

# The Uniform (Co-Irreducible) Dimension of Rings and Modules

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# Chapter 0

## Generalities

### 0.1 Semisimple modules

**Definition 0.1.1.** A non-zero  $R$ -module  $S$  is called *simple* if its only submodules are 0 and  $S$ .

**Proposition 0.1.2.** Let  $S$  be an  $R$ -module. The following statements are equivalent:

1.  $S$  is simple;
2. for every non-zero element  $x \in S$  we have  $S = xR$ ;
3.  $S \cong R/I$ , where  $I$  is a maximal right ideal.

**Lemma 0.1.3** (Schur). Let  $S$  and  $S'$  be simple  $R$ -modules and let  $f: S \rightarrow S'$  be a homomorphism of  $R$ -modules. Then  $f = 0$  or  $f$  is an isomorphism. In particular  $\text{End}_R(S)$  is a division ring.

**Definition 0.1.4.** Let  $M$  be an  $R$ -module and let  $(S_i)_{i \in I}$  be the family of all simple submodules of  $M$ . If  $M = \sum_{i \in I} S_i$ , then  $M$  is called *semisimple*.

**Proposition 0.1.5.** Let  $M$  be a semisimple  $R$ -module and  $N$  a submodule of  $M$ . Then there exists a subset  $J \subseteq I$  such that

1. the family  $(S_j)_{j \in J}$  is independent;
2.  $M = N \oplus (\bigoplus_{j \in J} S_j)$ .

**Corollary 0.1.6.** With the above notation, for the semisimple module  $M$  there exists  $J \subseteq I$  such that the family  $(S_j)_{j \in J}$  is independent and

$$M = \bigoplus_{i \in I} M_i.$$

**Corollary 0.1.7.** If  $M$  is a semisimple  $R$ -module and  $N$  a submodule of  $M$ , then both  $N$  and  $M/N$  are semisimple.

**Corollary 0.1.8.** A direct sum of semisimple modules is a semisimple module.

**Theorem 0.1.9.** *Let  $M$  be an  $R$ -module. The following statements are equivalent:*

1.  $M$  is semisimple;
2.  $M$  is isomorphic to a direct sum of simple modules;
3. every submodule of  $M$  is a direct summand of  $M$ ;
4. every short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

*splits.*

**Definition 0.1.10.** The sum of all simple submodules of  $M$  is called the *socle* of  $M$  and is denoted by  $\text{soc}(M)$ . If  $M$  has no simple submodule we put  $\text{soc}(M) = 0$ .

**Proposition 0.1.11.** *Let  $M$  and  $N$  be  $R$ -modules and  $f: M \rightarrow N$  a homomorphism. Then  $f(\text{soc}(M)) \subseteq \text{soc}(N)$ .*

**Proposition 0.1.12.** *Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . Then*

$$\text{soc}(N) = \text{soc}(M) \cap N.$$

**Proposition 0.1.13.** *If  $M = \bigoplus_{i \in I} M_i$ , then*

$$\text{soc}(M) = \bigoplus_{i \in I} \text{soc}(M_i).$$

**Proposition 0.1.14.** *Let  $R$  be a ring. Then the socle  $\text{soc}(R_R)$  is a two-sided ideal of  $R$ .*

## 0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings

**Definition 0.2.1.** Let  $R$  be a ring and  $M$  a right  $R$ -module. We say that  $M$  satisfies the *maximal condition* (resp. the *minimal condition*) if every non-empty set of submodules of  $M$ , ordered by inclusion, has a maximal (resp. minimal) element.

We say that  $M$  satisfies the *ascending* (resp. *descending*) *chain condition* if every ascending chain of submodules of  $M$

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

(resp. every descending chain

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots$$

) is stationary, that is, there exists  $n \geq 1$  such that  $M_n = M_{n+1} = \cdots$ .

**Proposition 0.2.2.** *Let  $M$  be an  $R$ -module. The following statements are equivalent:*

1.  $M$  satisfies the maximal (minimal) condition;
2.  $M$  satisfies the ascending (descending) chain condition.

**Definition 0.2.3.** An  $R$ -module  $M$  is called *noetherian* (resp. *artinian*) if it satisfies the maximal (resp. minimal) condition. The ring  $R$  is called right noetherian (resp. right artinian) if the right module  $R_R$  is noetherian (resp. artinian).

**Example 0.2.4.**

1.  $\mathbb{Z}$  is a noetherian ring but not artinian.
2. Every finite group is a noetherian and artinian  $\mathbb{Z}$ -module.
3. Every finite ring is noetherian and artinian.
4. The ring  $\mathbb{Z}[X_1, X_2, \dots, X_n, \dots]$  is neither noetherian nor artinian:

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \dots, X_n) \subsetneq \cdots$$

$$(X_1) \supsetneq (X_1^2) \supsetneq \cdots \supsetneq (X_1^k) \supsetneq \cdots$$

5. The Prüfer  $p$ -group  $\mathbb{Z}_{p^\infty}$  is an artinian but not noetherian  $\mathbb{Z}$ -module.

**Proposition 0.2.5.** Let  $N$  and  $P$  be submodules of  $M$  such that  $M = N + P$ . Then  $M$  is noetherian (artinian) if and only if  $N$  and  $P$  are noetherian (artinian).

**Proposition 0.2.6.** For an  $R$ -module  $M$  the following statements are equivalent:

1.  $M$  is noetherian;
2. every submodule of  $M$  is finitely generated.

**Proposition 0.2.7.** For an  $R$ -module  $M$  the following statements are equivalent:

1.  $M$  is artinian;
2. for every family  $(X_i)_{i \in I}$  of submodules of  $M$  there exists a finite subset  $J \subseteq I$  such that

$$\bigcap_{i \in I} X_i = \bigcap_{j \in J} X_j.$$

## 0.3 Finite length modules

**Definition 0.3.1.** Let  $M$  be a non-zero right  $R$ -module. A *composition series* or *Jordan–Hölder series* of  $M$  is a finite strictly ascending chain of submodules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = M$$

such that  $X_{i+1}/X_i$  is a simple module for  $0 \leq i \leq n-1$ . The integer  $n$  is called the *length* of the series, and the modules  $X_{i+1}/X_i$  are called its *factors*.

**Proposition 0.3.2.** Let  $M$  be an  $R$ -module. The following statements are equivalent:

1.  $M$  has a composition series;
2.  $M$  is noetherian and artinian.

**Proposition 0.3.3.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*be a short exact sequence of right  $R$ -modules. Then  $M$  has a composition series if and only if both  $M'$  and  $M''$  have composition series.*

If

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M$$

are two composition series of  $M$ , we say that they are *equivalent* if  $n = p$  and there exists a bijection  $\sigma : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$  such that

$$M_{i+1}/M_i \cong M_{\sigma(i)+1}/M_{\sigma(i)} \quad (0 \leq i \leq n-1).$$

**Theorem 0.3.4** (Jordan–Hölder). *If an  $R$ -module  $M$  has two composition series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M,$$

*then these two series are equivalent.*

**Definition 0.3.5.** An  $R$ -module  $M$  which admits a composition series is called a *module of finite length*. The length of its composition series is called the *length* of  $M$  and is denoted by  $l(M)$ . If  $M$  admits no composition series, we say that  $M$  has *infinite length* and we write  $l(M) = \infty$ .

**Proposition 0.3.6.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*be a short exact sequence of  $R$ -modules of finite length. Then*

$$l(M) = l(M') + l(M'').$$

**Corollary 0.3.7.** *Let  $M$  be an  $R$ -module of finite length and let  $N, L$  be submodules of  $M$ . Then:*

1.  $l(M) = l(N) + l(M/N)$ ;
2.  $l(N + L) + l(N \cap L) = l(N) + l(L)$ .

