

The Uniform (Co-Irreducible) Dimension of Rings and Modules

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Contents

0 Generalities	1
0.1 Semisimple modules	1
0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings	2
0.3 Finite length modules	3
0.4 The Jacobson radical	5
0.5 Semisimple rings	6
1 Essential submodules	7
2 Injective Modules	13
2.1 Injective Module	13
2.2 Injective envelopes	20
3 Direct Sums of Uniform (Co-irreducible) Modules	25
4 Ore domains and Goldie rings	33
5 The Osofsky–Smith Theorem	39
6 PCQI Rings	45

Chapter 0

Generalities

0.1 Semisimple modules

Definition 0.1.1. A non-zero R -module S is called *simple* if its only submodules are 0 and S .

Proposition 0.1.2. Let S be an R -module. The following statements are equivalent:

1. S is simple;
2. for every non-zero element $x \in S$ we have $S = xR$;
3. $S \cong R/I$, where I is a maximal right ideal.

Lemma 0.1.3 (Schur). Let S and S' be simple R -modules and let $f: S \rightarrow S'$ be a homomorphism of R -modules. Then $f = 0$ or f is an isomorphism. In particular $\text{End}_R(S)$ is a division ring.

Definition 0.1.4. Let M be an R -module and let $(S_i)_{i \in I}$ be the family of all simple submodules of M . If $M = \sum_{i \in I} S_i$, then M is called *semisimple*.

Proposition 0.1.5. Let M be a semisimple R -module and N a submodule of M . Then there exists a subset $J \subseteq I$ such that

1. the family $(S_j)_{j \in J}$ is independent;
2. $M = N \oplus (\bigoplus_{j \in J} S_j)$.

Corollary 0.1.6. With the above notation, for the semisimple module M there exists $J \subseteq I$ such that the family $(S_j)_{j \in J}$ is independent and

$$M = \bigoplus_{i \in I} M_i.$$

Corollary 0.1.7. If M is a semisimple R -module and N a submodule of M , then both N and M/N are semisimple.

Corollary 0.1.8. A direct sum of semisimple modules is a semisimple module.

Theorem 0.1.9. Let M be an R -module. The following statements are equivalent:

1. M is semisimple;
2. M is isomorphic to a direct sum of simple modules;
3. every submodule of M is a direct summand of M ;
4. every short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits.

Definition 0.1.10. The sum of all simple submodules of M is called the *socle* of M and is denoted by $\text{soc}(M)$. If M has no simple submodule we put $\text{soc}(M) = 0$.

Proposition 0.1.11. Let M and N be R -modules and $f: M \rightarrow N$ a homomorphism. Then $f(\text{soc}(M)) \subseteq \text{soc}(N)$.

Proposition 0.1.12. Let M be an R -module and N a submodule of M . Then

$$\text{soc}(N) = \text{soc}(M) \cap N.$$

Proposition 0.1.13. If $M = \bigoplus_{i \in I} M_i$, then

$$\text{soc}(M) = \bigoplus_{i \in I} \text{soc}(M_i).$$

Proposition 0.1.14. Let R be a ring. Then the socle $\text{soc}(R_R)$ is a two-sided ideal of R .

0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings

Definition 0.2.1. Let R be a ring and M a right R -module. We say that M satisfies the *maximal condition* (resp. the *minimal condition*) if every non-empty set of submodules of M , ordered by inclusion, has a maximal (resp. minimal) element.

We say that M satisfies the *ascending* (resp. *descending*) *chain condition* if every ascending chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

(resp. every descending chain

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots$$

) is stationary, that is, there exists $n \geq 1$ such that $M_n = M_{n+1} = \cdots$.

Proposition 0.2.2. Let M be an R -module. The following statements are equivalent:

1. M satisfies the maximal (minimal) condition;
2. M satisfies the ascending (descending) chain condition.

Definition 0.2.3. An R -module M is called *noetherian* (resp. *artinian*) if it satisfies the maximal (resp. minimal) condition. The ring R is called right noetherian (resp. right artinian) if the right module R_R is noetherian (resp. artinian).

Example 0.2.4.

1. \mathbb{Z} is a noetherian ring but not artinian.
2. Every finite group is a noetherian and artinian \mathbb{Z} -module.
3. Every finite ring is noetherian and artinian.
4. The ring $\mathbb{Z}[X_1, X_2, \dots, X_n, \dots]$ is neither noetherian nor artinian:

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \dots, X_n) \subsetneq \cdots$$

$$(X_1) \supsetneq (X_1^2) \supsetneq \cdots \supsetneq (X_1^k) \supsetneq \cdots$$

5. The Prüfer p -group \mathbb{Z}_{p^∞} is an artinian but not noetherian \mathbb{Z} -module.

Proposition 0.2.5. Let N and P be submodules of M such that $M = N + P$. Then M is noetherian (artinian) if and only if N and P are noetherian (artinian).

Proposition 0.2.6. For an R -module M the following statements are equivalent:

1. M is noetherian;
2. every submodule of M is finitely generated.

Proposition 0.2.7. For an R -module M the following statements are equivalent:

1. M is artinian;
2. for every family $(X_i)_{i \in I}$ of submodules of M there exists a finite subset $J \subseteq I$ such that

$$\bigcap_{i \in I} X_i = \bigcap_{j \in J} X_j.$$

0.3 Finite length modules

Definition 0.3.1. Let M be a non-zero right R -module. A *composition series* or *Jordan–Hölder series* of M is a finite strictly ascending chain of submodules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = M$$

such that X_{i+1}/X_i is a simple module for $0 \leq i \leq n - 1$. The integer n is called the *length* of the series, and the modules X_{i+1}/X_i are called its *factors*.

Proposition 0.3.2. Let M be an R -module. The following statements are equivalent:

1. M has a composition series;
2. M is noetherian and artinian.

Proposition 0.3.3. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of right R -modules. Then M has a composition series if and only if both M' and M'' have composition series.

If

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M$$

are two composition series of M , we say that they are *equivalent* if $n = p$ and there exists a bijection $\sigma : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$ such that

$$M_{i+1}/M_i \cong M_{\sigma(i)+1}/M_{\sigma(i)} \quad (0 \leq i \leq n-1).$$

Theorem 0.3.4 (Jordan–Hölder). *If an R -module M has two composition series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M,$$

then these two series are equivalent.

Definition 0.3.5. An R -module M which admits a composition series is called a *module of finite length*. The length of its composition series is called the *length* of M and is denoted by $l(M)$. If M admits no composition series, we say that M has *infinite length* and we write $l(M) = \infty$.

Proposition 0.3.6. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R -modules of finite length. Then

$$l(M) = l(M') + l(M'').$$

Corollary 0.3.7. *Let M be an R -module of finite length and let N, L be submodules of M . Then:*

1. $l(M) = l(N) + l(M/N);$
2. $l(N + L) + l(N \cap L) = l(N) + l(L).$

Corollary 0.3.8. *Let M be an R -module of finite length and let M_1, M_2, \dots, M_n be submodules of M such that*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Then

$$l(M) = \sum_{i=1}^n l(M_i).$$

0.4 The Jacobson radical

The Jacobson radical of a module

Definition 0.4.1. Let M be an R -module. The intersection of all maximal submodules of M is called the *Jacobson radical* of M and is denoted by $\text{Rad}(M)$. If M has no maximal submodules, we adopt the convention $\text{Rad}(M) = M$.

Remark 0.4.2. If M is a finitely generated R -module, then $\text{Rad}(M) \neq M$.

Proposition 0.4.3. Let M be an R -module. Then

$$\text{Rad}(M) = \bigcap_{\substack{f:M \rightarrow S \\ S \text{ simple}}} \ker(f) = \bigcap_{\substack{f:M \rightarrow X \\ X \text{ semisimple}}} \ker(f).$$

Proposition 0.4.4. Let $f : M \rightarrow N$ be a homomorphism of R -modules. Then $f(\text{Rad}(M)) \subseteq \text{Rad}(N)$. If, in addition, f is an epimorphism and $\ker(f) \subseteq \text{Rad}(M)$, then $f(\text{Rad}(M)) = \text{Rad}(N)$.

Corollary 0.4.5. For every R -module M we have $\text{Rad}(M/\text{Rad}(M)) = 0$.

Corollary 0.4.6. If M is a semisimple R -module, then $\text{Rad}(M) = 0$.

Corollary 0.4.7. If $M = \bigoplus_{i \in I} M_i$, then

$$\text{Rad}(M) = \bigoplus_{i \in I} \text{Rad}(M_i).$$

Proposition 0.4.8. Let M be an R -module with $\text{Rad}(M) \neq M$. Then

$$\text{Rad}(M) = \bigcap \{ L \leq M \mid L \text{ is a superfluous submodule} \}.$$

Proposition 0.4.9 (Nakayama's Lemma). Let M be a finitely generated R -module and N a submodule of M . If $N + \text{Rad}(M) = M$, then $N = M$. (In other words, $\text{Rad}(M)$ is the largest superfluous submodule of M .)

The Jacobson radical of a ring

Let R be a ring. We consider the left ideal $\text{Rad}(_R R)$, the intersection of all maximal left ideals of R , and the right ideal $\text{Rad}(R_R)$, the intersection of all maximal right ideals of R .

Proposition 0.4.10.

1. $\text{Rad}(R_R)$ is a two-sided ideal.
2. $\text{Rad}(R_R) = \{ r \in R \mid 1 - ar \in U(R) \text{ for all } a \in R \}$.
3. $\text{Rad}(R_R) = \text{Rad}(_R R)$.

Definition 0.4.11. The two-sided ideal $\text{Rad}(R_R) = \text{Rad}(_R R)$ is called the *Jacobson radical* of the ring R and is denoted by $\text{Rad}(R)$.

Proposition 0.4.12.

1. If J is a left (resp. right or two-sided) ideal such that $1 - x$ is invertible for every $x \in J$, then $J \subseteq \text{Rad}(R)$.
2. If J is a nil left (resp. right or two-sided) ideal, then $J \subseteq \text{Rad}(R)$.

Proposition 0.4.13. Let $\varphi : R \rightarrow S$ be a surjective ring homomorphism. Then $\varphi(\text{Rad}(R)) \subseteq \text{Rad}(S)$. If $\ker(\varphi) \subseteq \text{Rad}(R)$, then $\varphi(\text{Rad}(R)) = \text{Rad}(S)$.

Proposition 0.4.14. If $(R_i)_{i \in I}$ is a family of rings, then

$$\text{Rad}\left(\prod_{i \in I} R_i\right) = \prod_{i \in I} \text{Rad}(R_i).$$

Proposition 0.4.15. Let M be an R -module. Then $M \text{ Rad}(R) \subseteq \text{Rad}(M)$.

Theorem 0.4.16. If R is an artinian ring, then $\text{Rad}(R)$ is nilpotent.

0.5 Semisimple rings

Theorem 0.5.1. For a ring R the following statements are equivalent:

1. every non-zero right R -module is semisimple;
2. R as a right R -module is semisimple;
3. R is artinian and $\text{Rad}(R) = 0$.

Definition 0.5.2. A ring R satisfying any (hence all) of the above conditions is called a *semisimple ring*.

Proposition 0.5.3. Let R be a semisimple ring and M a non-zero R -module. The following statements are equivalent:

1. M has finite length;
2. M is noetherian;
3. M is artinian.

Theorem 0.5.4. Let R be a right artinian ring and M a non-zero right R -module. The following statements are equivalent:

1. M has finite length;
2. M is noetherian;
3. M is artinian.

Corollary 0.5.5 (Hopkins). A right artinian ring (respectively a left artinian ring) is right noetherian (respectively left noetherian).

Chapter 1

Essential submodules

Definition 1.1. Let M be a right R -module. A submodule N of M is called *essential* (or we say that M is an essential extension of N) if $N \cap N' \neq 0$ for every non-zero submodule N' of M . In this case we write $N \trianglelefteq M_R$.

A homomorphism of right R -modules $f : M \rightarrow N$ is called *essential* if $\text{Im } f$ is an essential submodule of N (i.e. $\text{Im } f \trianglelefteq N_R$).

Example.

1. $n\mathbb{Z} \trianglelefteq \mathbb{Z}_{\mathbb{Z}}$ for every $n \geq 1$.
2. Every submodule of the Prüfer group \mathbb{Z}_{p^∞} is essential.

Remark 1.2. Let M be a right R -module and N a submodule of M . Then $N \trianglelefteq M_R$ if and only if for every $x \in M$, $x \neq 0$, there exists $r \in R$ such that $xr \in N \setminus \{0\}$.

Proof. “ \Rightarrow ” Let $x \in M \setminus \{0\}$. Since $0 \neq xR \leq M_R$ and $N \trianglelefteq M_R$, we have $xR \cap N \neq 0$; hence there exists $r \in R$ with $xr \in N \setminus \{0\}$.

“ \Leftarrow ” Let $N' \leq M_R$, $N' \neq 0$. For every $x \in N' \setminus \{0\}$ there exists $r \in R$ such that $xr \in N \setminus \{0\}$, so $N \cap N' \neq 0$. Thus $N \trianglelefteq M_R$. \square

Definition 1.3. A monomorphism of right R -modules $f : N_R \rightarrow M_R$ is called *essential* if $\text{Im } f \trianglelefteq M_R$. It is immediate that, if N is a submodule of M , then the canonical inclusion $i_N : N \rightarrow M$ is an essential monomorphism if and only if $N \trianglelefteq M_R$.

