spot. Table 6.8 shows quotes implied Black volatilities and bp volatilities as observed on 22.07.2013. Corresponding premium quoted in basis points are also quoted, Table 6.9. The quotes are from 23.07.2013. Finally, the market quotes volatilities for the shifted Black model. The corresponding quotes can be found in Table 6.10. The shifts are also given in this table.

6.4.3 The swaption smile/skew

For the valuation of Swaptions to different than the ATM strike we have to model the volatilities for different strikes or infer these values from the market. Standard

Table 6.8 *EUR ATM swaption straddles – Black volatilities (EONIA discounting) as of 22.07.2013 (top) and normal volatilities (bottom)*

Mat/T	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M	88.8	86.0	83.5	68.1	56.8	49.0	43.6	39.5	36.3	33.6	26.0	23.6	22.6	21.9
2M	86.5	88.4	81.8	65.6	55.2	47.8	42.7	38.7	35.5	33.1	26.5	24.5	23.5	22.5
3M	91.7	87.3	80.2	64.8	54.6	47.4	42.4	38.6	35.6	33.3	26.9	24.9	24.0	23.1
6M	97.1	84.4	73.6	61.9	53.6	47.2	42.6	39.2	36.4	34.3	28.1	26.3	25.4	24.5
9M	97.1	79.7	69.0	59.2	51.8	46.4	42.4	39.3	36.9	34.9	28.9	27.1	26.2	25.4
1Y	93.0	74.2	65.0	56.5	50.8	45.9	42.3	39.4	37.1	35.1	29.5	27.8	27.0	26.3
18M	87.5	65.7	57.7	51.2	47.1	43.0	39.9	37.5	35.5	33.8	28.9	27.5	26.9	26.3
2Y	82.5	60.6	52.7	47.5	44.0	40.4	37.7	35.6	33.8	32.4	28.2	27.1	26.7	26.2
3Y	60.4	48.3	44.0	40.7	38.0	35.5	33.7	32.2	31.0	30.1	27.2	26.4	26.4	26.1
4Y	48.4	40.4	37.6	35.4	33.6	32.1	30.9	29.9	29.1	28.5	26.4	25.9	26.1	25.9
5Y	40.9	35.3	33.5	31.9	30.6	29.5	28.7	28.1	27.7	27.3	25.8	25.4	25.7	25.5
7Y	31.5	28.1	27.0	26.3	25.8	25.4	25.2	25.1	25.0	25.0	24.0	23.4	23.5	23.3
10Y	24.6	23.0	22.7	22.6	22.8	22.9	23.1	23.3	23.5	23.6	22.8	22.0	21.6	21.0
15Y	23.0	22.6	23.1	23.4	23.7	23.9	24.0	24.3	24.4	24.2	22.5	20.8	19.6	18.9
20Y	24.5	24.3	24.5	24.5	24.6	24.7	24.8	24.8	24.8	24.3	21.3	19.0	17.9	17.1
25Y	25.2	25.1	25.1	24.9	24.7	24.3	24.0	23.6	23.3	22.7	19.3	17.1	16.3	15.4
30Y	23.9	22.7	22.4	22.0	21.5	21.1	20.7	20.3	20.1	19.7	16.8	15.2	14.6	13.9
1M	28.3	46.8	57.6	60.5	62.3	–	63.5	–	–	63.2	59.8	57.1	55.1	53.4
2M	28.9	49.6	58.4	60.0	62.0	–	63.3	–	–	63.0	61.2	59.4	57.5	55.0
3M	31.9	50.8	59.1	61.1	62.8	–	63.9	–	–	64.1	62.6	60.8	59.0	56.6
6M	38.0	54.1	60.2	63.9	66.4	–	67.7	–	–	68.6	66.7	65.2	63.1	60.6
9M	42.2	56.1	62.1	66.4	68.8	–	70.7	–	–	72.0	69.9	67.9	65.8	63.4
1Y	44.8	57.7	64.4	68.7	71.8	–	73.8	–	–	74.8	72.7	70.4	68.5	66.2
18M	52.5	62.7	68.7	72.3	75.2	–	76.2	–	–	76.6	73.6	71.2	69.4	67.4
2Y	63.0	70.2	73.7	76.1	78.0	–	78.0	–	–	77.5	74.1	71.7	70.0	68.0
3Y	75.8	78.8	80.3	80.7	80.9	–	80.0	–	–	79.3	75.3	72.5	71.3	69.9
4Y	83.1	82.0	81.8	81.6	81.7	–	81.3	–	–	80.3	75.7	73.0	71.8	70.7
5Y	84.5	82.2	82.0	81.9	81.9	–	81.4	–	–	80.6	75.6	72.5	71.4	70.3
7Y	81.3	78.8	78.6	78.4	78.6	–	78.3	–	–	77.4	71.6	67.5	66.0	65.2
10Y	76.9	74.5	74.1	73.8	73.8	–	73.3	–	–	72.3	66.2	61.6	59.5	58.1
15Y	68.9	67.3	67.5	67.1	66.6	–	65.9	–	–	64.3	57.9	53.5	51.2	49.9
20Y	61.2	61.1	60.8	60.5	60.2	–	59.7	–	–	57.8	52.0	47.8	46.1	44.4
25Y	57.2	57.2	57.1	56.7	56.2	–	55.4	–	–	54.0	48.2	44.1	42.7	40.9
30Y	54.2	53.8	53.8	53.5	52.9	–	52.0	–	–	50.8	44.9	41.3	39.8	38.4

quotes for such volatilities are given relative to the current ATM strike. Table 6.11 shows the volatility quotes for strikes $ATM \pm 150$, $ATM \pm 100$, $ATM \pm 50$, $ATM \pm 25$ for the Gamma Swaptions and the Vega Swaptions. Corresponding quotes in term of the shifted Black model are also available.

6.5 Transforming volatility

The advent of the multi-curve approach to derivatives pricing makes it necessary to consider different volatilities for interest rates depending on different forward rates. We summarize the results from Kienitz (2014) here.

Table 6.9 *EUR ATM swaption straddles – implied spot premium mids (EONIA discounting) as of 22.07.2013*

Mat/ T	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M	6.5	21.5	39.5	55.0	70.0	84.0	97.5	110	122	133	175	208	238	262
2M	9.5	32.5	57.0	78.0	99.5	119	138	156	173	189	255	309	353	384
3M	12.5	40.5	70.0	95.5	121	145	169	191	211	231	314	382	437	477
6M	21.5	60.5	101	141	181	217	252	286	318	349	472	576	659	720
9M	29.0	76.5	126	178	228	275	320	363	406	445	601	731	836	916
1Y	35.5	91.0	151	212	274	330	384	436	486	532	718	870	1000	1100
18M	51.0	121	196	272	350	417	483	546	606	662	886	1073	1235	1363
2Y	70.0	155	241	328	415	492	566	636	702	765	1020	1237	1426	1576
3Y	102.5	210	318	420	519	610	698	782	863	942	1249	1508	1754	1951
4Y	127.5	249	367	482	594	700	802	899	992	1079	1423	1723	2004	2238
5Y	142.5	273	403	529	651	766	877	984	1087	1183	1556	1875	2184	2434
7Y	154.5	295	434	569	701	826	947	1064	1177	1280	1665	1975	2283	2549
10Y	160.5	306	448	586	721	850	974	1094	1211	1319	1705	1999	2279	2509
15Y	151.0	290	430	563	689	812	931	1049	1159	1253	1599	1857	2085	2293
20Y	135.0	266	393	514	632	747	858	964	1067	1147	1456	1672	1892	2059
25Y	125.0	247	365	478	585	688	789	886	980	1057	1326	1516	1722	1859
30Y	115.5	226	335	437	534	626	716	801	885	959	1189	1365	1543	1678

Risk managers or traders often face the problem of extrapolating given market data. In this paper we consider the problem that data providers or brokers only quote cap and swaption volatilities for one underlying forward (e.g. 6M). For instance the standard quotes for EUR are caps quoted against $3m$ forward rates up to 2 years and against $6m$ forward rates from 2 years onwards. For swaptions the quoted ATM volatilities and the skew/smile is quoted against swap rates for different tenors and against $6m$ forwards.

We do not consider the inter-nor extrapolation of interest rates as considered for instance by Schloegl (2002). Thus, our assumption is that arbitrage-free models for the term structures of interest rates for standard and non-standard tenors exist. This includes LIBOR rates as well as swap rates. Our contribution is to define a method to transfer the volatilities quoted for standard tenors only to non-standard tenors. Thus, we are only considering the parametrization of arbitrage-free models using available but sparse market information. To make the issue clear: take as an example of arbitrage-free models for the 10 year swap rates one based on a floating leg with 3 month tenor and one for a floating leg with 6 month tenor. The market only quotes volatilities for the 10y swap rate based on the floating leg with 6 month tenor. Now, we have to find a way of using this information to derive a volatility for the 10 year swap rate based on the 3 month tenor floating leg. We establish a method for describing this transform of volatilities assuming that the corresponding underlying models are arbitrage free. Thus, our methodology does not allow arbitrage.

Thus, we consider the extrapolation problem of calculating cap and swaption volatilities for non-standard underlying forwards. This problem is even more complex after significant money market basis spreads emerged starting in 2007. Now, different curves for discounting and forwarding are used. This adds another dimension to the

Table 6.10 *Market quotes in terms of the volatility for a displaced diffusion (top). The corresponding displacement respectively shift parameters are also given (bottom). Quotes date from 22.07.2013*

ATM swaption straddles – shifted Black volatilities														
T	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1M	54.5	62.9	67.2	59.2	52.0	45.4	40.8	37.2	34.3	31.9	24.9	22.6	21.7	21.0
2M	54.0	65.1	66.3	57.2	50.7	44.5	40.0	36.5	33.6	31.5	25.4	23.5	22.5	21.6
3M	58.1	64.9	65.3	56.8	50.2	44.1	39.7	36.4	33.7	31.6	25.8	23.9	23.1	22.2
6M	64.0	64.1	61.0	54.8	49.6	44.1	40.1	37.0	34.6	32.7	26.9	25.3	24.4	23.5
9M	66.0	61.9	58.1	52.8	48.2	43.5	40.0	37.2	35.1	33.3	27.7	26.1	25.2	24.4
1Y	65.2	58.8	55.5	50.9	47.4	43.2	40.0	37.4	35.3	33.5	28.3	26.7	26.0	25.3
18M	65.0	54.1	50.5	46.7	44.3	40.7	37.9	35.7	33.9	32.4	27.8	26.4	25.9	25.3
2Y	64.7	51.5	47.0	43.8	41.6	38.4	36.0	34.0	32.4	31.1	27.2	26.1	25.7	25.2
3Y	51.8	42.9	40.3	38.1	36.3	34.0	32.3	30.9	29.8	29.0	26.2	25.5	25.5	25.2
4Y	43.2	36.7	34.9	33.4	32.3	30.8	29.7	28.8	28.1	27.5	25.5	25.0	25.2	25.0
5Y	37.1	32.5	31.3	30.3	29.5	28.5	27.7	27.1	26.7	26.4	24.9	24.5	24.8	24.6
7Y	29.2	26.2	25.5	25.1	25.0	24.6	24.4	24.3	24.3	24.2	23.2	22.6	22.7	22.5
10Y	23.1	21.6	21.6	21.7	22.1	22.2	22.3	22.5	22.8	22.9	22.0	21.2	20.8	20.2
15Y	21.5	21.1	21.8	22.4	22.8	23.0	23.2	23.4	23.5	23.3	21.6	20.0	18.9	18.2
20Y	22.6	22.4	22.8	23.2	23.6	23.7	23.8	23.8	23.7	23.2	20.5	18.2	17.2	16.4
25Y	23.1	22.9	23.3	23.5	23.6	23.2	22.9	22.6	22.3	21.7	18.5	16.4	15.7	14.8
30Y	21.9	20.8	20.9	20.8	20.6	20.2	19.9	19.5	19.3	18.9	16.2	14.7	14.0	13.4
Shifts for shifted Black ATM swaptions volatilities														
1M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
2M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
3M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
6M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
9M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
1Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
18M	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
2Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
3Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
4Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
5Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
7Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
10Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
15Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
20Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
25Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
30Y	0.20	0.20	0.17	0.13	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10

extrapolation problem. For what follows we denote by mM the $m-$month forward rate. If we consider two rates and if we assume $m < n$, then, mM refers to the rate relying on the shorter forward rate. For swap rates we use the notation $SR^{mM}_{T_a,T_b}$ to denote a swap rate with tenor $T_b - T_a$ and calculated against the $m-$month forward. Our contribution is as follows:

Table 6.11 *EUR smile/skew volatilities as of 22.07.2013 for gamma swaptions (top) and vega swaptions (bottom)*

	Receivers					Payers			
	−150	−100	−50	−25	ATM	+25	+50	+100	+150
1m2y	–	–	81.87	−0.86	87.62	5.69	7.79	10.82	13.69
1m5y	–	48.56	3.52	−0.54	56.80	3.40	5.90	9.73	12.98
1m10y	35.45	14.17	3.18	0.51	33.35	1.87	3.32	6.17	9.00
1m20y	25.14	12.63	4.32	1.49	23.30	−0.76	−0.60	1.05	3.13
1m30y	25.33	13.06	4.74	1.81	21.56	−0.10	0.19	2.00	4.22
3m2y	–	–	55.78	6.52	89.48	−1.01	−0.49	0.81	2.39
3m5y	–	44.99	5.72	1.40	56.03	0.32	1.25	3.20	5.11
3m10y	31.37	10.97	1.98	0.01	34.01	0.26	1.03	2.95	4.96
3m20y	22.50	10.96	3.61	1.27	25.41	−0.72	−0.72	0.48	2.18
3m30y	22.91	11.56	4.13	1.62	23.48	−0.69	−0.66	0.52	2.21
6m2y	–	–	35.55	6.91	87.30	−2.85	−3.85	−4.12	−3.43
6m5y	–	32.42	5.61	1.76	54.37	−0.63	−0.51	0.32	1.43
6m10y	28.86	10.79	2.53	0.56	34.70	−0.16	0.26	1.74	3.46
6m20y	19.06	9.05	2.95	1.07	26.57	−0.71	−0.90	−0.28	0.91
6m30y	19.34	9.54	3.38	1.36	24.62	−0.80	−1.07	−0.58	0.51
9m2y	–	0.00	22.49	4.84	81.68	−2.25	−3.07	−3.18	−2.46
9m5y	–	22.57	4.17	1.27	52.34	−0.74	−0.75	−0.21	0.63
9m10y	23.37	9.01	2.46	0.76	35.13	−0.49	−0.52	0.03	0.97
9m20y	16.05	7.56	2.54	0.98	27.27	−0.75	−1.18	−1.19	−0.59
9m30y	16.46	8.01	2.86	1.18	25.44	−0.79	−1.22	−1.23	−0.65

	Receivers					Payers			
	−150	−100	−50	−25	ATM	+25	+50	+100	+150
1y2y	–	–	14.49	3.28	75.54	−1.58	−2.04	−1.80	−0.94
1y5y	–	17.44	3.59	1.13	51.11	−0.79	−0.94	−0.63	0.03
1y10y	18.90	7.48	2.20	0.74	35.31	−0.65	−0.93	−0.96	−0.59
1y20y	13.17	6.15	2.16	0.93	27.86	−0.70	−1.31	−1.85	−1.77
1y30y	16.29	7.92	2.81	1.13	26.30	−0.81	−1.29	−1.42	−0.93
2y2y	–	34.42	4.68	1.32	60.86	−0.62	−0.91	−0.78	−0.23
2y5y	34.52	11.06	3.10	1.08	44.12	−0.91	−1.36	−1.42	−1.06
2y10y	12.87	5.90	2.10	0.85	32.54	−0.76	−1.34	−2.06	−2.39
2y20y	12.33	6.08	2.24	0.95	27.17	−0.66	−1.09	−1.43	−1.32
2y30y	11.54	5.76	2.22	0.98	26.20	−0.72	−1.31	−2.03	−2.30
5y2y	11.32	4.81	1.51	0.52	35.38	−0.56	−0.95	−1.37	−1.50
5y5y	8.43	3.77	1.08	0.23	30.65	−0.68	−0.96	−1.13	−1.03
5y10y	7.74	3.98	1.55	0.68	27.28	−0.53	−0.93	−1.44	−1.68
5y20y	7.75	3.96	1.51	0.63	25.35	−0.57	−1.00	−1.57	−1.84
5y30y	8.69	4.41	1.67	0.71	25.39	−0.59	−1.03	−1.57	−1.80
10y2y	5.89	2.98	1.09	0.44	22.87	−0.39	−0.66	−0.94	−0.97
10y5y	6.22	3.22	1.23	0.53	22.64	−0.39	−0.67	−0.94	−0.97
10y10y	6.18	3.28	1.33	0.62	23.50	−0.46	−0.82	−1.33	−1.62
10y20y	7.71	3.91	1.46	0.60	21.81	−0.52	−0.89	−1.32	−1.46
10y30y	7.08	3.45	1.19	0.43	20.81	−0.51	−0.81	−1.06	−1.04

(1) Algorithm for transforming cap volatilities for $m-$month forward rates to volatilities on $n-$month forward rates assuming a single curve for calculating the forwards.
(2) Extending the algorithm from (1) to a multi-curve setting taking the money market basis into account. To this end we use displaced diffusions.
(3) Algorithm for transforming cap volatilities for $n-$month forward rates to volatilities on $m-$month forward rates for single as well as multi-curve settings.
(4) Algorithm for transforming swaption volatilities for swap rates on $m-$month forwards to $n-$month forwards and vice versa.
(5) Transforming the smile/skew for cap and swaption volatilities to rates calculated on non-standard forward rates using (1)–(4) and the SABR model.

6.5.1 Transforming caplet volatilities

In this section we consider the transformation of cap volatilities to non-standard tenors. We consider a single curve framework and then transform the results obtained to a multi-curve setting. To achieve this we apply the methodology of displaced diffusions.

Single curve framework

In this subsection we assume that there is one single curve for discounting and forwarding. When we consider the multi-curve setting this case is relevant since we apply this method to the OIS curve.

From short periods to long periods

We consider the case where we have caplet volatilities for a given tenor of n month. For instance for the EUR currency 3 month cap volatilities are quoted up to 2 years. Now, we wish to find 6 month volatilities.

To this end we proceed as follows. Since we work in a single curve setting, we can calculate a given forward rate for a given time period $\tau_{1,2} := T_2 - T_1$ from the prices of zero coupon bonds using the standard methodology, thus,

$$F_{1,2}(t) := F(t, T_1, T_2) = \frac{1}{\tau_{1,2}} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1 \right) \tag{6.15}$$

We outline our procedure only for the $3M$ and $6M$ case. But the method can be extended to other tenors but the corresponding formulas get lengthy since we have to take into account the variance of k rates if we have that $n = km$ for $m < n$.

First, for volatilities $\sigma_{1,2}, \sigma_{2,3} > 0$ we assume that the dynamics of the 3 month rates is given by

$$dF_{1,2}(t) = \ldots dt + \sigma_{1,2} F_{1,2}(t) dW_1 \tag{6.16}$$

$$dF_{2,3}(t) = \ldots dt + \sigma_{2,3} F_{2,3}(t) dW_2 \tag{6.17}$$

$$\langle dW_1(t), dW_2(t) \rangle = \rho dt \tag{6.18}$$

Now, we write $F_{1,3}(t)$ in terms of $F_{1,2}(t)$ and $F_{2,3}(t)$. We take two different approaches. The first approach takes into account compounding on the rates where the second does not. Thus, we call the second method the swap method.

- Compounding

$$\tau_{1,3}F_{1,3}(t) = \tau_{1,2}F_{1,2}(t)(1+\tau_{2,3}F_{2,3}(t)) + \tau_{2,3}F_{2,3}(t)$$

- Swap

$$\tau_{1,3}F_{1,3}(t) = \tau_{1,2}F_{1,2}(t) + \tau_{2,3}F_{2,3}(t)$$

Furthermore, for $\sigma_{1,3}$ we also assume

$$dF_{1,3}(t) = \cdots dt + \sigma_{1,3}F_{1,3}(t)dW_3(t)$$

For both methods we apply the Ito formula and derive a stochastic dynamic for the rate $F_{1,3}$. We give the result for the compounding case. The dynamic for $F_{1,3}(t)$ is

$$\begin{aligned} dF_{1,3}(t) = (\ldots)dt &+ \frac{\sigma_{1,2}}{\tau_{1,3}}\big(\tau_{1,2}F_{1,2}(t) + \tau_{1,2}\tau_{2,3}F_{1,2}(t)F_{2,3}(t)\big)\,dW_1(t) \\ &+ \frac{\sigma_{2,3}}{\tau_{1,3}}\big(\tau_{2,3}F_{2,3}(t) + \tau_{1,2}\tau_{2,3}F_{1,2}(t)F_{2,3}(t)\big)\,dW_2(t) \end{aligned}$$

This enables us to compute the variance which is for $a = c, s$ given by

$$\sigma_{1,3}^2 = V_{1,a}(t)^2\sigma_{1,2}^2 + V_{2,a}(t)^2\sigma_{2,3}^2 + 2\rho V_{1,a}(t)\sigma_{1,2}V_{2,a}(t)\sigma_{2,3} \tag{6.19}$$

with

$$\begin{aligned} V_{j,c}(t) &:= \frac{\tau_{j,j+1}F_{j,j+1}(t) + \tau_{j,j+1}\tau_{j+1,j+2}F_{j,j+1}(t)F_{j+1,j+2}(t)}{\tau_{1,3}F_{1,3}(t)} \text{ (compound)} \\ V_{j,s}(t) &:= \frac{\tau_{j,j+1}F_{j,j+1}(t)}{\tau_{1,3}F_{1,3}(t)} \text{ (swap)} \end{aligned}$$

Using the standard freezing the drifts argument for the rates which is setting $F_{i,j}(t)$ to $F_{i,j}(0)$ we find

$$\sigma_{1,3}^2 \approx V_{1,a}(0)^2\sigma_{1,2}^2 + V_{2,a}(0)^2\sigma_{2,3}^2 + 2\rho V_{1,a}(0)\sigma_{1,2}V_{2,a}(0)\sigma_{2,3} \tag{6.20}$$

For examples see Figures 6.4 and 6.5.

Remark

(1) The general approximation formula allowing for time-dependent volatilities for the case given above in fact is

$$\nu_{j,j+2}^2 \approx V_{j,a}^2 \nu_{j,j+1}^2 + V_{j+1,a}^2 \nu_{j+1,j+2}^2 + 2\rho V_{j,a} V_{j+1,a} \nu_{j,j+1} \nu_{j+1,j+2}$$
$$\nu_{j,j+1}^2 = \frac{1}{\tau_{0,j}} \int_0^{T_j} \sigma_{j,j+1}^2(t)\, dt$$

but in practice we only work with constant volatilities since such volatilities are quoted on broker pages.

(2) For the general setting such that the n–month period is a decomposition of k rates $F_{j,j+k}$, $j = 1, \ldots, k$ of some period and volatilities $\sigma_{1,k}$ and correlation matrix $\left(\rho_{ij}\right)_{i,j}$, we find

$$\sigma_{1,k}^2 \approx \sum_{j=1}^{k} \sigma_j^2 V_{j,a}^2 + \sum_{i \neq j} \rho_{ij} \sigma_{i,a} \sigma_{j,a} V_{i,a} V_{j,a}, \quad a = c, s \tag{6.21}$$

From long to short periods

Market practitioners also face the problem of moving from long periods to short periods. For instance in the EUR market the caps are quoted against 6 month forward rates from 2 years onwards. Thus, we face the problem that we have to infer short period cap volatilities, for instance 3 month, from long periods, e.g. 6 month.