**Corollary 0.3.8.** *Let  $M$  be an  $R$ -module of finite length and let  $M_1, M_2, \dots, M_n$  be submodules of  $M$  such that*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

*Then*

$$l(M) = \sum_{i=1}^n l(M_i).$$



## 0.4 The Jacobson radical

### The Jacobson radical of a module

**Definition 0.4.1.** Let  $M$  be an  $R$ -module. The intersection of all maximal submodules of  $M$  is called the *Jacobson radical* of  $M$  and is denoted by  $\text{Rad}(M)$ . If  $M$  has no maximal submodules, we adopt the convention  $\text{Rad}(M) = M$ .

*Remark 0.4.2.* If  $M$  is a finitely generated  $R$ -module, then  $\text{Rad}(M) \neq M$ .

**Proposition 0.4.3.** Let  $M$  be an  $R$ -module. Then

$$\text{Rad}(M) = \bigcap_{\substack{f: M \rightarrow S \\ S \text{ simple}}} \ker(f) = \bigcap_{\substack{f: M \rightarrow X \\ X \text{ semisimple}}} \ker(f).$$

**Proposition 0.4.4.** Let  $f : M \rightarrow N$  be a homomorphism of  $R$ -modules. Then  $f(\text{Rad}(M)) \subseteq \text{Rad}(N)$ . If, in addition,  $f$  is an epimorphism and  $\ker(f) \subseteq \text{Rad}(M)$ , then  $f(\text{Rad}(M)) = \text{Rad}(N)$ .

**Corollary 0.4.5.** For every  $R$ -module  $M$  we have  $\text{Rad}(M/\text{Rad}(M)) = 0$ .

**Corollary 0.4.6.** If  $M$  is a semisimple  $R$ -module, then  $\text{Rad}(M) = 0$ .

**Corollary 0.4.7.** If  $M = \bigoplus_{i \in I} M_i$ , then

$$\text{Rad}(M) = \bigoplus_{i \in I} \text{Rad}(M_i).$$

**Proposition 0.4.8.** Let  $M$  be an  $R$ -module with  $\text{Rad}(M) \neq M$ . Then

$$\text{Rad}(M) = \bigcap \{ L \leq M \mid L \text{ is a superfluous submodule} \}.$$

**Proposition 0.4.9** (Nakayama's Lemma). Let  $M$  be a finitely generated  $R$ -module and  $N$  a submodule of  $M$ . If  $N + \text{Rad}(M) = M$ , then  $N = M$ . (In other words,  $\text{Rad}(M)$  is the largest superfluous submodule of  $M$ .)

### The Jacobson radical of a ring

Let  $R$  be a ring. We consider the left ideal  $\text{Rad}({}_R R)$ , the intersection of all maximal left ideals of  $R$ , and the right ideal  $\text{Rad}(R_R)$ , the intersection of all maximal right ideals of  $R$ .

**Proposition 0.4.10.**

1.  $\text{Rad}(R_R)$  is a two-sided ideal.
2.  $\text{Rad}(R_R) = \{ r \in R \mid 1 - ar \in U(R) \text{ for all } a \in R \}$ .
3.  $\text{Rad}(R_R) = \text{Rad}({}_R R)$ .

**Definition 0.4.11.** The two-sided ideal  $\text{Rad}(R_R) = \text{Rad}({}_R R)$  is called the *Jacobson radical* of the ring  $R$  and is denoted by  $\text{Rad}(R)$ .

**Proposition 0.4.12.**

1. If  $J$  is a left (resp. right or two-sided) ideal such that  $1 - x$  is invertible for every  $x \in J$ , then  $J \subseteq \text{Rad}(R)$ .
2. If  $J$  is a nil left (resp. right or two-sided) ideal, then  $J \subseteq \text{Rad}(R)$ .

**Proposition 0.4.13.** Let  $\varphi : R \rightarrow S$  be a surjective ring homomorphism. Then  $\varphi(\text{Rad}(R)) \subseteq \text{Rad}(S)$ . If  $\ker(\varphi) \subseteq \text{Rad}(R)$ , then  $\varphi(\text{Rad}(R)) = \text{Rad}(S)$ .

**Proposition 0.4.14.** If  $(R_i)_{i \in I}$  is a family of rings, then

$$\text{Rad}\left(\prod_{i \in I} R_i\right) = \prod_{i \in I} \text{Rad}(R_i).$$

**Proposition 0.4.15.** Let  $M$  be an  $R$ -module. Then  $M \text{Rad}(R) \subseteq \text{Rad}(M)$ .

**Theorem 0.4.16.** If  $R$  is an artinian ring, then  $\text{Rad}(R)$  is nilpotent.

## 0.5 Semisimple rings

**Theorem 0.5.1.** For a ring  $R$  the following statements are equivalent:

1. every non-zero right  $R$ -module is semisimple;
2.  $R$  as a right  $R$ -module is semisimple;
3.  $R$  is artinian and  $\text{Rad}(R) = 0$ .

**Definition 0.5.2.** A ring  $R$  satisfying any (hence all) of the above conditions is called a *semisimple ring*.

**Proposition 0.5.3.** Let  $R$  be a semisimple ring and  $M$  a non-zero  $R$ -module. The following statements are equivalent:

1.  $M$  has finite length;
2.  $M$  is noetherian;
3.  $M$  is artinian.

**Theorem 0.5.4.** Let  $R$  be a right artinian ring and  $M$  a non-zero right  $R$ -module. The following statements are equivalent:

1.  $M$  has finite length;
2.  $M$  is noetherian;
3.  $M$  is artinian.

**Corollary 0.5.5** (Hopkins). A right artinian ring (respectively a left artinian ring) is right noetherian (respectively left noetherian).

# Chapter 1

## Essential submodules

**Definition 1.1.** Let  $M$  be a right  $R$ -module. A submodule  $N$  of  $M$  is called *essential* (or we say that  $M$  is an essential extension of  $N$ ) if  $N \cap N' \neq 0$  for every non-zero submodule  $N'$  of  $M$ . In this case we write  $N \trianglelefteq M_R$ .

A homomorphism of right  $R$ -modules  $f : M \rightarrow N$  is called *essential* if  $\text{Im } f$  is an essential submodule of  $N$  (i.e.  $\text{Im } f \trianglelefteq N_R$ ).

**Example.**

1.  $n\mathbb{Z} \trianglelefteq \mathbb{Z}_{\mathbb{Z}}$  for every  $n \geq 1$ .
2. Every submodule of the Prüfer group  $\mathbb{Z}_{p^\infty}$  is essential.

*Remark 1.2.* Let  $M$  be a right  $R$ -module and  $N$  a submodule of  $M$ . Then  $N \trianglelefteq M_R$  if and only if for every  $x \in M$ ,  $x \neq 0$ , there exists  $r \in R$  such that  $xr \in N \setminus \{0\}$ .

*Proof.* “ $\Rightarrow$ ” Let  $x \in M \setminus \{0\}$ . Since  $0 \neq xR \leq M_R$  and  $N \trianglelefteq M_R$ , we have  $xR \cap N \neq 0$ ; hence there exists  $r \in R$  with  $xr \in N \setminus \{0\}$ .

“ $\Leftarrow$ ” Let  $N' \leq M_R$ ,  $N' \neq 0$ . For every  $x \in N' \setminus \{0\}$  there exists  $r \in R$  such that  $xr \in N \setminus \{0\}$ , so  $N \cap N' \neq 0$ . Thus  $N \trianglelefteq M_R$ .  $\square$

**Definition 1.3.** A monomorphism of right  $R$ -modules  $f : N_R \rightarrow M_R$  is called *essential* if  $\text{Im } f \trianglelefteq M_R$ . It is immediate that, if  $N$  is a submodule of  $M$ , then the canonical inclusion  $i_N : N \rightarrow M$  is an essential monomorphism if and only if  $N \trianglelefteq M_R$ .