Proposition 1.4. *A monomorphism $f : N_R \rightarrow M_R$ is essential if and only if for every right R -module M' and every $g \in \text{Hom}(M, M')$, the fact that $g \circ f$ is a monomorphism implies that g is a monomorphism.*

Proof. “ \Rightarrow ” Let g be as in the statement and suppose that $g \circ f$ is a monomorphism. Assume $\text{Ker } g \neq 0$. Take $x \in \text{Ker } g \cap \text{Im } f \setminus \{0\}$. Then $x = f(x')$ for some $x' \in N$ and $g(x) = 0$, so $g(f(x')) = 0$. Since $g \circ f$ is a monomorphism, it follows that $x' = 0$ and hence $x = 0$, a contradiction.

“ \Leftarrow ” If f is not an essential monomorphism, then there exists $N' \leq M_R$, $N' \neq 0$, such that $N' \cap \text{Im } f = 0$. Consider the canonical projection $\pi_{N'} : M \rightarrow M/N'$. If $x \in \text{Ker}(\pi_{N'} \circ f)$, then $f(x) \in N'$, so $f(x) = 0$, hence $x = 0$. Thus $\pi_{N'} \circ f$ is injective. By the hypothesis in the statement, this implies that $\pi_{N'}$ is injective, so $N' = 0$, a contradiction. \square

Corollary 1.5. *Let M be a right R -module and $N \leq M_R$. Then the following statements are equivalent:*

1. $N \trianglelefteq M_R$;
2. the inclusion $i_N : N \rightarrow M$ is an essential monomorphism;
3. for every $f \in \text{Hom}(M, M')$, with M' an arbitrary R -module, the fact that $f \circ i_N$ is a monomorphism implies that f is a monomorphism.

Proposition 1.6. *Let $f : N_R \rightarrow M_R$ and $g : M_R \rightarrow P_R$ be monomorphisms. Then $g \circ f$ is essential if and only if both g and f are essential.*

Proof. “ \Leftarrow ” Let $z \in P \setminus \{0\}$. Since g is essential, there exists $r \in R$ such that $zr \in \text{Im } g \setminus \{0\}$. Thus there exists $y \in M \setminus \{0\}$ with $zr = g(y)$. As f is essential, there exists $r' \in R$ such that $yr' \in \text{Im } f \setminus \{0\}$. Hence there exists $x \in N \setminus \{0\}$ with $yr' = f(x)$. But $zr' = g(y)r' = g(yr') = g(f(x))$. If $zr' = 0$, then $g(f(x)) = 0$, hence $x = 0$, a contradiction. Therefore $zr' \in \text{Im}(g \circ f)$ and $zr' \neq 0$, which shows that $g \circ f$ is essential.

“ \Rightarrow ” Let $y \in M \setminus \{0\}$. Since g is a monomorphism, $g(y) \neq 0$. Thus there exists $r \in R$ such that $g(yr) \in \text{Im } g \setminus \{0\}$ and $g(yr) \neq 0$. Hence there exists $x \in N \setminus \{0\}$ such that $g(yr) = g(f(x))$, therefore $yr = f(x) \in \text{Im } f$, which shows that f is an essential monomorphism.

If $z \in P \setminus \{0\}$, there exists $r \in R$ such that $zr \in \text{Im}(g \circ f)$ and $zr \neq 0$. Since $\text{Im}(g \circ f) \subseteq \text{Im } g$, we obtain $zr \in \text{Im } g$, so g is essential. \square

Proposition 1.7. *Let M be a right R -module and L_1, L_2, \dots, L_n submodules of M . Then:*

- 1) $\bigcap_{i=1}^n L_i$ is essential in M iff each L_i is essential in M .
- 2) If $L_1 \subseteq L_2$ and L_1 is essential in M , then L_2 is essential in M .

The proof is obvious.

Proposition 1.8. *Let K and L be submodules of M .*

- 1) If $K \subseteq L \subseteq M$, then $K \trianglelefteq M$ iff $K \trianglelefteq L$ and $L \trianglelefteq M$.
- 2) If $h : K_R \rightarrow M_R$ is a module morphism and $L \trianglelefteq M$, then $h^{-1}(L) \trianglelefteq K$.
- 3) If $L_1, L_2 \leq M_R$ with $K_1 \trianglelefteq L_1$ and $K_2 \trianglelefteq L_2$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.

Proof.

- 1) Apply 1.5 and 1.6.
- 2) Let U be a non-zero submodule of K .
 - (i) If $h(U) = 0$, then $U \subseteq \ker h \subseteq h^{-1}(L)$, so $U \cap h^{-1}(L) \neq 0$.
 - (ii) If $h(U) \neq 0$, then $h(U) \cap L \neq 0$. Hence there exists $u \in U$ with $h(u) \in L$, $h(u) \neq 0$, so $u \in U \cap h^{-1}(L)$ and $u \neq 0$.
- Thus $h^{-1}(L) \trianglelefteq K$.
- 3) Let $0 \neq X \leq L_1 \cap L_2$. Then $X \subseteq L_1$, so $0 \neq X \cap K_1 \leq L_1$. But $X \subseteq L_2$ implies $0 \neq (X \cap K_1) \cap L_2 = X \cap (K_1 \cap K_2)$. Hence $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.

Proposition 1.9. Let $(K_\lambda)_{\lambda \in \Lambda}$, $(L_\lambda)_{\lambda \in \Lambda}$ be families of submodules of M . If (K_λ) is independent in M and $K_\lambda \trianglelefteq L_\lambda$ for all $\lambda \in \Lambda$, then (L_λ) is independent in M and

$$\left(\bigoplus_{\lambda \in \Lambda} K_\lambda \right) \trianglelefteq \left(\bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

Proof.

Let $K_1 \trianglelefteq L_1$, $K_2 \trianglelefteq L_2$ with $K_1 \cap K_2 = 0$. By 1.8(3), $0 \trianglelefteq L_1 \cap L_2$, hence $L_1 \cap L_2 = 0$.

Let $\pi_1 : L_1 \oplus L_2 \rightarrow L_1$, $\pi_2 : L_1 \oplus L_2 \rightarrow L_2$ be the canonical projections. Since $K_1 \trianglelefteq L_1$ and $K_2 \trianglelefteq L_2$,

$$\pi_1^{-1}(K_1) = K_1 \oplus 0 \trianglelefteq L_1 \oplus L_2,$$

and

$$\pi_2^{-1}(K_2) = 0 \oplus K_2 \trianglelefteq L_1 \oplus L_2.$$

Hence

$$K_1 \oplus K_2 = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \trianglelefteq L_1 \oplus L_2.$$

Induction gives the finite case. For the general case, let $0 \neq m \in \bigoplus_{\lambda \in \Lambda} L_\lambda$. Then m lies in a finite direct sum $\bigoplus_{\lambda \in \Lambda_0} L_\lambda$ for some finite $\Lambda_0 \subseteq \Lambda$. Since $(\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \trianglelefteq (\bigoplus_{\lambda \in \Lambda_0} L_\lambda)$, there exists $r \in R$ with $rm \in (\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \setminus \{0\} \subseteq (\bigoplus_{\lambda \in \Lambda} K_\lambda)$. Thus

$$\left(\bigoplus_{\lambda \in \Lambda} K_\lambda \right) \trianglelefteq \left(\bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

Proposition 1.10. Let N be a submodule of M . Then there exists a submodule Q with $N \subseteq Q \subseteq M$ such that Q is a maximal essential extension of N inside M .

Proof.

Let

$$\mathfrak{S} = \{L \leq M ; N \subseteq L \subseteq M, N \trianglelefteq L\},$$

ordered by inclusion. Clearly $\mathfrak{S} \neq \emptyset$, since $N \in \mathfrak{S}$.

Let $(L_\lambda)_{\lambda \in \Lambda}$ be a totally ordered family of elements of \mathfrak{S} and put

$$L := \bigcup_{\lambda \in \Lambda} L_\lambda.$$

Clearly $L \leq M_R$.

Let $x \in L \setminus \{0\}$. Then there exists $\lambda_0 \in \Lambda$ with $x \in L_{\lambda_0}$. Since N is essential in L_{λ_0} , there exists $r \in R$ such that $rx \in N$ and $rx \neq 0$, hence L is an essential extension of N . Thus \mathfrak{S} is inductive and, by Zorn's lemma, \mathfrak{S} admits a maximal element Q which satisfies the required conditions.

Definition 1.11. Let M be a right R -module and $N \leq M_R$. A submodule $K \leq M_R$ is called a *complement* of N in M if K is a maximal submodule of M with the property that $K \cap N = 0$. A submodule $K \leq M_R$ is called a *complement submodule* of M if there exists $N \leq M_R$ such that K is a complement of N in M .

Remark 1.12. The set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid N \cap L = 0 \}$$

is inductive and, by applying Zorn's lemma, it follows that there exists a complement of N in M . In particular, 0 and M are complement submodules of M .

Proposition 1.13. *Let M_R , $N \leq M_R$ and $K \leq M_R$, with K a complement of N in M . There exists a complement Q of K in M such that $N \subseteq Q$. Moreover, Q is a maximal essential extension of N in M .*

Proof. It is easy to see that the set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid K \cap L = 0, N \subseteq L \}$$

is inductive, and Zorn's lemma guarantees the existence of Q .

Let L be a non-zero submodule of Q such that $L \cap N = 0$. Put $K_1 = L + K$. Clearly $K \subseteq K_1$. If $x \in N \cap (L + K)$, then $x = y + z$ with $y \in L$, $z \in K$. But $z = x - y \in Q$. Since $Q \cap K = 0$, we obtain $z = 0$ and hence $x = y$. From $L \cap N = 0$ it follows that $x = y = 0$, and therefore $N \cap (L + K) = 0$, which contradicts the fact that K is a complement of N in M . Thus $L \cap N \neq 0$ for every $0 \neq L \leq Q$, so Q is an essential extension of N .

Suppose that there exists $Q' \leq M_R$ with $N \trianglelefteq Q'$ and $Q \subsetneq Q'$. Since Q' is a complement of K , we have $Q' \cap K \neq 0$. But $N \cap (Q' \cap K) = 0$ and $0 \neq Q' \cap K \leq Q'$, contradicting $N \trianglelefteq Q'$. Hence Q is a maximal essential extension of N in M . \square

Definition 1.14. A submodule N of M_R is called *closed* if N has no proper (meaning different from N) essential extension in M .

Corollary 1.15. *Let M_R be a right R -module. The complement submodules of M coincide with the closed submodules of M .*

Proof. From 1.13 it follows immediately that every closed submodule of M is a complement submodule of M .

Conversely, let K be a complement submodule of M_R . Then there exists $N \leq M_R$ such that K is a complement of N in M . Assume that K has a proper essential extension in M ; that is, there exists $K' \leq M_R$ with $K \trianglelefteq K'$ and $K \subsetneq K'$. By the maximality of K we have $K' \cap N \neq 0$, and since $K \trianglelefteq K'$, it follows that

$$K \cap K' \cap N \neq 0,$$

contradiction. \square

Corollary 1.16. *Let N be a submodule of M_R . If K is a complement of N in M , then:*

$$1. (N + K) \trianglelefteq M_R.$$

$$2. \text{The canonical morphism } \pi_K \circ i_N : N \rightarrow M/K \text{ is an essential monomorphism.}$$

Proof. (1) Let $x \in M \setminus \{0\}$. If $x \notin K$, then $K + Rx \neq K$ and hence $N \cap (K + Rx) \neq 0$. Let $y \in N \cap (K + Rx)$, $y \neq 0$. There exist $z \in K$ and $r \in R$ such that $y = z + rx$. If $rx = 0$, then $y = z$ and, since $N \cap K = 0$, we obtain $y = 0$, a contradiction. Thus $rx \neq 0$ and, because $rx = y - z$, we have $rx \in N + K$, which shows that $(N + K) \trianglelefteq M_R$.

(2) We have $\text{Im}(\pi_K \circ i_N) = (N + K)/K$. Let L/K be a non-zero submodule of M/K . Then

$$\frac{N + K}{K} \cap \frac{L}{K} = \frac{(N + K) \cap L}{K} = \frac{N \cap L + K}{K}.$$

Since K is a complement of N , we have $N \cap L \neq 0$, and hence

$$\frac{N \cap L + K}{K} \neq 0,$$

which shows that $\pi_K \circ i_N$ is an essential monomorphism. \square

Chapter 2

Injective Modules

2.1 Injective Module

Let Q and M be two right R -modules. We say that Q is M -injective if for every monomorphism $u : M' \rightarrow M$ and every morphism $f : M' \rightarrow Q$, there exists $g : M \rightarrow Q$ such that $g \circ u = f$; that is, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M \\ & & f \downarrow & \nearrow g & \\ & & Q & & \end{array}$$

is commutative.

This property is equivalent to the condition that the map

$$\text{Hom}(u, Q) : \text{Hom}(M, Q) \longrightarrow \text{Hom}(M', Q)$$

is surjective for every monomorphism $u : M' \rightarrow M$. Since the functor $\text{Hom}(-, Q)$ is left exact, it follows that Q is M -injective if and only if $\text{Hom}(-, Q)$ is exact with respect to every short exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

The R -module Q is called *quasi-injective* (or *self-injective*) if it is Q -injective. If Q is M -injective for every R -module M , then Q is called *injective*.