However, in this case we face the problem of an under-determined system. To stay in the 3 month and 6 month example we have to find two 3 month volatilities from only one given 6 month volatility.

We have to stress that if we wish to transform from $6m$ volatility to $1m$ volatility we face an ill-determined problem not in two but in six dimensions.

Again, we wish to apply the approximation Equation (6.20) (or (6.21)). Since the quantities $V_{j,a}$ do not depend on the volatility but on the rates and the year fractions $\tau_{j,j+1}$ we have to consider the problem of finding solutions (x, y) subject to

$$c \approx ax^2 + by^2 + 2\rho abxy, \quad a, b, c \text{ given} \tag{6.22}$$

To determine a solution to our problem we have to find restrictions to the problem. To this end we propose to use the last available volatility for the short tenor and some functional form for the volatilities. An semi-analytic solution can be obtained if we assume that the two unknowns have the same value. Then, we can obtain the value from solving a quadratic equation.

If we are faced with the general setting Equation (6.21) the situation is worse. To this end we suggest using a functional form for a parsimonious representation of volatility

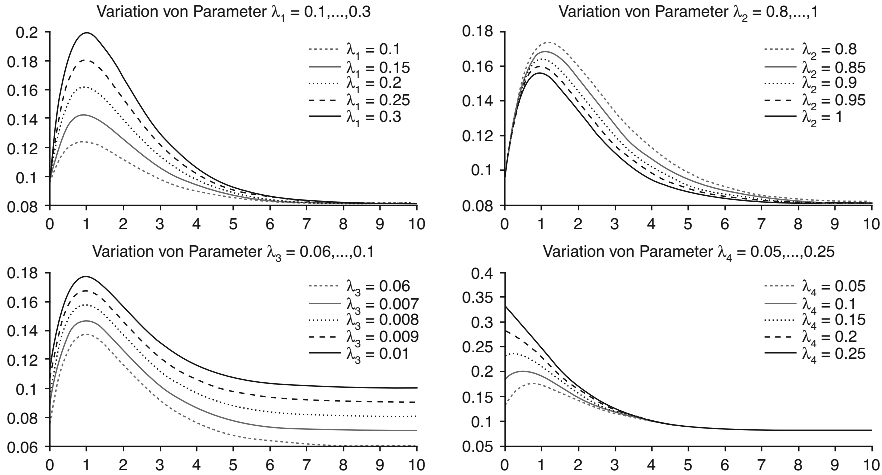


Figure 6.2 Some possible shapes for the volatility specified by Equation (6.23)

$\sigma(t)$. For real numbers $\gamma_1,\ldots,\gamma_5$ we take

$$\sigma(t) = \gamma_5 \left(((T_{i-1} - t)\gamma_1 + \gamma_4) \exp\left(-(T_{i-1} - t)\gamma_2 \right) + \gamma_3 \right) \tag{6.23}$$

Example

For the example we consider the function Equation (6.23) and show different shapes arising by using different parameters. This is Figure 6.2. Calculate the parameters $\gamma_1,\ldots,\gamma_4$ from the given volatility and use the Equation (6.21) together with the first given volatility as γ_5. For instance the starting volatility in the EUR cap example could be the last quoted volatility for $3M$ and the functional form is inferred from $6M$ volatilities.

If you do not wish to use a parametric form it would also be possible to bootstrap the volatilities and take the form of the resulting curve instead of the parsimonious structure.

Remark A simple rule of thumb if we have rates $F_{2,3}(t)$ and $F_{1,3}(t)$ and volatilities for the rates $\sigma_{2,3}$, $\sigma_{1,3}$, is that the normal volatility is the same for $F_{2,3}$ and $F_{1,3}$. This is the reason for traders considering the equation

$$\sigma_{1,2} F_{1,2} = \sigma_{2,3} F_{2,3} \tag{6.24}$$

Thus, by Equation (6.24) we could infer $\sigma_{1,3}$ from $\sigma_{2,3}$ and vice versa. However, this "*trick*" does not work if the money market basis spread is large.

The smile

We have only considered the case of a single volatility so far. However, the market has quoted different volatilities for different strikes and maturities. To this end we also have to transfer the whole smile structure. For achieving this goal we take an arbitrage-free model. In our case we focus on the market standard (displaced diffusion) SABR model for the forward rate $F(t)$, respectively $F(t)+b$.

$$dF(t) = a(t)F^{\beta}dW(t), \quad F(0) = F_0 \tag{6.25}$$

$$da(t) = \nu a(t)dZ(t), \quad a(0) = a_0 \tag{6.26}$$

$$\langle dW(t), dZ(t)\rangle = \rho dt \tag{6.27}$$

We base our consideration on West (2005a), Equation (22) which is

$$\frac{(1-\beta)^2 T}{24F^{2-2\beta}}a_0^3 + \frac{\rho\beta\nu T}{4F^{1-\beta}}a_0^2 + \left(1 + \frac{2-3\rho^2}{24}\right)a_0 - \sigma_{\text{ATM}}F^{1-\beta} = 0 \tag{6.28}$$

Equation (6.28) enables us to compute the SABR parameter a_0 from the other SABR parameters β, ν and ρ and the current ATM volatility level σ_{ATM}.

We apply Equation (6.28) as follows: first, we infer the SABR parameters for standard quoted rates, e.g. based on the 3 month rates.

Since we do not know much about the smile for options for non-standard rate tenors, e.g. 1 month rates, we wish to preserve the shape of the smile with respect to moneyness. To this end we fix the SABR parameters ρ, ν and β previously calibrated for the standard tenor where market quotes are available.

Then, we apply our proposed methodology for transforming the ATM volatility of a standard tenor to a non-standard one. Thus, we end up with ATM volatilities for standard tenors.

To build the whole smile we apply Equation (6.28) to calculate the only missing SABR parameter which is a_0. In this way we have setup an SABR model with the same smile shape for moneyness but with a transformed ATM volatility for non-standard tenor. Thus, we are able to price ITM and OTM derivatives for non-standard tenors. Figure 6.7 illustrates this method.

It is of course possible to use different skews for the different floating rate tenors. To this end the above procedure to transfer the skews can be adapted.

Multi-curve framework

We have considered the extrapolation problem in a single curve setting. But as outlined in previous chapters the market has adopted a multi-curve setting since August 2007. To this end we describe a method based on displaced diffusions for taking into account the new market paradigm.

From short to long periods

The methodology from above has to be extended. Simply relying on the methods described in the previous sections is not valid. If we take as an example $F_{1,2} = 0.47$, $F_{2,3} = 0.48$, $F_{1,3} = 0.82$, $\tau_{1,2} = 0.25$, $\tau_{2,3} = 0.25$, $\tau_{1,3} = 0.5$ and $\rho = 0.9$ we would transform the input volatilities of 72% into a $6M$ volatility of 41.56% which is not reasonable.

To account for the money market basis we denote

$$F_{1,2}^{\mathrm{OIS}}(t) := F^{\mathrm{OIS}}(t, T_1, T_2) := \text{forward calculated on the OIS curve} \tag{6.29}$$

$$F_{1,2}^{\mathrm{nM}}(t) := F^{\mathrm{nM}}(t, T_1, T_2) := \text{FRA rate retrieved from a Fwd curve} \tag{6.30}$$

We observe that for some given dates T_1 and T_2:

$$F_{1,2}^{\mathrm{nM}}(t) = F_{1,2}^{\mathrm{OIS}} + b_{1,2}^{\mathrm{nM}} \tag{6.31}$$

In Equation (6.31) the quantity b^{nM} denotes the money market basis spread which has to be added to the forward rate calculated on the OIS curve to get the correct nM forward rate.

Now, if we assume that $F_{1,2}^{nM}$ follows some log-normal process given by

$$dF_{1,2}^{\mathrm{nM}}(t) = \ldots dt + \sigma_{1,2} F_{1,2}^{\mathrm{nM}} dW(t) \tag{6.32}$$

The volatility $\sigma_{1,2}$ is quoted. But we also have (assuming deterministic money market basis spreads) a representation as a displaced diffusion dynamic. This dynamic is

$$dF_{1,2}^{\mathrm{nM}}(t) = \ldots dt + \sigma_{1,2}(F_{1,2}^{\mathrm{OIS}}(t) + b_{1,2}^{\mathrm{nM}}) dW(t) \tag{6.33}$$

Thus, we may interpret the volatility $\sigma_{1,2}$ as a displaced diffusion volatility σ_{DD} and b^{nM} as the displacement b. We convert the volatility to a Black volatility by using the displaced diffusion model from Rubinstein (1983).

For this special case let σ denote the ATM implied Black volatility. With respect to Equation (6.33) we set for some given displaced diffusion volatility $\sigma_{DD} = \beta\xi$ and define via the displacement coefficient b the variable β using $b = (1-\beta)/\beta F$ with the forward denoted by F. Then, using the new variables β and ξ we have

$$\sigma = \frac{2}{\sqrt{T}} \mathcal{N}^{-1}\left(\beta^{-1}\mathcal{N}\left(\frac{\xi\beta\sqrt{T}}{2}\right) - \frac{1-\beta}{2\beta}\right), \tag{6.34}$$

$$\sigma_{dd} = \xi\beta = \frac{2}{\sqrt{T}} \mathcal{N}^{-1}\left(\beta\mathcal{N}\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1-\beta}{2}\right) \tag{6.35}$$

> **Example**
> Let us give an example to illustrate the formulas Equations (6.34) and (6.35). We consider a forward rate $F = 0.03$, $T = 5$, $b = 0.0005$ and a displaced diffusion volatility $\sigma_{dd} = 0.2$. First, we take Equation (6.34). The variable β is determined by $b = (1-\beta)/\beta F$, that is $\beta = F/(F+b)$ and $\xi = \sigma/\beta$. To test the formulas we then take the result which we call σ_{atm} as an input and apply Equation (6.35) to get back the displace diffusion volatility $\sigma_{dd} = \beta \cdot \xi = 0.2$. We find the results:

$F = 0.03$	$F = 0.03$
$\sigma_{dd} = 0.2$	$\sigma_{atm} = 0.23347457$
$T = 1$	$T = 1$
$b = 0.005$	$b = 0.005$
$\beta = 0.857142857$	$\beta = 0.857142857$
$\xi = 0.233333333$	$\xi = 0.233333333$
$\sigma_{atm} = 0.23347457$	$\sigma_{dd} = 0.200000498$

Now, we show how we use the Equations (6.34) and (6.35). We proceed as follows assuming $n = km$:

(1) We first transform the displaced diffusion volatility with displacement b^{mM} to a Black volatility using Equation (6.34) which corresponds to the move from the forwarding curve to the OIS curve.
(2) We apply result Equation (6.20) to convert the volatilities.
(3) We transform the volatility back to a displaced diffusion volatility using Equation (6.35) now with a different displacement given by b^{nM} (and, thus, implicitly with another β).

This methodology does not face the problems occurring when simply applying the single curve methodology to convert volatilities. We present examples in 6.5.3. More examples can be found in Kienitz (2014).

From long to short periods

We could apply the same methodology as suggested for the one curve framework. First, we apply the transformation using the displacements and, then, we apply (6.24). After that we solve the now well determined problem and transform the resulting volatilities back to the corresponding tenor volatilitiy.

Figure 6.6 illustrates the application of the method for 3 month, 6 month and 12 month rates. We assume the 6 month is the standard tenor.

6.5.2 Transforming swaption volatilities

After considering the cap volatility case we now consider swaption volatilities.

Single curve framework

First, we work within a single curve framework and consider the swap rates $SR^{nM}_{T_a,T_b}$ and $SR^{mM}_{T_a,T_b}$ with dynamic

$$dSR^{mM}_{T_a,T_b}(t) = (\dots)dt + \sigma^{mM}_{T_a,T_b} SR^{mM}_{T_a,T_b}(t)dW_1(t) \tag{6.36}$$

$$dSR^{nM}_{T_a,T_b}(t) = (\dots)dt + \sigma^{nM}_{T_a,T_b} SR^{nM}_{T_a,T_b}(t)dW_2(t) \tag{6.37}$$

$$\langle dW_1(t), dW_2(t)\rangle = \rho dt \tag{6.38}$$

We conclude that $SR^{nM}_{T_a,T_b}(t) - SR^{mM}_{T_a,T_b}(t)$ is constant and, thus, $dSR^{nM}_{T_a,T_b}(t) - dSR^{mM}_{T_a,T_b}(t) = 0$. To this end we have that $0 = \sigma^{nM}_{T_a,T_b} SR^{nM}_{T_a,T_b}(t) - \sigma^{mM}_{T_a,T_b} SR^{mM}_{T_a,T_b}(t)$ and, therefore,

$$\sigma^{mM}_{T_a,T_b} = \sigma^{nM}_{T_a,T_b} \frac{SR^{nM}_{T_a,T_b}(t)}{SR^{mM}_{T_a,T_b}(t)} \tag{6.39}$$

Using Equation (6.39) leads to a linear dependence of the volatilities if the swap rates are given.

Example

We illustrate the relationship Equation (6.39) by an example. To this end take $SR^{nM}_{T_a,T_b}(0) = 0.0197$ and $SR^{mM}_{T_a,T_b}(0) = 0.0205$. The volatility $\sigma^{nM}_{T_a,T_b} = 0.2$. Then, we get

$$\sigma^{mM}_{T_a,T_b} = 0.2 \cdot \frac{0.0197}{0.0205} = 0.22102439$$

First approach

For the multi-curve setting we use the same method based on displaced diffusion dynamics but now for the swap rates. In the swaption case we do not have to apply the transformation within the OIS curve setting. We transform the volatilities from one displaced diffusion process into the other. To this end we apply Equation (6.34) and transform the volatility directly into another displaced diffusion volatility by using Equation (6.35).

Let us suppose we have two forward swap rates $SR_1 := S^{nM}_{T_a,T_b}$ and $S_2 := SR^{mM}_{T_a,T_b}$ on a given swap tenor starting at time T_a and having basis spreads b^{nM} and b^{mM}. For given $\sigma_1 := \sigma^{mM}_{T_a,T_b}$ and for $j = 1,2$ denoting $b_j = \frac{1-\beta_j}{\beta_j} SR_j$ and $\sigma_j = \xi_j \cdot \beta_j$. The OIS ATM volatility is denoted by σ. Then, we calculate $\sigma_2 := \sigma^{nM}_{T_a,T_b}$ by

$$\sigma = \frac{2}{\sqrt{T}} \mathcal{N}^{-1}\left(\beta^{-1}\mathcal{N}\left(\frac{\xi\beta\sqrt{T}}{2}\right) - \frac{1-\eta}{2\beta}\right) \tag{6.40}$$

$$\xi = \frac{2}{\beta\sqrt{T}} \mathcal{N}^{-1}\left(\beta\mathcal{N}\left(\frac{\sigma\sqrt{T}}{2}\right) - \frac{1-\eta}{2}\right) \tag{6.41}$$

$$\sigma_2 = \beta_2\xi_2 \tag{6.42}$$

The above formulas involve the calculation of the inverse of the cumulative normal distribution $\mathcal{N}^{-1}$. For convenience we give the Moro method, Moro (1995), to calculate function values for $\mathcal{N}^{-1}$.

Let us assume that we wish to calculate the value at x. To this end take

$a_0 = 2.50662823884$	$b_0 = -8.4735109309$	$c_1 = 0.337475482272615$
$a_1 = -18.61500062529$	$b_1 = 23.08336743743$	$c_2 = 0.976169019091719$
$a_2 = 41.39119773534$	$b_2 = -21.06224101826$	$c_3 = 0.160797971491821$
$a_3 = -25.44106049637$	$b_3 = 3.13082909833$	$c_4 = 2.76438810333863E-02$
		$c_5 = 3.8405729373609E-03$
		$c_6 = 3.951896511919E-04$
		$c_7 = 3.21767881768E-05$
		$c_8 = 2.888167364E-07$
		$c_9 = 3.960315187E-07$

Set $y = x - 0.5$

If $|y| < 0.42$ Then

$p = y^2$

$p = \frac{y\cdot(((a_3\cdot p+a_2)\cdot p+a_1)\cdot p+a0)}{(((b3\cdot p+b_2)\cdot p+b_1)\cdot p+b_0)\cdot p+1}$

Else

If $y > 0$ Then $p = \log(-\log(1-x))$

If $y \leq 0$ Then $p = \log(-\log(x))$

$p_1 = p\cdot(c_7 + p\cdot(c_8 + p\cdot c_9))$

$p_2 = p\cdot(c_4 + p\cdot(c_5 + p\cdot(c_6 + p_1)))$

$p = c_1 + p\cdot(c_2 + p\cdot(c_3 + p_2))$

If $y \leq 0$ Then $p = -p$

End If

The output and, thus, the value for $\mathcal{N}^{-1}(x)$ is p

In fact the conversion is simpler than for the cap case since we do not need to switch from a short to a long tenor or vice versa. We show some numerical examples for using Equation (6.40)–(6.42) in Section 6.5.3, we consider the linear case and the case illustrated by Equation 6.3.

The above calculation can be improved slightly by multiplying Equation (6.40) with the quotient of the annuity factor for $SR^{nM}_{T_a,T_b}$ and $SR^{mM}_{T_a,T_b}$ to adjust for the fact that one is taken on forwards with period nM and the other with period mM.
We have studied the impact of money market basis swap spreads and displayed the results in Figure 6.3.

Second approach

For the second approach we suppose that we have set up different forwarding curves. For the tenor nM the swap rate $SR^{nM}_{T_a,T_b}$ is given by

$$SR^{nM}_{T_a,T_b}(t) = \frac{\sum_{j=T_a+1}^{T_b} \tau^{nM}_{j-1,j} F^{nM}_{j-1,j}(t) DF^{OIS}(t,T_j)}{A^{nM}(t,T_a,T_b)}$$

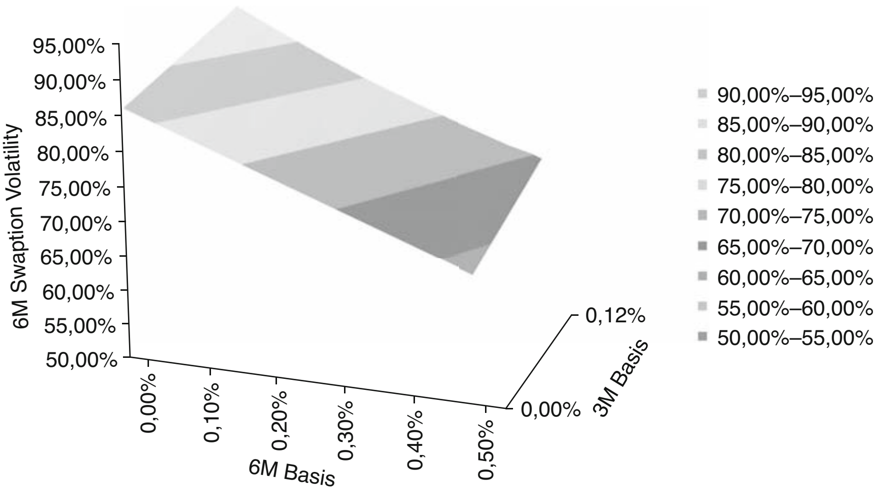


Figure 6.3 Swaption volatilities for $SR^{6M}_{2Y,12Y}$ calculated from volatilities for $SR^{3M}_{2Y,12Y}$ for different money market basis spreads

We consider the absolute moneyness given as the difference of the swap rate and the strike price.

$$\mathrm{AM}(SR, K) := SR - K \tag{6.43}$$

We wish to calculate the volatilities for the nM forward. We proceed as follows:

- Set up forwarding curves for forwards based on mM and nM rates and calculate the discount factors.
- Retrieve or calculate normal volatilities from given market quotes.
- Calculate the swaption price $C^{mM}_{T_a,T_b}(K,T)$ and annuity factors $A^{mM}_{T_a,T_b}$ and $A^{nM}_{T_a,T_b}$.
- Calculate the swaption price $C^{nM}_{T_a,T_b}(K,T)$ by

$$C^{nM}_{T_a,T_b}(K^{nM},T) = \frac{A^{mM}_{T_a,T_b}}{A^{nM}_{T_a,T_b}} C^{mM}_{T_a,T_b}(K,T)$$

  where the strike K^{nM} corresponds to the strike K leading to the same absolute moneyness, Equation (6.43).
- Calculate Black volatilities.

Remark In the last step we referred to Black volatilities but using for instance results from Hagan et al. (2002) we transform to either normal or displaced diffusion models once the displacement is known.

This methodology essentially relates the volatility for a given forward rate based on nM to one for a forward rate on mM at another strike. The strike is determined by equating the absolute moneyness.

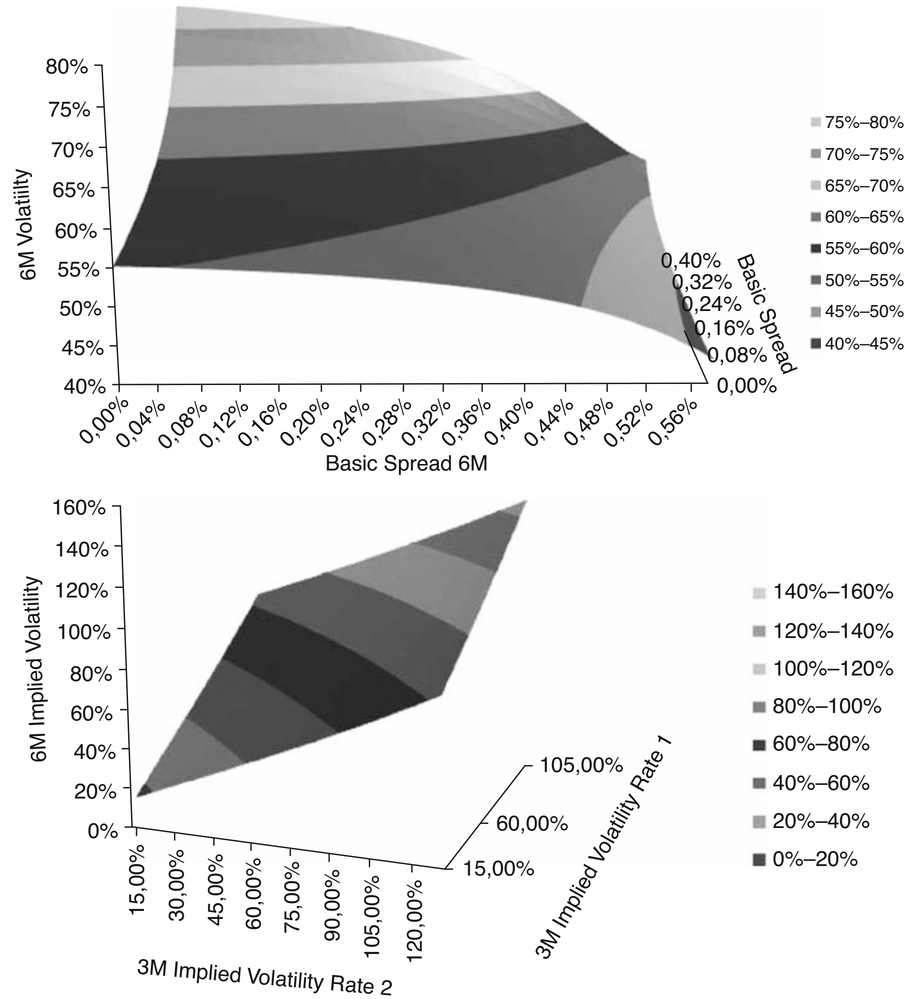


Figure 6.4 Implied volatility for caplets based on the 6 month forward rate calculated from volatilities for caplets based on the 3m forward rate. We plotted the dependence on the 3m basis and 6m basis with respect to OIS (top) and different volatilities for the used 3m rates (bottom)

6.5.3 Examples

In this section we give several examples for our proposed method. We first consider our proposed method for caplets. To this end we consider $F_{1,2}(0) = 0.5778\%$, $F_{2,3}(0) = 0.5978\%$, $F_{1,3}(0) = 0.8214\%$ with corresponding day fractions $\tau_{1,2} = 0.255556$, $\tau_{2,3} = 0.252778$, $\tau_{1,3} = 0.513889$. The basis spreads with respect to OIS are given by $b_{1,2} = 0.09\%$, $b_{2,3} = 0.09\%$, $b_{1,3} = 0.29\%$. For the volatilities we have chosen $\sigma_{1,2} = \sigma_{2,3} = 72\%$. We assume $T = 2$ and for the correlation where it is fixed we take $\rho = 90\%$. We choose to compound the LIBOR rates but we do not get very different numbers if we choose swap. The illustrating figures are 6.4 and 6.5.