**Proposition 1.4.** A monomorphism  $f : N_R \rightarrow M_R$  is essential if and only if for every right  $R$ -module  $M'$  and every  $g \in \text{Hom}(M, M')$ , the fact that  $g \circ f$  is a monomorphism implies that  $g$  is a monomorphism.

*Proof.* “ $\Rightarrow$ ” Let  $g$  be as in the statement and suppose that  $g \circ f$  is a monomorphism. Assume  $\text{Ker } g \neq 0$ . Take  $x \in \text{Ker } g \cap \text{Im } f \setminus \{0\}$ . Then  $x = f(x')$  for some  $x' \in N$  and  $g(x) = 0$ , so  $g(f(x')) = 0$ . Since  $g \circ f$  is a monomorphism, it follows that  $x' = 0$  and hence  $x = 0$ , a contradiction.

“ $\Leftarrow$ ” If  $f$  is not an essential monomorphism, then there exists  $N' \leq M_R$ ,  $N' \neq 0$ , such that  $N' \cap \text{Im } f = 0$ . Consider the canonical projection  $\pi_{N'} : M \rightarrow M/N'$ . If  $x \in \text{Ker}(\pi_{N'} \circ f)$ , then  $f(x) \in N'$ , so  $f(x) = 0$ , hence  $x = 0$ . Thus  $\pi_{N'} \circ f$  is injective. By the hypothesis in the statement, this implies that  $\pi_{N'}$  is injective, so  $N' = 0$ , a contradiction.  $\square$

**Corollary 1.5.** *Let  $M$  be a right  $R$ -module and  $N \leq M_R$ . Then the following statements are equivalent:*

1.  $N \leq M_R$ ;
2. the inclusion  $i_N : N \rightarrow M$  is an essential monomorphism;
3. for every  $f \in \text{Hom}(M, M')$ , with  $M'$  an arbitrary  $R$ -module, the fact that  $f \circ i_N$  is a monomorphism implies that  $f$  is a monomorphism.

**Proposition 1.6.** *Let  $f : N_R \rightarrow M_R$  and  $g : M_R \rightarrow P_R$  be monomorphisms. Then  $g \circ f$  is essential if and only if both  $g$  and  $f$  are essential.*

*Proof.* “ $\Leftarrow$ ” Let  $z \in P \setminus \{0\}$ . Since  $g$  is essential, there exists  $r \in R$  such that  $zr \in \text{Im } g \setminus \{0\}$ . Thus there exists  $y \in M \setminus \{0\}$  with  $zr = g(y)$ . As  $f$  is essential, there exists  $r' \in R$  such that  $yr' \in \text{Im } f \setminus \{0\}$ . Hence there exists  $x \in N \setminus \{0\}$  with  $yr' = f(x)$ . But  $zr' = g(y)r' = g(yr') = g(f(x))$ . If  $zr' = 0$ , then  $g(f(x)) = 0$ , hence  $x = 0$ , a contradiction. Therefore  $zr' \in \text{Im}(g \circ f)$  and  $zr' \neq 0$ , which shows that  $g \circ f$  is essential.

“ $\Rightarrow$ ” Let  $y \in M \setminus \{0\}$ . Since  $g$  is a monomorphism,  $g(y) \neq 0$ . Thus there exists  $r \in R$  such that  $g(yr) \in \text{Im } g \setminus \{0\}$  and  $g(yr) \neq 0$ . Hence there exists  $x \in N \setminus \{0\}$  such that  $g(yr) = g(f(x))$ , therefore  $yr = f(x) \in \text{Im } f$ , which shows that  $f$  is an essential monomorphism.

If  $z \in P \setminus \{0\}$ , there exists  $r \in R$  such that  $zr \in \text{Im}(g \circ f)$  and  $zr \neq 0$ . Since  $\text{Im}(g \circ f) \subseteq \text{Im } g$ , we obtain  $zr \in \text{Im } g$ , so  $g$  is essential.  $\square$

**Proposition 1.7.** *Let  $M$  be a right  $R$ -module and  $L_1, L_2, \dots, L_n$  submodules of  $M$ . Then:*

- 1)  $\bigcap_{i=1}^n L_i$  is essential in  $M$  iff each  $L_i$  is essential in  $M$ .
- 2) If  $L_1 \subseteq L_2$  and  $L_1$  is essential in  $M$ , then  $L_2$  is essential in  $M$ .

The proof is obvious.

**Proposition 1.8.** *Let  $K$  and  $L$  be submodules of  $M$ .*

- 1) If  $K \subseteq L \subseteq M$ , then  $K \leq M$  iff  $K \leq L$  and  $L \leq M$ .
- 2) If  $h : K_R \rightarrow M_R$  is a module morphism and  $L \leq M$ , then  $h^{-1}(L) \leq K$ .
- 3) If  $L_1, L_2 \leq M_R$  with  $K_1 \leq L_1$  and  $K_2 \leq L_2$ , then  $K_1 \cap K_2 \leq L_1 \cap L_2$ .

**Proof.**

- 1) Apply 1.5 and 1.6.
- 2) Let  $U$  be a non-zero submodule of  $K$ .
  - (i) If  $h(U) = 0$ , then  $U \subseteq \ker h \subseteq h^{-1}(L)$ , so  $U \cap h^{-1}(L) \neq 0$ .
  - (ii) If  $h(U) \neq 0$ , then  $h(U) \cap L \neq 0$ . Hence there exists  $u \in U$  with  $h(u) \in L$ ,  $h(u) \neq 0$ , so  $u \in U \cap h^{-1}(L)$  and  $u \neq 0$ .

Thus  $h^{-1}(L) \leq K$ .

- 3) Let  $0 \neq X \leq L_1 \cap L_2$ . Then  $X \subseteq L_1$ , so  $0 \neq X \cap K_1 \leq L_1$ . But  $X \subseteq L_2$  implies  $0 \neq (X \cap K_1) \cap L_2 = X \cap (K_1 \cap K_2)$ . Hence  $K_1 \cap K_2 \leq L_1 \cap L_2$ .

**Proposition 1.9.** *Let  $(K_\lambda)_{\lambda \in \Lambda}$ ,  $(L_\lambda)_{\lambda \in \Lambda}$  be families of submodules of  $M$ . If  $(K_\lambda)$  is independent in  $M$  and  $K_\lambda \leq L_\lambda$  for all  $\lambda \in \Lambda$ , then  $(L_\lambda)$  is independent in  $M$  and*

$$\left( \bigoplus_{\lambda \in \Lambda} K_\lambda \right) \leq \left( \bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

**Proof.**

Let  $K_1 \leq L_1$ ,  $K_2 \leq L_2$  with  $K_1 \cap K_2 = 0$ . By 1.8(3),  $0 \leq L_1 \cap L_2$ , hence  $L_1 \cap L_2 = 0$ .

