Notes

A Preprint

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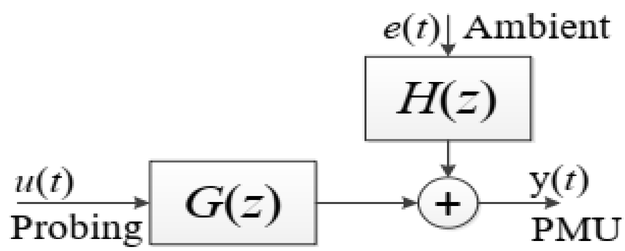
Abstract

This document provides background information on different topics related to the project.

1 Problem Formulation

We are considering the setup as defined in Fig. 1.



In this setting we have G defined as the true plant and H as the true noise model. The probing signal $u(t)$ can be chosen by the user and the measurement $y(t)$ can be used for this. The signal $e(t)$ is ambient noise and unknown except that is is assumed to be white Gaussian.

We want to solve the following optimization problem:

$$\begin{aligned} \min_{\Phi_u(i\omega)} \quad & \underbrace{\frac{c_1}{2\pi}\int_{-\pi}^{\pi}\Phi_u(i\omega)\mathrm{d}\omega}_{J_1} + \underbrace{\frac{c_2}{2\pi}\int_{-\pi}^{\pi}|G(i\omega)|\Phi_u(i\omega)\mathrm{d}\omega}_{J_2}, \\ \text{s.t.} \quad & \text{variance}(\zeta_i) < \varepsilon. \end{aligned} \tag{1}$$

The first term in the cost J_1 represents the power in the probing signal while he second term in the cost J_2 represents the deviation of the measured signal due to probing. The scalars c_1, c_2, ε are variables that can be chosen by he user. The constraint variance$(\zeta_i) < \varepsilon$ means that there is an upperbound on the variance of the damping coefficient of mode $i = 1, 2, \ldots, l$. Solving the problem defined in (1) results in a $\Phi_u^*(i\omega)$, the optimal spectrum of the probing signal that needs to be transformed to a time domain signal and consequently needs to be applied in the following system identification batch.

Define the class of input signals as:

$$u(t) = \sum_{r=1}^{M} A_r \cos(\omega_r t + \phi_r), \qquad \Phi_u(i\omega) = \frac{\pi}{2}\sum_{r=1}^{M} A_r^2\Big(\delta(\omega - \omega_r) + \delta(\omega + \omega_r)\Big). \tag{2}$$

The problem given in (1) can now be rewritten as:

$$\begin{aligned}\min_{A_r^2(r=1,2,,\ldots,,M)} \qquad & \frac{c_1}{2}\sum_{r=1}^{M} A_r^2 + \frac{c_2}{2}\sum_{r=1}^{M} A_r^2 |G_0(i\omega_r,\rho_0)|^2, \\ \text{s.t.} \qquad & \text{variance}(\zeta_i) < \varepsilon,\end{aligned} \tag{3}$$

with $G_0(i\omega_r, \rho_0)$ the true system and ρ_0 the true parameter vector that contains, among others, the damping coefficients of the critical modes. Obviously both of these are unknown, but the idea is to get estimations via system identification. The information matrix of the parameter vector ρ is defined as:

$$\begin{aligned}P_\rho^{-1} &= \left(\frac{N}{2\sigma_e^2}\sum_{r=1}^{M} \text{Re}\left\{F_u(i\omega_r,\rho_0)F_u^*(i\omega_r,\rho_0)\right\} A_r^2\right) + \left(\frac{N}{2\pi}\int_{-\pi}^{\pi} F_e(i\omega,\rho_0)F_e^*(i\omega,\rho_0)\mathrm{d}\omega\right), \\ &= \frac{N}{\sigma_e^2}\sum_{r=1}^{M} R_r A_r^2 + NX\end{aligned} \tag{4}$$

where:

- N is the number of data points used in the optimization,
- σ_e^2 is the variance of the ambient noise,
- P_ρ the covariance matrix of the parameter vector,
- X is the solution of the Lyapunov equation $X = CXC^T + DD^T$ with A, D state-space matrices of a realization of $F_e(i\omega)$.
- and the functions $F_u(i\omega, \rho_0)$ and $F_e(i\omega, \rho_0)$ are defined as:

$$\begin{aligned}F_u(i\omega_r,\rho_0) &= H^{-1}(i\omega_r,\rho)\left.\frac{\partial G(i\omega_r,\rho)}{\partial\rho}\right|_{\rho=\rho_0} \\ F_e(i\omega,\rho_0) &= H^{-1}(i\omega,\rho)\left.\frac{\partial H(i\omega,\rho)}{\partial\rho}\right|_{\rho=\rho_0}.\end{aligned} \tag{5}$$

Note that

$$P_\rho(i,i) = \text{var}(\zeta_i), \qquad \text{for} \quad i = 1,2,\ldots,n, \tag{6}$$

with n the number of critical nodes. We use the above to get the following LMI:

$$\begin{pmatrix} \varepsilon & e_i^T \\ e_i & P_\rho^{-1}\end{pmatrix} > 0, \tag{7}$$

with e_i a unit vector with one on its i^{th} element (corresponding to the i^{th} damping coefficient). This LMI defines a bound ε on the variance of the damping coefficient.