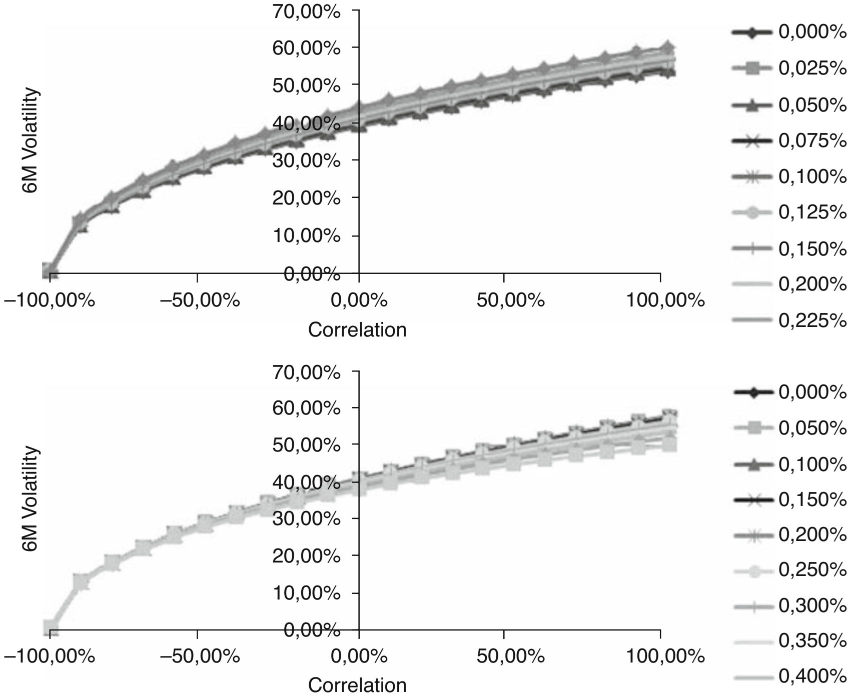


Figure 6.5 Implied volatility for caplets based on the 6 month forward rate calculated from volatilities for caplets based on the 3m forward rate. We plotted the dependence on the 3 month basis and correlation (left) and on the 6m basis and correlation (right)

Finally, we assume that the term structure for Caplets based on 6 month rates are given. Figure 6.6 shows the corresponding calculated Caplet volatilities based on 3 month and 12 month rates.

For the swaption examples we consider the swap rates $SR^{3m}_{2Y,12Y}(0) = 1.5878\%$ and $SR^{6m}_{2Y,12Y}(0) = 1.8952\%$. We assume that the corresponding basis spreads with respect to OIS are $b^{3m}_{2Y,12Y} = 0.2\%$ and $b^{6m}_{2Y,12Y} = 0.14\%$. The volatility of the swap rate based on the 3 month forwards is $\sigma_1 = 82\%$.

The final example illustrates the methodology for transferring the shape of the smile to non-standard tenors. To this end we have chosen $\beta = 0.7$, $\rho = 0.3$, $\nu = 0.83$ and $T = 1$. The forward rates are $F_{6m}(0) = 3\%$ and $F_{3m}(0) = 2.7\%$. We take the volatilities $\sigma_{6M} = 43\%$, $\sigma_{3M} = 41\%$, $\sigma_{3M} = 38\%$. The corresponding SABR parameters are calculated using Equation (6.28) and are given by $a = 0,140640168$, $a = 0,130030807$, $a = 0,120661502$

6.6 Bond options

Bond options are options to buy or sell a bond at a given price. The market uses an option pricing model such as the Black model to compute options prices for bond

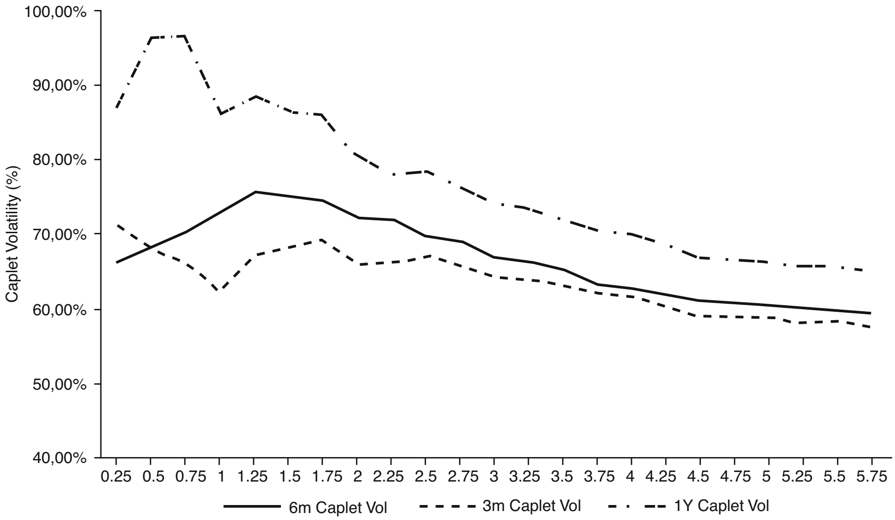


Figure 6.6 Transformation of a given term structure of volatility for 6m caplets. The volatilities for the 3m and the 12m case are derived using the proposed methods

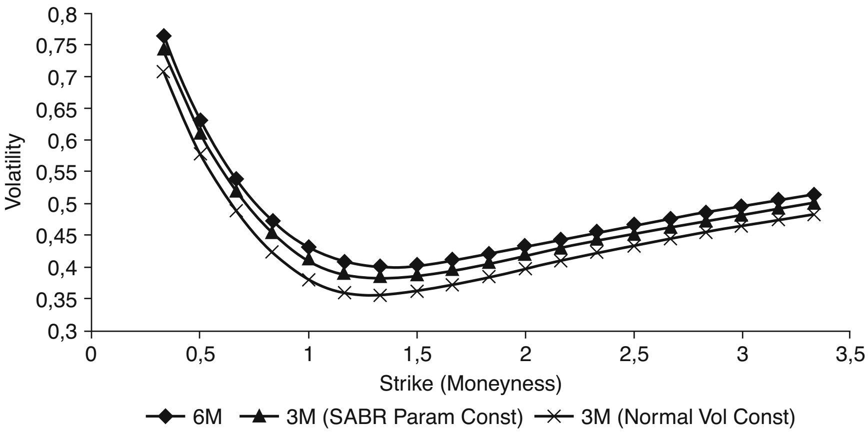


Figure 6.7 The transformed smile for two different ATM volatilities for $SR^{3M}_{2Y,12Y}(0)$ obtained from $SR^{6M}_{2Y,12Y}(0)$ values and the application of Equation (6.28).

options. The only unknown for applying such a pricing model is the implied volatility. However, since the bond and the swap markets are closely connected because one can use swaps as hedges for the interest rate risk of a bond we can rely on information from the swaption market. We have seen that a swaption is an option on yield and the corresponding implied volatilities are yield volatilities. For pricing a bond option this yield volatility cannot directly be used. We have to transform the yield volatility into a price volatility since the market quotes for bonds is the price. Let us call the implied volatility for a swaption on swap with tenor T be $\sigma_{\text{Swaption},T}$. The implied yield of a bond is denoted by y_B. Then, the normal volatility would be

$$\sigma_{\text{Swaption},T} \cdot y_B$$

Then, we denote by D_B the duration of the bond. Finally, the price volatility is given by

$$\sigma_{\text{Price},T} = \sigma_{\text{Swaption},T} \cdot y_B \cdot D_B \tag{6.44}$$

This is the implied volatility, Equation (6.44), we have to use when we use the Black formula to price bond options. Corresponding examples for this conversion can be found on the corresponding spreadsheet.

For the reader's convenience we give another useful formula. Let the forward be f and be a function of yield y. We have $f = F(y)$. Furthermore, denote by Y the function which gives the forward yield when a value for the forward is inserted. Thus, $Y(f) = y$. Assume a model for the forward

$$df(t) = \sigma_f f(t) dW(t)$$

and that $Y(f) = y$. Then, we obtain the following dynamics for y:

$$dy(t) = \sigma_f F'(y(t)) dW(t) + \cdots dt$$

We wish to find a value σ_y such that

$$dy(t) = \sigma_y y(t) dW(t) + \cdots dt$$

The latter can be solved by *equivalent volatility* techniques and we have

$$\frac{1}{\sigma_y} \int_{Y(K)}^{Y(f)} \frac{1}{y} dy = \frac{1}{\sigma_F} \int_K^F \frac{F'(y)}{F(y)} dy$$

Thus, we have

$$\sigma_y = \sigma_F \left| \frac{\log(F) - \log(K)}{F - K} \right|$$

To get the price and yield relations we can resort to the results in Chapter 3 and find for the forward:

$$F = -\text{Accrued} + \frac{1}{(1+\frac{y}{k})^b} \sum_{j=1}^{N} \frac{C}{k} \frac{1}{(1+\frac{y}{k})^j}$$

The length of the coupon period is the length of the accrual period plus b and we assume that k is the number of coupon periods per year.

6.7 Reading list

We give the reading list on options and the basic valuation formulas in the next chapter, Chapter 7.

7 Options II

7.1 Introduction and objectives

In this chapter we consider the pricing and risk management of a very popular product which is called a *Constant Maturity Swap* or *CMS* for short. We have introduced the corresponding financial instruments in Chapter 4, Section 4.4.2. Despite the fact that such contracts seem to be plain vanilla since only short term rates and swap rates are involved our analysis shows that they are much more complex. In fact we find that CMS depend not only on the yield curve but also on the implied volatility of swaptions. And even worse they not only depend on the ATM volatility but also on the ITM and OTM volatilities. Furthermore, a CMS with say a 10Y tenor and a maturity of 5Y depends on the given yield curve up to 14Y. This is because the final coupon in four years time is the then 10Y swap rate which is calculated using the rates up to 14 years. This instrument is an exotic derivative in disguise and we have to take many risks into account for analyzing a pricing and hedging mechanism. For the convenience of the reader we have created a spreadsheet which illustrates the pricing of CMS and even the corresponding derivatives CMS Caplets and Floorlets. Due to the popularity as the building blocks of many structured swaps we also consider CMS spreads and the corresponding Call and Put options in this chapter.

Change of measure

We consider a measurable space $(\Omega, \mathcal{F})$. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures on Ω with the same null sets in $\mathcal{F}$. This is equivalent to a non-negative, $\mathcal{F}$-measurable function f such that

$$\mathbb{P}(A) = \int_A f d\mathbb{Q}, \text{ for all } A \in \mathcal{F}$$

We call $f = \frac{d\mathbb{P}}{d\mathbb{Q}}$ the *Radon-Nikodym* derivative and write

$$\mathbb{E}_{\mathbb{P}}[X] = \mathbb{E}_{\mathbb{Q}}\left[X \frac{d\mathbb{P}}{d\mathbb{Q}}\right]$$

For pricing purposes for CMS Caplets we use the T-Forward measure introduced in Section 1.5.1. The value of a single CMS payment is

$$V(t) = \mathbb{E}_{\mathbb{Q}}[SR_{T_{i-1},T_{i-1+nY}}(T_{i-1})\tau_i D(t, T_i)]$$

First, we know by the definition of the T-Forward measure that for $t < T_i$

$$V(t) = D(t, T_i)\tau_i \mathbb{E}_{T_i}[SR_{i-1,i-1+nY}(T_{i-1})]$$

But the swap rate SR is not a martingale with respect to the T_i-Forward measure. Using the annuity measure instead we calculate:

$$\begin{aligned} V(0) &= D(0, T_i)\tau_i \mathbb{E}_{T_i}[SR_{i-1,i-1+nY}(T_{i-1})] \\ &= A(0)\tau_i \mathbb{E}_{\mathbb{A}}\left[SR_{i-1,i-1+nY}(T_{i-1})\frac{D(T_{i-1}, T_i)}{A(T_{i-1})}\right] \end{aligned} \tag{7.1}$$

For two random variables X and Y let us denote the covariance of X and Y with respect to some measure $\mathbb{Q}$ by $\mathbb{C}_{\mathbb{Q}}[X, Y]$. Then, Equation (7.1) can be written as

$$\tau_i D(0, T_i)\left(SR_{T_{i-1},T_{i-1}+nY}(0) + \frac{A(0)}{D(0, T_i)}\mathbb{C}_{\mathbb{A}}\left[SR_{T_{i-1},T_{i-1}+nY}(T_{i-1}), \frac{D(T_{i-1}, T_i)}{A(T_{i-1})}\right]\right) \tag{7.2}$$

The second summand in the brackets is the convexity adjustment and we observe that the CMS depends on the current yield curve and the covariance between the swap rate and a function of zero bonds under the swap measure $\mathbb{A}$. This suggests that the CMS is a product depending on volatility and correlation.

7.2 CMS caps and floors

Vanilla swaps offer an easy and straightforward way to trade the steepness of the curve. Paying one maturity, say 10Y, against another, say 2Y (eventually PV01 weighted) is a zero-cost curve steepener position. The steeper the curve the bigger the difference between the 10Y and the 2Y rate. If time passes towards maturity the trades ages. The 10Y rate will not stay a 10Y rate. For instance if one year has passed the remaining life of the trade is 9Y and the value of the floating leg of the swap corresponds to a 9Y swap. This phenomenon is known as *roll-down the curve*. If we assume that nothing changes but only time goes by the trade generates profit and loss. This is known as the *roll-down* and it can be significant if the curve is steep or inverted. It might be an idea to combine standard curve trades with swaptions. The implicit cost of a break-even improvement is the conditional nature of the swaption trade. By conditional nature we mean that the swaption either ends up in or out of the money. The view on the curve has to prove right should the swaption expire in the money. Curve views via swaptions have to take into account the swaption volatility cube. The worst thing is that the view has proven wrong and the options expire out of the money.

Standard curve trades via swaps are independent of the market direction. The conditional nature of swaption trades can be an advantage when the market context implies a clear directionality of the curve slope. Over a long period, the yield curve slope tends to be negatively correlated with short rates, flattening in a bear market when central banks are hiking rates, and steepening in a bull market during monetary easing. However, this relationship is not stable. For instance, when sticky central bank expectations limit volatility at the shortend, the yield curve tends to steepen in a bear market for instance and tends to flatten in a bull market.

When we have established our pricing mechanism the reader can prove the fact that a position in a CMS is an implicit steeping position which is not affected very much by parallel movements of the curve. The spreadsheet accompanying this book can be used to play around and analyze the situation.

Curve steepeners via CMS receiver swaps are attractive when volatility is low and the implied volatility smile is flat. Curve flatteners via CMS payer swaps are attractive on the other hand when volatility is high and the implied volatility smile is convex. The sensitivity of CMS swaps to volatility and smile is typically marginal compared to the sensitivity to the yield curve. The CMS swap market is restricted to some liquid points. For example the Euro CMS swaps mostly trade for 5Y, 10Y, 15Y, 20Y and 30Y expiries, and 2Y, 5Y, 10Y, 20Y and 30Y tenors. The CMS swap market in USD is less active and less liquid. Options occur since for instance we need a regulatory Cap to qualify for a certain accounting category, avoid negative coupons or make the product cheaper for the investor (sell upside = sell cap).

Let us first consider Caps and Floors on the swap rate. The Cap, respectively Floor is an option which caps, respectively floors a given rate on a given time period. As we have seen in the last chapter usually such options are on short term floating rates and the time period is determined by the length of the short term floating rate period Δ. For instance the period could be $3M$ or $6M$. To define the CMS Cap/Floor we consider a time period of Δ for which a rate with tenor nY is capped/floored. The difference between Δ and nY is significant. To this end let $SR_{m,n}(t)$ be the forward swap rate starting at T_m with floating rate tenor Δ. The payoff of a CMS Caplet at time T for a period of Δ is

$$\left(SR_{m,m+nY}(T) - K\right)^+$$

In practice we are given a schedule of dates, $\tilde{\mathcal{T}}$ corresponding to some short period floating rate and an nY Swap Rate $SR_{i,i+nY}(\tilde{T}_i)$, $\tilde{T}_i \in \tilde{\mathcal{T}}$. A CMS Cap/Floor is the sum of all CMS Caplets/Floorlets and the payoff is given by

$$P^{\text{CMS Cap}}(t) = \sum_i \tilde{\tau}_i \left(SR_{i,i+nY}(\tilde{T}_i) - K\right)^+$$

$$P^{\text{CMS Floor}}(t) = \sum_i \tilde{\tau}_i \left(K - SR_{i,i+nY}(\tilde{T}_i)\right)^+$$

As remarked earlier the rationale for considering such options can be selling the upside to cheapen a contract. It can also have regulatory issues (IFRS) or ensure positive coupons (CMS–x bp) in a trade.

The value of this contract is obtained by taking expected values with respect to the $\tilde{T}_i$-forward measures

$$V^{\text{CMS Cap}}(t) = \sum_i \tilde{\tau}_i \mathbb{E}_{\tilde{T}_i}\left[\left(SR_{i,i+nY}(\tilde{T}_i) - K\right)^+\right], \tag{7.3}$$

$$V^{\text{CMS Floor}}(t) = \sum_i \tilde{\tau}_i \mathbb{E}_{\tilde{T}_i}\left[\left(K - SR_{i,i+nY}(\tilde{T}_i)\right)^+\right] \tag{7.4}$$

Equations (7.3) and (7.4) show that the period for which we take the Caplet/Floorlet is not the same as the period for the swap rate. The period of the Caplets/Floorlets is usually much smaller than the tenor of the Swap Rate. Typical values are $3M$ and $6M$ for the time period for the Caplet/Floorlet and $2Y$, $5Y$, $10Y$, $20Y$ or $30Y$ for the tenor of the Swap Rate. This observation has an immense impact on the pricing and leads to the so called *Convexity Adjustment* which measures the difference between swap rates and CMS rates.

Having introduced CMS Caps and Floors we see that it is possible to value a standard CMS by the *Put-Call-Parity*. We have

$$\sum_i \underbrace{\mathbb{E}_{\tilde{T}_i}[SR_{i,i+nY}(\tilde{T}_i)]}_{=:CMS_{i,nY}(0)} - K\sum \tau_i D(T_i) = V^{\text{CMS Cap}}(t) - V^{\text{CMS Floor}}(t) \tag{7.5}$$

We observe that if we are also able to price CMS Caps/Floors by Equation (7.5) we are able to price CMS. In fact the market prices the corresponding options and infers the price of the corresponding swap. It is market practice to consider CMS quoted against a short tenor floating rate for instance $3M$. This turns Equation (7.5) into

$$\sum_i \mathbb{E}_{\tilde{T}_i}[SR_{i,i+nY}(\tilde{T}_i)] - \sum \tilde{\tau}_i \left(L(\tilde{T}_{i-1}, \tilde{T}_{i-1}, \tilde{T}_i) + s\right) D(0, \tilde{T}_i) = 0 \tag{7.6}$$

In the sequel we consider

- Theoretical issues and review of concepts
- Valuation of CMS Caplets/Floorlets and, thus, CMS Caps/Floors
- Pricing of Constant maturity swaps (CMS) using Equation (7.5)

For deriving the prices we proceed by introducing a replication method. This replication uses other financial instruments, swaptions in this case, to build CMS Caplets/Floorlets. Being able to synthetically build the Caplets/Floorlets we can treat CMS Caps/Floors. In fact, each Caplet/Floorlet can have its specific strike value.

7.2.1 CMS math

In this subsection we consider the pricing of CMS Caps/Floors. As a by product we get the price of a CMS. Before we start we observe market data that suggests that the risk neutral distributions applicable for pricing are not symmetric and can be fat tailed.

Several methods for computing the Convexity Adjustment exist. We give three different models here and discuss their weaknesses and strengths.

- The Black model
  Assumes a logarithmic normal distribution of the swap rate at maturity.
- The Hagan approach
  This assumes a linear swap rate model and applies replication within this model.
- The replication approach
  Setting up a weighted portfolio of swaptions to replicate the CMS option with no further modeling assumptions.

In the sequel we consider the replication approach in detail but let us briefly consider the first two approaches. By definition of the covariance with respect to the annuity measure (swap measure) we get:

$$\begin{aligned}
&\frac{A(0)}{D(0,t_i)}\mathbb{C}_A\left[SR_{i-1,i-1+nY}(T_{i-1}),\frac{D(t_{i-1},t_i)}{A(t_{i-1})}\right]\\
&\quad=\mathbb{E}_A\left[\left(SR_{i-1,i-1+nY}(T_{i-1})-SR_{i-1,i-1+nY}(0)\right)\left(\frac{D(t_{i-1},t_i)/A(t_{i-1})}{D(0,t_i)/A(0)}-1\right)\right]
\end{aligned}$$

In Hagan (2003) suggests restricting the moments of the yield curve by expressing the quotient $D(\cdot,\cdot)/A(\cdot)$ in terms of a function G of the swap rate $SR_{\cdot,\cdot}$. Let us assume there exists a function G and we consider

$$f(x):=[x-K]\left(\frac{G(x)}{G(SR_{i-1,i-1+nY}(0))}-1\right)$$

Using this assumption for the CMS rate and setting

$$f(x)=[x-SR_{i-1,i-1+nY}(0)]\left(\frac{G(x)}{G(SR_{i-1,i-1+nY}(0))}-1\right)$$

we observe for the CMS rate:

$$\frac{1}{A(0)}\left[\int_{SR}^{\infty}PS(x)f''(x)dx-\int_{-\infty}^{SR}RS(x)f''(x)dx\right]$$

In the latter equation we denote by $PS(x)$ the values of payer swaptions and by $RS(x)$ receiver swaptions with strike x. Thus, if all the market values for payer swaptions (PS) and receiver swaptions (RS) at any strike level x are available it would be possible to compute the adjustment. The Black approach can be obtained as a special case where the dynamics of the swap rate is determined by Geometric Brownian motion. Then, we find

$$\frac{G'(SR(0,t_{i-1}))}{G(SR_{i-1,i-1+nY}(0))}SR^2_{i-1,i-1+nY}(0)\left[\exp\left(\sigma^2_{i-1}t_{i-1}\right)-1\right]$$

We observed that this approach leads to totally wrong numbers if volatilities are very high and rates are very low which was for instance the case after August 2007 and is the current market situation.