Let  $\pi_1 : L_1 \oplus L_2 \rightarrow L_1$ ,  $\pi_2 : L_1 \oplus L_2 \rightarrow L_2$  be the canonical projections. Since  $K_1 \leq L_1$  and  $K_2 \leq L_2$ ,

$$\pi_1^{-1}(K_1) = K_1 \oplus 0 \leq L_1 \oplus L_2,$$

and

$$\pi_2^{-1}(K_2) = 0 \oplus K_2 \leq L_1 \oplus L_2.$$

Hence

$$K_1 \oplus K_2 = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \leq L_1 \oplus L_2.$$

Induction gives the finite case. For the general case, let  $0 \neq m \in \bigoplus_{\lambda \in \Lambda} L_\lambda$ . Then  $m$  lies in a finite direct sum  $\bigoplus_{\lambda \in \Lambda_0} L_\lambda$  for some finite  $\Lambda_0 \subseteq \Lambda$ . Since  $(\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \leq (\bigoplus_{\lambda \in \Lambda_0} L_\lambda)$ , there exists  $r \in R$  with  $rm \in (\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \setminus \{0\} \subseteq (\bigoplus_{\lambda \in \Lambda} K_\lambda)$ . Thus

$$\left( \bigoplus_{\lambda \in \Lambda} K_\lambda \right) \leq \left( \bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

**Proposition 1.10.** *Let  $N$  be a submodule of  $M$ . Then there exists a submodule  $Q$  with  $N \subseteq Q \subseteq M$  such that  $Q$  is a maximal essential extension of  $N$  inside  $M$ .*

**Proof.**

Let

$$\mathfrak{S} = \{L \leq M ; N \subseteq L \subseteq M, N \leq L\},$$

ordered by inclusion. Clearly  $\mathfrak{S} \neq \emptyset$ , since  $N \in \mathfrak{S}$ .

Let  $(L_\lambda)_{\lambda \in \Lambda}$  be a totally ordered family of elements of  $\mathfrak{S}$  and put

$$L := \bigcup_{\lambda \in \Lambda} L_\lambda.$$

Clearly  $L \leq M_R$ .

Let  $x \in L \setminus \{0\}$ . Then there exists  $\lambda_0 \in \Lambda$  with  $x \in L_{\lambda_0}$ . Since  $N$  is essential in  $L_{\lambda_0}$ , there exists  $r \in R$  such that  $xr \in N$  and  $xr \neq 0$ , hence  $L$  is an essential extension of  $N$ . Thus  $\mathfrak{S}$  is inductive and, by Zorn's lemma,  $\mathfrak{S}$  admits a maximal element  $Q$  which satisfies the required conditions.

**Definition 1.11.** Let  $M$  be a right  $R$ -module and  $N \leq M_R$ . A submodule  $K \leq M_R$  is called a *complement* of  $N$  in  $M$  if  $K$  is a maximal submodule of  $M$  with the property that  $K \cap N = 0$ . A submodule  $K \leq M_R$  is called a *complement submodule* of  $M$  if there exists  $N \leq M_R$  such that  $K$  is a complement of  $N$  in  $M$ .

*Remark 1.12.* The set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid N \cap L = 0 \}$$

is inductive and, by applying Zorn's lemma, it follows that there exists a complement of  $N$  in  $M$ . In particular,  $0$  and  $M$  are complement submodules of  $M$ .

**Proposition 1.13.** *Let  $M_R$ ,  $N \leq M_R$  and  $K \leq M_R$ , with  $K$  a complement of  $N$  in  $M$ . There exists a complement  $Q$  of  $K$  in  $M$  such that  $N \subseteq Q$ . Moreover,  $Q$  is a maximal essential extension of  $N$  in  $M$ .*

*Proof.* It is easy to see that the set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid K \cap L = 0, N \subseteq L \}$$

is inductive, and Zorn's lemma guarantees the existence of  $Q$ .

Let  $L$  be a non-zero submodule of  $Q$  such that  $L \cap N = 0$ . Put  $K_1 = L + K$ . Clearly  $K \subseteq K_1$ . If  $x \in N \cap (L + K)$ , then  $x = y + z$  with  $y \in L$ ,  $z \in K$ . But  $z = x - y \in Q$ . Since  $Q \cap K = 0$ , we obtain  $z = 0$  and hence  $x = y$ . From  $L \cap N = 0$  it follows that  $x = y = 0$ , and therefore  $N \cap (L + K) = 0$ , which contradicts the fact that  $K$  is a complement of  $N$  in  $M$ . Thus  $L \cap N \neq 0$  for every  $0 \neq L \leq Q$ , so  $Q$  is an essential extension of  $N$ .

Suppose that there exists  $Q' \leq M_R$  with  $N \leq Q'$  and  $Q \subsetneq Q'$ . Since  $Q'$  is a complement of  $K$ , we have  $Q' \cap K \neq 0$ . But  $N \cap (Q' \cap K) = 0$  and  $0 \neq Q' \cap K \leq Q'$ , contradicting  $N \leq Q'$ . Hence  $Q$  is a maximal essential extension of  $N$  in  $M$ .  $\square$

**Definition 1.14.** A submodule  $N$  of  $M_R$  is called *closed* if  $N$  has no proper (meaning different from  $N$ ) essential extension in  $M$ .

**Corollary 1.15.** *Let  $M_R$  be a right  $R$ -module. The complement submodules of  $M$  coincide with the closed submodules of  $M$ .*

*Proof.* From 1.13 it follows immediately that every closed submodule of  $M$  is a complement submodule of  $M$ .

Conversely, let  $K$  be a complement submodule of  $M_R$ . Then there exists  $N \leq M_R$  such that  $K$  is a complement of  $N$  in  $M$ . Assume that  $K$  has a proper essential extension in  $M$ ; that is, there exists  $K' \leq M_R$  with  $K \leq K'$  and  $K \subsetneq K'$ . By the maximality of  $K$  we have  $K' \cap N \neq 0$ , and since  $K \leq K'$ , it follows that

$$K \cap K' \cap N \neq 0,$$

contradiction.  $\square$

**Corollary 1.16.** *Let  $N$  be a submodule of  $M_R$ . If  $K$  is a complement of  $N$  in  $M$ , then:*

1.  $(N + K) \leq M_R$ .
2. The canonical morphism  $\pi_K \circ i_N : N \rightarrow M/K$  is an essential monomorphism.

*Proof.* (1) Let  $x \in M \setminus \{0\}$ . If  $x \notin K$ , then  $K + Rx \neq K$  and hence  $N \cap (K + Rx) \neq 0$ . Let  $y \in N \cap (K + Rx)$ ,  $y \neq 0$ . There exist  $z \in K$  and  $r \in R$  such that  $y = z + rx$ . If  $rx = 0$ , then  $y = z$  and, since  $N \cap K = 0$ , we obtain  $y = 0$ , a contradiction. Thus  $rx \neq 0$  and, because  $rx = y - z$ , we have  $rx \in N + K$ , which shows that  $(N + K) \trianglelefteq M_R$ .

(2) We have  $\text{Im}(\pi_K \circ i_N) = (N + K)/K$ . Let  $L/K$  be a non-zero submodule of  $M/K$ . Then

$$\frac{N + K}{K} \cap \frac{L}{K} = \frac{(N + K) \cap L}{K} = \frac{N \cap L + K}{K}.$$

Since  $K$  is a complement of  $N$ , we have  $N \cap L \neq 0$ , and hence

$$\frac{N \cap L + K}{K} \neq 0,$$

which shows that  $\pi_K \circ i_N$  is an essential monomorphism. □





# Chapter 2

## Injective Modules

### 2.1 Injective Module

Let  $Q$  and  $M$  be two right  $R$ -modules. We say that  $Q$  is  $M$ -*injective* if for every monomorphism  $u : M' \rightarrow M$  and every morphism  $f : M' \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g \circ u = f$ ; that is, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M \\ & & \downarrow f & \swarrow g & \\ & & Q & & \end{array}$$

is commutative.

This property is equivalent to the condition that the map

$$\text{Hom}(u, Q) : \text{Hom}(M, Q) \longrightarrow \text{Hom}(M', Q)$$

is surjective for every monomorphism  $u : M' \rightarrow M$ . Since the functor  $\text{Hom}(-, Q)$  is left exact, it follows that  $Q$  is  $M$ -injective if and only if  $\text{Hom}(-, Q)$  is exact with respect to every short exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

The  $R$ -module  $Q$  is called *quasi-injective* (or *self-injective*) if it is  $Q$ -injective. If  $Q$  is  $M$ -injective for every  $R$ -module  $M$ , then  $Q$  is called *injective*.