2.1.1 Proposition

Proposition 2.1.1. *Let Q and M be two R -modules. The following statements are equivalent:*

1. *Q is M -injective.*
2. *For every submodule N of M and every morphism $f : N \rightarrow Q$, there exists $g : M \rightarrow Q$ such that $g|_N = f$.*
3. *For every essential submodule N of M and every morphism $f : N \rightarrow Q$, there exists $g : M \rightarrow Q$ such that $g|_N = f$.*

Proof. Implications (1) \Rightarrow (2) and (2) \Rightarrow (3) are obvious.

(2) \Rightarrow (1). Let M'_R , $0 \longrightarrow M' \xrightarrow{u} M$ and $f : M' \rightarrow Q$. Then $u(M') \leq M$. Consider $i : u(M') \rightarrow M$ the canonical injection and $\bar{u} : M' \rightarrow u(M')$ the isomorphism induced by u . There exists $g : M \rightarrow Q$ such that $g \circ i = f \circ \bar{u}^{-1}$. Hence

$$g \circ i \circ \bar{u} = f \quad \text{and therefore} \quad g \circ u = f.$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{\bar{u}} & u(M') & \xhookrightarrow{i} & M \\ & & f \downarrow & & & & \\ & & Q & & \xleftarrow{g} & & \end{array}$$

(3) \Rightarrow (2). Let N be a submodule of M and K a complement of N in M . Then $(N \oplus K) \trianglelefteq M$. Define $h : N \oplus K \rightarrow Q$ by $h(n+k) = f(n)$ for all $n \in N, k \in K$. Since $N \cap K = 0$, the map h is well-defined. There exists $g : M \rightarrow Q$ such that $g|_{N \oplus K} = h$, and hence $g|_N = h|_N = f$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \hookrightarrow & N \oplus K & \hookrightarrow & M \\ & & f \downarrow & & \nearrow h & & \\ & & Q & & \xleftarrow{g} & & \end{array}$$

□

Proposition 2.1.2. *Let $(M_\alpha)_{\alpha \in \Lambda}$ be a family of R -modules and M an R -module. Then $\prod_{\alpha \in \Lambda} M_\alpha$ is M -injective if and only if each M_α is M -injective.*

Proof. Let N be a submodule of M . Put $P = \prod_{\alpha \in \Lambda} M_\alpha$ and let $\pi_\alpha : P \rightarrow M_\alpha$ be the canonical projections for all $\alpha \in \Lambda$.

" \Leftarrow " Given a morphism $f : N \rightarrow P$, the morphisms $\pi_\alpha \circ f : N \rightarrow M_\alpha$ can be extended to $g_\alpha : M \rightarrow M_\alpha$. There exists $g : M \rightarrow P$ such that $g|_N = f$.

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \hookrightarrow & M \\ & & f \downarrow & \nearrow g & \\ & & P & \xleftarrow{\pi_\alpha} & M_\alpha \\ & & \pi_\alpha \downarrow & \nearrow g_\alpha & \end{array}$$

" \Rightarrow " Let $\forall \alpha \in \Lambda$ and $f : N \rightarrow M_\alpha$. Considering the canonical inclusion $\varepsilon_\alpha : M_\alpha \rightarrow P$, since P is M -injective, there exists $g : M \rightarrow P$ which extends $\varepsilon_\alpha \circ f : N \rightarrow P$. Then $\varepsilon_\alpha : M_\alpha \rightarrow P$ extends f and hence M_α is M -injective.

$$\begin{array}{ccccc}
0 & \longrightarrow & N & \hookrightarrow & M \\
& & f \downarrow & & \swarrow \\
& & M_\alpha & \xrightarrow{g} & \\
& & \varepsilon_\alpha \downarrow & & \pi_\alpha g \\
& & P & & \\
& & \pi_\alpha \downarrow & & \\
& & M_\alpha & &
\end{array}$$

□

Corollary 2.1.3.

1. Let $(Q_\alpha)_{\alpha \in \Lambda}$ be a family of R -modules. Then $\prod_{\alpha \in \Lambda} Q_\alpha$ is injective if and only if each Q_α is injective for every $\alpha \in \Lambda$.
2. The module $Q_1 \oplus Q_2$ is an injective R -module if and only if each Q_i is injective for $i = 1, 2$. In particular, a direct summand of an injective module is injective.

Proposition 2.1.4. Let Q be an R -module.

1. If $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ is an exact sequence of R -modules and Q is M -injective, then Q is M' -injective and M'' -injective.
2. If $(M_\alpha)_{\alpha \in \Lambda}$ is a family of submodules of M such that $M = \sum_{\alpha \in \Lambda} M_\alpha$ and Q is M_α -injective for every α , then Q is M -injective.
3. Let $(N_\alpha)_{\alpha \in \Lambda}$ be a family of R -modules. Then Q is $\bigoplus_{\alpha \in \Lambda} N_\alpha$ -injective if and only if Q is N_α -injective for every $\alpha \in \Lambda$.

Proof. To show that Q is M' -injective, we consider N a submodule of M' and $\varphi : N \rightarrow Q$ a morphism of R -modules. Since Q is M -injective, there exists $\phi : M \rightarrow Q$ such that $g \circ f|_N = \varphi$, and hence $\psi \circ f : M' \rightarrow Q$ is a morphism which extends φ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & M' & \xrightarrow{f} & M \\
& & \varphi \downarrow & & \swarrow \psi & & \\
& & Q & & & &
\end{array}$$

Let $h : L \rightarrow M''$ be a monomorphism. We may assume, without loss of generality, that $M' \leq M$ and $M'' = M/M'$. Since $L \cong h(L) \leq M''$, there exist $P \leq M$, $M' \subseteq P$ such that $h(L) = P/M'$ and hence $L \cong P/M'$. We obtain the commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow h \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0
\end{array}$$

Since Q is M -injective, applying the functor $\text{Hom}(-, Q)$ we obtain the commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(M'', Q) & \longrightarrow & \text{Hom}(M, Q) & \longrightarrow & \text{Hom}(M', Q) \longrightarrow 0 \\
 & & h^* \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Hom}(L, Q) & \longrightarrow & \text{Hom}(P, Q) & \longrightarrow & \text{Hom}(M', Q) \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

We obtain that $h^* = \text{Hom}(h, Q)$ is an epimorphism, which shows that Q is M'' -injective.

2) Let N be a submodule of M and $f : N \rightarrow Q$ a morphism of R -modules. Consider the set

$$\mathfrak{S} = \{(L, h) \mid N \leq L \leq M, h : L \rightarrow Q, h|_N = f\}.$$

Since $(N, f) \in \mathfrak{S}$, we have $\mathfrak{S} \neq \emptyset$. Define on \mathfrak{S} the order relation $(L_1, h_1) \preccurlyeq (L_2, h_2)$ if and only if $L_1 \leq L_2$ and $h_2|_{L_1} = h_1$. One checks that \mathfrak{S} is inductive and, by Zorn's lemma, there exists a maximal element (L_0, g_0) of \mathfrak{S} . To show that $L_0 = M$ it is enough to prove that $M_\alpha \leq L_0$ for every $\alpha \in \Lambda$.

Consider the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & L_0 \cap M_\alpha & \xrightarrow{i_\alpha} & M_\alpha \\
 & & i_0 \downarrow & & \swarrow h_\alpha \\
 & & L_0 & & \\
 & & g_0 \downarrow & & \\
 & & Q & &
 \end{array}$$

from which it follows that there exists $h_\alpha : M_\alpha \rightarrow Q$ such that $h_\alpha \circ i_\alpha = g_0 \circ i_\alpha$. Define $h^* : L_0 + M_\alpha \rightarrow Q$ by $h^*(l + m_\alpha) = g_0(l) + h_\alpha(m_\alpha)$, for all $l \in L_0, m_\alpha \in M_\alpha$. If $l + m_\alpha = 0$, then $l = -m_\alpha \in L_0 \cap M_\alpha$ and hence $h^*(l + m_\alpha) = g_0(l) + h_\alpha(l)$, which shows that h^* is well defined. Thus $(L_0 + M_\alpha, h^*) \in \mathfrak{S}$, and since $(L_0, g_0) \preccurlyeq (L_0 + M_\alpha, h^*)$, by the maximality of (L_0, g_0) we obtain $L_0 = L_0 + M_\alpha$, i.e. $M_\alpha \leq L_0$ for every $\alpha \in \Lambda$.

3) “ \Rightarrow ” Since $N_\alpha \leq N$ and Q is N -injective, it follows that Q is N_α -injective for every $\alpha \in \Lambda$.

“ \Leftarrow ” Let $N'_\alpha = i_\alpha(N_\alpha)$. Since Q is N_α -injective and $N'_\alpha \cong N_\alpha$, we see that Q is N'_α -injective. Now apply (2). \square

Corollary 2.1.5.

1. The module $Q_1 \oplus Q_2$ is a quasi-injective R -module if and only if each Q_i is Q_j -injective for all $i, j = 1, 2$. In particular, a direct summand of a quasi-injective module is quasi-injective.

2. The module Q^n is quasi-injective over R if and only if Q is quasi-injective.

Corollary 2.1.6. Let Q and M be two R -modules. Then Q is M -injective if and only if Q is mR -injective for every $m \in M$.

Proof. The implication “ \Rightarrow ” is clear.

For “ \Leftarrow ”, since $M = \sum_{m \in M} mR$, it follows from 2.1.4(2) that Q is M -injective. \square

Theorem 2.1.7 (Baer’s criterion). For an R -module Q the following statements are equivalent:

1. Q is injective.
2. Q is R -injective.
3. For every right ideal I of R and every morphism $f : I \rightarrow Q$ there exists $x \in Q$ such that $f(a) = xa$ for all $a \in I$.

Proof. The implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Let M be an R -module and $x \in M$. Since $\varphi_x : R \rightarrow xR$, $\varphi_x(a) = xa$ for all $a \in R$, is a surjective morphism of R -modules, it follows that $R/\text{Ker } \varphi_x \cong xR$. As Q is R -injective, 2.1.4(1) implies that Q is $R/\text{Ker } \varphi_x$ -injective, and hence Q is xR -injective for every $x \in M$. Therefore, by 2.1.6 we obtain that Q is M -injective.

(2) \Rightarrow (3). Let I be a right ideal of R and $f : I \rightarrow Q$. There exists $g : R \rightarrow Q$ such that $g|_I = f$. Put $x = g(1) \in Q$. Then $f(a) = g(a) = ag(1) = xa$ for all $a \in I$.

(3) \Rightarrow (2). Suppose that for a morphism $f : I \rightarrow Q$ there exists $x \in Q$ with $f(a) = xa$ for all $a \in I$. Define $g : R \rightarrow Q$ by $g(r) = xr$ for all $r \in R$. Then clearly $g|_I = f$. \square

Definition 2.1.8.

1. An R -module Q is called *divisible* if for every $y \in Q$ and every $a \in R$ which is not a zero divisor, there exists $x \in Q$ such that $ax = y$. It is easily seen that any factor module of a divisible module is divisible.
2. A commutative integral domain is called a *PID ring* (principal ideal domain) if every ideal of it is principal.

Proposition 2.1.9.

1. Every injective module is divisible.
2. Let R be a PID.
 - (i) If Q is an R -module, then Q is injective if and only if it is divisible.
 - (ii) If I is a non-zero ideal of R , then R/I is a quasi-injective R -module. In particular, \mathbb{Z}_n is a quasi-injective \mathbb{Z} -module for every $n \geq 1$.

Proof.

1) Let Q be a divisible R -module, $y \in Q$ and $a \in R$ a non-zero divisor. Define $f : aR \rightarrow Q$ by $f(ax) = yx$ for all $x \in R$. Since a is not a zero divisor, f is well defined. By Baer's criterion there exists $x \in Q$ such that $f(\lambda) = x\lambda$ for all $\lambda \in aR$. In particular $y = f(a) = f(a \cdot 1) = xa$; hence Q is divisible.

2)(i) The implication " \Rightarrow " follows immediately from (1).

" \Leftarrow " Let Q be a divisible R -module and I a right ideal of R . Then there exists $a \in R$ with $I = aR$. Given a morphism of R -modules $f : I \rightarrow Q$, choose $x \in Q$ with $f(a) = xa$. For any $r \in R$, $f(ar) = f(a)r = xar$, so by Baer's criterion Q is injective.

2)(ii) Let $I = aR$ and $J = bR$ be non-zero ideals of R with $I \subseteq J$, and let $f : J/I \rightarrow R/I$ be a morphism of R -modules. There exists $c \in R$ such that $a = bc$. Write $f(\widehat{b}) = \widehat{x} \in R/I$. Then $xc \in I = aR$, so there is $a_1 \in R$ with $xc = aa_1$, and hence $x = ba_1$.

Define $g : R/I \rightarrow R/I$ by $g(\widehat{r}) = \widehat{a_1 r}$ for all $r \in R$. Then g is an R -module homomorphism and $g(\widehat{b}) = \widehat{x}$, so $g|_{J/I} = f$. Thus R/I is quasi-injective. In particular, for $R = \mathbb{Z}$ and $I = n\mathbb{Z}$ we obtain that \mathbb{Z}_n is a quasi-injective \mathbb{Z} -module for every $n \geq 1$. \square

Corollary 2.1.10. *An abelian group G is an injective \mathbb{Z} -module if and only if G is divisible.*

Corollary 2.1.11.