This leads to the final optimization problem:

$$\begin{aligned}\min_{A_r^2(r=1,2,,\ldots,,M)} \qquad & \frac{c_1}{2}\sum_{r=1}^{M} A_r^2 + \frac{c_2}{2}\sum_{r=1}^{M} A_r^2 |G_0(i\omega_r,\rho_0)|^2, \\ \text{s.t.} \qquad & \begin{pmatrix} \varepsilon & e_i^T \\ e_i & P_\rho^{-1}\end{pmatrix} > 0, \quad \text{and} \quad A_r^2 \geq 0, \quad \text{for} \quad r = 1,2,\ldots,M.\end{aligned} \tag{8}$$

Indeed the true parameter vector ρ_0 will be estimated using system identification techniques. We need the following expression: P_ρ, hence we need $\hat{G}(i\omega,\hat{\rho})$, $\hat{H}(i\omega,\hat{\rho})$, $\frac{\partial\hat{G}(i\omega,\rho)}{\partial\rho}$, $\frac{\partial\hat{H}(i\omega,\rho)}{\partial\rho}$. Note that the parameter vector should contain the critical damping coefficients, which is not standard. In other words, we need to find a parameterization of $G(i\omega), H(i\omega)$ in terms of, among others, the critical damping coefficients.

2 Parameterization ARMAX in terms of the damping coefficients

The parameterization presented in (**??**) is in terms of the parameter vector θ. However, we would like to have a parameterization in terms of, among others, the damping coefficients of the ARMAX model. These damping coefficients are defined as ζ_i for $i = 1, 2, \ldots, n_a$ and hence should be contained in the complete parameter vector ρ. In other words, we want a following transfer function representation

$$\begin{aligned} Y(z) = & \underbrace{\frac{\theta_{n_a+1} z^{n_b-1} + \theta_{n_a+2} z^{n_b-2} + \cdots + \theta_{n_a+n_b}}{z^{n_a} + \theta_1 z^{n_a-1} + \cdots + \theta_{n_a}}}_{G(z,\theta)} U(z) + \ldots \\ & \ldots \underbrace{\frac{z^{n_c} + \theta_{n_a+n_b+1} z^{n_c-1} + \cdots + \theta_{n_a+n_b+n_c}}{z^{n_a} + \theta_1 z^{n_a-1} + \cdots + \theta_{n_a}}}_{H(z,\theta)} E(z), \end{aligned} \tag{9}$$

with $z \in \mathbb{C}$, parameter vector $\theta = \begin{pmatrix} \theta_1 & \theta_2 & \ldots & \theta_{n_a} & \theta_{n_a+1} & \theta_{n_a+n_b+n_c} \end{pmatrix}^T \in \mathbb{R}^{n_a+n_b+n_c}$, $Y(z), U(z), E(z)$ the z-transform of the output, input and noise, respectively. The positive scalars n_a, n_b, n_c are variables that need to be defined by the user and depend on the application under consideration. The above equation can be written as:

$$\begin{aligned} Y(z) = & \underbrace{\frac{\theta_{n_a+1} z^{n_b-1} + \theta_{n_a+2} z^{n_b-2} + \cdots + \theta_{n_a+n_b}}{\prod_{i=1, i\neq j}^{n_i} \left(z^2 - 2e^{-\zeta_i w_{n,i} h} \cos(w_{n,i}\sqrt{1-\zeta_i^2} h) z + e^{-2\zeta_i w_{n,i} h} \right) \prod_{j=1, j\neq i}^{n_r} \left(z - \operatorname{sign}(z_r) e^{-\omega_{n,j} h} \right)}}_{G(z,\theta)} U(z) + \ldots \\ & \ldots \underbrace{\frac{z^{n_c} + \theta_{n_a+n_b+1} z^{n_c-1} + \cdots + \theta_{n_a+n_b+n_c}}{\prod_{i=1, i\neq j}^{n_i} \left(z^2 - 2e^{-\zeta_i w_{n,i} h} \cos(w_{n,i}\sqrt{1-\zeta_i^2} h) z + e^{-2\zeta_i w_{n,i} h} \right) \prod_{j=1, j\neq i}^{n_r} \left(z - \operatorname{sign}(z_r) e^{-\omega_{n,j} h} \right)}}_{H(z,\theta)} E(z), \end{aligned} \tag{10}$$