#### 2.1.1 Proposition

**Proposition 2.1.1.** *Let  $Q$  and  $M$  be two  $R$ -modules. The following statements are equivalent:*

1.  $Q$  is  $M$ -injective.
2. For every submodule  $N$  of  $M$  and every morphism  $f : N \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g|_N = f$ .
3. For every essential submodule  $N$  of  $M$  and every morphism  $f : N \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g|_N = f$ .

*Proof.* Implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are obvious.

(2)  $\Rightarrow$  (1). Let  $M'_R$ ,  $0 \longrightarrow M' \xrightarrow{u} M$  and  $f : M' \rightarrow Q$ . Then  $u(M') \leq M$ . Consider  $i : u(M') \rightarrow M$  the canonical injection and  $\bar{u} : M' \rightarrow u(M')$  the isomorphism induced by  $u$ . There exists  $g : M \rightarrow Q$  such that  $g \circ i = f \circ \bar{u}^{-1}$ . Hence

$$g \circ i \circ \bar{u} = f \quad \text{and therefore} \quad g \circ u = f.$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{\bar{u}} & u(M') & \xhookrightarrow{i} & M \\ & & \downarrow f & & & \nearrow g & \\ & & Q & & & & \end{array}$$

(3)  $\Rightarrow$  (2). Let  $N$  be a submodule of  $M$  and  $K$  a complement of  $N$  in  $M$ . Then  $(N \oplus K) \trianglelefteq M$ . Define  $h : N \oplus K \rightarrow Q$  by  $h(n + k) = f(n)$  for all  $n \in N$ ,  $k \in K$ . Since  $N \cap K = 0$ , the map  $h$  is well-defined. There exists  $g : M \rightarrow Q$  such that  $g|_{N \oplus K} = h$ , and hence  $g|_N = h|_N = f$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \hookrightarrow & N \oplus K & \hookrightarrow & M \\ & & \downarrow f & & \nearrow h & \nearrow g & \\ & & Q & & & & \end{array}$$

□

**Proposition 2.1.2.** *Let  $(M_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules and  $M$  an  $R$ -module. Then  $\prod_{\alpha \in \Lambda} M_\alpha$  is  $M$ -injective if and only if each  $M_\alpha$  is  $M$ -injective.*

*Proof.* Let  $N$  be a submodule of  $M$ . Put  $P = \prod_{\alpha \in \Lambda} M_\alpha$  and let  $\pi_\alpha : P \rightarrow M_\alpha$  be the canonical projections for all  $\alpha \in \Lambda$ .

"  $\Leftarrow$  " Given a morphism  $f : N \rightarrow P$ , the morphisms  $\pi_\alpha \circ f : N \rightarrow M_\alpha$  can be extended to  $g_\alpha : M \rightarrow M_\alpha$ . There exists  $g : M \rightarrow P$  such that  $g|_N = f$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \hookrightarrow & M \\ & & \downarrow f & & \nearrow g \\ & & P & & \\ & & \downarrow \pi_\alpha & & \nearrow g_\alpha \\ & & M_\alpha & & \end{array}$$

"  $\Rightarrow$  " Let  $\forall \alpha \in \Lambda$  and  $f : N \rightarrow M_\alpha$ . Considering the canonical inclusion  $\varepsilon_\alpha : M_\alpha \rightarrow P$ , since  $P$  is  $M$ -injective, there exists  $g : M \rightarrow P$  which extends  $\varepsilon_\alpha \circ f : N \rightarrow P$ . Then  $\varepsilon_\alpha : M_\alpha \rightarrow P$  extends  $f$  and hence  $M_\alpha$  is  $M$ -injective.

$$\begin{array}{ccccc}
0 & \longrightarrow & N & \hookrightarrow & M \\
& & \downarrow f & & \nearrow g \\
& & M_\alpha & & \\
& & \downarrow \varepsilon_\alpha & & \searrow \pi_\alpha g \\
& & P & & \\
& & \downarrow \pi_\alpha & & \\
& & M_\alpha & & 
\end{array}$$

□

**Corollary 2.1.3.**

1. Let  $(Q_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules. Then  $\prod_{\alpha \in \Lambda} Q_\alpha$  is injective if and only if each  $Q_\alpha$  is injective for every  $\alpha \in \Lambda$ .
2. The module  $Q_1 \oplus Q_2$  is an injective  $R$ -module if and only if each  $Q_i$  is injective for  $i = 1, 2$ . In particular, a direct summand of an injective module is injective.

**Proposition 2.1.4.** *Let  $Q$  be an  $R$ -module.*

1. If  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  is an exact sequence of  $R$ -modules and  $Q$  is  $M$ -injective, then  $Q$  is  $M'$ -injective and  $M''$ -injective.
2. If  $(M_\alpha)_{\alpha \in \Lambda}$  is a family of submodules of  $M$  such that  $M = \sum_{\alpha \in \Lambda} M_\alpha$  and  $Q$  is  $M_\alpha$ -injective for every  $\alpha$ , then  $Q$  is  $M$ -injective.
3. Let  $(N_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules. Then  $Q$  is  $\bigoplus_{\alpha \in \Lambda} N_\alpha$ -injective if and only if  $Q$  is  $N_\alpha$ -injective for every  $\alpha \in \Lambda$ .

*Proof.* To show that  $Q$  is  $M'$ -injective, we consider  $N$  a submodule of  $M'$  and  $\varphi : N \rightarrow Q$  a morphism of  $R$ -modules. Since  $Q$  is  $M$ -injective, there exists  $\psi : M \rightarrow Q$  such that  $\psi \circ f|_N = \varphi$ , and hence  $\psi \circ f : M' \rightarrow Q$  is a morphism which extends  $\varphi$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & M' & \xrightarrow{f} & M \\
& & \downarrow \varphi & & \nearrow \psi & & \nearrow \\
& & Q & & & & 
\end{array}$$

Let  $h : L \rightarrow M''$  be a monomorphism. We may assume, without loss of generality, that  $M' \leq M$  and  $M'' = M/M'$ . Since  $L \cong h(L) \leq M''$ , there exist  $P \leq M$ ,  $M' \subseteq P$  such that  $h(L) = P/M'$  and hence  $L \cong P/M'$ . We obtain the commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow h \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0
\end{array}$$

Since  $Q$  is  $M$ -injective, applying the functor  $\text{Hom}(-, Q)$  we obtain the commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(M'', Q) & \longrightarrow & \text{Hom}(M, Q) & \longrightarrow & \text{Hom}(M', Q) \longrightarrow 0 \\
 & & \downarrow h^* & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Hom}(L, Q) & \longrightarrow & \text{Hom}(P, Q) & \longrightarrow & \text{Hom}(M', Q) \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

We obtain that  $h^* = \text{Hom}(h, Q)$  is an epimorphism, which shows that  $Q$  is  $M''$ -injective.

2) Let  $N$  be a submodule of  $M$  and  $f : N \rightarrow Q$  a morphism of  $R$ -modules. Consider the set

$$\mathfrak{S} = \{(L, h) \mid N \leq L \leq M, h : L \rightarrow Q, h|_N = f\}.$$

Since  $(N, f) \in \mathfrak{S}$ , we have  $\mathfrak{S} \neq \emptyset$ . Define on  $\mathfrak{S}$  the order relation  $(L_1, h_1) \preceq (L_2, h_2)$  if and only if  $L_1 \leq L_2$  and  $h_2|_{L_1} = h_1$ . One checks that  $\mathfrak{S}$  is inductive and, by Zorn's lemma, there exists a maximal element  $(L_0, g_0)$  of  $\mathfrak{S}$ . To show that  $L_0 = M$  it is enough to prove that  $M_\alpha \leq L_0$  for every  $\alpha \in \Lambda$ .