To value the product using this approach a continuous strip of traded cash-settled swaptions is needed to determine the convexity adjustment. But we observe that swaptions are not liquid for all strikes and may even not be available for all tenors. The application of the above formula is appealing since the CMS pricing is then in line with pricing swaptions. To use the formula we need a fully specified volatility surface for the corresponding swap tenor. However, we know that smile modeling for in-the-money (ITM) and out-of-the money (OTM) swaptions is an issue since market data is raw and we have to pose model assumptions to be able to extrapolate/interpolate all values.

Theoretically we know how to price CMS caplets, floorlets and compute the convexity correction. Now, we wish to illustrate how this is done in practice. We use a replication strategy with finitely many pre-defined swaption strikes to price the corresponding Caplets and Floorlets. We consider the following ansatz for the replication strategy:

$$
\begin{aligned}
&\left[SR_{N,N+nY}(T_N)-K\right]^+-\left[K-SR_{N,N+nY}(T_N)\right]^+\\
&=\left(\sum_{j=1}^{N_C}\omega_j\left[SR_{N,N+nY}(T_N)-K_j\right]^+-\sum_{j=1}^{N_F}\omega_j\left[K-SR_{N,N+nY}(T_N)\right]^+\right)\sum_{i=1}^{N}\frac{1}{SR^{t_i}_{N,N+nY}(0)}
\end{aligned}
$$

The quantities N_C and N_F denote the numbers of payer swaptions, respectively receiver swaptions in the replication of the Caplet, respectively Floorlet.

Static hedging
Consider a monotonically increasing function

$$
\begin{aligned}
f: I &\to \mathbb{R}\\
S &\mapsto f(S)
\end{aligned}
$$

with $f(K_0)=0$, $K_0\in I$. It is a fact that f can be approximated by Call option payoffs:

$$
f(S)\approx\sum_{i=1}^{N}\omega(K_i)(S-K_i)^+
$$

The weights ω_i, $i=1,\ldots,N$ are given by

$$
\omega_0=f'(K_0);\omega_1=f''(K_1-K_0);\ldots;\omega_i=f''(K_i)(K_{i+1}-K_i),
$$

In the limit we have a full replication

$$f(S) = f'(K_0)(S-K_0)^+ + \int_{K_0}^{\infty} f''(S-K)^+ dK$$

In case the function f is monotonically decreasing and $f(K_0) = 0$ we have

$$f(S) \approx \sum_{i=1}^{N} \tilde{\omega}_i(K_i)(K_i - S)^+,$$

with weights

$$\tilde{\omega}_0 = -f'(K_0); \tilde{\omega}_1 = -f''(K_0 - K_1); \ldots; \tilde{\omega}_N = -f''(K_N)(K_{N-1} - K_N),$$

which than leads to a replication formula applicable for Put payoffs and we find for the limit

$$f(S) = -f'(K_0)(K_0 - S)^+ + \int_0^{K_0} f''(K)(K-S)^+ dK$$

The expectation of the swap rate with respect to the T_{k+1} forward measure is

$$\mathbb{E}_{T_{k+1}}[SR_{k,k+nY}(T_k)] = \frac{A_k(0)}{D(0,T_{k+1})} \mathbb{E}_{\mathbb{A}}\left[SR^2_{k,k+nY}(T_k) \frac{D(T_k, T_{k+1})}{A_k(T_k)} \right]$$

In the final equation the second factor can be approximated and differentiated with respect to the swap rate and, then, replication is applied.

In fact we cheated when stating the replication formula for CMS. It is market standard to use cash settled swaptions instead of physically delivered swaptions. To this end the replication formula should involve a change of numeraire. The details can be found in Kienitz and Wetterau (2012). We have mapped the replication to a spreadsheet which can be consulted for any further analysis.

Let us give the pseudo code for calculating the replication weights. To this end we need the following information as the user input:

- Strike – this is the Caplet, respectively Floorlet strike
- frq – the frequency of the Caplet/Floorlet
- nP – the number of periods ($nP = frq \cdot$ SwapTenor)
- nK – the number of replication strikes

- pDelay – the payment delay
- $K()$ replication strikes – the array of replication strikes

The outputs are the replication weights stored into the array $\omega()$. For the algorithm given below we take the counting variables n and i. To calculate a sum by using a loop we need to store the actual value of the sum and in the next loop we add the current value of a summand to the current value of the sum. This current value we denote by *tmpS*. Finally, *tmpS* is the value of the sum. Furthermore, we need two functions. The first function gives the payoff of a CMS option taking as input the swap rate (SR), the strike (K), the payment delay ($pDelay$) and if it is a Call or a Put option. Thus, we have

$$CMS\_Pay = \begin{cases} \max(SR - K, 0)/(1 + pDelay \cdot SR) & \text{, if Call} \\ \max(K - SR, 0)/(1 + pDelay \cdot SR) & \text{, if Put} \end{cases}$$

and the other function takes additionally the factor $f = \frac{1}{SR} \cdot \left(1 - \left(1 + \frac{SR}{frq}\right)^{-nPeriods}\right)$ with number of periods (nP). We have

$$CS\_Swaption\_Pay = \begin{cases} \max(SR - K, 0) \cdot f, & \text{if Call} \\ \max(K - SR, 0) \cdot f, & \text{if Put} \end{cases}$$

Now, the algorithms applied for the replication of a CMS Caplet, respectively CMS Floorlet can be specified. For the CMS Caplet the algorithm is given by:

Algorithm (CMS Caplet)
% The first weight (index 0) is special
$\omega(0)$ = CMS_Pay($K(1)$, K, pDelay, 0) / CS_Swpt_Pay($K(1)$, $K(0)$, frq, nP, Call)
for $n = 1$ To $nK - 2$ step 1
    weight = CMS_Pay($K(n+1)$, K, pDelay, 0)
    tmpS = 0
    for $i = 0$ to $n - 1$ step 1
        tmpS = tmpS + $\omega(i)\cdot$ CS_Swpt_Pay($K(n+1)$, $K(i)$, frq, nP, Call)
    next i
    weight = (weight – tmpS) / CS_Swpt_Pay($K(n+1)$, $K(n)$, frq, nP, Call)
    $\omega(n)$ = weight next n

Table 7.1 *Example replication weights calculation*

$\omega(0) = 12.61\%$	$\omega(1) = 0.55\%$	$\omega(2) = 0.56\%$	$\omega(3) = 0.57\%$	$\omega(4) = 0.57\%$
$\omega(5) = 0.58\%$	$\omega(6) = 0.59\%$	$\omega(7) = 0.59\%$	$\omega(8) = 0.60\%$	$\omega(9) = 0.60\%$
$\omega(10) = 0.61\%$	$\omega(11) = 0.61\%$	$\omega(12) = 0.61\%$	$\omega(13) = 0.62\%$	$\omega(14) = 0.62\%$
$\omega(15) = 0.63\%$	$\omega(16) = 0.63\%$	$\omega(17) = 0.63\%$	$\omega(18) = 0.64\%$	$\omega(19) = 0.64\%$
$\omega(20) = 0.64\%$	$\omega(21) = 0.64\%$	$\omega(22) = 0.65\%$	$\omega(23) = 0.65\%$	$\omega(24) = 0.65\%$
$\omega(25) = 0.65\%$	$\omega(26) = 0.65\%$	$\omega(27) = 0.65\%$	$\omega(28) = 0.65\%$	$\omega(29) = 0.66\%$
$\omega(30) = 0.66\%$	$\omega(31) = 0.66\%$	$\omega(32) = 0.66\%$	$\omega(33) = 0.66\%$	$\omega(34) = 0.66\%$
$\omega(35) = 0.66\%$	$\omega(36) = 0.66\%$	$\omega(37) = 0.66\%$	$\omega(38) = 0.66\%$	$\omega(39) = 0.66\%$
$\omega(40) = 0.65\%$	$\omega(41) = 0.65\%$	$\omega(42) = 0.65\%$	$\omega(43) = 0.65\%$	$\omega(44) = 0.65\%$
$\omega(45) = 0.65\%$	$\omega(46) = 0.65\%$	$\omega(47) = 0.65\%$	$\omega(48) = 0.64\%$	$\omega(49) = 0.00\%$

and for the CMS Floorlet we have

Algorithm (CMS) Floorlet

```
% The first weight (index nK − 1) is special
ω(nK − 1) = CMS_Pay(K(nK − 2), K, pDelay, 1)/
                CS_Swpt_Pay(K(nK − 2), K(nK − 1), frq, nP, Floor)
for n = nK − 2 to 1 Step −1
   if K(n − 1) = 0 then
      weight = 0
   else
      weight = CMS_Pay(K(n − 1), K, pDelay, 1)
      tempSum = 0
      for i = nK − 1 To 4n + 1 Step -1
         tmpS = tmpS + ω(i)· CS_Swpt_Pay(K(n − 1), K(i), frq, nP, Floor)
      next i
      weight = (weight - tmpS) / CS_Swpt_Pay(K(n − 1), K(n), frq, nP, Floor)
   end if
   ω(n) = weight
next n
```

We take as an input to the algorithm a swap rate of $SR = 4.46\%$, $frq = 1$, $nP = 10$ and $pDelay = 0.5$. The input strikes are given by

$$4.46\% + 0.5 \cdot j, \quad j = 0, 1, \ldots, 49\%$$

We find the following weights $\omega()$ given in Table 7.1. The replication method determines the hedging cost at the beginning of the trade and bid-ask spreads are minimized since in principle the replicating position can be bought at trade inception. That is of course not done in practice but around the strikes one can buy a portfolio partially replicating the position and rehedge dynamically if necessary. The pricing is model dependent since the whole volatility smile is used for calculation.

Before we show how the risk can be further analyzed we consider the replication approach in a bit more detail. To this end we start by taking Figure 7.1. The top of this figure shows the payoff of a CMS Caplet and three replicating portfolios with 1, 4, 8

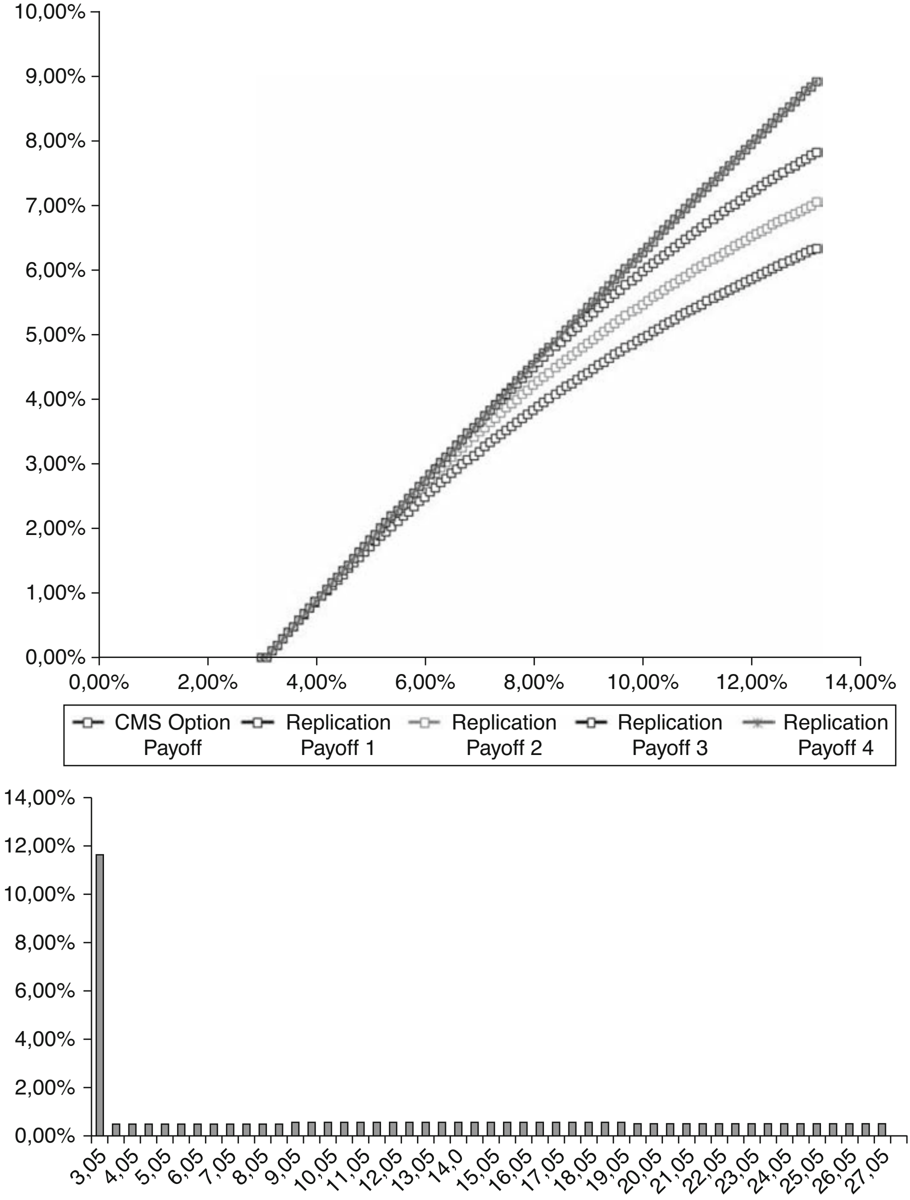


Figure 7.1 Terminal payoff of an ATM on a 10Y CMS replicated with 1, 4, 8 and 50 payer/receiver swaptions with strike ATM (top) and the corresponding weights (bottom). We considered the Caplet (top), the Floorlet (mid) and the Swaplet (bottom)

and 50 payer swaptions. For given strikes K_i the replicating portfolios are obtained by calculating the weights as outlined for the static hedging concept. It is clear that if we add more replicating instruments we better approximate the payoff of the CMS Caplet. For the replicating portfolio with 50 swaptions we have also displayed the calculated

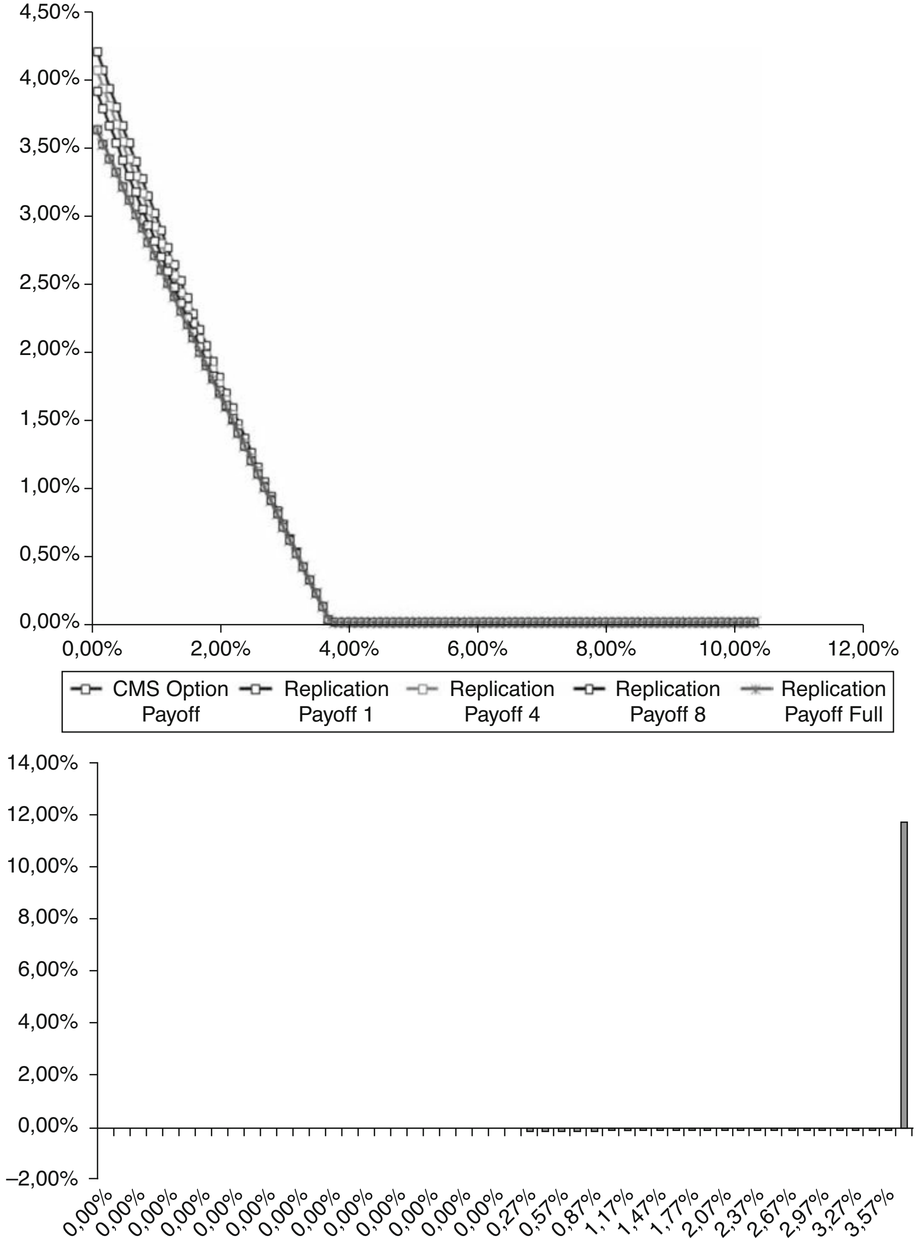


Figure 7.1 (*Continued*)

weights on the top right. We observe that the weight for the ATM swaption is highest. In the middle of Figure 7.1 we analogously consider the replication portfolio for a CMS Floorlet. Again, the approximation quality increases if we take a bigger number of replicating instruments. The result for the static hedging weights taking 50 receiver swaptions is displayed. To actually price the CMS we apply the Put-Call parity by

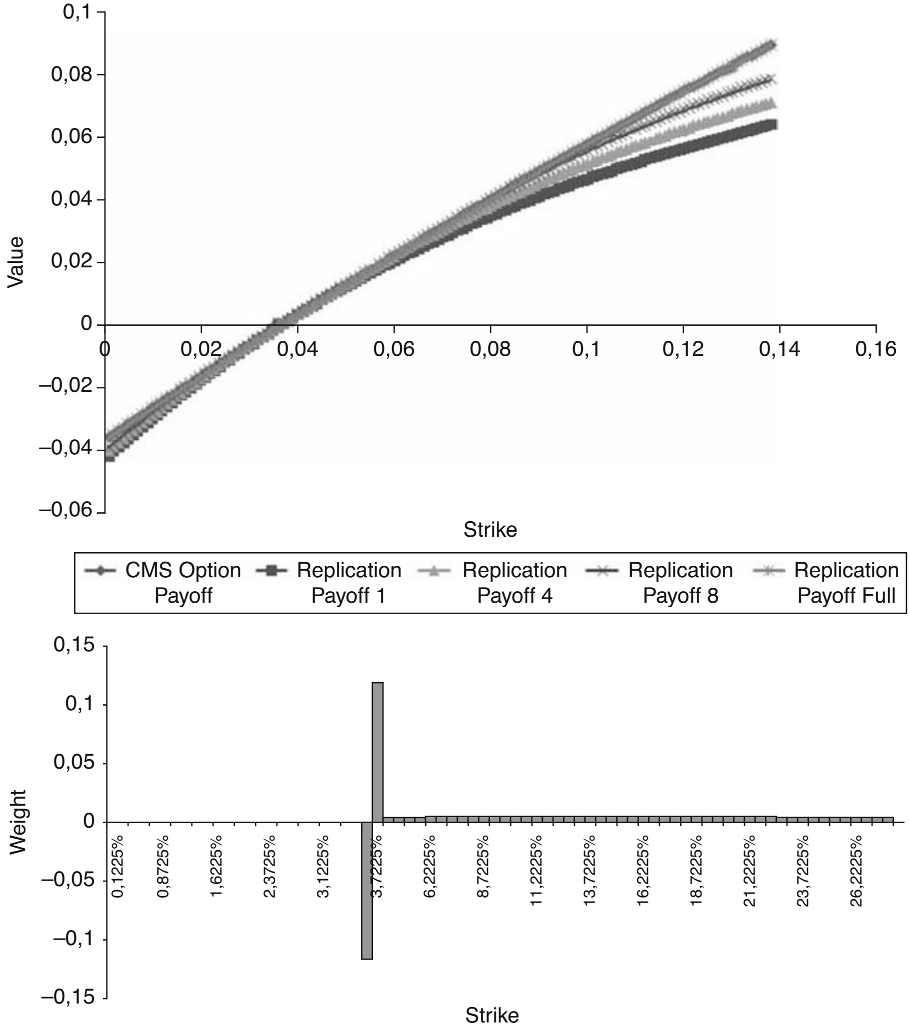


Figure 7.1 (*Continued*)

adding the corresponding replicating portfolios. This result is displayed at the bottom of Figure 7.1.

If we apply the replication method to all CMS Caplets and Floorlets in a CMS contract we get the corresponding values for the CMS at each Caplet/Floorlet period. Figure 7.2 displays the difference of the swap rate and the CMS rate which we called the convexity adjustment. It is clearly visible that the convexity adjustment is positive and that the longer the time to maturity the bigger is the adjustment. Furthermore in, Figure 7.3 we show the convexity adjustment for different CMS rates corresponding to 2y, 5y, 10y and 20y. Here we observe that the difference between the corresponding swap rates and the CMS rates depends on the underlying swap tenor. The rule of thumb is that the longer the swap tenor the higher the convexity adjustment.