with z_r the real valued pole. We assume $H(z)$ to be proper, i.e., $n_c = n_a$. Define a new parameter vector:

$$\theta = \begin{pmatrix} \omega_{n,1} & \ldots & \omega_{n,n_r} & \omega_{n,n_r+1} & \zeta_{n_r+1} & \ldots & \omega_{n,n_r+n_i} & \zeta_{n_r+n_i} & \theta_{2n_i+n_r+1} \ldots & \theta_{2n_i+n_r+n_b+n_c} \end{pmatrix}^T \in \mathbb{R}^{2n_i+n_r+n_b+n_c},$$
$$\theta = \begin{pmatrix} \rho^T & \tilde{\theta}^T \end{pmatrix}^T,$$

with n_i the number of complex pole pairs and n_r the number of real valued poles and the relation $2n_i + n_r = n_a$. Define a two row vector containing a monomial basis:

$$m(z) = \begin{pmatrix} z^{n_a} & z^{n_a-1} & \ldots & z & 1 \end{pmatrix} \in \mathbb{R}^{1 \times n_a+1} \tag{11}$$

and characteristic polynomial as:

$$p(z, \rho) = \prod_{i=1, i\neq j}^{n_i} \left(z^2 - 2e^{-\zeta_i w_{n,i} h} \cos(w_{n,i}\sqrt{1-\zeta_i^2} h) z + e^{-2\zeta_i w_{n,i} h} \right) \prod_{j=1, j\neq i}^{n_r} \left(z - \operatorname{sign}(z_r) e^{-\omega_{n,j} h} \right), \tag{12}$$

so that we can write:

$$G(z,\theta) = m(z) \underbrace{\begin{pmatrix} 0_{n_a-n_b} \\ \theta_{n_a+1} \\ \vdots \\ \theta_{n_a+n_b} \end{pmatrix}}_{B_1(\tilde{\theta})} p(z,\rho)^{-1}, \qquad H(z,\theta) = m(z) \underbrace{\begin{pmatrix} 1 \\ \theta_{n_a+n_b+1} \\ \vdots \\ \theta_{n_a+n_b+n_c} \end{pmatrix}}_{B_2(\tilde{\theta})} p(z,\rho)^{-1}. \tag{13}$$

The derivative of $G(z)$ with respect to the parameter vector is then defined as follows:

$$\frac{\partial G(z,\theta)}{\partial \theta} = \begin{pmatrix} \frac{\partial G(z,\theta)}{\partial \omega_{n,1}} \\ \vdots \\ \frac{\partial G(z,\theta)}{\partial \omega_{n,n_r}} \\ \frac{\partial G(z,\theta)}{\partial \omega_{n,n_r+1}} \\ \frac{\partial G(z,\theta)}{\partial \zeta_{n_r+1}} \\ \vdots \\ \frac{\partial G(z,\theta)}{\partial \omega_{n,n_r+n_i}} \\ \frac{\partial G(z,\theta)}{\partial \zeta_{n_r+n_i}} \\ \hline \frac{\partial G(z,\theta)}{\partial \theta_{n_a+1}} \\ \frac{\partial G(z,\theta)}{\partial \theta_{n_a+2}} \\ \vdots \\ \frac{\partial G(z,\theta)}{\partial \theta_{n_a+n_b}} \\ \hline \frac{\partial G(z,\theta)}{\partial \theta_{n_a+n_b+1}} \\ \vdots \\ \frac{\partial G(z,\theta)}{\partial \theta_{n_a+n_b+n_c}} \end{pmatrix} = \begin{pmatrix} \frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,1}} m(z) B_1(\tilde{\theta}) \\ \vdots \\ \frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r}} m(z) B_1(\tilde{\theta}) \\ \frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r+1}} m(z) B_1(\tilde{\theta}) \\ \frac{\partial p(z,\rho)^{-1}}{\partial \zeta_{n_r+1}} m(z) B_1(\tilde{\theta}) \\ \vdots \\ \frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r+n_i}} m(z) B_1(\tilde{\theta}) \\ \frac{\partial p(z,\rho)^{-1}}{\partial \zeta_{n_r+n_i}} m(z) B_1(\tilde{\theta}) \\ \hline p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+1}} \\ p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+2}} \\ \vdots \\ p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+n_b}} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,1}} m(z) B_1(\tilde{\theta}) \\ \vdots \\ -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r}} m(z) B_1(\tilde{\theta}) \\ -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r+1}} m(z) B_1(\tilde{\theta}) \\ -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \zeta_{n_r+1}} m(z) B_1(\tilde{\theta}) \\ \vdots \\ -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r+1+n_i}} m(z) B_1(\tilde{\theta}) \\ -p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \zeta_{n_r+1+n_i}} m(z) B_1(\tilde{\theta}) \\ \hline p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+1}} \\ p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+2}} \\ \vdots \\ p(z,\rho)^{-1} m(z) \frac{\partial B_1(\tilde{\theta})}{\partial \theta_{n_a+n_b}} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}$$