Consider the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & L_0 \cap M_\alpha & \xrightarrow{i_\alpha} & M_\alpha \\
 & & \downarrow i_0 & & \searrow h_\alpha \\
 & & L_0 & & \\
 & & \downarrow g_0 & & \\
 & & Q & & 
 \end{array}$$

from which it follows that there exists  $h_\alpha : M_\alpha \rightarrow Q$  such that  $h_\alpha \circ i_\alpha = g_0 \circ i_\alpha$ . Define  $h^* : L_0 + M_\alpha \rightarrow Q$  by  $h^*(l + m_\alpha) = g_0(l) + h_\alpha(m_\alpha)$ , for all  $l \in L_0, m_\alpha \in M_\alpha$ . If  $l + m_\alpha = 0$ , then  $l = -m_\alpha \in L_0 \cap M_\alpha$  and hence  $h^*(l + m_\alpha) = g_0(l) + h_\alpha(l)$ , which shows that  $h^*$  is well defined. Thus  $(L_0 + M_\alpha, h^*) \in \mathfrak{S}$ , and since  $(L_0, g_0) \preceq (L_0 + M_\alpha, h^*)$ , by the maximality of  $(L_0, g_0)$  we obtain  $L_0 = L_0 + M_\alpha$ , i.e.  $M_\alpha \leq L_0$  for every  $\alpha \in \Lambda$ .

3) “ $\Rightarrow$ ” Since  $N_\alpha \leq N$  and  $Q$  is  $N$ -injective, it follows that  $Q$  is  $N_\alpha$ -injective for every  $\alpha \in \Lambda$ .

“ $\Leftarrow$ ” Let  $N'_\alpha = i_\alpha(N_\alpha)$ . Since  $Q$  is  $N_\alpha$ -injective and  $N'_\alpha \cong N_\alpha$ , we see that  $Q$  is  $N'_\alpha$ -injective. Now apply (2).  $\square$

### Corollary 2.1.5.

1. The module  $Q_1 \oplus Q_2$  is a quasi-injective  $R$ -module if and only if each  $Q_i$  is  $Q_j$ -injective for all  $i, j = 1, 2$ . In particular, a direct summand of a quasi-injective module is quasi-injective.

2. The module  $Q^n$  is quasi-injective over  $R$  if and only if  $Q$  is quasi-injective.

**Corollary 2.1.6.** *Let  $Q$  and  $M$  be two  $R$ -modules. Then  $Q$  is  $M$ -injective if and only if  $Q$  is  $mR$ -injective for every  $m \in M$ .*

*Proof.* The implication “ $\Rightarrow$ ” is clear.

For “ $\Leftarrow$ ”, since  $M = \sum_{m \in M} mR$ , it follows from 2.1.4(2) that  $Q$  is  $M$ -injective.  $\square$

**Theorem 2.1.7** (Baer’s criterion). *For an  $R$ -module  $Q$  the following statements are equivalent:*

1.  $Q$  is injective.
2.  $Q$  is  $R$ -injective.
3. For every right ideal  $I$  of  $R$  and every morphism  $f : I \rightarrow Q$  there exists  $x \in Q$  such that  $f(a) = xa$  for all  $a \in I$ .

*Proof.* The implication (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (1). Let  $M$  be an  $R$ -module and  $x \in M$ . Since  $\varphi_x : R \rightarrow xR$ ,  $\varphi_x(a) = xa$  for all  $a \in R$ , is a surjective morphism of  $R$ -modules, it follows that  $R/\text{Ker } \varphi_x \cong xR$ . As  $Q$  is  $R$ -injective, 2.1.4(1) implies that  $Q$  is  $R/\text{Ker } \varphi_x$ -injective, and hence  $Q$  is  $xR$ -injective for every  $x \in M$ . Therefore, by 2.1.6 we obtain that  $Q$  is  $M$ -injective.

(2)  $\Rightarrow$  (3). Let  $I$  be a right ideal of  $R$  and  $f : I \rightarrow Q$ . There exists  $g : R \rightarrow Q$  such that  $g|_I = f$ . Put  $x = g(1) \in Q$ . Then  $f(a) = g(a) = ag(1) = xa$  for all  $a \in I$ .

(3)  $\Rightarrow$  (2). Suppose that for a morphism  $f : I \rightarrow Q$  there exists  $x \in Q$  with  $f(a) = xa$  for all  $a \in I$ . Define  $g : R \rightarrow Q$  by  $g(r) = xr$  for all  $r \in R$ . Then clearly  $g|_I = f$ .  $\square$

**Definition 2.1.8.**

1. An  $R$ -module  $Q$  is called *divisible* if for every  $y \in Q$  and every  $a \in R$  which is not a zero divisor, there exists  $x \in Q$  such that  $ax = y$ . It is easily seen that any factor module of a divisible module is divisible.
2. A commutative integral domain is called a *PID ring* (principal ideal domain) if every ideal of it is principal.

**Proposition 2.1.9.**

1. Every injective module is divisible.
2. Let  $R$  be a PID.
  - (i) If  $Q$  is an  $R$ -module, then  $Q$  is injective if and only if it is divisible.
  - (ii) If  $I$  is a non-zero ideal of  $R$ , then  $R/I$  is a quasi-injective  $R$ -module. In particular,  $\mathbb{Z}_n$  is a quasi-injective  $\mathbb{Z}$ -module for every  $n \geq 1$ .

*Proof.*

1) Let  $Q$  be a divisible  $R$ -module,  $y \in Q$  and  $a \in R$  a non-zero divisor. Define  $f : aR \rightarrow Q$  by  $f(ax) = yx$  for all  $x \in R$ . Since  $a$  is not a zero divisor,  $f$  is well defined. By Baer's criterion there exists  $x \in Q$  such that  $f(\lambda) = x\lambda$  for all  $\lambda \in aR$ . In particular  $y = f(a) = f(a \cdot 1) = xa$ ; hence  $Q$  is divisible.

2)(i) The implication " $\Rightarrow$ " follows immediately from (1).

" $\Leftarrow$ " Let  $Q$  be a divisible  $R$ -module and  $I$  a right ideal of  $R$ . Then there exists  $a \in R$  with  $I = aR$ . Given a morphism of  $R$ -modules  $f : I \rightarrow Q$ , choose  $x \in Q$  with  $f(a) = xa$ . For any  $r \in R$ ,  $f(ar) = f(a)r = xar$ , so by Baer's criterion  $Q$  is injective.

2)(ii) Let  $I = aR$  and  $J = bR$  be non-zero ideals of  $R$  with  $I \subseteq J$ , and let  $f : J/I \rightarrow R/I$  be a morphism of  $R$ -modules. There exists  $c \in R$  such that  $a = bc$ . Write  $f(\widehat{b}) = \widehat{x} \in R/I$ . Then  $xc \in I = aR$ , so there is  $a_1 \in R$  with  $xc = aa_1$ , and hence  $x = ba_1$ .

Define  $g : R/I \rightarrow R/I$  by  $g(\widehat{r}) = \widehat{a_1 r}$  for all  $r \in R$ . Then  $g$  is an  $R$ -module homomorphism and  $g(\widehat{b}) = \widehat{x}$ , so  $g|_{J/I} = f$ . Thus  $R/I$  is quasi-injective. In particular, for  $R = \mathbb{Z}$  and  $I = n\mathbb{Z}$  we obtain that  $\mathbb{Z}_n$  is a quasi-injective  $\mathbb{Z}$ -module for every  $n \geq 1$ .  $\square$

**Corollary 2.1.10.** *An abelian group  $G$  is an injective  $\mathbb{Z}$ -module if and only if  $G$  is divisible.*

**Corollary 2.1.11.**

1.  $\mathbb{Q}$  and  $\mathbb{Z}_{p^\infty}$  are injective  $\mathbb{Z}$ -modules.
2. Any direct sum of injective  $\mathbb{Z}$ -modules is an injective  $\mathbb{Z}$ -module.
3. Any factor group of an injective  $\mathbb{Z}$ -module is injective.