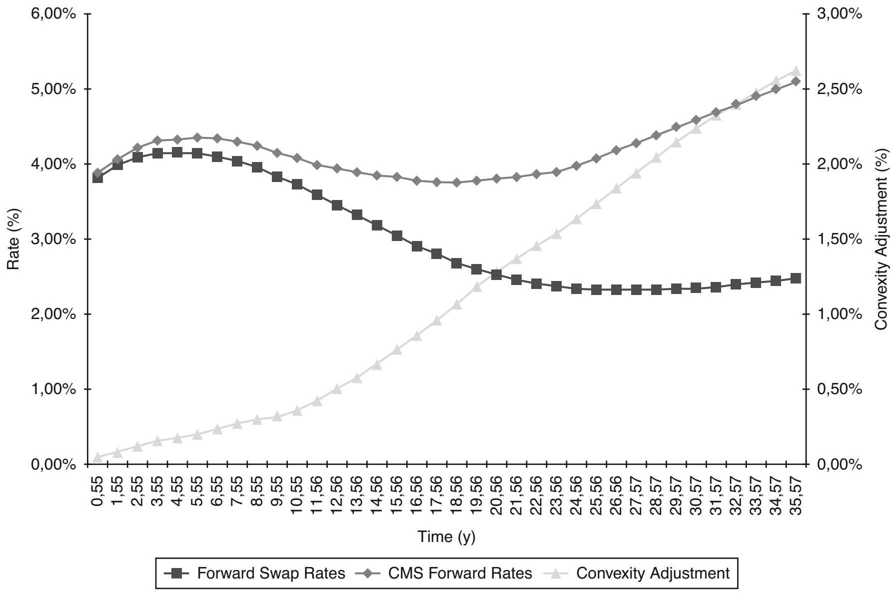


Figure 7.2 Convexity adjustment for 10y CMS index

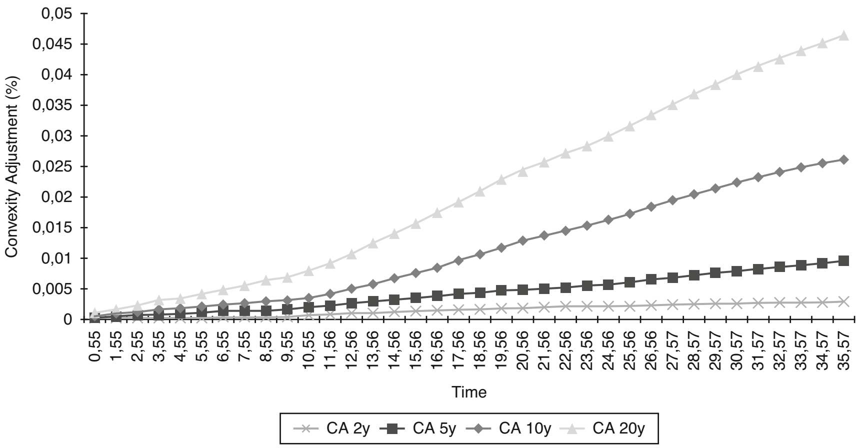


Figure 7.3 Convexity adjustment for 2y, 5y, 10y and 20y CMS

Using the replication all risks and their interactions can be studied. We find that the following risks drive the price of a CMS:

- Risk – The Curve
- Risk – The Basis

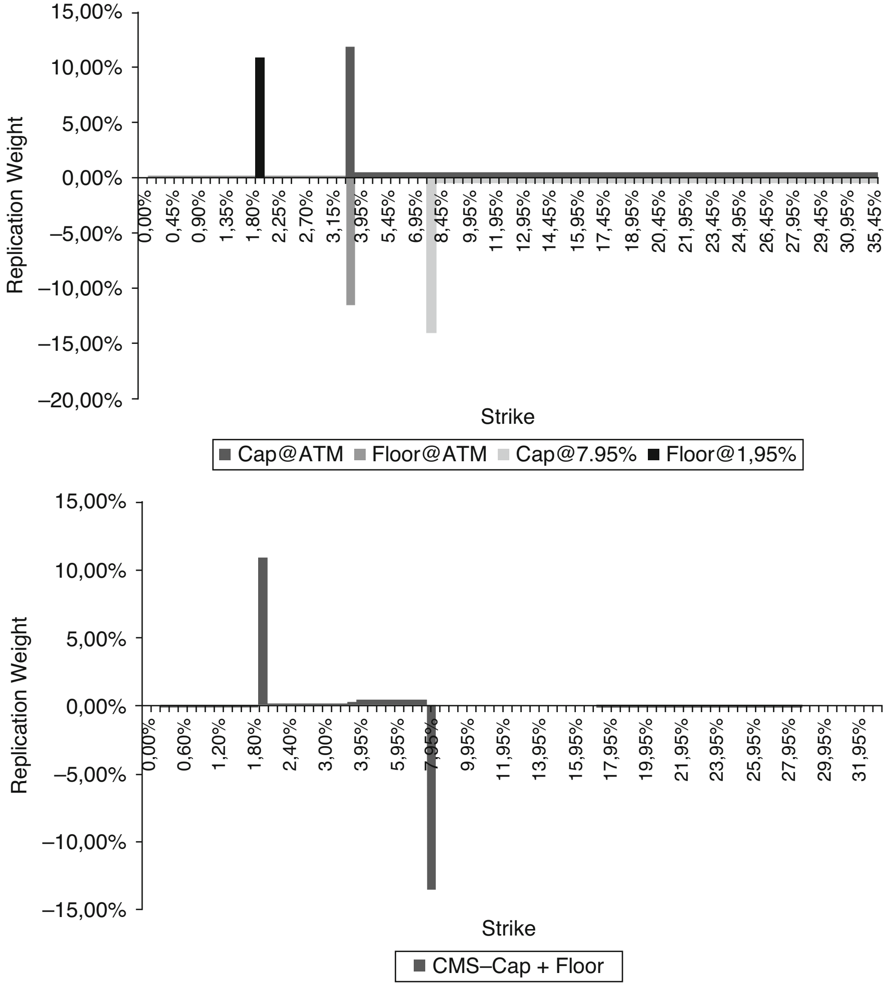


Figure 7.4 Replicating portfolio for CMS with cap at 7.95 % and floor 1.95% (top) and the net position (bottom)

- Risk – The Volatility
- Risk – The Smile

The reader is able to study the impact of the risk factors by considering the spreadsheet for illustrating CMS pricing. In the sequel we consider some issues which are analyzed by using this spreadsheet.

Some analysis by varying the input to the replication shows that the convexity adjustment differs between the different indexes (2Y, 5Y, 10Y, 20Y and 30Y) and that longer dated tenors, Figure 7.3, have a longer duration and more convexity. Thus, the deviation from a linear payoff which causes higher convexity adjustments is necessary. CMS are exposed to non-parallel shifts of the curve whereas usual swaps are sensitive to parallel shifts. Receiver CMS can be seen as steepener and Payer CMS as flattener trades of the yield curve.

The dependence on the whole swaption smile is shown in Figure 7.4. The replication portfolio consists of a CMS Cap with all strikes fixed at 7.95% and a CMS Floor with all strikes fixed at 1.95%. Thus, the replicating portfolio for the Cap depends on the strike range $[7.95\%, \infty]$ and for the Floor $[0, 1.95\%]$. The net position is also shown. Depending on the current level of rates the moneyness of the replicating swaptions is affected.

As a matter of fact trading activity in CMS also affects the shape of the yield curve. Many structured products are based on such indexes and Hedge flows from such products result in trades on the CMS underlyings which change the shape of the yield curve.

The overall level of volatility is illustrated by Figure 7.5. We applied a shift of 1% to all volatilities and calculated the price of a CMS contract on a 10Y Swap Rate. The higher volatility level leads to a bigger convexity adjustment and, thus, to a significant price difference.

The dependence on the volatility is even more delicate. First, the dependency on volatility is with respect to the convexity adjustment used to adjust the rates and, second, on the corresponding option. This double dependence impacts CMS option prices. For CMS Caps, increasing the volatility impacts the CMS convexity adjustment. It is increased. This leverages the volatility effect. The double dependence for a CMS Floor is different. Since for replication we are long volatility and at the same time short volatility this leads to a more balanced or de-leveraged dependence.

Once we are able to price CMS Options it is possible to compute the implied CMS volatility. This volatility can be seen as a mixture of swaption volatilites and can be used to efficiently price CMS Options. Using this methodology in practice shows that the differences to the swaption smile are relatively small. This makes it possible to distinguish vega risk from rates risk (see later)

7.2.2 CMS quotes

Finally, we give examples of CMS quotes in the EUR market. First, we observe by considering Table 7.2 that only special tenors are quoted. Usually we find quotes for $2Y$, $5Y$, $10Y$, $20Y$ and $30Y$. The quotes are with respect to the floating schedule of the $3M$, respectively $6M$ EURIBOR. The discounting convention is as usual the OIS discounting which is EONIA for the EUR market.

7.3 CMS spread options

There are other financial instruments based on CMS of particular interest for interest rate practitioners. Using *Constant Maturity Spreads* it is possible to trade the yield curve movement. We have shown that and CMS can be applied to trade the height of the yield curve and CMS Spreads can be applied to trade the steepness of the curve. Even trading the slope of the curve is possible when trading the spread of two CMS Spreads. Such contracts are called *CMS Flys*. Figure 7.6 illustrates the height, steepness

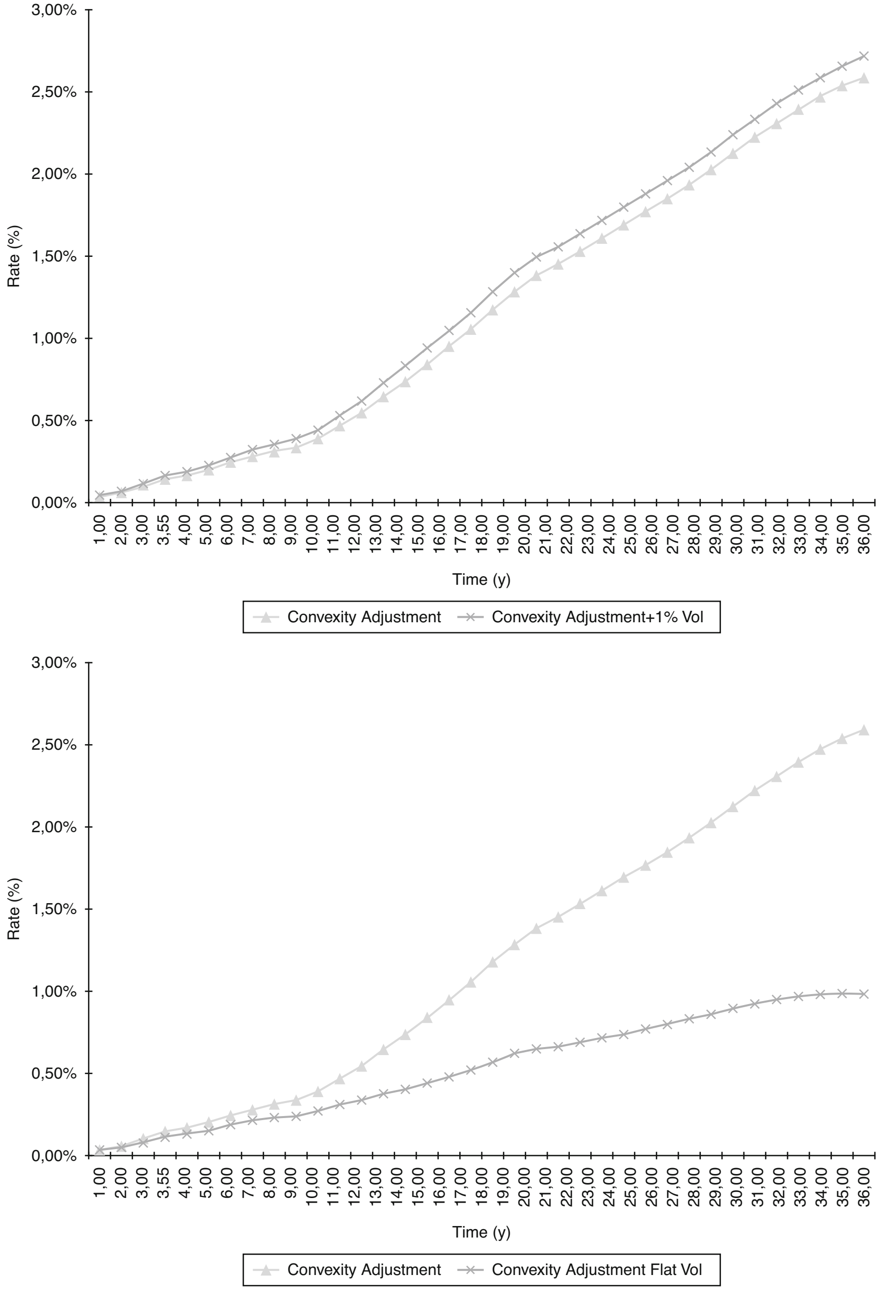


Figure 7.5 Convexity adjustment of a 10Y swap rate for a given volatility cube and a 1% shift of the whole cube (top) and the difference of the adjustment calculated with a smile and with flat volatility (bottom)

Table 7.2 *CMS swap prices quoted against 3M EURIBOR (top)/6M EURIBOR (bottom) and with EONIA discounting as of 22.07.2013*

T	2Y tenor	5Y tenor	10Y tenor	20Y tenor	30Y tenor
5Y Swaps	46.9/52.9	98.5/107.5	159.2/169.2	180.5/200.5	176.4/201.4
10Y Swaps	43.6/49.6	85.7/94.7	123.3/133.3	127.8/147.8	129.2/154.2
15Y Swaps	36.5/42.5	66.5/75.5	94.6/104.6	93.1/113.1	100.6/125.6
20Y Swaps	31.1/37.1	55.6/64.6	76.1/96.1	81.8/101.8	88.4/118.4
5Y Swaps	34.1/40.1	85.7/94.7	146.4/156.4	167.8/187.8	163.7/188.7
10Y Swaps	32.4/38.4	74.6/83.6	112.2/122.2	116.7/136.7	118.2/143.2
15Y Swaps	27.1/33.1	57.1/66.1	85.3/95.3	83.8/103.8	91.3/116.3
20Y Swaps	23.1/29.1	47.7/56.7	68.2/88.2	73.9/93.9	80.5/110.5

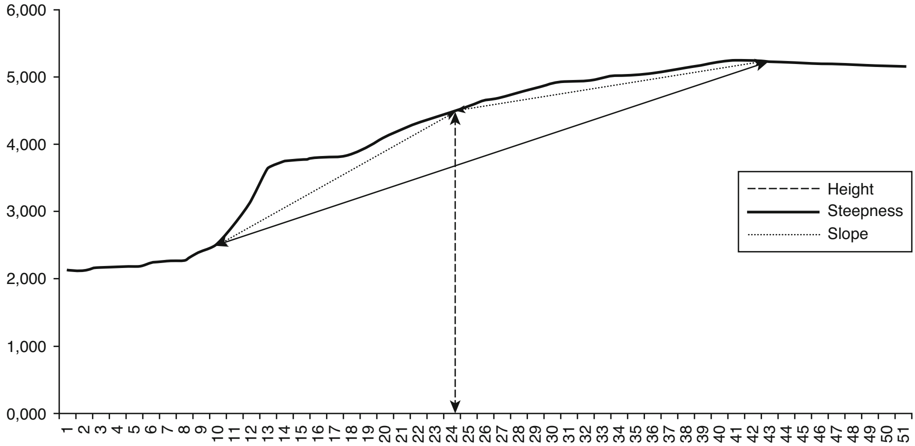


Figure 7.6 Trading the curve – height, steepness, slope

and the slope of a curve. This suggests that it is possible to trade the shapes by entering into a position of CMS contracts with different maturities.

In this subsection we wish to outline the risks involved in CMS Spreads and, especially, the risks when considering CMS Spread Options. To this end we review the risks embedded in CMS spread options and outline some approaches to value CMS spread options without using term structure models. For the different approaches we consider the strengths and weakness. We have to take into account the following risk factors:

- Yield curve movement
- Marginal volatility
- Marginal volatility smile
- Correlation
- Correlation smile

Let us consider the difference between two swap rates with different tenors. CMS spread options are derivatives directly linked to spreads between two swap rates with constant maturity. It is a way to express views on a precise curve slope even for long time horizons. To this end they are typical components of structured interest-rate products in a form of embedded linear/digital caps and floors and are particularly well-suited for expressing medium to long term views on the yield curve. Like all options, they require an upfront premium payment (different from curve trades which are at zero cost!) As we have already remarked the price of a spread option is determined by the joint dynamics of the two underlying rates. Approximately (omitting vol smiles), it is sensitive to the spread volatility and hence to the correlation between underlying rates and to their respective volatilities (marginal vols). For a given level of marginal vols, spread options are cheap when correlation is high. On the other hand, for a given level of correlation, spread option prices are the most attractive when marginal vols are low and when the ratio of the lower vol to higher vol is equal to the correlation parameter. Figure 7.7 outlines the basic setting.

For fixed correlation spread volatility is a function of marginal vols. Let both marginal vols be equal (e.g. $3bp/d$) and correlation is 1, then, spread vol is 0. Same level of marginal vols, spread vol moves to $0.95bp/d$ ($1.34bp/d$) if correlation drops to 0.95 (0.90). Correlation is constant at 0.95, then, spread vol is minimal at $0.94bp/d$ if the ratio of marginal vols equals 0.95, this is $\sigma_1 = 2.85bp/d$, $\sigma_2 = 3bp/d$. Spread vol up to $1.27bp/d$ if σ_1 moves down to $2bp$ ($-0.85bp/d$) or up to $3.70bp/d$ ($0.85bp/d$). Spread vol tends, to some extent, to be positively correlated to the slope of the yield curve (spread vol low – curve flat and high – curve steep). Spread volatility tends – to some extent – to be positively correlated to the slope of the curve. Tends to be low if the curve is flat Tends to be high if the curve is steep Suggests to combine delta and vega exposure in spread option trades Steepening views: Long Spread caps. Flattening views: Short Spread caps.

The price of a CMS spread option depends in the first order on the interest rate curve, in the second order on marginal vols and correlation (i.e. spread vol), and in the third order on vol smiles. All these parameters can help to improve entry levels of a curve trade with CMS spread options, but we can also play spread vol/correlation in outright or relative value trades. Strategies related directly to spread volatility or spread volatility smile are the most common relative value trades. They are similar to standard option strategies, the only difference comes from the underlying. We can distinguish between implied vs. realized spread vol, high-strike vs. low-strike spread vol, one currency vs. another currency spread vol or spread vol skew, spread vol roll-down trades or forward trades (taking advantage of a flat term structure). Correlation trades are an alternative to spread vol strategies. The idea is to keep only a pure correlation exposure by vega hedging (and delta hedging). Spread options are suited to trade implied correlation whereas a combination of three swaption straddles allows trading of realized correlation. Finally, arbitrage opportunities, like trading digital options against spreads of spread options, arise due to one-way hedging flows of exotics with a CMS spread component. Implied spread vol tends to be higher than realized spread volatility.

Implied and realized spread vol can differ significantly, in particular for long expiries. 2Y expiry implied vol on EUR 10Y–2Y spread at $1.60bp/d$, compared to only $1bp/d$ for realized vol estimated over 6M. The 10Y expiry implied vol stands at $1.50bp/d$, much

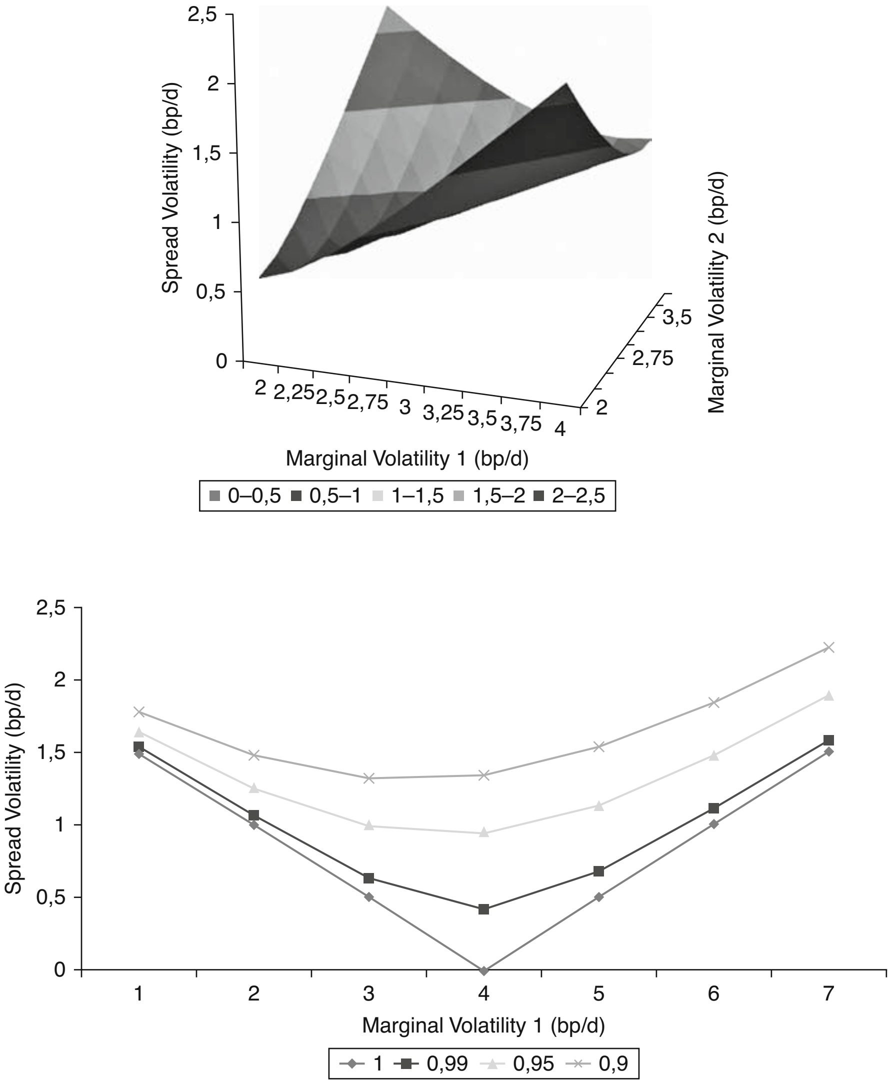


Figure 7.7 Spread volatility as a function of marginals (top) and as a function of marginal volatilities and correlation (bottom)

higher than the realized, at around $0.50bp/d$. 95% of CMS spread option were trades related to hedging flows from exotic desks.

7.3.1 CMS spread option math

In this section we consider the pricing of CMS Spread options. Since the spread can have negative values we need some appropriate modeling assumption. One method is to use the Normal Model, see Section 6.2.2, and assume that the distribution of the spread at a given time is normal. The only thing which is left is to determine the correct volatility. Take two random variables X and Y. We find for the variance of $X - Y$ that it is given by $\mathbb{V}(X-Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\mathbb{E}(XY)$. Thus, for the spread we are able to calculate the spread volatility using

$$(\sigma_1 S_1)^2 + (\sigma_2 S_2)^2 - 2 \cdot \sigma_1 \sigma_2 S_1 S_2$$

There is one parameter $\rho \in (-1,1)$ which affects the price of the spread option.

CMS spread option – Copula pricing methods

Let us consider the difference between two swap rates with given tenors $S(t) := SR_{T,T+nY}(t) - SR_{T,T+mY}(t)$. We assume that we have already set up a volatility surface such that we can apply the replication method considered in the last section to calculate the convexity adjustment. Thus,

$$SR_{T,T+kY}(t) = SR_{T_i,T_i+kY}(0) + CA(0,t,T_i), \quad k = n, m$$

Now, we wish to link the dynamics – in fact the finite dimensional distributions at given time points – of the convexity adjusted swap rates. To this end we use a statistical tool called *copula*.

A copula C is a function

$$C : [0,1]^d \to [0,1]$$

having the following properties:

- C is increasing
- $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$ for all $i = 1,\ldots,d$
- For all $\underline{a}$, $\underline{b} \in [0,1]^d$ such that $a_i \leq b_i$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\ldots i_d} C(u_{1i_1},\ldots,u_{di_d}) \geq 0$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all i.