1. \mathbb{Q} and \mathbb{Z}_{p^∞} are injective \mathbb{Z} -modules.
2. Any direct sum of injective \mathbb{Z} -modules is an injective \mathbb{Z} -module.
3. Any factor group of an injective \mathbb{Z} -module is injective.

Lemma 2.1.12. *Let A, S, T be rings and let ${}_A M_S, {}_A N_T$ be bimodules. Then $\text{Hom}_A(M, N)$ has a structure of left S -module and right T -module given by*

$$(s \cdot f)(x) = f(xs), \quad (f \cdot t)(x) = f(x)t,$$

for $s \in S$, $t \in T$, $x \in M$, $f \in \text{Hom}_A(M, N)$.

Proof. Let $a, b \in A$ and $x, y \in M$. Then

$$(s \cdot f)(ax+by) = f((ax+by)s) = f((ax)s) + f((by)s) = af(xs) + bf(ys) = a(s \cdot f)(x) + b(s \cdot f)(y),$$

so $s \cdot f \in \text{Hom}_A(M, N)$. Similarly $f \cdot t \in \text{Hom}_A(M, N)$.

For $s, s' \in S$ and $f, g \in \text{Hom}_A(M, N)$ we have

$$(s \cdot (f + g))(x) = (f + g)(xs) = f(xs) + g(xs) = (s \cdot f)(x) + (s \cdot g)(x), \quad (1)$$

$$((s + s') \cdot f)(x) = f(x(s + s')) = f(xs + xs') = f(xs) + f(xs') = (s \cdot f)(x) + (s' \cdot f)(x), \quad (2)$$

$$((ss') \cdot f)(x) = f(xss') = f((xs)s') = (s' \cdot f)(xs) = (s \cdot (s' \cdot f))(x), \quad (3)$$

$$(1_S \cdot f)(x) = f(x1_S) = f(x). \quad (4)$$

From (1)–(4) we see that $\text{Hom}_A(M, N)$ is a left S -module. A similar computation shows that $\text{Hom}_A(M, N)$ is a right T -module. Moreover, $((s \cdot f) \cdot t)(x) = (s \cdot f)(xt) = f(xts)t$ and $(s \cdot (f \cdot t))(x) = f(xts)t$, so the two actions commute and $\text{Hom}_A(M, N)$ is an S – T bimodule. \square

Proposition 2.1.13 (Eckmann–Schopf). *Let Q be a divisible abelian group. Then the left R -module $\text{Hom}_{\mathbb{Z}}(R, Q)$ is injective.*

Proof. By the previous lemma, $\text{Hom}_{\mathbb{Z}}(R, Q)$ carries a structure of left R -module given by

$$(r \cdot f)(a) = f(ar), \quad \forall a, r \in R, f \in \text{Hom}_{\mathbb{Z}}(R, Q).$$

Let I be a left ideal of R and $h : I \rightarrow \text{Hom}_{\mathbb{Z}}(R, Q)$ a morphism of left R -modules. Define

$$\gamma : \mathbb{Z}I \longrightarrow \mathbb{Z}Q, \quad \gamma(a) = h(a)(1).$$

Then γ is a morphism of \mathbb{Z} -modules. Since Q is \mathbb{Z} -injective, there exists $\tilde{\gamma} : \mathbb{Z}R \rightarrow \mathbb{Z}Q$ such that $\tilde{\gamma}|_I = \gamma$. For $a \in I$ and $r \in R$ we have

$$(a \cdot \tilde{\gamma})(r) = \tilde{\gamma}(ra) = h(ra)(1) = (r \cdot h(a))(1) = h(a)(r),$$

hence $h(a) = a \cdot \tilde{\gamma}$ for all $a \in I$. By Baer's criterion, $\text{Hom}_{\mathbb{Z}}(R, Q)$ is an injective left R -module. \square

Proposition 2.1.14. *Every left R -module M can be embedded into an injective left R -module.*

Proof. There exists a free abelian group $\mathbb{Z}^{(A)}$ and a surjective \mathbb{Z} -morphism $f : \mathbb{Z}^{(A)} \rightarrow M$. Hence

$$\mathbb{Z}M \cong \mathbb{Z}^{(A)} / \ker f \subseteq \mathbb{Q}^{(A)} / \ker f,$$

and therefore there is a divisible abelian group G with $\mathbb{Z}M \subseteq \mathbb{Z}G$. Applying the functor $\text{Hom}_{\mathbb{Z}}(R, -)$ we obtain a monomorphism

$${}_R M \cong \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, G).$$

Since G is divisible, Proposition 2.1.13 implies that $\text{Hom}_{\mathbb{Z}}(R, G)$ is an injective left R -module. Thus M embeds into an injective left R -module. \square

Proposition 2.1.15. *Let Q be an R -module. Then Q is injective if and only if every short exact sequence*

$$0 \longrightarrow Q \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

splits.

Proof. “ \Rightarrow ”. Assume Q is injective. Exactness implies that f is a monomorphism. By injectivity of Q there exists $h : M \rightarrow Q$ with $hf = \text{id}_Q$, hence the sequence splits.

“ \Leftarrow ”. By Proposition 2.1.14 there is an injective R -module Q' and a monomorphism $i : Q \rightarrow Q'$. Consider the exact sequence

$$0 \longrightarrow Q \xrightarrow{i} Q' \longrightarrow Q'/i(Q) \longrightarrow 0.$$

By the hypothesis this sequence splits, so Q is a direct summand of Q' . A direct summand of an injective module is injective, therefore Q is injective. \square

2.2 Injective envelopes

Definition 2.2.1. Let M be an R -module. A pair (E, i) is called an *injective envelope* of M if E is injective and $i : M \rightarrow E$ is an essential monomorphism.

Proposition 2.2.2. Let Q be an injective R -module. Then every complement submodule of Q is a direct summand of Q .

Proof. Let K be a submodule of Q and N a complement of K in Q , that is, $K \cap N = 0$ and $K + N$ is an essential submodule of Q . Then $(K + N)/N \cong Q/N$. Define $g : (K + N)/N \rightarrow Q$ by

$$g((x + y) + N) = x, \quad x \in K, y \in N.$$

Since $K \cap N = 0$, g is well defined and injective. As Q is injective, there exists $h : Q/N \rightarrow Q$ with $h|_{(K+N)/N} = g$. Because $(K + N)/N \cong Q/N$ and g is a monomorphism, h is also a monomorphism. We have $K = \text{Im } g = h((K + N)/N) \subseteq h(Q/N)$. As K is a closed submodule, it follows that $K = h(Q/N)$. Since h is a monomorphism, $(K + N)/N = Q/N$, hence $K + N = Q$. Thus K is a direct summand of Q . \square

Theorem 2.2.3 (Eckmann–Schopf). Every R -module M has an injective envelope, unique up to isomorphism.

Proof. By Proposition 2.1.14 there exists an injective R -module Q with $M \leq Q$. Let E be a maximal essential extension of M in Q . Then E is a complement submodule of Q , and by the previous proposition E is injective. Hence (E, i) , where $i : M \hookrightarrow E$ is the inclusion, is an injective envelope of M .

For uniqueness, let (E_1, i_1) and (E_2, i_2) be two injective envelopes of M . Since E_2 is injective, there exists $f : E_1 \rightarrow E_2$ with $fi_1 = i_2$. The map i_2 is a monomorphism and i_1 is an essential monomorphism, so (using 1.4) f is a monomorphism. Thus $E_1 \cong f(E_1)$ and $E_2 = f(E_1) \oplus E_3$ for some submodule E_3 . But $i_2(M) \subseteq f(E_1)$, hence $i_2(M) \cap E_3 = 0$. Since i_2 is essential, we must have $E_3 = 0$, so $E_2 = f(E_1)$ and f is an isomorphism. \square

In practice we fix one representative of this isomorphism class and denote it by $E(M)$, with $M \trianglelefteq E(M)$.

Proposition 2.2.4. Let M be an R -module and $i : M \rightarrow Q$ a monomorphism with Q_R injective. The following statements are equivalent:

1. (Q, i) is an injective envelope of M ;
2. for every monomorphism $f : M \rightarrow Q'$ with Q' injective, there exists a monomorphism $g : Q \rightarrow Q'$ such that $gi = f$.

Proof. (1) \Rightarrow (2). Let $f : M \rightarrow Q'$ be a monomorphism with Q' injective. By injectivity of Q' there is $u : Q \rightarrow Q'$ with $ui = f$. Because i is an essential monomorphism and Q' is injective, the image $u(Q)$ is a complement of $f(M)$; by the definition of injective envelope this forces u to be a monomorphism. Set $g = u$.

(2) \Rightarrow (1). Let $(E(M), j)$ be an injective envelope of M . Applying (2) to $f = j$ we obtain a monomorphism $g : Q \rightarrow E(M)$ with $gi = j$. Since j is an essential monomorphism, it follows that i is also essential; hence (Q, i) is an injective envelope of M . \square

Proposition 2.2.5. *For any family of right R -modules M_1, M_2, \dots, M_n we have*

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

Proof. By 1.9, $\bigoplus_{i=1}^n E(M_i)$ is an essential extension of $\bigoplus_{i=1}^n M_i$. Moreover,

$$\bigoplus_{i=1}^n E(M_i) \cong \prod_{i=1}^n E(M_i),$$

so by 2.1.3 the module $\bigoplus_{i=1}^n E(M_i)$ is injective. By uniqueness of injective envelopes we obtain

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

\square

Theorem 2.2.6. *Let Q and M be two R -modules. Then Q is M -injective if and only if $f(M) \subseteq Q$ for every $f \in \text{Hom}(E(M), E(Q))$.*

Proof. “ \Rightarrow ” Let $f \in \text{Hom}(E(M), E(Q))$ and set

$$K := \{m \in M \mid f(m) \in Q\}.$$

Since Q is M -injective, there exists a morphism $\bar{f} : M \rightarrow Q$ such that $\bar{f}|_K = f|_K$. We claim that

$$Q \cap (\bar{f} - f)(M) = 0.$$

Take $x \in Q$ and $m \in M$ with $x = (\bar{f} - f)(m)$. Then

$$f(m) = \bar{f}(m) - x \in Q,$$

so $m \in K$. Hence

$$x = \bar{f}(m) - f(m) = f(m) - f(m) = 0,$$

and therefore $Q \cap (\bar{f} - f)(M) = 0$. Since Q is an essential submodule of $E(Q)$, it follows that $(\bar{f} - f)(M) = 0$. Thus $f(M) = \bar{f}(M) \subseteq Q$.

“ \Leftarrow ” As $E(Q)$ is injective, it is enough to work with $f \in \text{Hom}(M, E(Q))$. Let N be a submodule of M and $g : N \rightarrow Q$ a morphism of R -modules. Because $E(Q)$ is injective, there exists $\tilde{g} : M \rightarrow E(Q)$ such that $\tilde{g}|_N = i \circ g$, where $i : Q \rightarrow E(Q)$ is the canonical injection. By hypothesis, $\tilde{g}(M) \subseteq Q$, so identifying \tilde{g} with its corestriction to Q we obtain a morphism $h : M \rightarrow Q$ with $h|_N = g$. Therefore Q is M -injective. \square

Corollary 2.2.7. *An R -module Q is quasi-injective if and only if $f(Q) \subseteq Q$ for every $f \in \text{End}(E(Q))$.*

Theorem 2.2.8 (Matlis–Bass). *Let R be a ring. Then R is right noetherian if and only if, for every simple right R -module S_i ($i \geq 1$),*

$$Q := \bigoplus_{i=1}^{\infty} E(S_i)$$

is an injective right R -module.

Proof. “ \Rightarrow ” Let L be a right ideal of R ,

$$Q = \bigoplus_{i=1}^{\infty} E(S_i)$$

and let $f : L \rightarrow Q$ be a morphism of right R -modules. There exist elements $a_1, \dots, a_n \in L$ such that

$$L = a_1R + a_2R + \cdots + a_nR.$$

Clearly, there is $m \geq 1$ with $f(a_k) \in \bigoplus_{j=1}^m E(S_j)$ for all $k = 1, \dots, n$, hence

$$\Im f \subseteq \bigoplus_{j=1}^m E(S_j).$$

Since $\bigoplus_{j=1}^m E(S_j)$ is injective, there exists $g : R \rightarrow \bigoplus_{j=1}^m E(S_j)$ such that $g|_L = f$. Let $\bar{f} = i \circ g$, where $i : \bigoplus_{j=1}^m E(S_j) \rightarrow Q$ is the canonical injection. Then $\bar{f}|_L = f$, so Q is injective.