ε

The derivative of $H(z)$ with respect to the parameter vector is then defined as follows:

$$
\frac{\partial H(z,\theta)}{\partial \theta} = \left(\begin{array}{c}
\frac{\partial H(z,\theta)}{\partial \omega_{n,1}} \\
\vdots \\
\frac{\partial H(z,\theta)}{\partial \omega_{n,n_r}} \\
\frac{\partial H(z,\theta)}{\partial \omega_{n,n_r+1}} \\
\frac{\partial H(z,\theta)}{\partial \zeta_{n_r+1}} \\
\vdots \\
\frac{\partial H(z,\theta)}{\partial \omega_{n,n_r+n_i}} \\
\frac{\partial H(z,\theta)}{\partial \zeta_{n_r+n_i}} \\
\hline
\frac{\partial H(z,\theta)}{\partial \theta_{n_a+1}} \\
\frac{\partial H(z,\theta)}{\partial \theta_{n_a+2}} \\
\vdots \\
\frac{\partial H(z,\theta)}{\partial \theta_{n_a+n_b}} \\
\hline
\frac{\partial H(z,\theta)}{\partial \theta_{n_a+n_b+1}} \\
\vdots \\
\frac{\partial H(z,\theta)}{\partial \theta_{n_a+n_b+n_c}}
\end{array}\right) = \left(\begin{array}{c}
\frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,1}} m(z) B_2(\tilde{\theta}) \\
\vdots \\
\frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r}} m(z) B_2(\tilde{\theta}) \\
\frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r+1}} m(z) B_2(\tilde{\theta}) \\
\frac{\partial p(z,\rho)^{-1}}{\partial \zeta_{n_r+1}} m(z) B_2(\tilde{\theta}) \\
\vdots \\
\frac{\partial p(z,\rho)^{-1}}{\partial \omega_{n,n_r+n_i}} m(z) B_2(\tilde{\theta}) \\
\frac{\partial p(z,\rho)^{-1}}{\partial \zeta_{n_r+n_i}} m(z) B_2(\tilde{\theta}) \\
\hline
0 \\
\vdots \\
0 \\
\hline
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+1}} \\
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+2}} \\
\vdots \\
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+n_c}}
\end{array}\right) = \left(\begin{array}{c}
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,1}} m(z) B_2(\tilde{\theta}) \\
\vdots \\
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r}} m(z) B_2(\tilde{\theta}) \\
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r+1}} m(z) B_2(\tilde{\theta}) \\
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \zeta_{n_r+1}} m(z) B_2(\tilde{\theta}) \\
\vdots \\
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \omega_{n,n_r+1+n_i}} m(z) B_2(\tilde{\theta}) \\
-p(z,\rho)^{-2} \frac{\partial p(z,\rho)}{\partial \zeta_{n_r+1+n_i}} m(z) B_2(\tilde{\theta}) \\
\hline
0 \\
\vdots \\
0 \\
\hline
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+1}} \\
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+2}} \\
\vdots \\
p(z,\rho)^{-1} m(z) \frac{\partial B_2(\tilde{\theta})}{\partial \theta_{n_a+n_b+n_c}}
\end{array}\right)
$$

with

$$
\frac{\partial p(z,\rho)}{\partial \omega_{n,i}} = \begin{cases} \mathrm{sign}(z_r) h e^{-\omega_{n,i} h}, & \text{for} \quad i = 1, \ldots, n_r, \\ f_1(\rho), & \text{for} \quad i = n_i + 1, \ldots, n_i + n_r, \end{cases} \tag{14}
$$

and

$$
\frac{\partial p(z,\rho)}{\partial \zeta_i} = f_2(\rho), \quad \text{for} \quad i = n_i + 1, \ldots, n_i + n_r, \tag{15}
$$

A Appendix