Using the definition of a copula the joint cumulative probability distribution and the density are given in terms of the copula by:

$$F_C(x_1,\ldots,x_d) = C\left(F_{X_1}(x_1),\ldots,F_{X_d}(x_d)\right) \tag{7.7}$$

$$f_C(x_1,\ldots,x_d) = c\left(F_{X_1}(x_1),\ldots,F_{X_d}(x_d)\right)\prod_{i=1}^{d} f_{X_i}(x_i) \tag{7.8}$$

For continuous distributions the representation is unique, see Sklar (1959) or Nelsen (2006). Furthermore, we have for two random variables X and Y with cumulative distribution F_1 and F_2 that the joint distribution F is specified by a copula C:

$$
\begin{aligned}
\mathbb{P}(X \le x|Y = y) &= \lim_{h\to 0} \mathbb{P}(X \le x|y \le Y \le y+h)\\
&= \lim_{h\to 0} \frac{F(x,y+h) - F(x,y)}{F_2(y+h) - F_2(y)}\\
&= \lim_{h\to 0} \frac{C(F_1(x),F_2(y+h)) - C(F_1(x),F_2(y))}{F_2(y+h) - F_2(y)}\\
&= \lim_{h\to 0} \frac{C(F_1(x),F_2(y)+h) - C(F_1(x),F_2(y))}{\Delta_h}\\
&= \frac{\partial}{\partial u_2} \; C(u_1,u_2)|_{(F_1(x),F_2(y))}
\end{aligned}
$$

Thus, conditional probability, respectively expectation can be computed by differentiation.

$$
\begin{aligned}
&D_1 C(F(X),F(Y)) \text{ is a version of } \mathbb{P}(Y \le y|X)\\
&D_2 C(F(X),F(Y)) \text{ is a version of } \mathbb{P}(X \le x|Y)
\end{aligned}
$$

Since the copula is nothing but a probability distribution on the n dimensional cube for a copula C we can consider the density c given by

$$
c(u_1,\dots,u_d) = \frac{\partial^d C(u_1,\dots,u_d)}{\partial u_1 \cdots \partial u_d} \tag{7.9}
$$

If f denotes the density of the multivariate cumulative distribution function F from above, then, f and c are related by:

$$
f(x_1,\dots,x_d) = c(F_1(x_1),\dots,F_d(x_d)) \prod_{n=1}^{d} f_n(x_n) \tag{7.10}
$$

One nice property of copulas is ordering. Given two copulae C_1 and C_2, we say C_1 is smaller than C_2, denoted by $C_1 \prec C_2$ if for $u \in [0,1]^d$ we have

$$
C_1(u_1,\dots,u_d) \prec C_2(u_1,\dots,u_d) \tag{7.11}
$$

In fact there are functions which represent maximum possible dependence, minimum possible dependence and independence. The following functions are called *lower*

Fréchet bound, *upper Fréchet bound* and independence copula.

$$C^{-}(u) = \max\left(\sum_{i=1}^{d} u_i - 1, 0\right) \tag{7.12}$$

$$C^{+}(u) = \min(u_i, i = 1, \ldots, d) \tag{7.13}$$

$$C^{\perp}(u) = \prod_{i=1}^{d} u_i \tag{7.14}$$

The corresponding copulas are shown in Figure 7.8.

The concept of tail dependence relates to the amount of dependence in the upper quadrant or the lower quadrant of a bivariate distribution.

It is a concept that is relevant to dependence in extreme values. Furthermore, tail dependence between two random variables X and Y is a copula property and hence the amount of tail dependence is invariant under strictly increasing transformations of X and Y

Suppose (X, Y) is a random vector with marginals F_1 and F_2 and joint distribution $F = C(F_1, F_2)$, where C is a copula. Then,

$$\lambda_U = \lim_{u \to 1-} \mathbb{P}[Y > F_2^{-1}(u) | X > F_1^{-1}(u)] \tag{7.15}$$

is called the *coefficient of upper tail dependence* provided the limit exists:

$$\lambda_U = \lim_{u \to 1-} \frac{1 - 2u + C(u, u)}{1 - u} \tag{7.16}$$

We can define the *coefficient of lower tail dependence* in the same way:

$$\lambda_L = \lim_{u \to 0+} \frac{C(u, u)}{u} \tag{7.17}$$

The *comonotonic copula* is given by Equation (7.13). This copula is also called the *upper Frechet bound.*

For any copula C we have $C \prec C^{-}$.

Pricing using copulas

We give an overview of the different possibilities of implementing CMS spread pricing based on copulas. Let p be the payoff of an option on d assets where the joint distribution is given by the marginals F_{X_i}, the densities f_{X_i}, $i = 1, \ldots, d$ and a copula C.

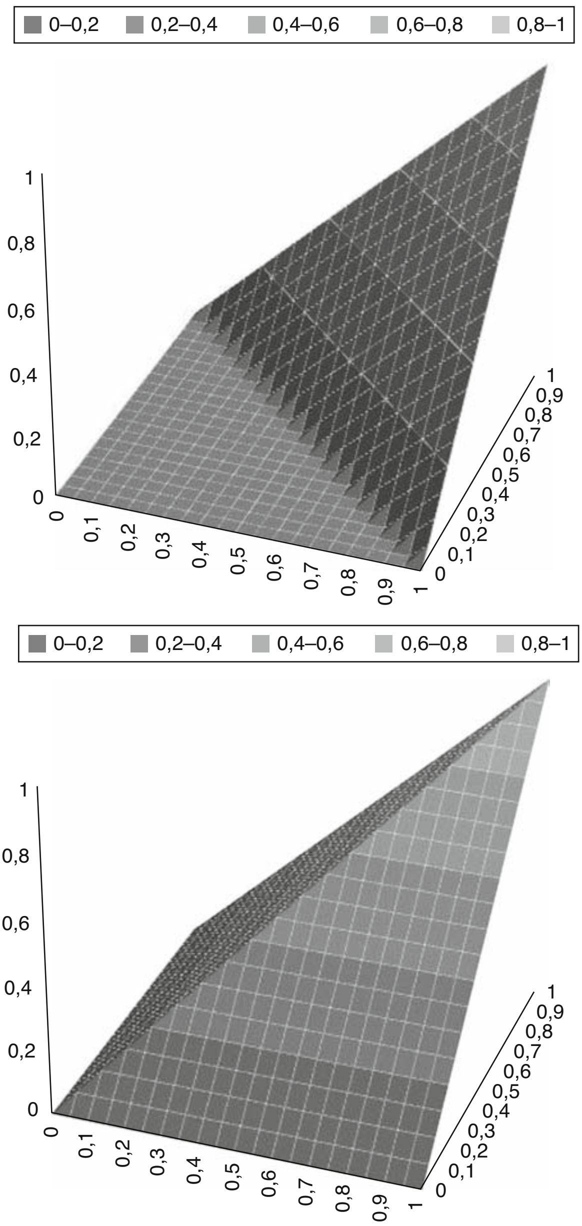


Figure 7.8 The comonotonic (top), countermonotonic (mid) and the independence copula (bottom)

The price of a European option with payoff p is given by:

$$\int_D p(x)c(F_X(x))\prod_{i=1}^{d} f_i(x_i)dx \tag{7.18}$$

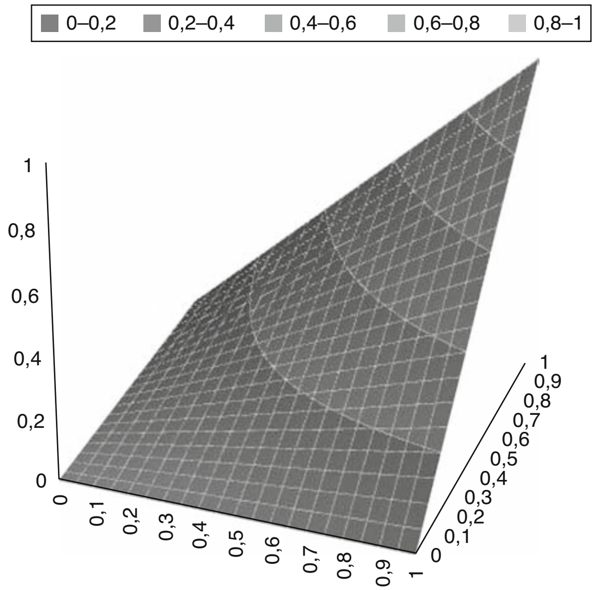


Figure 7.8 (*Continued*)

Using the general pricing formula (7.18) we compute the price of a spread option by setting $d = 2$ and $p(x_1, x_2) = (x_1 - x_2 - K)^+$:

$$\int_0^\infty \int_0^\infty p(x_1, x_2) c\left(F_{X_1}(x_1), F_{X_2}(x_2)\right) f_{X_1}(x_1) f_{X_2}(x_2) dx_1 \, dx_2 \tag{7.19}$$

For the implementation the cumulative distributions F_{X_i} as well as the densities f_{X_i} and the copula density c have to be evaluated numerically. Furthermore, the integral has to be truncated to apply some numerical integration scheme.

Another standard approach is to apply a transform to restrict the domain of integration to a bounded region. In this case $[0, 1] \times [0, 1]$. The corresponding formula is

$$\int_0^1 \int_0^1 \left(F_{X_1}^{-1}(u_1) - F_{X_2}^{-1}(u_2) - K\right)^+ c(u_1, u_2) du_1 \, du_2 \tag{7.20}$$

For implementation the inverse cumulative distributions $F_{X_i}^{-1}$ and the densities f_{X_i} have to be evaluated numerically. For integration we can apply an adaptive integration method.

- It is well known that the applications Equations (7.19) and (7.20) should rely on stable integration methods.
- Numerical evaluation of the marginal densities, the cumulative density or the inverse cumulative density has to be performed and should be stable and fast.
- The integration can be reduced to one dimensional integrals which is computationally much more efficient.

Another method is to use the conditional distribution. The payoff of the spread option can be written in integral form:

$$(S_1(T) - S_2(T) - K)^+ = \int_0^\infty 1_{\{S_1(T) > x+K\}} 1_{\{S_2(T) < x\}} dx \tag{7.21}$$

Using (7.21) and taking expectations we find:

$$C(K,T) = D(0,T) \int_0^\infty \mathbb{P}[S_1(T) > x+K, S_2(T) < x] dx \tag{7.22}$$

Finally, equation (7.22) is expressed in terms of the copula C

$$\mathbb{P}[S_1(T) > x+K, S_2(T) < x] = F_{X_2}(x+K) - C\big(F_{X_1}(x+K), F_{X_2}(x)\big) \tag{7.23}$$

Super and sub hedging

In the following we wish to apply a super- and sub-hedge argument for pricing CMS Spread options. Let $S = S_1 - S_2$ and K_1, K_2 such that $K = K_1 + K_2$. We consider the following lower and upper bound on the payoff of a spread option:

$$(S_1 - K_1)^+ - (S_2 - K_2)^+ \leq (S - K)^+ \leq (S_1 - K_1)^+ + (K_2 - S_2)^+ \tag{7.24}$$

Thus, it is possible to hedge a spread option by trading in the marginal underlyings S_1 and S_2. We have, taking expectations:

$$C(K_1,T) - C(K_2,T) \leq C(S,K) \leq C(K_1,T) + P(K_2,T) \tag{7.25}$$

Let us consider the mid strike $M = (K_1 - K_2)/2$ and the average strike $A = (K_1 + K_2)/2$. Using $K_1 = A + M$ and $K_2 = A - M$ we have

$$\begin{aligned} \mathbb{E}[(S-K)^+] = \mathbb{E}[(S_1 - K_1)^+] - \mathbb{E}[(S_2 - K_2))^+] \\ + \int_{-\infty}^{A} \mathbb{P}(S_1 > x+M, S_2 < x-M)\, dx \\ + \int_{A}^{\infty} \mathbb{P}(S_1 < x+M, S_2 > x-M)\, dx \end{aligned}$$

Using the same notation we find another equation for the price of the spread option:

$$\mathbb{E}[(S-K)^+] = \mathbb{E}[(S_1 - (A+M))^+] + \mathbb{E}[((A-M) - S_2)^+]$$

$$
\begin{aligned}
&-\int_{-\infty}^{A} \mathbb{P}(S_1 < x+M, S_2 < x-M)\, dx \\
&-\int_{A}^{\infty} \mathbb{P}(S_1 > x+M, S_2 > x-M)\, dx.
\end{aligned}
$$

The probabilities from Equations (7.22), (7.26) and (7.26)

$$
\begin{aligned}
&\mathbb{P}(S_1 > x+M, S_2 < x-M); \mathbb{P}(S_1 < x+M, S_2 > x-M) \\
&\mathbb{P}(S_1 < x+M, S_2 < x-M); \mathbb{P}(S_1 > x+M, S_2 > x-M)
\end{aligned}
$$

can be expressed using the copula C and digital options denoted by *Dig*, see Nelsen (2006).

For the sub-hedge we have, using Equation (7.26):

$$
\begin{aligned}
&\mathbb{E}[(S-K)^{+}] = C(K_1,T;S_1) - C(K_2,T;S_2) \\
&+\int_{-\infty}^{A} Dig(x-M;S_2) - C(Dig(x+M;S_1), Dig(x-M);S_2)dx \\
&+\int_{A}^{\infty} Dig(x+M;S_1) - C(Dig(x+M;S_1), Dig(x-M;S_2))dx
\end{aligned}
$$

If we expect the assets to be positively correlated the integrals should be small!

For the super-hedge there is a corresponding formula:

$$
\begin{aligned}
&\mathbb{E}[(S-K)^{+}] = C(K_1,T;S_1) - P(K_2,T;S_2) \\
&-\int_{-\infty}^{A} C(Dig(x+M;S_1), Dig(x-M;S_2))dx \\
&-\int_{A}^{\infty} 1 - Dig(x+M;S_1) - Dig(x-M;S_2) \\
&\qquad -C(Dig(x+M;S_1), Dig(x-M;S2))dx
\end{aligned}
$$

If we expect the assets to be negatively correlated the integrals should be small. Bounds for CMS Spread option prices can then be obtained using the copulas C^{+} and C^{-}.

Some copulas for CMS spread pricing

Let ρ be a symmetric positive definite matrix with $\rho_{ii} = 1$, $i = 1,\dots,d$. We denote by $F_{\mathcal{N}_d}$ the cumulative normal distribution and by $F_{\mathcal{N}_d}^{-1}$ its inverse. Then, the *Gaussian copula* is defined by:

$$
C^{\text{Gauss}} = (u_1,\dots,u_d;\rho) = F_{\mathcal{N}_d}^{\rho}\left(F_{\mathcal{N}_1}^{-1}(u_1),\dots,F_{\mathcal{N}_1}^{-1}(u_d)\right) \tag{7.26}
$$

The copula density is given by:

$$c^{\text{Gauss}}(u_1,\ldots,u_d;\rho) = \frac{1}{\det(\rho)} \exp\left(-\frac{1}{2}x^\top(\rho^{-1}-1)x\right)$$

where $x_i = F^{-1}_{\mathcal{N}_d}(u_i)$.

The Gaussian copula exhibits no upper tail dependence, no lower tail dependence and is symmetric.

Let ρ be a symmetric positive definite matrix with $\rho_{ii} = 1$, $i = 1,\ldots,d$. Let t_ν denote the cumulative distribution function of the t-distribution with ν degrees of freedom and t_ν^{-1} its inverse, then the *t-copula*, is given by:

$$C^{\text{t}}_{\rho,\nu}(u_1,\ldots,u_d) = t_{\rho,\nu}\left(t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_d)\right) \tag{7.27}$$

$$c^{\text{t}}_{\rho,\nu} = \frac{1}{\sqrt{\det(\rho)}} \frac{\Gamma\left(\frac{\nu+d}{2}\right)\Gamma\left(\frac{\nu}{2}\right)^d\left(1+\frac{1}{\nu}x^\top\rho^{-1}x\right)^{(d-\nu)/2}}{\Gamma\left(\frac{\nu+1}{2}\right)^d\Gamma\left(\frac{\nu}{2}\right)\prod_{n=1}^d\left(1+x_n^2/\nu\right)^{(d-\nu)/2}}$$

where $x_n = t_\nu^{-1}(u_n)$.

The t-copula exhibits upper tail dependence and lower tail dependence. For illustrations showning the Gauss and the t-Copula see Figure 7.9.

For $\theta_1,\theta_2 \in [0,1]$ and denoting by C_{G} the Gaussian copula with correlation $\rho \in (0,1)$ the Power Gauss copula, see Andersen and Piterbarg (2010c) is given by:

$$C_{\text{PG}}(u_1,u_2) = u_1^{1-\theta_1}u_2^{1-\theta_2}C_{\text{G}}(u_1^{\theta_1},u_2^{\theta_2})$$

The correlation ρ is used to move the height of the implied volatility smile whereas the parameters θ_1 and θ_2 control the slope and the curvature of the smile. Choosing $\theta_1 = \theta_2 = 1$ leads to the (symmetric) Gaussian copula.

For $\theta_1,\theta_2 \in [0,1]$ and denoting by C_{T} the t-Copula with correlation $\rho \in (0,1)$ the Power Gauss Copula, see Anderser and Piterbarg (201c), is given by:

$$C_{\text{PT}}(u_1,u_2) = u_1^{1-\theta_1}u_2^{1-\theta_2}C_T(u_1^{\theta_1},u_2^{\theta_2},\rho)$$

The correlation ρ is used to move the height of the implied volatility smile whereas the parameters θ_1 and θ_2 control the slope and the curvature of the smile. Choosing $\theta_1 = \theta_2 = 1$ leads to the (symmetric) Gaussian copula.

Copulas based on the skewed normal and skewed t-distributions have been proposed by Kainth (2010). The authors applied the skewed normal copula and the skewed t-copula to CMS spread options and argued that the skewed t-copula gives better fit to market observed data than the skewed normal copula. In general asymmetric copulas give much better fit than symmetric ones. To calibrate a CMS Spread pricing model based on copulas we take

(a)

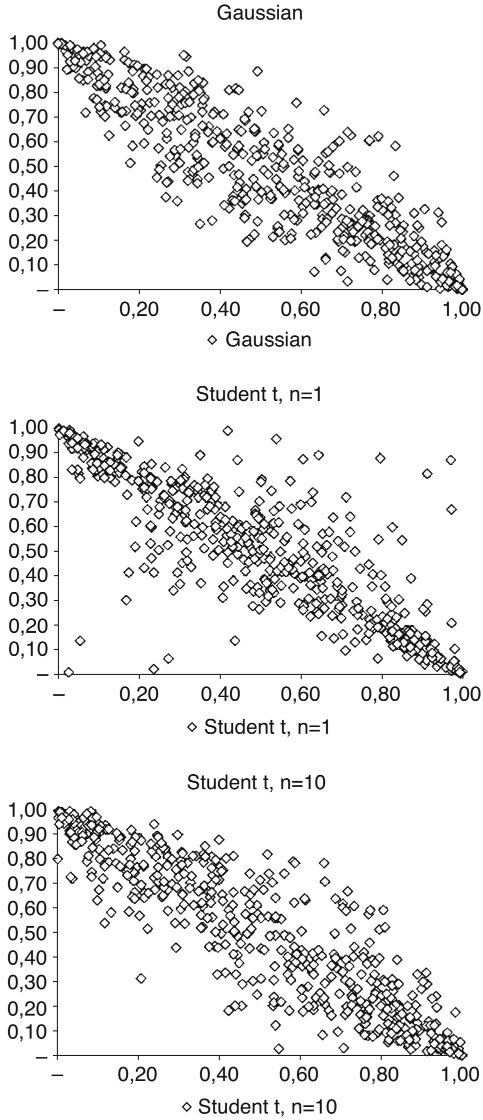


Figure 7.9 Gaussian copula (top), t-copula with one degree of freedom (mid) and t-copula with 10 degrees of freedom (bottom) for different correlations, $\rho = -0.9$ (a), 0.0 (b) and 0 (c)

- CMS Caplet prices across different strikes
- Swaption prices across different strikes
- CMS prices
- CMS Spread option prices (across different strikes)

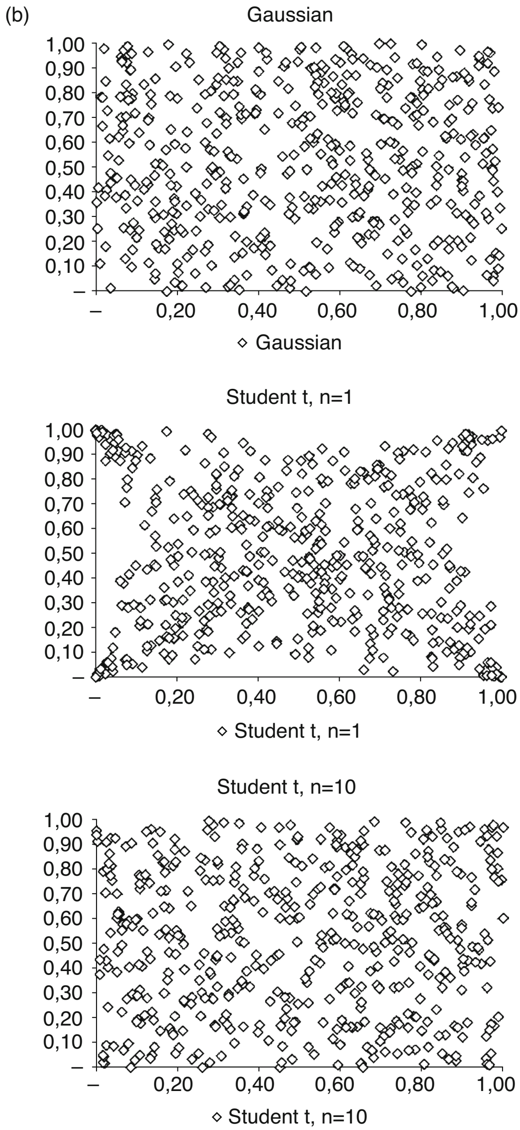


Figure 7.9 (*Continued*)

Often it is observed that market participants have black box solutions or some simple models without fine tuning (interpolation, extrapolation, no-arbitrage constraints, etc.) of use off the shelf formulas (for implied vol or copulae). Even worse the resulting volatility structures are applied to the pricing exotic derivatives.

Let us summarize the approach by observing that all option prices have to be computed using numerical integration methods and the compuational time as well as the complexity increases with regard to the chosen copula model. But with the

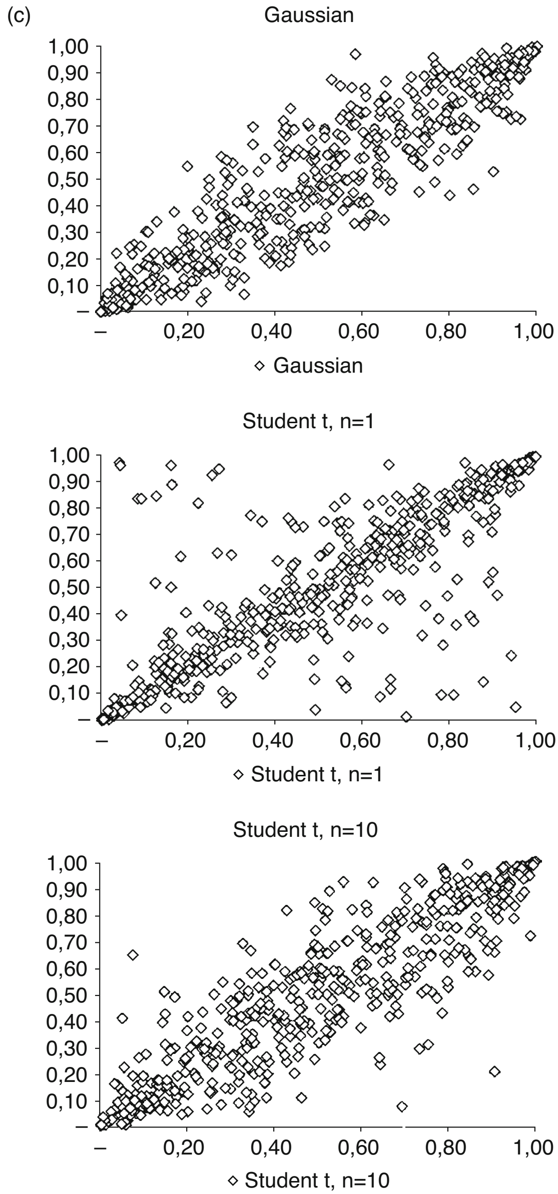


Figure 7.9 (*Continued*)

parameters we have we might achieve a more reasonable fit to observed market prices. We illustrate the pricing using different copulas in Figure 7.10. We observe that many different shapes of the smile can be fitted using different copulas.