**Lemma 2.1.12.** *Let  $A, S, T$  be rings and let  ${}_A M_S, {}_A N_T$  be bimodules. Then  $\text{Hom}_A(M, N)$  has a structure of left  $S$ -module and right  $T$ -module given by*

$$(s \cdot f)(x) = f(xs), \quad (f \cdot t)(x) = f(x)t,$$

for  $s \in S, t \in T, x \in M, f \in \text{Hom}_A(M, N)$ .

*Proof.* Let  $a, b \in A$  and  $x, y \in M$ . Then

$$(s \cdot f)(ax + by) = f((ax + by)s) = f((ax)s) + f((by)s) = af(xs) + bf(ys) = a(s \cdot f)(x) + b(s \cdot f)(y),$$

so  $s \cdot f \in \text{Hom}_A(M, N)$ . Similarly  $f \cdot t \in \text{Hom}_A(M, N)$ .

For  $s, s' \in S$  and  $f, g \in \text{Hom}_A(M, N)$  we have

$$(s \cdot (f + g))(x) = (f + g)(xs) = f(xs) + g(xs) = (s \cdot f)(x) + (s \cdot g)(x), \quad (1)$$

$$((s + s') \cdot f)(x) = f(x(s + s')) = f(xs + xs') = f(xs) + f(xs') = (s \cdot f)(x) + (s' \cdot f)(x), \quad (2)$$

$$((ss') \cdot f)(x) = f(xss') = f((xs)s') = (s' \cdot f)(xs) = (s \cdot (s' \cdot f))(x), \quad (3)$$

$$(1_S \cdot f)(x) = f(x1_S) = f(x). \quad (4)$$

From (1)–(4) we see that  $\text{Hom}_A(M, N)$  is a left  $S$ -module. A similar computation shows that  $\text{Hom}_A(M, N)$  is a right  $T$ -module. Moreover,  $((s \cdot f) \cdot t)(x) = (s \cdot f)(xt) = f(xts)t$  and  $(s \cdot (f \cdot t))(x) = f(xts)t$ , so the two actions commute and  $\text{Hom}_A(M, N)$  is an  $S$ – $T$  bimodule.  $\square$

**Proposition 2.1.13** (Eckmann–Schopf). *Let  $Q$  be a divisible abelian group. Then the left  $R$ -module  $\text{Hom}_{\mathbb{Z}}(R, Q)$  is injective.*

*Proof.* By the previous lemma,  $\text{Hom}_{\mathbb{Z}}(R, Q)$  carries a structure of left  $R$ -module given by

$$(r \cdot f)(a) = f(ar), \quad \forall a, r \in R, f \in \text{Hom}_{\mathbb{Z}}(R, Q).$$

Let  $I$  be a left ideal of  $R$  and  $h : I \rightarrow \text{Hom}_{\mathbb{Z}}(R, Q)$  a morphism of left  $R$ -modules. Define

$$\gamma : \mathbb{Z}I \longrightarrow \mathbb{Z}Q, \quad \gamma(a) = h(a)(1).$$

Then  $\gamma$  is a morphism of  $\mathbb{Z}$ -modules. Since  $Q$  is  $\mathbb{Z}$ -injective, there exists  $\tilde{\gamma} : \mathbb{Z}R \rightarrow \mathbb{Z}Q$  such that  $\tilde{\gamma}|_I = \gamma$ . For  $a \in I$  and  $r \in R$  we have

$$(a \cdot \tilde{\gamma})(r) = \tilde{\gamma}(ra) = h(ra)(1) = (r \cdot h(a))(1) = h(a)(r),$$

hence  $h(a) = a \cdot \tilde{\gamma}$  for all  $a \in I$ . By Baer's criterion,  $\text{Hom}_{\mathbb{Z}}(R, Q)$  is an injective left  $R$ -module.  $\square$

**Proposition 2.1.14.** *Every left  $R$ -module  $M$  can be embedded into an injective left  $R$ -module.*

*Proof.* There exists a free abelian group  $\mathbb{Z}^{(A)}$  and a surjective  $\mathbb{Z}$ -morphism  $f : \mathbb{Z}^{(A)} \rightarrow M$ . Hence

$$\mathbb{Z}M \cong \mathbb{Z}^{(A)} / \ker f \subseteq \mathbb{Q}^{(A)} / \ker f,$$

and therefore there is a divisible abelian group  $G$  with  $\mathbb{Z}M \subseteq \mathbb{Z}G$ . Applying the functor  $\text{Hom}_{\mathbb{Z}}(R, -)$  we obtain a monomorphism

$${}_R M \cong \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, G).$$

Since  $G$  is divisible, Proposition 2.1.13 implies that  $\text{Hom}_{\mathbb{Z}}(R, G)$  is an injective left  $R$ -module. Thus  $M$  embeds into an injective left  $R$ -module.  $\square$

**Proposition 2.1.15.** *Let  $Q$  be an  $R$ -module. Then  $Q$  is injective if and only if every short exact sequence*

$$0 \longrightarrow Q \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

*splits.*

*Proof.* “ $\Rightarrow$ ”. Assume  $Q$  is injective. Exactness implies that  $f$  is a monomorphism. By injectivity of  $Q$  there exists  $h : M \rightarrow Q$  with  $hf = \text{id}_Q$ , hence the sequence splits.

“ $\Leftarrow$ ”. By Proposition 2.1.14 there is an injective  $R$ -module  $Q'$  and a monomorphism  $i : Q \rightarrow Q'$ . Consider the exact sequence

$$0 \longrightarrow Q \xrightarrow{i} Q' \longrightarrow Q'/i(Q) \longrightarrow 0.$$

By the hypothesis this sequence splits, so  $Q$  is a direct summand of  $Q'$ . A direct summand of an injective module is injective, therefore  $Q$  is injective.  $\square$

## 2.2 Injective envelopes

**Definition 2.2.1.** Let  $M$  be an  $R$ -module. A pair  $(E, i)$  is called an *injective envelope* of  $M$  if  $E$  is injective and  $i : M \rightarrow E$  is an essential monomorphism.

**Proposition 2.2.2.** Let  $Q$  be an injective  $R$ -module. Then every complement submodule of  $Q$  is a direct summand of  $Q$ .

*Proof.* Let  $K$  be a submodule of  $Q$  and  $N$  a complement of  $K$  in  $Q$ , that is,  $K \cap N = 0$  and  $K + N$  is an essential submodule of  $Q$ . Then  $(K + N)/N \cong Q/N$ . Define  $g : (K + N)/N \rightarrow Q$  by

$$g((x + y) + N) = x, \quad x \in K, y \in N.$$

Since  $K \cap N = 0$ ,  $g$  is well defined and injective. As  $Q$  is injective, there exists  $h : Q/N \rightarrow Q$  with  $h|_{(K+N)/N} = g$ . Because  $(K + N)/N \cong Q/N$  and  $g$  is a monomorphism,  $h$  is also a monomorphism. We have  $K = \text{Im } g = h((K + N)/N) \subseteq h(Q/N)$ . As  $K$  is a closed submodule, it follows that  $K = h(Q/N)$ . Since  $h$  is a monomorphism,  $(K + N)/N = Q/N$ , hence  $K + N = Q$ . Thus  $K$  is a direct summand of  $Q$ .  $\square$

**Theorem 2.2.3** (Eckmann–Schopf). Every  $R$ -module  $M$  has an injective envelope, unique up to isomorphism.

*Proof.* By Proposition 2.1.14 there exists an injective  $R$ -module  $Q$  with  $M \leq Q$ . Let  $E$  be a maximal essential extension of  $M$  in  $Q$ . Then  $E$  is a complement submodule of  $Q$ , and by the previous proposition  $E$  is injective. Hence  $(E, i)$ , where  $i : M \hookrightarrow E$  is the inclusion, is an injective envelope of  $M$ .