“ \Leftarrow ” Suppose that R is not right noetherian. Then there exists a strictly ascending chain of finitely generated right ideals:

$$L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq \cdots.$$

By Krull’s lemma, for every $n \geq 1$ there exists a maximal submodule $M_n \subsetneq L_n$ such that

$$L_{n-1} \subseteq M_n \quad \text{for all } n \geq 2.$$

Set

$$L := \bigcup_{k=1}^{\infty} L_k, \quad \pi_k : L_k \longrightarrow L_k/M_k$$

for the canonical projections, and put

$$E_k := E(L_k/M_k) \quad \text{for each } k \geq 1.$$

Then

$$E := \bigoplus_{k=1}^{\infty} E_k$$

is injective, and

$$f : L \longrightarrow E, \quad f(a) = \sum_{k=1}^{\infty} \pi_k(a)$$

is well defined. There exists an element $x \in E_1 \oplus \cdots \oplus E_n$ such that $f(a) = xa$ for all $a \in L$. It follows that $\pi_k(a) = 0$ for every $k \geq n + 1$, that is, $a \in M_k$ for all $k \geq n + 1$. Hence

$$L \subseteq M_{n+1} \subsetneq L_{n+1} \subseteq M_{n+2} \subsetneq L_{n+2} \subseteq \cdots \subseteq L,$$

a contradiction. Therefore R is right noetherian. \square

Chapter 3

Direct Sums of Uniform (Co-irreducible) Modules

Proposition 3.1. *Let M be an R -module and $E(M)$ its injective envelope. Then the following statements are equivalent:*

1. *$E(M)$ is indecomposable.*
2. *If L and K are non-zero submodules of M , then $L \cap K \neq 0$.*
3. *If $x, y \in M \setminus \{0\}$, then there exist $a, b \in R$ such that $0 \neq xa = yb$.*
4. *M is an essential extension of each of its non-zero submodules.*

Proof. (2) \Rightarrow (1). Assume that $E(M) = L' \oplus K'$ with L' and K' non-zero submodules of $E(M)$. Since $M \trianglelefteq E(M)$, we have $L = L' \cap M \neq 0$ and $K = K' \cap M \neq 0$, hence

$$L \cap K = (L' \cap K') \cap M = 0,$$

which contradicts (2).

(1) \Rightarrow (2). Assume there exist non-zero submodules $L, K \leq M$ such that $L \cap K = 0$. Then $E(L) \leq E(M)$ and $K \cap E(L) = 0$ (otherwise, since $L \subseteq E(L)$, we would have $L \cap K \neq 0$). The short exact sequence

$$0 \longrightarrow E(L) \longrightarrow E(M) \longrightarrow E(M)/E(L) \longrightarrow 0$$

splits because $E(L)$ is injective, hence $E(M) \cong E(L) \oplus E(M)/E(L)$. It follows that

$$0 = K \cap E(L) = K \cap E(M) = K,$$

a contradiction.

The equivalences (2) \Leftrightarrow (3) and (2) \Leftrightarrow (4) are straightforward. \square

Definition 3.2. An R -module M satisfying any of the equivalent conditions above is called *uniform* (or *co-irreducible*). Let M be an R -module and N a proper submodule of M . If M/N is uniform, we say that N is *irreducible in M* . Clearly, N is irreducible in M if and only if from an equality $N = P \cap Q$ with $P, Q \leq M$ it follows that $N = P$ or $N = Q$.

Example.

1. Every simple module is clearly uniform.
2. \mathbb{Z} is a uniform \mathbb{Z} -module.
3. By Proposition 3.1, an injective module is indecomposable if and only if it is uniform. In particular, if M is a uniform R -module, then $E(M)$ is uniform. Hence the \mathbb{Z} -modules \mathbb{Q} and \mathbb{Z}_{p^∞} are uniform.
4. The \mathbb{Z} -module \mathbb{Z}_n is uniform if and only if n is a power of a prime. If $n = p^k$ with $p \geq 2$ and $k \geq 1$, then $\langle p^m \rangle \subseteq \langle p^i \rangle \cap \langle p^j \rangle$ for all $i, j \in \{1, \dots, k-1\}$, where $m = \min(i, j)$, so \mathbb{Z}_{p^k} is uniform. Conversely, assume that \mathbb{Z}_n is uniform and write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ as the factorisation of n into primes. If $s \geq 2$, then

$$\langle p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} \rangle + \mathbb{Z} \quad \text{and} \quad \langle p_s^{\alpha_s} \rangle + \mathbb{Z}$$

are non-zero submodules whose intersection is zero, a contradiction. Hence $s = 1$ and n is a prime power.

Proposition 3.3. *Let M be an R -module and N a proper submodule of M . If $x \in M \setminus N$, then there exists an irreducible submodule $P \leq M$ such that $N \subseteq P$ and $x \notin P$.*

Proof. Set

$$\mathfrak{S} = \{N' \leq M \mid N \subseteq N' \text{ and } x \notin N'\}.$$

The set \mathfrak{S} , ordered by inclusion, is inductive, so by Zorn's Lemma it has a maximal element P . We show that P is irreducible in M . Assume that $P = U \cap V$ with $U, V \leq M$ and $P \subsetneq U$, $P \subsetneq V$. Since $x \notin P$, we have $x \notin U$ or $x \notin V$; suppose $x \notin U$. Then $U \in \mathfrak{S}$ and $P \subsetneq U$, contradicting the maximality of P . Hence P is irreducible. \square

Corollary 3.4. *Let M be an R -module and N a proper submodule of M . Then N is the intersection of irreducible submodules of M .*

Proof. This follows immediately from the previous proposition. \square

Proposition 3.5. *Let M be a uniform R -module. Then:*

1. Every non-zero submodule of M is uniform.
2. Every essential extension of M is uniform.

Proof. 1) This follows immediately from 3.1.

2) Let $M \trianglelefteq E$ be an essential extension of M . If E_1, E_2 are non-zero submodules of E , then $M \cap E_1 \neq 0$ and $M \cap E_2 \neq 0$, hence

$$(M \cap E_1) \cap (M \cap E_2) \neq 0,$$

and therefore $E_1 \cap E_2 \neq 0$. Thus E is uniform. \square

Lemma 3.6. *Let M be an R -module and $(M_\alpha)_{\alpha \in \Lambda}$ a family of independent submodules. If N is a submodule of M such that*

$$N \cap \left(\sum_{\alpha \in \Lambda} M_\alpha \right) \neq 0,$$

then there exists a non-zero submodule of some M_α that is isomorphic to a submodule of N .

Proof. If $\text{Card } \Lambda = 1$, the statement is clear.

If $\text{Card } \Lambda = 2$, set $P = N \cap (M_1 + M_2)$. If $N \cap M_1 = 0$, then $P \cong (P + M_1)/M_1$ and

$$(P + M_1)/M_1 \leq (M_1 + M_2)/M_1 \cong M_2.$$

Hence P is isomorphic to a submodule of M_2 . If Λ is finite, arguing by induction reduces the proof to this case.

If Λ is infinite, choose $x \in N \cap (\sum_{\alpha \in \Lambda} M_\alpha)$, $x \neq 0$. Then there exist $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$xR \cap (M_{\alpha_1} + M_{\alpha_2} + \dots + M_{\alpha_n}) \neq 0,$$

and we are reduced to the case where Λ is finite. \square

Remark 3.7. Let M be a right R -module and let Ω be the set of uniform submodules of M . It may happen that $\Omega = \emptyset$. Set

$$\mathcal{S} = \{\Omega' \subseteq \Omega \mid \sum_{N \in \Omega'} N \text{ is a direct sum}\}.$$

The partially ordered set (\mathcal{S}, \subseteq) is inductive and, by Zorn's lemma, \mathcal{S} has a maximal element Ω_0 . Put $S = \bigoplus_{N \in \Omega_0} N$. We say that S is a maximal direct sum of uniform submodules of M .

Proposition 3.8. *Let M be an R -module and S a maximal direct sum of uniform submodules of M . If N is a non-zero submodule of M , then the following statements are equivalent:*

1. $S \cap N \neq 0$;
2. N contains a uniform submodule.

Proof. Implication (1) \Rightarrow (2) follows from Lemma 3.6.

Implication (2) \Rightarrow (1) follows from the maximality of S . \square

Theorem 3.9. *Let M be an R -module and S a maximal direct sum of uniform submodules of M . There exists a submodule K , maximal among the submodules of M that contain no uniform submodule, with the properties:*

1. $S + K$ is a direct sum;
2. $(S \oplus K) \trianglelefteq M$.

Proof. 1. Let

$$\mathcal{S} = \{ L \leq M \mid \forall L' \leq L, L' \text{ is not uniform} \}.$$

Then $\mathcal{S} \neq \emptyset$ because $0 \in \mathcal{S}$ and \mathcal{S} is inductive. By Zorn's lemma, \mathcal{S} has a maximal element K . By Proposition 3.8 we have $S \cap K = 0$.

2. Suppose $N \cap (S \oplus K) = 0$, where N is a non-zero submodule of M . Then $(N+K) \cap S = 0$, and by Proposition 3.8, $N+K$ contains no uniform submodule, which contradicts the maximality of K . Hence $(S \oplus K) \trianglelefteq M$. \square

Definition 3.10. Let M be an R -module. A finite intersection of submodules

$$\bigcap_{i \in I} N_i \quad (I \text{ finite})$$

is called *reduced* if, for every $i \in I$, we have

$$\bigcap_{i \in I} N_i \neq \bigcap_{\substack{j \in I \\ j \neq i}} N_j.$$

It is clear that if we have a finite intersection $N = \bigcap_{i \in I} N_i$ of irreducible submodules, then it can always be written as a reduced intersection of irreducible submodules.

Theorem 3.11 (Kurosh–Ore Theorem). *Let M be an R -module and N a submodule of M . Assume that N has finite reduced intersections:*

$$N = N_1 \cap N_2 \cap \cdots \cap N_m = L_1 \cap L_2 \cap \cdots \cap L_n,$$

where each N_i and each L_j is an irreducible submodule of M . Then $m = n$.

Proof. Let $N'_i = N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_m$. Then $N \subseteq N'_i$ and $N = N_i \cap N'_i$. For each j , set $P_j = N_i \cap L_j$. Then

$$N \subseteq P_j \subseteq N'_i \quad \text{and} \quad P_j \subseteq L_j.$$

Hence

$$N \subseteq \bigcap_{j=1}^n P_j \subseteq \bigcap_{j=1}^n L_j = N,$$

so the intersection of the P_j is exactly N .

Since N_i is irreducible in M , it is irreducible in $N_i + N'_i$, and therefore

$$\frac{N'_i}{N'_i \cap N_i} \cong \frac{N_i + N'_i}{N_i}$$

is uniform (co-irreducible), showing that N is irreducible in N'_i .

Thus from $N = \bigcap_{j=1}^n P_j$, there exists some j with $N = P_j = N_i \cap L_j$.

Set $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$, and choose $i_1 \in I$. Then for some $j_1 \in J$,

$$N = L_{j_1} \cap \left(\bigcap_{i \neq i_1} N_i \right).$$

Since the intersection is finite we may reduce it step by step, maintaining at each step at least one L_{j_k} that does not disappear. We obtain a decreasing chain

$$J \supseteq J_1 \supseteq \dots \supseteq J_r,$$

with $r \leq m$, such that for the final set $J' \subseteq J$,

$$N = \bigcap_{j \in J'} L_j.$$

Because the intersection $L_1 \cap \dots \cap L_n$ is reduced, we must have $J' = J$. Since J' has at most r elements, $n \leq r \leq m$. A symmetric argument gives $m \leq n$. \square

Theorem 3.12. *Let M be an R -module and let $(N_i)_{i=1}^m$ and $(L_j)_{j=1}^n$ be two independent families of uniform (co-irreducible) submodules such that*

$$\bigoplus_{i=1}^m N_i \subseteq M \quad \text{and} \quad \bigoplus_{j=1}^n L_j \subseteq M$$

are essential in M . Then $m = n$.

Proof. For each k , set $N'_k = \bigoplus_{i \neq k} N_i$. Then $N_k \cap N'_k = 0$.

Consider

$$\mathfrak{S}_k = \{P \leq M \mid N_k \cap P = 0 \text{ and } N_k \subseteq P\}.$$

This set is inductive, so by Zorn's lemma it has a maximal element P_k . If $Q \subseteq Q' \subseteq M$ and $P_k \subseteq Q \subseteq Q'$, then the irreducibility of N_k forces $P_k = Q = Q'$. Thus each P_k is co-irreducible.

For any k ,

$$N = \bigoplus_{k=1}^m N_k = N_k \oplus N'_k.$$

By the modular law,

$$N \cap P_k = (N_k \oplus N'_k) \cap P_k = N_k + (N'_k \cap P_k),$$

so $N \cap P_k = N_k$. Hence

$$N \cap \bigcap_{k=1}^m P_k = \bigcap_{k=1}^m (N \cap P_k) = \bigcap_{k=1}^m N_k = 0.$$

But since N is essential in M , this implies $\bigcap_{k=1}^m P_k = 0$.

Moreover, $N_i = \bigcap_{k \neq i} P_k$, so each such intersection is reduced.

Repeating the argument for (L_j) , we obtain irreducible submodules (Q_j) with $\bigcap_j Q_j = 0$.

Applying the Kurosh–Ore theorem gives $m = n$. \square

Definition 3.13. An R -module M is said to have *finite uniform (co-irreducible) dimension* if there exists a finite independent family $(N_i)_{i=1}^n$ of uniform submodules of M such that

$$\bigoplus_{i=1}^n N_i \trianglelefteq M.$$

In this case the integer n is called the *uniform (co-irreducible) dimension* of M and we write $\dim M = n$.