The corresponding Single Look quotes are displayed in Table 7.4.

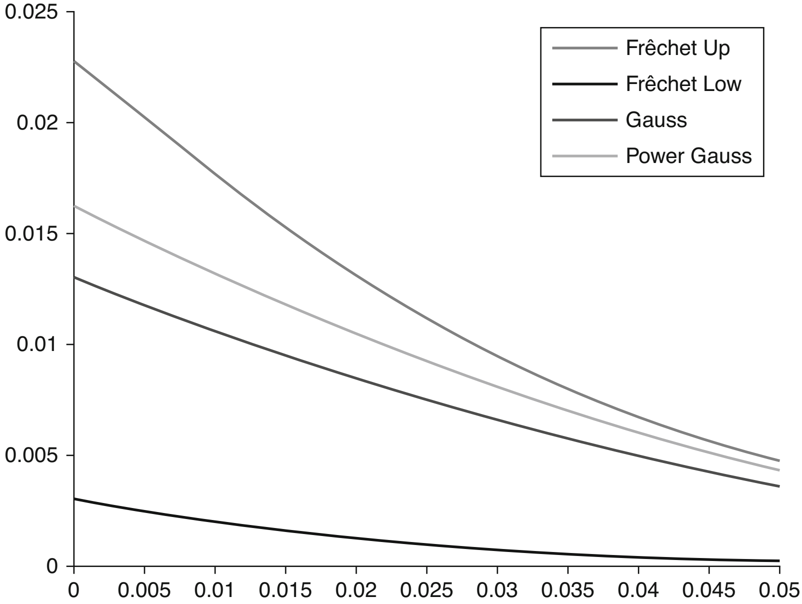


Figure 7.10 Pricing the CMS spread option using different copulas

7.3.2 CMS spread option quotes

Nowadays there are market quotes available for some standardized CMS and CMS Spread options. We have already given examples for market quotes for CMS options. For CMS Spread Caps and Floors the market quote premium and volatilities for different strikes. The quoted volatility is used as the volatility in a Black type formula for each constituting CMS Spread Caplet/Floorlet. The quoted volatility is called flat volatility. There are also market quotes, called Single Look, which are CMS Spread Call/Put options. An example for Caps and Floors is shown in Table 7.3.

Finally, we also give an example for the quotation in terms of the premium. This is Table 7.5.

7.4 Reading list

The option pricing methods outlined in the last two chapters and many theoretical and practical concepts can be found in Brigo and Mercurio (2006) as well as the three books, Andersen and Piterbarg (2010a,b,c). The scope of the books is much wider than the part illustrated in the last two chapters and include many more aspects of interest rate modeling such as volatility and term structure models. The basic options can be found in their Chapter 5. Furthermore, in depth analysis of valuation methods including finite difference methods, trees, transforms and Monte Carlo methods are introduced and detailed (Chapters 1–3). The basic instruments including standard options such as Caps, Floors or Swaptions are covered in Bianchetti (2010) with

Table 7.3 *CMS spread caps and floors for EUR and 10/2 (top), 30/2 (mid) and 30/10 (bottom) as of 22.07.2013. For all cases EONIA discounting is applied*

T	FWD	ATM	Flr −0.25	Flr −0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
1y	1.36	22.3	0.3	0.3	0.4	84.8	66.3	48.1	30.8	6.8
2y	1.35	71.2	1.7	2.1	2.5	198.5	156.3	115.5	77.7	24.0
3y	1.29	135.3	6.1	7.3	8.3	300.8	236.5	175.3	119.8	42.0
4y	1.21	211.6	15.2	17.6	19.6	388.2	303.6	224.6	154.4	58.5
5y	1.13	295.2	27.1	31.3	34.7	464.1	361.2	266.5	184.0	73.4
7y	0.99	487.2	63.8	73.0	80.1	595.9	461.8	341.5	239.4	104.8
10y	0.78	831.2	159.7	179.9	195.6	744.1	577.4	432.7	313.0	156.5
15y	0.54	1444.6	388.7	433.4	467.6	951.4	748.4	576.0	435.2	249.8
20y	0.45	1987.0	592.7	659.4	709.1	1188.3	946.8	740.9	571.2	340.4
1y	1.86	26.1	0.0	0.0	0.0	0.1	103.3	84.6	66.0	31.5
2y	1.77	88.0	1.0	1.3	1.5	2.2	228.7	186.4	145.6	73.5
3y	1.64	170.6	4.6	5.6	6.4	9.0	330.2	267.1	207.6	107.1
4y	1.50	266.8	11.8	14.5	16.6	23.5	409.7	329.5	255.2	132.8
5y	1.37	372.4	23.6	28.8	32.9	45.9	475.8	380.9	294.2	153.6
7y	1.16	600.2	61.6	73.8	83.2	111.6	586.1	466.8	359.7	190.0
10y	0.92	956.4	149.2	175.1	194.5	251.8	715.2	567.9	437.9	235.4
15y	0.69	1530.1	334.9	385.3	422.4	528.7	905.2	720.6	559.4	310.8
20y	0.64	2037.3	486.8	554.6	604.3	745.5	1148.2	923.4	727.5	424.4
1y	0.50	13.2	0.1	0.2	0.3	20.3	6.6	1.2	0.2	0.0
2y	0.42	45.9	2.1	3.4	4.8	42.7	16.5	5.2	2.0	0.4
3y	0.35	89.5	7.9	12.0	16.0	61.3	25.9	10.0	4.5	1.3
4y	0.29	139.6	18.6	26.5	33.9	77.8	34.9	15.1	7.6	2.6
5y	0.24	192.0	32.7	44.9	56.1	92.6	43.3	20.3	11.0	4.4
7y	0.17	303.8	68.9	90.3	109.4	123.4	61.9	32.2	19.2	8.8
10y	0.14	477.2	123.3	158.2	187.9	184.5	99.8	55.3	33.4	14.7
15y	0.15	811.6	221.6	276.7	321.9	339.7	211.4	133.4	86.9	38.8
20y	0.19	1172.7	319.9	392.1	450.0	537.0	366.1	252.4	176.8	87.8

respect to the new modeling assumptions including CSA. This is one of the early fundamental papers on the subject. Standard references for valuation of CMS include Hagan (2003), Amblard and Lebuchoux (2000)–Mercurio and Pallavicini (2006) and Berrahoui (2004). These papers apply the replication approach for the valuation as outlined in the last chapter. The copula methodology is described in the monographs Sklar (1959) and Nelsen (2006) from a mathematical perspective. The copula theory has many applications in mathematical finance and studying the books is worthwhile. The copula is a useful tool to model dependency and, thus, it is also applied to CMS spread options pricing. Using copula theory handy formulas for such options have been derived. An interesting aspect of the copula modeling of CMS spread options – the CMS triangle arbitrage - was observed and published in McLoud (2011). It is worth taking a look at this paper to realize that the copula theory is not suited for fitting the observed market prices. If the reader is interested in implementing such models including the applications to CMS spread we refer to Kienitz and Wetterau (2012)

Table 7.4 *CMS spread single look options for EUR and 10/2 (top), 30/2 (mid) and 30/10 (bottom) as of 22.07.2013. For all cases EONIA discounting is applied*

T	FWD	ATM	Flr −0.25	Flr −0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
2w	1.34	8.3	0.0	0.0	0.0	108.8	83.8	58.8	33.8	0.3
1m	1.34	11.9	0.0	0.0	0.0	109.2	84.2	59.2	34.3	1.1
3m	1.35	20.9	0.0	0.0	0.0	110.0	85.1	60.4	36.6	4.6
6m	1.37	30.1	0.2	0.3	0.4	112.1	87.7	63.7	41.1	9.3
9m	1.37	37.3	0.5	0.7	1.0	113.7	89.6	66.2	44.4	12.9
1y	1.37	42.5	0.6	0.8	1.2	114.1	90.2	67.3	46.2	15.5
18m	1.34	50.6	1.2	1.7	2.4	112.0	88.8	66.8	46.8	18.0
2y	1.27	55.5	1.5	2.1	3.1	106.2	83.7	62.7	44.1	17.8
3y	1.05	66.0	4.1	5.5	7.4	89.9	69.5	51.2	35.9	15.8
4y	0.89	72.3	7.3	9.3	12.1	79.2	60.4	44.2	31.1	14.7
5y	0.83	79.3	8.7	11.3	15.0	75.6	58.4	43.6	31.6	15.7
6y	0.73	91.3	16.3	19.4	23.5	74.6	58.4	44.7	33.9	19.8
7y	0.56	95.8	21.1	24.9	29.8	65.5	51.2	39.5	30.6	19.4
8y	0.39	99.5	26.4	30.8	36.6	56.9	44.6	35.1	28.1	19.4
9y	0.28	102.1	30.2	34.9	41.2	52.7	41.8	33.7	27.7	20.1
10y	0.11	107.1	33.8	40.7	49.3	47.5	38.2	30.7	24.6	16.2
12y	−0.05	108.4	40.6	47.3	55.9	42.9	35.6	30.0	25.7	19.9
15y	0.07	108.2	37.5	44.1	51.8	47.5	39.5	32.8	27.2	18.9
20y	0.20	98.4	30.3	36.3	43.2	47.5	40.2	33.6	27.8	18.5
2w	1.91	9.5	0.0	0.0	0.0	166.0	141.0	116.0	91.0	41.0
1m	1.91	13.5	0.0	0.0	0.0	165.6	140.6	115.6	90.6	40.7
3m	1.89	23.9	0.0	0.0	0.0	164.0	139.0	114.0	89.0	40.5
6m	1.86	34.9	0.0	0.0	0.0	161.3	136.4	111.5	87.0	41.5
9m	1.83	44.2	0.0	0.1	0.1	158.3	133.6	109.2	85.3	42.7
1y	1.80	52.7	0.2	0.3	0.5	155.6	131.3	107.4	84.4	44.1
18m	1.68	62.1	0.9	1.2	1.7	145.1	121.4	98.4	76.7	40.6
2y	1.53	69.1	1.8	2.3	3.1	131.9	108.8	86.9	66.8	36.0
3y	1.20	81.7	3.8	5.5	7.9	105.5	85.6	67.5	51.7	27.9
4y	0.95	88.1	6.9	9.8	13.7	87.7	70.2	54.8	41.7	22.2
5y	0.77	92.8	10.7	14.6	19.7	76.2	60.6	47.2	35.9	19.5
6y	0.59	97.6	15.4	20.4	26.8	66.8	53.1	41.4	31.7	17.8
7y	0.40	100.2	20.6	26.6	34.2	57.5	45.6	35.6	27.3	15.6
8y	0.26	102.1	25.1	32.0	40.5	51.5	40.9	32.0	24.8	14.4
9y	0.16	101.8	27.8	35.2	44.1	47.2	37.5	29.4	22.8	13.5
10y	0.07	105.0	32.5	40.4	49.7	45.0	36.1	28.7	22.6	13.7
12y	0.06	101.3	32.0	39.5	48.4	43.2	34.9	27.8	22.0	13.5
15y	0.33	103.4	27.6	33.6	40.7	54.9	45.8	37.9	31.1	20.3
20y	0.54	87.7	18.2	22.6	28.0	53.3	45.1	38.2	32.4	23.5
2w	0.57	5.2	0.0	0.0	0.0	32.3	7.7	0.0	0.0	0.0
1m	0.56	7.3	0.0	0.0	0.0	31.4	7.8	0.1	0.0	0.0
3m	0.54	12.2	0.0	0.0	0.0	29.3	8.4	0.7	0.0	0.0
6m	0.50	17.3	0.0	0.0	0.2	26.4	8.6	1.5	0.2	0.0

Table 7.4 *(Continued)*

T	FWD	ATM	Flr −0.25	Flr −0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
9m	0.46	22.2	0.1	0.3	1.0	24.8	9.2	2.5	0.6	0.1
1y	0.43	26.6	0.4	0.9	2.3	24.2	10.0	3.5	1.3	0.2
18m	0.34	32.3	1.2	2.4	5.2	21.2	9.7	4.2	1.9	0.5
2y	0.26	36.2	2.0	4.1	8.5	18.6	9.0	4.3	2.1	0.6
3y	0.15	44.1	5.4	9.1	15.9	17.3	9.4	5.2	3.0	1.2
4y	0.06	46.3	8.4	12.9	20.7	14.7	8.2	4.9	3.1	1.5
5y	−0.06	51.0	13.3	18.8	28.2	13.0	8.0	5.3	3.8	2.2
6y	−0.14	55.6	16.7	23.7	34.6	12.8	8.1	5.5	3.9	2.3
7y	−0.16	58.3	18.3	25.8	36.8	13.5	8.6	5.6	3.9	2.1
8y	−0.13	60.2	18.1	25.5	35.9	15.5	9.7	6.2	4.0	1.9
9y	−0.12	62.1	18.6	26.1	36.3	16.9	10.7	6.7	4.2	1.7
10y	−0.04	64.8	17.8	24.8	34.1	21.0	13.8	8.7	5.4	2.0
12y	0.10	66.9	15.9	21.9	29.6	27.7	19.5	13.4	8.9	3.8
15y	0.27	76.0	17.3	22.7	29.3	38.6	30.0	22.8	17.0	9.1
20y	0.35	68.4	14.3	18.8	24.5	37.4	29.5	23.0	17.6	9.8

Table 7.5 *Forward premiums CMS spread single look for EUR and 10/2 as of 22.07.2013. EONIA discounting is applied*

T	FWD	ATM	Flr −0.25	Flr −0.10	Flr 0.00	Cap 0.25	Cap 0.50	Cap 0.75	Cap 1.00	Cap 1.50
2w	1.34	8.3	0.0	0.0	0.0	108.8	83.8	58.8	33.8	0.3
1m	1.34	11.9	0.0	0.0	0.0	109.2	84.2	59.2	34.3	1.1
3m	1.35	20.9	0.0	0.0	0.0	110.0	85.1	60.4	36.6	4.6
6m	1.37	30.1	0.2	0.3	0.4	112.2	87.7	63.8	41.1	9.3
9m	1.37	37.4	0.5	0.7	1.0	113.8	89.7	66.3	44.4	13.0
1y	1.37	42.6	0.6	0.8	1.2	114.2	90.4	67.4	46.2	15.5
18m	1.34	50.7	1.2	1.7	2.4	112.3	89.1	67.0	46.9	18.0
2y	1.27	55.8	1.5	2.1	3.1	106.7	84.1	63.0	44.3	17.9
3y	1.05	66.7	4.1	5.5	7.5	90.8	70.2	51.7	36.2	16.0
4y	0.89	73.9	7.5	9.5	12.4	80.9	61.7	45.1	31.7	15.1
5y	0.83	82.3	9.0	11.8	15.6	78.4	60.6	45.2	32.8	16.3
6y	0.73	96.5	17.2	20.5	24.9	78.9	61.8	47.3	35.8	21.0
7y	0.56	103.6	22.8	26.9	32.2	70.8	55.3	42.8	33.1	21.0
8y	0.39	110.2	29.3	34.1	40.5	63.0	49.4	38.9	31.2	21.5
9y	0.28	116.3	34.4	39.8	46.9	60.0	47.7	38.3	31.5	22.9
10y	0.11	125.6	39.6	47.8	57.8	55.7	44.9	36.0	28.9	18.9
12y	-0.05	135.2	50.6	59.0	69.8	53.5	44.4	37.4	32.1	24.9
15y	0.07	148.4	51.5	60.5	71.1	65.1	54.2	45.0	37.4	25.9
20y	0.20	155.0	47.7	57.2	68.1	74.9	63.3	53.0	43.8	29.1

where also Matlab source code is found. This also includes code for the Black-Scholes, the Normal and the Displaced Diffusion model. In addition many highly sophisticated algorithms for everyday life can be found there. The source code is included and free of charge.

III COUNTERPARTY CREDIT RISK ADJUSTMENTS

8 Adjustments

8.1 Introduction and objectives

An introduction to interest rates, derivatives and options would not be complete without covering the basics of adjustments. We have already described the major changes in the modern financial markets. We described the observations after August 2007 and outlined the new market practice. One of the issues which was identified as the main drivers for the crisis was counterparty credit risk. In this chapter we take the definition of Credit Exposure given in Chapter 1, Section 1.4 to build our exposition of the different adjustments to consider adjustments with respect to credit risk. We cover CVA in Section 8.2, bilateral CVA (BCVA) and DVA in Section 8.3 and, finally, FVA in Section 8.4. Our examples to illustrate the concept use future scenarios of the yield curve together with a swap contract to illustrate the effects and for providing an example for the calculation.

8.2 The credit value adjustment (CVA)

Let us give some further illustration of what impacts the credit value adjustment without going into the tedious details of implementation and mathematics. We illustrate the EE and NEE for a fixed against float IRS, Figure 8.1. The corresponding data can be found in Table 8.1 at the end of this chapter. In our exposition we follow Gregory (2012). Having discussed the exposure of a party to a default of a counterparty we will now link this exposure to the default probability. This linkage is useful to calculate a price for counterparty creidt risk. As usual there are numerous levels of sophistication when performing the calculation. A very rough diversion is the calculation on the *trade level* and on the *firm level*. First, calculating on trade level is just a single trade but the firm level is really challenging since per trade the whole credit exposure can be altered. In principle the whole netting set has to be evaluated and the corresponding exposure has to be determined.

The price for counterparty credit risk is referred to as *credit value adjustment* or *CVA* for short. To keep this exposition understandable and to work out the main aspects of CVA we consider the default of the counterparty only first and suppose that a fair value pricing of all the contracts in a given netting set is possible. Furthermore, we currently ignore *Wrong Way Risk* here which means that we assume the credit exposure is independent of the time to default.

For the credit value adjustment we detail the derivation since a similar one is used in the next two sections. Let us consider two parties A and B. Furthermore, we assume that both parties have agreed to certain netting rules which leads us to consider the

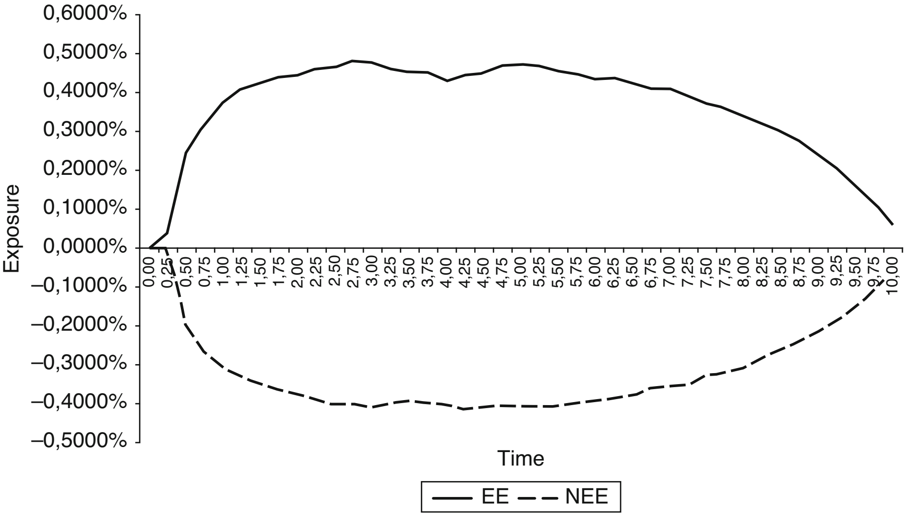


Figure 8.1 Exposure example of a swap. EE and NEE is shown

credit value adjustment per netting set. Suppose the value of the contracts forming the netting set and maturing before T is $V(t,T)$. Note that discounting is implicit here. This value only holds in the case where we assume that the probability of default of both parties is 0. First, we consider the possibility that party B can default and denote the (now risky) value of the netting set by $V_B^{\text{risky}}(t,T)$. Clearly, we have

$$V(t,T) \geq V_B^{\text{risky}}(t,T)$$

If τ_B denotes the time when party B defaults we split all possible outcomes into two scenarios which can be written as

$$1 = \underbrace{1_{\{\tau_B > T\}}}_{\text{No-default of party B before or at T}} + \underbrace{1_{\{\tau_B \leq T\}}}_{\text{Default of party B before T}}$$

Let R be the stochastic recovery rate, then, the payoff value of the netting set is

$$P^{\text{risky}}(t,T) = 1_{\{\tau_B > T\}} V(t,T) \\ + 1_{\{\tau_B \leq T\}} \left(\underbrace{V(t,\tau_B)}_{(1)} + \underbrace{R \max(V(\tau_B,T),0)}_{(2)} + \underbrace{\min(V(\tau_B,T),0)}_{(3)} \right) \tag{8.1}$$

Equation (8.1) means we have split the value of the netting set at time τ_B into three parts. Part (1) is the value just prior to default of party B, the recovery value at τ_B

Table 8.1 Calculating of CVA, BCVA, DVA and FVA using the default probability and simulated EE and NEE.

