Problem 8(a)

Srividya Bobba 33899207

Prove that there is no Universal Reg Exp (URE) using the Pumping Lemma.

Definitions

A universal regular expression over the alphabet $\Sigma = \{a, b, \cup, *, (,), \varepsilon, \emptyset, \$\}$ is a regular expression U such that:

→ for every regular expression R over the alphabet $\{a, b\}$ and every string $x \in \{a, b\}^*$,

→ R matches $x \iff U$ matches $R\$x$

→ The symbol \$ serves as a ~~sp~~ seperator between the regular expression R and the string x.

Assumption

Assume, by contradiction, that the URE, U, exists that can match $R\$x$ for any regular expression R and any string x such that:

→ $x \in L(R)$ if and only if U matches $R\$x$, where $L(R)$ is the language generated by R.

Problem 8(a) part 2

Pumping Lemma

The pumping Lemma states that if a language L is regular, then there exists a constant p (which is the pumping length) such that any string $w \in L$ with $|w| \geq p$ can be split as $w = xyz$ where:

1. $xy^iz \in L$ for any $i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

If U exists, then the language $L_U = \{R\$x \mid x \in L(R)\}$ must be regular.

Consider the regular expression $R = a^*b$

Let $R = a^*b$, which matches any string n occurrences of ~~they~~ the symbol a followed by exactly one b.

Consider the string $x = a^nb$, when $n \geq p$.

According to ~~ur~~ assumption, U must match the string $R\$x = a^*b\$a^nb$.

Since L_U is assumed to be regular, by the Pumping Lemma, the string $R\$x = a^*b\$a^nb$ can be split into three parts xyz.

Problem 8(a) part 3

Let y be a substring of $x = a^n b$ containing only a's (since $|xy| \le p$). The pumping lemma implies that we can "pump" y, meaning we can repeat the string y any number of times, and the resulting string will ~~be~~ still belong to Lu.

Let's pump y, for $i = 2$, we would get a string of the form $a^{n+|y|}b$, which does not match the regular expression $R = a^*b$, because the number of a's exceeds n, violating the constraint of R that b must occur immediately after a^n.

Thus, the pumped string $xy^2z = a^{n+|y|}b$ cannot belong to the language Lu, contradicting the Pumping Lemma, which states that the string must remain in the language after pumping.

Conclusion

This contradiction shows that the language Lu cannot be regular, meaning there is no regular expression U that can act as a URE. Therefore, no URE exists.

Problem 8(b)

Prove that there is no Universal Context-Free Grammar (UCFG) using Pumping lemma

Definitions

A UCFG over the alphabet $\Sigma = \{a, b, S, T, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \rightarrow, \varepsilon, ;, \$\}$ is a context-free grammar U such that:

→ for every context-free grammar G over the alphabet $\{a, b\}$ and every string $x \in \{a, b\}^*$,

- G generates $x \Leftrightarrow$ U generates $G\$x$

→ the symbol \$ serves as a seperator between the encoded context-free grammar G and the string x.

Assumption

Assume, by contradiction, that a UCFG, U, exists. This UCFG would generate the language:

$$Lu = \{G\$x \mid x \in L(G)\},$$

where G is a context-free grammar and $L(G)$ is the language generated by G.

Problem 8(b) part 2

The pumping lemma for context-free languages states that if a language L is context-free, then there exists a constant p (pumping length) such that any string $z \in L$ with $|z| \geq p$ can be split into 5 parts $z = uvwxy$ where:

1. $uv^iwx^iy \in L$ for any $i \geq 0$,
2. $|vx| > 0$ (atleast one of v or x is non-empty)
3. $|vwx| \leq p$

Let us consider a ~~simple~~ context-free grammar G that generates palindromes over the alphabet $\{a, b\}$:

$$G = \{ S \to aSa \mid bSb \mid \varepsilon \}.$$

this means that G generates all strings that read the same forward and backwards over $\{a, b\}$.

~~Consider the string $x = a^n b^n$~~

According to assumption, the UCF-G, U, must be able to distinguish between strings that belong to the language G and those that don't. particularly:

→ U should generate $G\$y$ for any palindrome y
→ U should not generate for any non-palindrome x

Problem 8(b) part 3

Consider the string $z = G\$a^nb^n$, which is in assumed language L_u (since $x = a^nb^n \notin L(G)$, U should reject this string).

Assuming L_u is context-free, apply pumping lemma for the string z. The pumping lemma guarantees that we can "pump" the parts v and x, meaning that for any $i \geq 0$, the string uv^iwx^iy must still belong to L_u.

when v and/or x string as pumped:

→ string $z = G\$a^nb^n$ contains specific encoding of G followed by non-palindromic string (a^nb^n)

→ pumping will either increase or decrease number of a's or b's in string $x = a^nb^n$. This results in strings like $a^{n+k}b^n$ or a^nb^{n+k}, for some $k \geq 0$.

However, none of these pumped strings will be valid palindromes, meaning none of these strings belong to the language generated by G. More specifically, U, should reject these strings, but pumping lemma implies that uv^iwx^iy must still be in L_u, which leads to a contradiction.

Problem 8(b) part 4.

Conclusion

The contradiction shows that the language L_{ii} cannot be context-free. Therefore, no UCFG exists.