Theorem 3.14. Let M be an R -module. The following statements are equivalent:

1. M has finite uniform (co-irreducible) dimension.
2. M satisfies the ascending chain condition for direct sums.

Proof. (2) \Rightarrow (1). Let \mathcal{S}' and \mathcal{S} be two finite families of independent submodules of M . We say that \mathcal{S}' is a *refinement* of \mathcal{S} if every submodule in \mathcal{S}' is contained in some submodule from \mathcal{S} . It is easy to check that refinement defines a partial order on the set of families of submodules of M . Since M satisfies (2), there exists a family \mathcal{S}_0 of independent submodules of M which is maximal with respect to this order.

Take $N \in \mathcal{S}_0$. If N is not uniform, then there exist non-zero submodules $P, Q \leq N$ with $P \cap Q = 0$. From \mathcal{S}_0 we can then construct a proper refinement of \mathcal{S}_0 , a contradiction. Hence every submodule in \mathcal{S}_0 is uniform, so M contains a uniform submodule. If L is a non-zero submodule of M , then L also satisfies (2), and the same argument shows that L contains a uniform submodule.

Now let

$$S = \bigoplus_{i \in I} N_i$$

be a maximal direct sum of uniform submodules of M and let $N \leq M$ be non-zero. Since M satisfies the ascending chain condition on direct sums, the index set I is finite. By Proposition 3.8 we have $S \cap N \neq 0$, so $S \trianglelefteq M$. By definition, M has finite uniform dimension.

(1) \Rightarrow (2). Let

$$S = \bigoplus_{i=1}^n N_i$$

be a direct sum of uniform submodules with $S \trianglelefteq M$. If N is a non-zero submodule of M , then $S \cap N \neq 0$, and by Lemma 3.6 the submodule N contains a uniform submodule.

Let $(L_j)_{j=1}^m$ be an independent family of submodules of M . From the previous paragraph, each L_j contains a uniform submodule P_j . Clearly, the family $(P_j)_{j=1}^m$ is independent. By Theorem 3.12 we deduce $m \leq n$. Thus every independent family of submodules has bounded finite length, which is equivalent to the ascending chain condition for direct sums. \square

Remark 3.15. Let M be an R -module. Then:

1. If $N \trianglelefteq M$, then M has finite uniform (co-irreducible) dimension if and only if N has finite uniform dimension, and in this case $\dim M = \dim N$.

2. If $M = \bigoplus_{i=1}^n N_i$, then M has finite uniform dimension if and only if each N_i has finite uniform dimension for all $i = 1, \dots, n$. Moreover,

$$\dim M = \sum_{i=1}^n \dim N_i.$$

Chapter 4

Ore domains and Goldie rings

Definition 4.1. An integral domain D is called a *right Ore domain* (respectively a left Ore domain) if $aD \cap bD \neq 0$ (respectively $Da \cap Db \neq 0$) for all $a, b \in D \setminus \{0\}$.

Proposition 4.2. Let D be an integral domain. Then D is a right Ore domain if and only if D contains a uniform right ideal. In particular, this condition holds whenever D satisfies the ascending chain condition on direct sums of right ideals.

Proof. “ \Rightarrow ” If D is a right Ore domain, then the right module D_D is uniform.

“ \Leftarrow ” Let X be a uniform right ideal of D and $x \in X \setminus \{0\}$. Consider the map

$$\varphi_x : D_D \longrightarrow X, \quad \varphi_x(a) = xa.$$

This is an injective homomorphism of right D -modules. Hence D_D is uniform and therefore D is a right Ore domain. \square

Example 4.3. Let K be a commutative field and y an indeterminate. Let $K(y)$ be the field of fractions of the polynomial ring $K[y]$. If x is another indeterminate, we consider $K(y)[x]$, the set of polynomials in the indeterminate x with coefficients in $K(y)$. On this set we define two operations: addition is the usual addition of polynomials, and multiplication in $K(y)[x]$ is defined by the commutation rule

$$xf(y) = f(y^2)x, \quad f \in K(y).$$

It is straightforward to check that $K(y)[x]$ with these two operations is a unital ring. We write $A = K(y)[x]$. Then A is a non-commutative integral domain.

Every element of A can be written uniquely in the form

$$P(x) = f_n(y)x^n + f_{n-1}(y)x^{n-1} + \cdots + f_1(y)x + f_0(y),$$

where n is a natural number, $f_0(y), f_1(y), \dots, f_n(y) \in K(y)$ and $f_n(y) \neq 0$. We set $\deg P(x) = n$.

If $P(x), Q(x) \in A$ with $Q(x) \neq 0$, then there exist $S(x), R(x) \in A$ such that

$$P(x) = S(x)Q(x) + R(x) \tag{*}$$

with either $R(x) = 0$ or $\deg R(x) < \deg Q(x)$. From $(*)$ it follows that A is a left principal ideal ring. Hence A satisfies the ascending chain condition on direct sums of left ideals, and therefore A is a left Ore domain.

We now show that $xA \cap yA = 0$. Let $\alpha \in xA \cap yA$. Then

$$\alpha = x(f_n(y)x^n + \cdots + f_0(y)) = yx(g_m(y)x^m + \cdots + g_0(y)),$$

so that

$$f_n(y^2)x^{n+1} + \cdots + f_0(y^2)x = y(g_m(y^2)x^{m+1} + \cdots + g_0(y^2)x),$$

and hence $m = n$. Thus

$$f_n(y^2) = yg_m(y^2), \dots, f_0(y^2) = yg_0(y^2),$$

which holds if and only if $f_n(y) = \cdots = f_1(y) = f_0(y) = 0$, and therefore $\alpha = 0$. Consequently, A is not a right Ore domain.

GOLDIE RINGS

Definition 4.4. A right (respectively left) ideal I is called a *right annihilator ideal* (respectively a left annihilator ideal) if there exists a non-empty subset X of R such that $I = \text{ann}_r(X)$ (respectively $I = \text{ann}_l(X)$).

A ring R is called a *right Goldie ring* if R satisfies the ascending chain condition on right annihilator ideals and there is no infinite direct sum of non-zero right ideals in R .

Example 4.5.

1. If R is right noetherian, then R is a right Goldie ring. In particular, \mathbb{Z} is a Goldie ring.
2. If R is an integral domain, then R is a right Goldie ring if and only if R is a right Ore domain.
3. From Example 4.3 it follows that there exist rings which are Goldie on the left but not on the right.

Proposition 4.6. *Let R be a ring. Then R satisfies the ascending chain condition on right (respectively left) annihilator ideals if and only if R satisfies the descending chain condition for left (respectively right) annihilator ideals.*

Proof. “ \Rightarrow ” Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \dots$ be a descending sequence of left annihilator ideals. For each $j \geq 1$ there exists a non-empty subset $X_j \subseteq R$ such that $I_j = \text{ann}_l(X_j)$.

We thus obtain an ascending sequence of right annihilator ideals

$$\text{ann}_r(I_1) \subseteq \text{ann}_r(I_2) \subseteq \cdots \subseteq \text{ann}_r(I_n) \subseteq \dots$$

Hence there exists $m \geq 1$ such that $\text{ann}_r(I_m) = \text{ann}_r(I_{m+k})$ for all $k \geq 1$. It follows that

$$\text{ann}_\ell(\text{ann}_r(I_m)) = \text{ann}_\ell(\text{ann}_r(I_{m+k})) \quad \text{for all } k \geq 1,$$

that is, $I_m = I_{m+k}$ for all $k \geq 1$.

Using the relations $\text{ann}_\ell(\text{ann}_r(\text{ann}_\ell(X))) = \text{ann}_\ell(X)$ and $\text{ann}_r(Y) \subseteq \text{ann}_r(X)$ whenever $X \subseteq Y$, the converse implication “ \Leftarrow ” is proved in the same way. \square

Proposition 4.7. *Let R be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If I and J are right ideals with $I \subseteq J$, then there exists an element $b \in R$ such that $bJ \neq 0$ and $bJ \cap I = 0$.*

Proof. By the preceding lemma there exists a minimal element a in the set of left annihilator ideals such that $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(I)$. Then $aJ \neq 0$; since R is semiprime we have $aJ \cdot aJ \neq 0$, so there exist $x \in J$ and $a \in aJ$ with $b = xa$ such that $xaJ \neq 0$. Clearly $bJ \neq 0$.

It remains to show that $bJ \cap I = 0$. Take $\lambda \in bJ \cap I$; then $\lambda = b\mu = xa\mu \in I$ for some $\mu \in J$. Since $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\lambda)$, we obtain

$$\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\lambda) \subseteq \text{ann}_\ell(\mu).$$

Because $a \in aJ \subseteq aR$, we have

$$xa \in a \quad \text{and hence} \quad xaJ \subseteq aJ. \quad (4.1)$$

On the other hand, $xaJ \subseteq I$, while $\lambda \in I$ and $a \subseteq \text{ann}_r(I)$; therefore

$$a \subseteq \text{ann}_\ell(\mu). \quad (4.2)$$

Since $xaJ \neq 0$, we also have

$$a \not\subseteq \text{ann}_\ell(J). \quad (4.3)$$

From (4.1), (4.2) and (4.3) we obtain $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\mu) \cap a$. If $\lambda \neq 0$, then $xa\mu \notin \text{ann}_\ell(\mu)$, and, since $xa \in a$, it follows that $xa\mu \neq 0$, contradicting the minimality of a . Thus $\lambda = 0$, and hence $bJ \cap I = 0$. \square

Corollary 4.8. *If R is a right Goldie semiprime ring, then R satisfies the descending chain condition on right annihilator ideals.*

Proof. Assume that there exists a strictly descending sequence of right annihilator ideals

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \dots$$

Then for every $n \geq 1$ we have $\text{ann}_\ell(I_n) \neq \text{ann}_\ell(I_{n+1})$. By Proposition 4.7, for each $n \geq 1$ there exists a right ideal K_n such that $K_n \subseteq I_n$ and $K_n \cap I_{n+1} \neq 0$. Hence the sum $\sum_{n \geq 1} K_n$ is an infinite direct sum of non-zero right ideals, which contradicts the right Goldie property of R . \square

Proposition 4.9. *Let R be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If $x, y \in R$ are such that xR and yR are essential in R_R , then xyR is essential.*

Proof. Let $I \leq R_R$ with $I \neq 0$. Define

$$J = \{I : x\} = \{a \in R \mid xa \in I\}.$$

Then J is a right ideal of R , and $xJ = xR \cap I \neq 0$ because $xR \leq R_R$. Since $\text{ann}_r(x) \subseteq J$, $xJ \neq 0$ and $\text{ann}_\ell(x) = 0$, we obtain

$$\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\text{ann}_r(x)).$$

By Proposition 4.7 there exists a right ideal $K \neq 0$ such that $K \subseteq J$ and $K \cap \text{ann}_r(x) = 0$.

Now set

$$L = \{K : y\} = \{a \in R \mid ya \in K\}.$$

Since $yR \trianglelefteq R_R$, we have $yL = yR \cap K \neq 0$. If $xyL = 0$, then $yL \subseteq \text{ann}_r(x)$ and hence $yL \subseteq \text{ann}_r(x) \cap K = 0$, a contradiction. Thus $xyL \neq 0$.

Because $xyL \subseteq xK \subseteq xJ \subseteq xR \cap I$, it follows that $0 \neq xyL \subseteq xyR \cap I$, showing that $xyR \trianglelefteq R_R$. \square

Proposition 4.10. *Let R be a ring which satisfies the ascending chain condition on right annihilator ideals. Then, for every $a \in R$ there exists an integer $k \geq 0$ such that*

$$\text{ann}_r(a^n) = \text{ann}_r(a^m) \quad \text{for all } n \geq 0 \text{ and } m \geq k.$$

Proof. We have the ascending chain of right annihilator ideals

$$\text{ann}_r(a) \subseteq \text{ann}_r(a^2) \subseteq \cdots \subseteq \text{ann}_r(a^n) \subseteq \cdots.$$

Let $k \geq 0$ be such that $\text{ann}_r(a^k) = \text{ann}_r(a^m)$ for every $m \geq k$. If $\lambda \in \text{ann}_r(a^n) \cap a^m R$, then $a^n \lambda = 0$ and $\lambda = a^m \mu$, hence $a^{m+n} \mu = 0$. Since $m+n \geq k$, it follows that $\mu \in \text{ann}_r(a^k)$ and thus $a^k \mu = 0$. As $m \geq k$, we may write $\lambda = a^{m-k} a^k \mu = 0$, and therefore

$$\text{ann}_r(a^n) \cap a^m R = 0.$$

\square

Corollary 4.11. *Assume the hypotheses of Proposition 4.10. If $xR \trianglelefteq R_R$, then x is a regular element of R .*

Proof. If $\text{ann}_r(x) \neq 0$, then $0 = \text{ann}_r(R) \neq \text{ann}_r(xR)$, so there exists a right ideal $I \neq 0$ such that $I \cap xR = 0$, a contradiction. Hence $\text{ann}_r(x) = 0$.