T_i	S_i^B	S_i^A	p_i^B	p_i^A	EE_i	NEE_i	D_i	CVA_i	$BCVA_i^1$	$BCVA_i^2$	DVA_i	FCA_i	FBA_i	FVA_i
0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
0.25	0.994	0.995	0.006	0.005	0	0	0.991	0.01	0.01	0	0	0.01	0	0.01
0.5	0.988	0.989	0.006	0.005	0.002	−0.002	0.981	0.06	0.059	−0.071	−0.072	0.059	−0.079	−0.02
0.75	0.981	0.984	0.006	0.005	0.003	-0.003	0.972	0.076	0.075	−0.095	−0.097	0.074	−0.106	−0.031
1	0.975	0.979	0.006	0.005	0.004	−0.003	0.962	0.088	0.086	−0.106	−0.109	0.086	−0.118	−0.032
1.25	0.969	0.974	0.006	0.005	0.004	−0.003	0.951	0.095	0.092	−0.111	−0.115	0.092	−0.124	−0.032
1.5	0.963	0.968	0.006	0.005	0.004	−0.004	0.94	0.096	0.093	−0.116	−0.12	0.093	−0.129	−0.036
1.75	0.957	0.963	0.006	0.005	0.004	−0.004	0.924	0.097	0.094	−0.116	−0.122	0.093	−0.129	−0.036
2	0.951	0.958	0.006	0.005	0.004	−0.004	0.919	0.097	0.093	−0.118	−0.125	0.093	−0.132	−0.039
2.25	0.945	0.953	0.006	0.005	0.005	−0.004	0.913	0.099	0.095	−0.121	−0.128	0.094	−0.134	−0.039
2.5	0.939	0.948	0.006	0.005	0.005	−0.004	0.902	0.099	0.093	−0.122	−0.13	0.093	−0.136	−0.042
2.75	0.934	0.943	0.006	0.005	0.005	−0.004	0.892	0.1	0.094	−0.118	−0.127	0.094	−0.131	−0.037
3	0.928	0.938	0.006	0.005	0.005	−0.004	0.89	0.098	0.092	−0.119	−0.128	0.092	−0.132	−0.04
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	0.839	0.861	0.005	0.005	0.004	−0.004	0.76	0.065	0.056	−0.074	−0.088	0.056	−0.082	−0.026
7.25	0.834	0.856	0.005	0.005	0.004	−0.004	0.758	0.061	0.052	−0.071	−0.086	0.052	−0.079	−0.027
7.5	0.829	0.852	0.005	0.005	0.004	−0.003	0.75	0.057	0.049	−0.065	−0.079	0.049	−0.073	−0.024
7.75	0.824	0.847	0.005	0.005	0.004	−0.003	0.735	0.054	0.046	−0.062	−0.076	0.046	−0.069	−0.023
8	0.819	0.842	0.005	0.005	0.003	−0.003	0.725	0.05	0.042	−0.058	−0.071	0.042	−0.065	−0.022
8.25	0.814	0.838	0.005	0.005	0.003	−0.003	0.713	0.046	0.039	−0.052	−0.064	0.038	−0.058	−0.02
8.5	0.809	0.833	0.005	0.004	0.003	−0.003	0.712	0.043	0.036	−0.047	−0.058	0.036	−0.052	−0.016
8.75	0.804	0.829	0.005	0.004	0.003	−0.002	0.703	0.039	0.032	−0.042	−0.053	0.032	−0.047	−0.015
9	0.799	0.825	0.005	0.004	0.002	−0.002	0.695	0.033	0.027	−0.037	−0.046	0.027	−0.041	−0.014
9.25	0.794	0.82	0.005	0.004	0.002	−0.002	0.685	0.028	0.023	−0.031	−0.039	0.023	−0.034	−0.012
9.5	0.789	0.816	0.005	0.004	0.002	−0.001	0.682	0.022	0.018	−0.024	−0.031	0.018	−0.027	−0.009
9.75	0.784	0.811	0.005	0.004	0.001	−0.001	0.671	0.015	0.012	−0.017	−0.022	0.012	−0.019	−0.007
10	0.779	0.807	0.005	0.004	0.001	−0.001	0.665	0.008	0.006	−0.009	−0.012	0.006	−0.01	−0.004

represented by (2) if the market value is positive, and respectively (3) if it is negative. Taking the expected value with respect to the risk neutral measure $\mathbb{Q}$ of Equation (8.1) we find

$$V_B^{\text{risky}}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[P_B^{\text{risky}}(t,T)\right]$$

This equation can be simplified such that it reveals further information about what is going on. We find

$$V_B^{\text{risky}}(t,T) = V(t,T) - \mathbb{E}_{\mathbb{Q}}\left[(1-R)1_{\{\tau_B \leq T\}} \max\left(V(\tau_B,T),0\right)\right] \tag{8.2}$$

The second part of Equation (8.2) is the credit value adjustment. We thus define

$$\text{CVA}(t,T) = \mathbb{E}_{\mathbb{Q}}\left[(1-R)1_{\{\tau_B \leq T\}} \max\left(V(\tau_B,T),0\right)\right] \tag{8.3}$$

$$= \mathbb{E}_{\mathbb{Q}}\left[(1-R)1_{\{\tau_B \leq T\}} \max\left(V(\tau_B,T),0\right)\frac{D(t)}{D(\tau))}\right] \tag{8.4}$$

Using further simplifying assumptions such as independence of the recovery from all other variables and denying wrong way risk we could approximate the Equations (8.3) and (8.4) by

$$(1-\bar{R}_B)\mathbb{E}_{\mathbb{Q}}\left[\int_t^T \underbrace{V(u,T)\frac{D(t)}{D(u)}}_{EE_d(u,T)} dF_B(t,u)\right] \tag{8.5}$$

Here $\bar{R}_B$ is the expected recovery value for party B and F_B the cumulative distribution of the default time τ_B. Taking a set of discrete time points $T_1, T_2, \ldots, T_N = T$ the discrete approximation of the credit value adjustment Equation (8.5) is

$$CVA(t,T) \approx \text{CVA}(t,T) \approx (1-\bar{R}_B)\sum_{i=1}^{N} \text{EE}_d(t,T_i)\left(F_B(t,T_i) - F_B(t,T_{i-1})\right) \tag{8.6}$$

Equation (8.6) is a handy expression for calculating the credit value adjustment. We remark that it is not as easy as it seems at first sight. This is due to the fact that we have to revalue the whole netting set to get the CVA when a new trade is done. Furthermore, we need the exposure at several future time points, thus, we face the problem of running a simulation within a simulation. To overcome this difficulty advanced simulation methods have to be set up or the whole calculation has to be simplified. This can for instance be done by only taking into account benchmark instruments where pricing can be done using (semi-) analytic approaches.

8.3 Bilateral CVA (BCVA) and debit value adjustment (DVA)

In the last section we considered the case of two parties having agreed to some netting agreement and one of the parties defaults. Now, we wish also to take into account the default of party A. To this end two other scenarios can occur. We have

- Parties A and B do not default before T
- Party B defaults and party A does not default before T
- Party A defaults and party B does not default before T
- Parties A and B do not default before T

Let us denote the default time of party A by τ_A and denote $\tau := \tau_A \wedge \tau_B$, then, it can be shown that the *bilateral credit value adjustment* or *BCVA* for short is given by

$$\begin{aligned} \mathrm{BCVA}(t,T) = \mathbb{E}\big[&(1-R_B)\max(V(\tau,T),0)1_{\{\tau\le T\}}1_{\{\tau=\tau_B\}} \\ &+(1-R_A)\max(V(\tau,T),0)1_{\{\tau\le T\}}1_{\{\tau=\tau_A\}}\big] \end{aligned} \tag{8.7}$$

The derivation is analogous to the derivation for the case of a single parties default.

The discrete approximation using a set of time points $T_1, T_2, \ldots, T_N = T$ of Equation 8.7 is

$$\begin{aligned} \mathrm{BCVA}(t,T) \approx (1-\bar{R}_B)&\sum_{i=1}^{N}\mathrm{EE}(T_i)F_A^C(0,T_{i-1})(F_B(T_i)-F_B(T_{i-1})) \\ +(1-\bar{R}_A)&\sum_{i=1}^{N}\mathrm{NEE}(T_i)F_B^C(0,T_{i-1})(F_A(T_i)-F_A(T_{i-1})) \end{aligned} \tag{8.8}$$

From the derivation and the Equations (8.7) and (8.8) we find three interesting facts. Risky derivatives can be worth more than a risk free one. This is true because of the second term in the equations. This term can be significantly large making the overall adjustment positive. Furthermore, assuming that there are many parties the overall counterparty risk in this market is 0. In our two party case the formula applied is symmetric. Thus, seen from the point of view of the other party the adjustment is of the same absolute value but with a different sign. Thus, taking the sum the adjustments cancel out. This can easily be generalized to a finite number of parties. Lastly, it seems possible that risk mitigation techniques regularly applied increase BCVA.

However, it is observed in everyday market practice that the market considers BCVA, respectively DVA as part of the derivative pricing methodology. Another feature which seems to make DVA necessary is to agree on a price since taking only CVA into account would lead to different prices, one seen from perspective of party A and one from party B. Nevertheless, there might be other issues explaining the necessity of some offsetting term for CVA and participants have to observe what the general market practice will be in the future.

Having introduced the bilateral credit value adjustment, then, the *debit value adjustment* or *DVA* for short is nothing but the amount representing the negative contribution and, thus, using the NEE instead of EE. Thus, we have

$$\mathrm{DVA}(t,T) = \mathbb{E}\left[(1-R_A)\max\left(V(\tau,T),0\right)1_{\{\tau\leq T\}}1_{\{\tau=\tau_A\}}\right] \tag{8.9}$$

The discrete approximation is then

$$\mathrm{DVA}(t,T) \approx \mathrm{BCVA}(t,T) - (1-\bar{R}_B)\sum_{i=1}^{N}\mathrm{EE}(T_i)F_A^C(0,T_{i-1})(F_B(T_i)-F_B(T_{i-1})) \tag{8.10}$$

Thus, Equations (8.9) and (8.10) identify the second term of Equations (8.7) as the debit value adjustment. Finally we wish to illustrate the effect of collateralization on the CVA. The collateral management process is determined by many parameters, for instance the remargining period. To illustrate the effect on CVA we took the exposure for a fixed against float interest rate swap and together with this exposure we also simulated the value of the collateral. The effects of partially offsetting the exposure by the collateral at a pre-defined period and varying this period are illustrated in Figure 8.2. This shows that it is possible to partly manage the exposure by rearranging the mechanism of collateralization.

8.4 The funding value adjustment (FVA)

There are un-collateralized trades and trades hedged with different collateral agreements or non-hedged (collateralized) trades. Due to the mismatch of the collateral a desk faces different funding requirements for posting collateral. To reflect the risk of such funding requirements we consider further adjustments. Such adjustments can be positive as well as negative and, thus, mean costs or benefits. The funding adjustments are closely linked to exposure. Negative exposure means that we do not have to post collateral or we do not face losses due to default of the party we trade with. This leads to a funding benefit. On the other hand if we have a positive exposure we need to post collateral, respectively we lose if the counterparty defaults and thus this leads to funding costs. With $\Delta_i = T_i - T_{i-1}$

$$\mathrm{FVA}(t,T) \approx \sum_{i=1}^{N}\mathrm{EE}(T_i)F_A^C(0,T_{i-1})F_B(0,T_{i-1})\left(F_B(T_{i-1,i}) - F_L(T_{i-1},T_i)\right)\Delta_i \tag{8.11}$$

The part including F_B is called *Funding Cost Adjustment* or *FCA*. The part where the lending spread F_L is involved is called *Funding Benefit Adjustment* or *FBA* for short. The spread refers to the cost for borrowing, respectively lending of money and is the spread above the corresponding OIS rate for the same period. This is fundamentally different to the spread we used for calculating DVA. The latter uses the credit default swap spread or some other quantity related to credit.

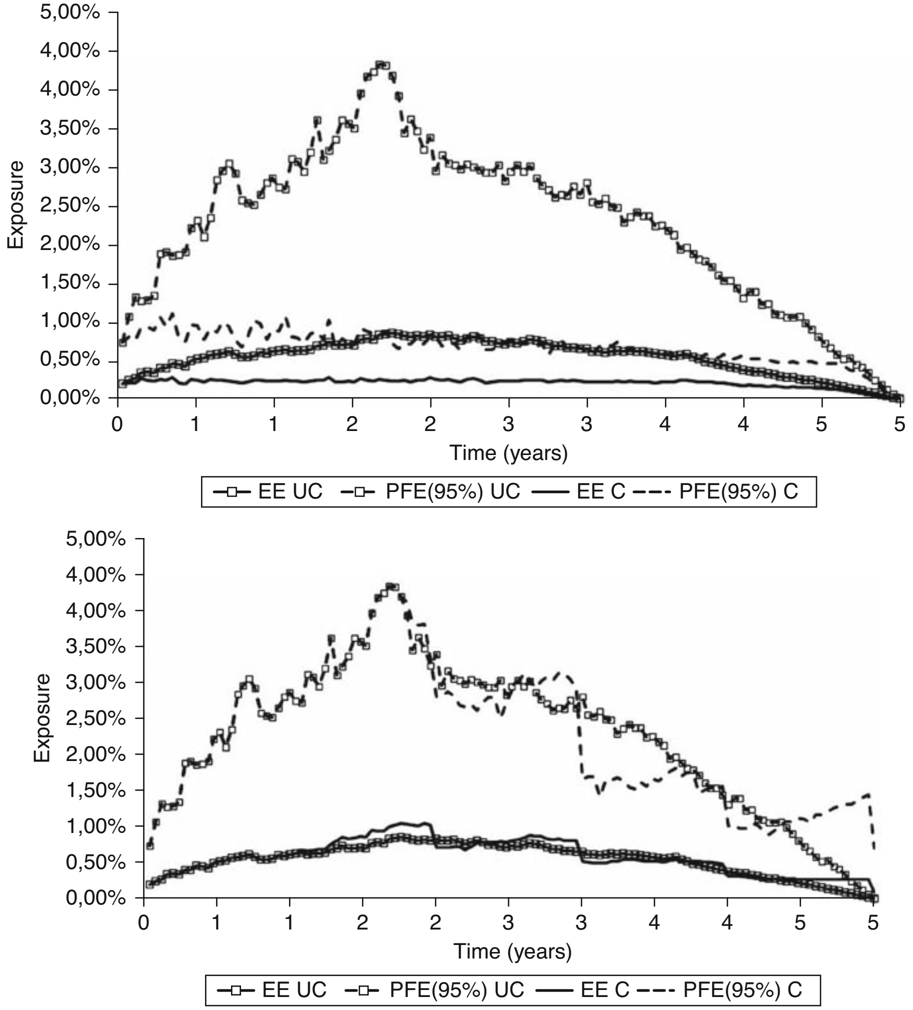


Figure 8.2 Impact of collateralization on CVA (top). Increasing the remargining period (bottom)

We can see the FVA as complementing CVA and DVA considerations since with respect to Equation (8.11) we observe that the probabilities F_A^C and F_B^C are involved. This means this is an adjustment which only contributes when there is no default.

8.5 CVA, DVA, FVA – what to do?

Finally, we ask ourselves the question of how to account for all the adjustments. Does it make sense to consider all the adjustments? There is no single answer to this question since the answer depends on the perspective. First, let us take a closer look at what we have done so far. We expressed the bilateral credit value adjustment by the equation $BCVA = CVA + DVA$. Sometimes people tend to see the second summand, DVA, as a kind of funding benefit. This is wrong for several reasons and we refer to Burgard and Kjaer (2012) for a deeper analysis. The main difference is that funding only occurs in the case of no default while the CVA and the DVA are only applicable in case of a default. Furthermore, the applied spreads for funding and credit adjustments stem from different sources. However, the problem of double counting by mixing things up is not only theoretically justified. For instance in Burgard and Kjaer (2012) the authors identify an asymmetry of unsecured borrowing and lending. The funding value adjustment can also be written as the sum of two parts. We have $FVA = FCA + FBA$. The first summand are the funding costs and the second the benefits. Thus, FBA is somehow related to DVA since it allows us to reduce the adjustment.

If we now try to answer the question from the beginning we say it depends on the view of the bank. If the bank thinks it can monetize its own credit quality it might be reasonable to include all adjustments. Also the view is reasonable that an institution cannot sell CDS protection on themselves and, thus, cannot benefit from taking DVA into account. The view that leaving out DVA but being able to realize a funding benefit is reasonable. Since regulations take CVA and DVA in a fair value approach or mark-to-market it is well defined and tractable. Funding is more problematic and several accounting issues come into play. Thus, banks have taken the viewpoint that calculating BCVA and adding a funding cost is most appropriate.

Finally, we remark that there is no single answer and several lines of reasoning are feasible. The BCVA+FCA calculation is quite popular and favored by market participants.

8.5.1 Example

To illustrate the formulas for computing the corresponding adjustments we consider the time schedule $\mathcal{T} = 0, 0.5, 1, \ldots, 5$. For all that follows we refer to Table 8.1, below. As an example we have taken the exposure calculated for a fixed against floating interest rate swap. We consider two parties A and B having a credit spread $CS_A = 150bp$, $CS_B = 100bp$ and recovery rates of $R_A = 30\%$ and $R_B = 60\%$. Starting at 0 with a survival probability of $p_0 = 1$ and assuming a constant hazard rate, the survival probability s_i (column 2) at time T_i (column 1) can be computed by $p_{i-1} \cdot \exp((T_i - T_{i-1})h)$. We have chosen the hazard rates for the two parties to be $CS/10000/(1 - R)$. This leads to the rates $h_A = 0.025$ and $h_B = 0.02143$. The probability of default in this setting is $p_i = s_i - s_{i-1}$ (column 3). The expected exposure EE_i and expected negative exposure NEE_i are assumed given and can for instance be calculated using a stochastic model (columns 6 and 7). Finally, the discount factor is also given in column 8. The CVA can then be calculated by multiplying columns 4, 6 and 8 times $(1 - R_B)$. The result is displayed in basis points. Summing up all entries leads to the CVA for the given exposure. The CVA at each time step is displayed in Figure 8.4.

Now, for the BCVA calculations we have to consider the NEE which is column 7. The first term of the BCVA from Equation (8.8) is now obtained by multiplying columns 3, 4, 6, 8 and $(1 - R_B)$ and the second by multiplying columns 2, 5, 7, 8 and $(1 - R_A)$. As we observed for the CVA summing up and adding both results gives the BCVA for the given exposure. Figure 8.3 shows the timely development of the BCVA.

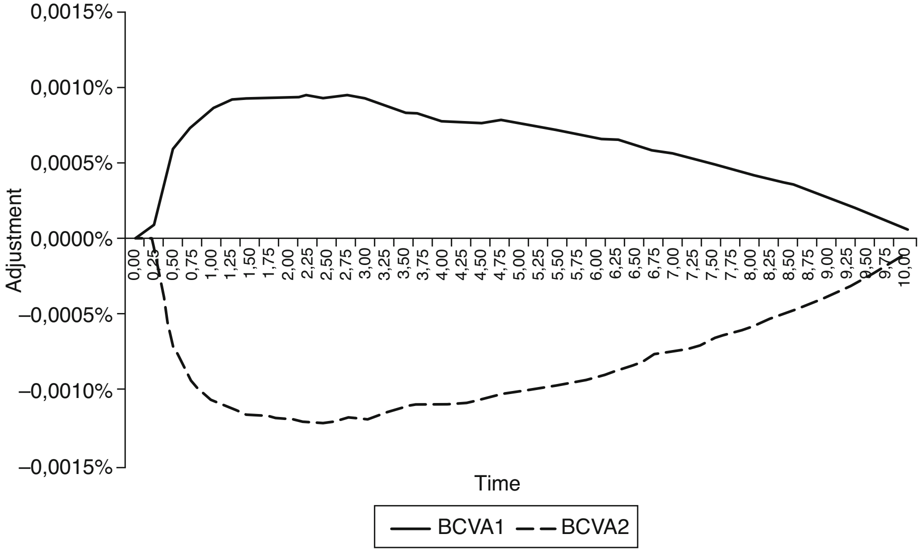


Figure 8.3 Bilateral CVA

The DVA at each time step is obtained by multiplying the columns 5, 7, 8 and $(1 - R_A)$. Summing up leads to the overall DVA for the given exposure. The DVA for all time steps is displayed in Figure 8.4.

Finally the FVA. To calculate the FCA we have to multiply columns 2, 3, 6, 8 and the time period which is 0.25 in our case. This result is then multiplied by the lending spread D_L in percent. For calculating FBA columns 2 ,3, 7, 8 and the time period have to be multiplied. This is then multiplied by the borrowing spread D_B. The FVA is simply the sum of FCA and FBA. Figure 8.5 shows the timely development of FCA, FBA and FVA. We have displayed all the results as graphs in Figures 8.3, 8.4 and 8.5.

8.6 Reading list

The literature on adjustments to account for counterparty credit risk is pretty much growing. There are many papers and books discussing whether such adjustments are necessary and how to implement algorithms to be able to calculate the adjustments. At the current stage there seems to be still considerable diversity regarding the application of adjustments. This means the valuation and management practice of derivatives trading is still in a development stage and there does not seem one unique approach but nevertheless it seems that some "best practices" are settling.

For further reading we suggest the monograph Gregory (2012). We have taken the definition of the adjustments CVA, DVA, BCVA and FVA from this resource. The author expands on the recent developments and accounts for many of the subtle aspects of counterparty credit risk and the adjustments considered in this chapter. Since a full treatment of all the peculiarities of adjustments necessary to cope with the subject is out of scope of this book we had to rely on the basic facts. Apart from

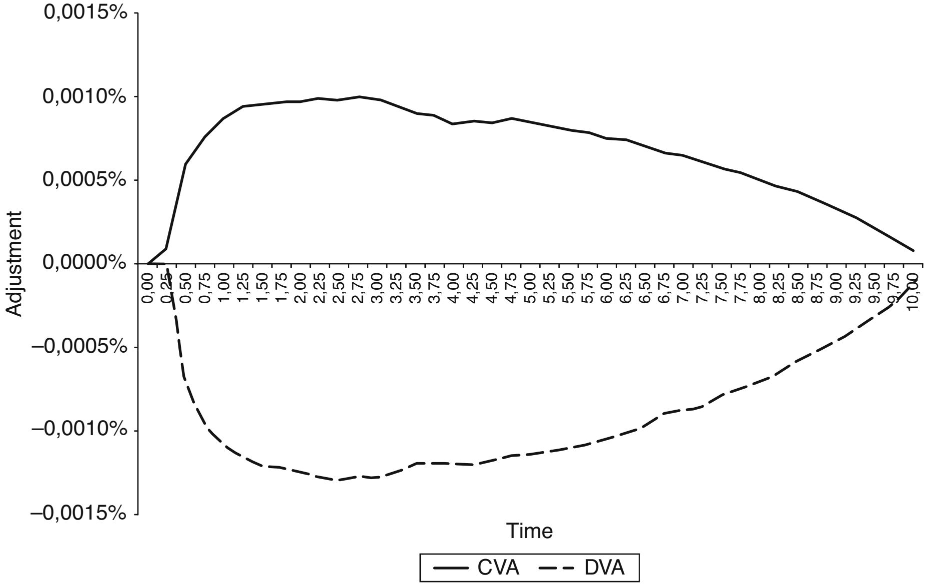


Figure 8.4 CVA and DVA

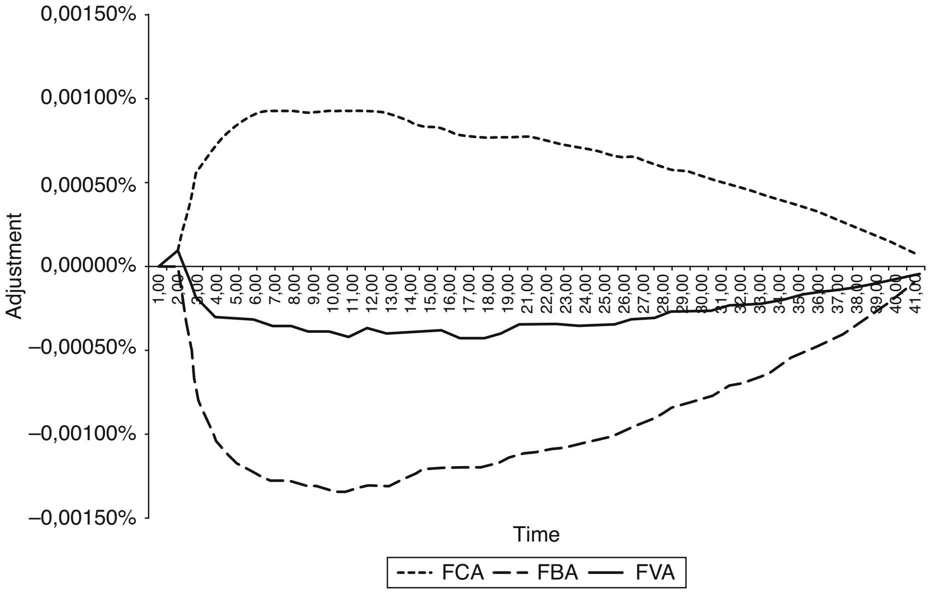


Figure 8.5 FVA example of a swap. The adjustments FCA, FBA and FVA are shown

Gregory's book, other publications include Kenyon and Stamm (2012) or Pallavicini and Perini and Brigo (2011) or one of the pioneering papers Burgard and Kjaer (2012) on the subject.

There is still an elevated debate on how to use the adjustments. Since CVA and DVA have been discussed recently this debate is about FVA. In Hull and White (2012) the authors argue that FVA is not a cost of a derivative and they earned agreement on this hypothesis but also disagreement especially from practitioners. For a view discussing the Hull and White approach with both arguments from the theoretical as well as from the practical side we suggest reading Burgard and Kjaer (2012a). For an application to the funding cost adjustment to the balance sheet we suggest Burgard and Kjaer (2012b).

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