By Proposition 4.9 we have $x^n R \trianglelefteq R_R$ for every $n \geq 1$. Applying Proposition 4.10, we obtain $k \geq 0$ such that $\text{ann}_r(x^n) \cap x^m R = 0$ for all $m \geq k$. In particular $\text{ann}_r(x) = 0$, so x is a regular element of R . \square

Corollary 4.12. *Let R be a right Goldie and semiprime ring. If $\text{ann}_r(x) = 0$ for some $x \in R$, then $xR \trianglelefteq R_R$ and x is regular.*

Proof. Let I be a non-zero right ideal of R and suppose $I \cap xR = 0$. Then the sum

$$\sum_{n \geq 1} x^n I$$

is direct. Indeed, for $p = 1$ we have $x^p I \cap \sum_{n \neq p} x^n I = xI \cap x^p I$. If $y \in xI \cap x^p I$, then $y = x\lambda = x^2 \mu$ for some $\lambda, \mu \in I$. Hence $\lambda - x\mu \in \text{ann}_r(x)$ and so $\lambda = x\mu$, which implies $\lambda \in I \cap xR = 0$. Thus $\lambda = 0$ and consequently $y = 0$. Therefore $xI \cap x^2 I = 0$, and a fortiori $xI \cap x^p I = 0$ for $p \neq 1$. Hence $x^p I \cap \sum_{n \neq p} x^n I = 0$, so the sum $\sum_{n \geq 1} x^n I$ is direct, a contradiction.

Thus $xR \trianglelefteq R_R$. By Corollary 4.11 we conclude that x is a regular element of R . \square

Proposition 4.13. *Let R be a right Goldie and semiprime ring. Then:*

- (a) *Every two-sided right annihilator ideal of R contains a minimal two-sided right annihilator ideal.*
- (b) *There exists a finite direct sum of non-zero minimal two-sided right annihilator ideals of R which is essential in R .*

Proof. (a) This follows from Corollary 4.8.

(b) Let $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ be a maximal direct sum where I_1, \dots, I_n are non-zero minimal two-sided right annihilator ideals of R . Let $K \leq R_R$, $K \neq 0$ with $I \cap K = 0$. Since $KI \subseteq I \cap K = 0$, we have $K \subseteq \text{ann}_r(I)$. As R is semiprime, $I \cap \text{ann}_r(I) = 0$, hence $I \text{ann}_r(I) = 0$ and thus $\text{ann}_l(I) \subseteq \text{ann}_r(I)$, so in particular $\text{ann}_r(I) \neq 0$. Again, R being semiprime implies $I \cap \text{ann}_l(I) = 0$, where $\text{ann}_l(I)$ is a two-sided ideal. By (a) there exists a non-zero two-sided right annihilator ideal J , minimal with respect to $J \subseteq \text{ann}_r(I)$ and $J \cap I = 0$, which contradicts the maximality of I as a direct sum. Hence $I \trianglelefteq R$. \square

Proposition 4.14. *Let R be a right Goldie and prime ring. If I is an essential right ideal of R , then I contains a regular element of R .*

Proof. Choose $a \in I$ such that $\text{ann}_r(a)$ is minimal in the set $\{\text{ann}_r(x) \mid x \in I\}$. Let $J \leq R_R$, $J \neq 0$ with $aR \cap J = 0$. Since $I \trianglelefteq R_R$, we have $I \cap J \neq 0$; we may assume that J is a non-zero right ideal of R contained in I and still satisfying $aR \cap J = 0$.

Let $x \in J$. If $\lambda \in \text{ann}_r(a+x)$, then $a\lambda + x\lambda = 0$, so $x\lambda = -a\lambda \in aR \cap J$, hence $x\lambda = 0$ and consequently $a\lambda = 0$. Thus $\lambda \in \text{ann}_r(a) \cap \text{ann}_r(x)$. Since the inclusion

$$\text{ann}_r(a) \cap \text{ann}_r(x) \subseteq \text{ann}_r(a+x)$$

is obvious, we obtain $\text{ann}_r(a+x) = \text{ann}_r(a) \cap \text{ann}_r(x)$. By the minimality of $\text{ann}_r(a)$ it follows that $\text{ann}_r(a) \subseteq \text{ann}_r(x)$ and hence $x \text{ann}_r(a) = 0$, so $J \text{ann}_r(a) = 0$. As R is prime, we must have $\text{ann}_r(a) = 0$.

By Corollary 4.12 we deduce $J = 0$, a contradiction. Therefore $aR \trianglelefteq R_R$, and Corollary 4.11 implies that a is regular in R . \square

Chapter 5

The Osofsky–Smith Theorem

Definition 5.1. An R -module M is called a *CS-module* if every submodule which is a complement of M is a direct summand of M . The module M is called *CS-complete* if M/N is a CS-module for every $N \leq M$.

Remark 5.2. From 2.2.2 it follows that every injective module is a CS-module.

Proposition 5.3. *For an R -module M the following statements are equivalent:*

1. *M is a CS-module.*
2. *Every maximal essential extension of a submodule of M is a direct summand of M .*
3. *Every submodule of M is essential in a direct summand of M .*

Proof. (1) \Rightarrow (2) Let N be a submodule of M and Q a maximal essential extension of N in M (Proposition 1.13). Then Q is a closed submodule, and by 1.15 it follows that Q is a direct summand of M .

(2) \Rightarrow (3) is immediate from Proposition 1.13.

(3) \Rightarrow (1) Let K be a complement submodule of M . There exists a direct summand Q of M such that $K \trianglelefteq Q$. Since K is a closed submodule, we obtain $K = Q$, and therefore K is a direct summand of M . \square

Proposition 5.4. *Every quasi-injective module is a CS-module.*

Proof. Let M be a quasi-injective R -module and N a submodule of M . Then there exists $E_2 \leq E(M)$ such that $E(M) = E_1 \oplus E_2$, where $E_1 = E(N)$. Let $\pi_i : E(M) \rightarrow E_i$, $i = 1, 2$, be the canonical projections. Since $\pi_i(M) \leq M$ for all $i = 1, 2$ (Corollary 2.2.7), it follows that

$$M = (M \cap E_1) \oplus (M \cap E_2).$$

Clearly $N \trianglelefteq (M \cap E_1)$, so N is essential in a direct summand of M . By the previous proposition we conclude that M is a CS-module. \square

Lemma 5.5. *Let X be a cyclic CS-module with the property that $S = \text{soc}(X) \trianglelefteq X_R$, where S is not finitely generated, but every finitely generated submodule of S is a direct summand of X . If every cyclic submodule of X is a CS-module, then X/S is not a CS-module.*

Proof. Assume that X/S is a CS-module. Since S is not finitely generated, we can write

$$S = \bigoplus_{i \geq 1} S_i,$$

with $S_i \leq X$ and S_i not finitely generated for every $i \geq 1$.

For every $i \geq 1$ there exists a complement submodule D_i of X such that $S_i \trianglelefteq D_i$. Then D_i is a direct summand of X , hence D_i is cyclic for all $i \geq 1$, and therefore $S_i \subsetneq D_i$.

Since X/S is a CS-module, there exists a direct summand \overline{E} of X such that

$$\sum_{i \geq 1} \frac{D_i + S}{S} = \frac{D}{S} \trianglelefteq \overline{E},$$

where $D := \bigoplus_{i \geq 1} D_i$. Let E be a cyclic submodule of X such that $\overline{E} = (E + S)/S$. We obtain

$$\frac{D}{S} \leq \frac{E + S}{S}.$$

Since $E \cap S \leq S$ and S is semisimple, there exists $T \leq S$ with

$$S = (E \cap S) \oplus T,$$

and hence

$$E + S = (E \cap S) \oplus T.$$

Suppose there exists $i \geq 1$ such that $D_i \cap E = 0$. Considering the canonical projection $\pi : E \oplus T \rightarrow T$, we have $D_i \cap \ker \pi = 0$, which shows that $\pi|_{D_i}$ is a monomorphism and therefore D_i is semisimple. But $S_i \trianglelefteq D_i$, which implies $S_i = D_i$, a contradiction. Hence $D_i \cap E \neq 0$ for all $i \geq 1$, and consequently $S_i \cap E \neq 0$ for all $i \geq 1$.

Since $S \trianglelefteq X_R$, there exist simple modules V_i with $V_i \leq S_i \cap E$ for all $i \geq 1$. Let $V := \bigoplus_{i \geq 1} V_i$. The module E is a CS-module because E is cyclic. There exists a direct summand L of X such that $V \trianglelefteq L$. Note that L is cyclic and therefore $V \subsetneq L$. For $n \geq 1$ set

$$P_n := \left(\bigoplus_{i=1}^n D_i \right) \cap L.$$

Then

$$P_n \cap S = L \cap \left(\bigoplus_{i=1}^n S_i \right) = V \cap \left(\bigoplus_{i=1}^n S_i \right) = \bigoplus_{i=1}^n V_i,$$

and hence $P_n \cap S$ is a finitely generated submodule of S .

It follows that $P_n \cap S$ is a direct summand of X . Since $S \trianglelefteq X_R$, we have

$$P_n := \left(\bigoplus_{i=1}^n D_i \right) \cap L \leq S \quad \text{for all } n \geq 1,$$

which shows that $D \cap L \leq S$.

Using the modular distributive law we obtain

$$\frac{D}{S} \cap \frac{L+S}{S} = \frac{D \cap (L+S)}{S} = \frac{S+D \cap L}{S} = 0.$$

Since $\frac{D}{S} \trianglelefteq \frac{E+S}{S}$, it follows that $L+S = S$, hence $L \leq S$. Thus L is semisimple and, as $V \trianglelefteq L$, we get $V = L$, a contradiction. Therefore X/S is not a CS-module. \square

Theorem 5.6 (Osofsky–Smith Theorem). *Let R be a ring and M a cyclic R -module such that mR is a completely CS-module for every $m \in M$. Then M is a finite direct sum of uniform modules.*

Proof. Every indecomposable CS-module is uniform, so it is enough to show that M is a finite direct sum of indecomposable modules.

Assume that M does not admit such a decomposition. Then there exist submodules

$$M = A_1 \oplus B_1, \quad 0 \neq A_1, B_1 \leq M,$$

with B_1 not a direct sum of indecomposable modules. Hence we can write

$$B_1 = A_2 \oplus B_2, \quad 0 \neq A_2, B_2 \leq B_1,$$

where B_2 is again not a direct sum of indecomposable modules.

Iterating this construction we obtain

$$0 \neq A_i, B_i \leq M \quad (i \geq 1)$$

with

$$M = A_1 \oplus \cdots \oplus A_n \oplus B_n \quad (n \geq 1)$$

and

$$A_{n+1} \oplus A_{n+2} \oplus \cdots \leq B_n \quad (n \geq 1).$$

Since each A_i is cyclic, choose for every $i \geq 1$ a maximal submodule $C_i \leq A_i$. Set

$$C := \bigoplus_{i \geq 1} C_i, \quad P := M/C, \quad S := \bigoplus_{i \geq 1} \frac{A_i}{C_i}.$$

Then S is a countably generated semisimple submodule of P .

For every $n \geq 1$, by the modular distributive law we have

$$(A_1 \oplus \cdots \oplus A_n \oplus C) \cap (B_n + C) = C + (A_1 \oplus \cdots \oplus A_n) \cap B_n = C + \bigoplus_{k \geq n+1} C_k = C.$$

Hence

$$P = \frac{M}{C} = \frac{A_1 \oplus \cdots \oplus A_n \oplus C}{C} \oplus \frac{B_n + C}{C} \quad \text{for all } n \geq 1.$$

If $T \leq S$ is a finitely generated submodule, then for some $n \geq 1$ we have

$$T = \frac{A_1 \oplus \cdots \oplus A_n + C}{C},$$

so every finitely generated submodule of S is a direct summand of P .

Now let X be a submodule of P with $S \trianglelefteq X \leq P$ such that X is CS and every finitely generated submodule of S is a direct summand of X . If $\text{soc}(X)$ denotes the socle of X , then $S \trianglelefteq \text{soc}(X)$, and since $S \trianglelefteq X$ we get $S = \text{soc}(X)$.

Thus the hypotheses of the previous lemma are satisfied, so X/S is not a CS-module, a contradiction. \square

Corollary 5.7. *Let R be a ring such that every cyclic R -module is a CS-module. Then every cyclic R -module is a finite direct sum of uniform modules.*

Proof. Let M_R be a cyclic module, $m \in M$ and $N \leq mR$. Then the factor module mR/N is cyclic and, by hypothesis, a CS-module. By the Osofsky–Smith Theorem it follows that mR , hence M , is a finite direct sum of uniform modules. \square

Corollary 5.8. *Let N be an R -module such that every factor module of every cyclic submodule of N is injective. Then N is semisimple.*

Proof. Take $0 \neq x \in N$ and set $C = xR \leq N$. By hypothesis, every factor module of a cyclic submodule of C is injective; hence every cyclic submodule of C is completely CS. By the Osofsky–Smith Theorem we have

$$C = xR = \bigoplus_{i=1}^n U_i$$

for certain uniform submodules U_i .

Since

$$U_i \cong C / \bigoplus_{j \neq i} U_j,$$

each U_i is injective and therefore indecomposable for all $i = 1, \dots, n$ (see Proposition 3.1).

Let $y \in U_i$. Then $yR \cong yR/0$ is injective, hence the exact sequence

$$0 \longrightarrow yR \longrightarrow U_i \longrightarrow U_i/yR \longrightarrow 0$$

splits. By 2.1.15 it follows that yR is a direct summand of U_i , and thus $yR = U_i$ or $yR = 0$ for every $y \in U_i$. We obtain that U_i is simple and therefore $C = xR$ is semisimple for every $x \in N$. Hence N is semisimple. \square

Corollary 5.9 (Osofsky). *If R is a ring such that every cyclic R -module is injective, then R is semisimple Artinian.*

Proof. Observing that we can apply the previous corollary with $N = R$, we obtain that R is semisimple. \square

Corollary 5.10. *Let R be a ring such that every singular cyclic R -module is injective. Then every singular R -module is semisimple.*

Proof. Let M be a singular R -module. Submodules and factor modules of a singular module are singular. In particular, if $x \in M \setminus \{0\}$, every factor module of any cyclic submodule of xR is injective. By Corollary 5.8 it follows that xR is semisimple for every $x \in M \setminus \{0\}$, which shows that M is semisimple. \square

Lemma 5.11. *Let S be a simple R -module. Then S is singular if and only if $S \cdot \text{soc}(R) = 0$.*

Proof. Let M be a maximal right ideal of R . We show that $M \trianglelefteq R_R$ if and only if $\text{soc}(R) \subseteq M$.

The necessity is clear if we take into account that

$$\text{soc}(R) = \bigcap_{L \trianglelefteq R} L,$$

where the intersection runs over all maximal right ideals L of R_R .

Conversely, if M is not essential in R , there exists $0 \neq a \in R$ such that $aR \cap M = 0$. Using the maximality of M we get $aR \oplus M = R$, hence $aR \cong R/M$, which shows that aR is simple and therefore $aR \subseteq \text{soc}(R)$, so $aR \subseteq M$, a contradiction.

Since S is a simple singular R -module, for every $x \in S$, $\text{ann}_R(x)$ is an essential maximal ideal of R ; from the above observation it follows that $\text{soc}(R) \subseteq \text{ann}_R(x)$ for all $x \in S$, which is equivalent to $S \cdot \text{soc}(R) = 0$. \square

Corollary 5.12. *The following statements are equivalent:*

1. *Every singular cyclic R -module is injective.*
2. *Every singular R -module is injective.*

Proof. (1) \Rightarrow (2). Assume that $Z(R_R) \neq 0$ and let $a \in Z(R_R) \setminus \{0\}$. Then aR is a singular cyclic R -module, hence injective. It follows that aR is a direct summand of R , so there exists $e = e^2 \in R \setminus \{0\}$ such that $aR = eR$ and hence $e \in Z(R_R)$, a contradiction since $\text{ann}_r(e) \cap eR = 0$. Thus R is nonsingular, and the injective envelope of a singular module is singular (see A. W. Goldie, *Torsion free modules and rings*, J. Algebra (1964), 268–287).

Let M be a singular R -module. Since $E(M)$ is a singular R -module, Corollary 5.10 yields that $E(M)$ is semisimple. Hence M is a direct summand of $E(M)$ and therefore $M = E(M)$, so M is injective.

(2) \Rightarrow (1). This implication is obvious. \square

Corollary 5.13 (Goodearl). *If every singular cyclic R -module is injective, then:*

1. *every simple $R/\text{soc}(R)$ -module is injective;*
2. *$R/\text{soc}(R)$ is noetherian.*

Proof. (1) Let S be a simple $R/\text{soc}(R)$ -module. Since $S \cdot \text{soc}(R) = 0$, Lemma 5.11 shows that S is a simple singular R -module and hence a simple injective R -module. It follows that S is a simple injective $R/\text{soc}(R)$ -module. \square

(2) Let $E = \bigoplus_{i=1}^{\infty} S_i$ in $\text{Mod-}R/\text{soc}(R)$, where S_i is a simple $R/\text{soc}(R)$ -module for every $i \geq 1$. Then $E = \bigoplus_{i=1}^{\infty} S_i$ in $\text{Mod-}R$, with each S_i a simple singular R -module. It follows that E is a singular R -module. By Corollary 5.12, E is an injective R -module and hence an injective $R/\text{soc}(R)$ -module. Applying the Matlis–Bass theorem, we obtain that $R/\text{soc}(R)$ is noetherian.

Definition 5.14. A ring R is called a *right PCI-ring* if every right R -module which is not isomorphic to R_R is injective.

Corollary 5.15 (Damiano). *If R is a right PCI-ring, then R is right noetherian.*

Proof. By Corollary 5.13 we know that $R/\text{soc}(R)$ is noetherian. Assume that $\text{soc}(R) \neq 0$. Let C be a cyclic $R/\text{soc}(R)$ -module. Then C is a proper cyclic R -module, because $C \cdot \text{soc}(R) = 0$ whereas $R \cdot \text{soc}(R) = \text{soc}(R) \neq 0$. Hence C is an injective R -module and therefore an injective $R/\text{soc}(R)$ -module. By Osofsky's theorem (Corollary 5.9), it follows that $R/\text{soc}(R)$ is a semisimple ring.

Assume that $\text{soc}(R) \neq R$. Then there exist $n \geq 1$ and simple submodules $W_i \leq R/\text{soc}(R)$, $i = 1, \dots, n$, such that

$$R/\text{soc}(R) = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Choose $\hat{y} \in W_n$. Then

$$W_n \cong \hat{y} R/\text{soc}(R) = \frac{yR + \text{soc}(R)}{\text{soc}(R)} \cong \frac{yR}{yR \cap \text{soc}(R)},$$

so $yR/(yR \cap \text{soc}(R))$ is simple.

We show that $\text{soc}(yR)$ has finite length. Otherwise we could write

$$\text{soc}(yR) = S \oplus T \oplus U$$

with S, T, U of infinite length. Clearly yR/S is a cyclic R -module. Since S is not a direct summand of yR , we have $yR/S \not\cong R$, hence yR/S is injective.

As $T \oplus U \cong \text{soc}(yR)/S \leq yR/S$, there is an injective homomorphism

$$0 \longrightarrow T \oplus U \longrightarrow yR/S,$$

and we can assume that $E(T) \oplus E(U)$ is a submodule of yR/S . There exists $K \leq yR/S$ such that

$$yR/S = E(T) \oplus E(U) \oplus K.$$

Hence

$$\frac{yR}{\text{soc}(yR)} \cong \frac{yR/S}{(S \oplus T \oplus U)/S} \cong \frac{E(T)}{T} \oplus \frac{E(U)}{U} \oplus K,$$

which contradicts the fact that $yR/\text{soc}(yR)$ is simple.

Thus $\text{soc}(yR)$ has finite length. Since every simple R -module is injective, $\text{soc}(yR)$ is an injective R -module and hence a direct summand of yR . Therefore there exists a simple module U such that

$$yR = \text{soc}(yR) \oplus U,$$

and so $y \in \text{soc}(R)$, which shows that $R = \text{soc}(R)$, i.e. R is semisimple. \square

Chapter 6

PCQI Rings

Definition 6.1. Let M be a right R -module. We say that M is *proper* if $M \not\cong R_R$.

A ring R is called a *right PCQI ring* if every proper cyclic right R -module is quasi-injective.

Lemma 6.2. Let R be a right PCQI ring. Then every cyclic right R -module has finite uniform (co-irreducible) dimension.

Proof. Let X be a cyclic right R -module.

Step 1. We show that X/E has finite uniform (co-irreducible) dimension for every $E \trianglelefteq X$.

Let $E \trianglelefteq X$. We prove that X/E satisfies the hypotheses of the Osofsky–Smith theorem. Consider a cyclic submodule Y/E of X/E and a submodule $Z/E \leq Y/E$. Then

$$(Y/E)/(Z/E) \cong Y/Z.$$

Since $Z \trianglelefteq Y$ (Proposition 1.8), we have $Y/Z \neq R_R$ (otherwise Y/Z would be projective and hence Z would be a direct summand of Y , a contradiction). As R is a right PCQI ring, the module Y/Z is quasi-injective and therefore, by Proposition 5.4, Y/Z is a CS-module.

Thus every cyclic submodule of X/E is completely CS. By the Osofsky–Smith theorem it follows that X/E has finite uniform (co-irreducible) dimension.

Step 2. The factor module X/S has finite uniform (co-irreducible) dimension, where $S = \text{soc}(X)$.

Let $T \leq X$ be such that, by Proposition 1.16, $(S \oplus T) \trianglelefteq X$. From Step 1 we obtain that $X/(S \oplus T)$ has finite uniform (co-irreducible) dimension. Put

$$\bar{T} = (S \oplus T)/S.$$

Then $(X/S)/\bar{T}$ has finite uniform (co-irreducible) dimension. Since $T \cong \bar{T}$, it remains to prove that T has finite uniform (co-irreducible) dimension.

Assume that T does not have finite uniform (co-irreducible) dimension. Then T contains an essential submodule

$$V = \bigoplus_{i \geq 1} V_i,$$

with $V_i \neq 0$ for all $i \geq 1$. As $\text{soc}(T) = S \cap T = 0$, we have $\text{soc}(V) = 0$. Hence for every $i \geq 1$ there exists a proper non-zero submodule $W_i \trianglelefteq V_i$, $W_i \neq V_i$ (otherwise, if for some $i \geq 1$ the module V_i had no proper essential submodules, then $\text{soc}(V_i) = V_i$, contradicting $\text{soc}(V) = 0$).

Let

$$W = \bigoplus_{i \geq 1} W_i.$$

By Proposition 1.9 we obtain $W \trianglelefteq V$ and therefore $(S + W) \trianglelefteq (S + V)$. Since $V \trianglelefteq T$, we have $(S + W) \trianglelefteq (S + T) \trianglelefteq X$, hence $(S + W) \trianglelefteq X$. Applying Step 1 to $S + W$ we deduce that $X/(S \oplus W)$ has finite uniform (co-irreducible) dimension.

On the other hand,

$$\bigoplus_{i \geq 1} V_i/W_i = V/W \leq (S \oplus V)/(S \oplus W) \leq X/(S \oplus W),$$

which contradicts the fact that $X/(S \oplus W)$ has finite uniform (co-irreducible) dimension. Therefore T must have finite uniform (co-irreducible) dimension.

Step 3. The socle $S = \text{soc}(X)$ is finitely generated.

Assume that S is not finitely generated. Then we can write $S = H \oplus K$, where neither H nor K is finitely generated. Since H cannot be a direct summand of X , we have $X/H \neq R_R$ (if $X = H \oplus H'$ then $H \cong X/H'$, and as X/H' is cyclic, this is impossible), hence X/H is quasi-injective. Let $\bar{K} = (K \oplus H)/H$. Then

$$X/S \cong (X/H)/\bar{K},$$

and by Step 2 the factor $(X/H)/\bar{K}$ has finite uniform (co-irreducible) dimension. Since $\bar{K} \cong K$ and K is an infinite direct sum of simple modules, $(X/H)/\bar{K}$ cannot have finite uniform (co-irreducible) dimension (see Lemma 1 in [?]), a contradiction.

We deduce that $S = \text{soc}(X)$ is finitely generated and therefore S has finite uniform (co-irreducible) dimension.

Combining Steps 2 and 3 we conclude that X has finite uniform (co-irreducible) dimension. \square

Theorem 6.3. *Let R be a prime right PCQI ring. Then R is right artinian or a right Ore domain.*

Proof. We first show that R is right nonsingular. Suppose that $Z(R_R) \neq 0$. By the previous lemma, R has finite uniform (co-irreducible) dimension, hence there exists a co-irreducible submodule $U \leq R_R$. Set

$$V := U \cap Z(R_R) \neq 0.$$

Since R is prime, we have $V^2 \neq 0$. Choose $a \in V$ with $aV \neq 0$. As V is co-irreducible, there exists $0 \neq x \in V \cap \text{ann}_r(a)$. Then $xR \neq R_R$, so xR is quasi-injective. Let $E(V)$ be the injective hull of V . Since $xR \trianglelefteq V$, we have $E(V) = E(xR)$ and xR is a fully invariant submodule of $E(V)$. In particular, $V(xR) \subseteq xR$. Hence

$$(aV)(xR) = a(VxR) \subseteq axR = 0.$$

But $aV \neq 0$ and $xR \neq 0$, contradicting the primeness of R .

Thus $Z(R_R) = 0$. By the previous lemma, R has finite uniform (co-irreducible) dimension. Since R is right nonsingular, it follows that R is a right Goldie ring. If R_R is co-irreducible, then R is either a division ring or a right Ore domain, and the theorem is proved in this case.

Assume from now on that R_R is not co-irreducible. Consider

$$U_1 \oplus U_2 \oplus \cdots \oplus U_m = U \quad (m \geq 2)$$

a direct sum of co-irreducible submodules with $U \trianglelefteq R_R$. Take $a_1 \in U_1 \setminus \{0\}$. Clearly $a_1R \neq R_R$, hence a_1R is quasi-injective. Since R is a prime right Goldie ring, for each $i = 2, \dots, m$ there exists $a_i \in U_i$ such that $a_iR \cong a_1R$. Then

$$A = a_1R \oplus a_2R \oplus \cdots \oplus a_mR$$

is quasi-injective. We have $A \trianglelefteq U$, hence $A \trianglelefteq R_R$. By Proposition 4.14, A contains a regular element $c \in R$. Since $cR \leq A$, the module A is cR -injective, and as $cR \cong R_R$ we conclude that A is injective.

Therefore the exact sequence

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0$$

splits, and from $A \trianglelefteq R_R$ we obtain $A = R$. Hence R is right self-injective. A prime right Goldie, right self-injective ring is simple artinian; in particular, R is right artinian.

Consequently, R is either a right Ore domain or a simple artinian ring, so in all cases R is right artinian or a right Ore domain. \square