For uniqueness, let  $(E_1, i_1)$  and  $(E_2, i_2)$  be two injective envelopes of  $M$ . Since  $E_2$  is injective, there exists  $f : E_1 \rightarrow E_2$  with  $f i_1 = i_2$ . The map  $i_2$  is a monomorphism and  $i_1$  is an essential monomorphism, so (using 1.4)  $f$  is a monomorphism. Thus  $E_1 \cong f(E_1)$  and  $E_2 = f(E_1) \oplus E_3$  for some submodule  $E_3$ . But  $i_2(M) \subseteq f(E_1)$ , hence  $i_2(M) \cap E_3 = 0$ . Since  $i_2$  is essential, we must have  $E_3 = 0$ , so  $E_2 = f(E_1)$  and  $f$  is an isomorphism.  $\square$

In practice we fix one representative of this isomorphism class and denote it by  $E(M)$ , with  $M \leq E(M)$ .

**Proposition 2.2.4.** Let  $M$  be an  $R$ -module and  $i : M \rightarrow Q$  a monomorphism with  $Q_R$  injective. The following statements are equivalent:

1.  $(Q, i)$  is an injective envelope of  $M$ ;
2. for every monomorphism  $f : M \rightarrow Q'$  with  $Q'$  injective, there exists a monomorphism  $g : Q \rightarrow Q'$  such that  $gi = f$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $f : M \rightarrow Q'$  be a monomorphism with  $Q'$  injective. By injectivity of  $Q'$  there is  $u : Q \rightarrow Q'$  with  $ui = f$ . Because  $i$  is an essential monomorphism and  $Q'$  is injective, the image  $u(Q)$  is a complement of  $f(M)$ ; by the definition of injective envelope this forces  $u$  to be a monomorphism. Set  $g = u$ .



(2)  $\Rightarrow$  (1). Let  $(E(M), j)$  be an injective envelope of  $M$ . Applying (2) to  $f = j$  we obtain a monomorphism  $g : Q \rightarrow E(M)$  with  $gi = j$ . Since  $j$  is an essential monomorphism, it follows that  $i$  is also essential; hence  $(Q, i)$  is an injective envelope of  $M$ .  $\square$

**Proposition 2.2.5.** *For any family of right  $R$ -modules  $M_1, M_2, \dots, M_n$  we have*

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

*Proof.* By 1.9,  $\bigoplus_{i=1}^n E(M_i)$  is an essential extension of  $\bigoplus_{i=1}^n M_i$ . Moreover,

$$\bigoplus_{i=1}^n E(M_i) \cong \prod_{i=1}^n E(M_i),$$

so by 2.1.3 the module  $\bigoplus_{i=1}^n E(M_i)$  is injective. By uniqueness of injective envelopes we obtain

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

$\square$

**Theorem 2.2.6.** *Let  $Q$  and  $M$  be two  $R$ -modules. Then  $Q$  is  $M$ -injective if and only if  $f(M) \subseteq Q$  for every  $f \in \text{Hom}(E(M), E(Q))$ .*

*Proof.* “ $\Rightarrow$ ” Let  $f \in \text{Hom}(E(M), E(Q))$  and set

$$K := \{m \in M \mid f(m) \in Q\}.$$

Since  $Q$  is  $M$ -injective, there exists a morphism  $\bar{f} : M \rightarrow Q$  such that  $\bar{f}|_K = f|_K$ . We claim that

$$Q \cap (\bar{f} - f)(M) = 0.$$

Take  $x \in Q$  and  $m \in M$  with  $x = (\bar{f} - f)(m)$ . Then

$$f(m) = \bar{f}(m) - x \in Q,$$

so  $m \in K$ . Hence

$$x = \bar{f}(m) - f(m) = f(m) - f(m) = 0,$$

and therefore  $Q \cap (\bar{f} - f)(M) = 0$ . Since  $Q$  is an essential submodule of  $E(Q)$ , it follows that  $(\bar{f} - f)(M) = 0$ . Thus  $f(M) = \bar{f}(M) \subseteq Q$ .

“ $\Leftarrow$ ” As  $E(Q)$  is injective, it is enough to work with  $f \in \text{Hom}(M, E(Q))$ . Let  $N$  be a submodule of  $M$  and  $g : N \rightarrow Q$  a morphism of  $R$ -modules. Because  $E(Q)$  is injective, there exists  $\tilde{g} : M \rightarrow E(Q)$  such that  $\tilde{g}|_N = i \circ g$ , where  $i : Q \rightarrow E(Q)$  is the canonical injection. By hypothesis,  $\tilde{g}(M) \subseteq Q$ , so identifying  $\tilde{g}$  with its corestriction to  $Q$  we obtain a morphism  $h : M \rightarrow Q$  with  $h|_N = g$ . Therefore  $Q$  is  $M$ -injective.  $\square$

**Corollary 2.2.7.** *An  $R$ -module  $Q$  is quasi-injective if and only if  $f(Q) \subseteq Q$  for every  $f \in \text{End}(E(Q))$ .*

**Theorem 2.2.8** (Matlis–Bass). *Let  $R$  be a ring. Then  $R$  is right noetherian if and only if, for every simple right  $R$ -module  $S_i$  ( $i \geq 1$ ),*

$$Q := \bigoplus_{i=1}^{\infty} E(S_i)$$

*is an injective right  $R$ -module.*

*Proof.* “ $\Rightarrow$ ” Let  $L$  be a right ideal of  $R$ ,

$$Q = \bigoplus_{i=1}^{\infty} E(S_i)$$

and let  $f : L \rightarrow Q$  be a morphism of right  $R$ -modules. There exist elements  $a_1, \dots, a_n \in L$  such that

$$L = a_1 R + a_2 R + \dots + a_n R.$$

Clearly, there is  $m \geq 1$  with  $f(a_k) \in \bigoplus_{j=1}^m E(S_j)$  for all  $k = 1, \dots, n$ , hence

$$\Im f \subseteq \bigoplus_{j=1}^m E(S_j).$$

Since  $\bigoplus_{j=1}^m E(S_j)$  is injective, there exists  $g : R \rightarrow \bigoplus_{j=1}^m E(S_j)$  such that  $g|_L = f$ . Let  $\bar{f} = i \circ g$ , where  $i : \bigoplus_{j=1}^m E(S_j) \rightarrow Q$  is the canonical injection. Then  $\bar{f}|_L = f$ , so  $Q$  is injective.

“ $\Leftarrow$ ” Suppose that  $R$  is not right noetherian. Then there exists a strictly ascending chain of finitely generated right ideals:

$$L_1 \subsetneq L_2 \subsetneq \dots \subsetneq L_n \subsetneq \dots$$

By Krull’s lemma, for every  $n \geq 1$  there exists a maximal submodule  $M_n \subsetneq L_n$  such that

$$L_{n-1} \subseteq M_n \quad \text{for all } n \geq 2.$$

Set

$$L := \bigcup_{k=1}^{\infty} L_k, \quad \pi_k : L_k \longrightarrow L_k/M_k$$

for the canonical projections, and put

$$E_k := E(L_k/M_k) \quad \text{for each } k \geq 1.$$

Then

$$E := \bigoplus_{k=1}^{\infty} E_k$$

is injective, and

$$f : L \longrightarrow E, \quad f(a) = \sum_{k=1}^{\infty} \pi_k(a)$$

is well defined. There exists an element  $x \in E_1 \oplus \cdots \oplus E_n$  such that  $f(a) = xa$  for all  $a \in L$ . It follows that  $\pi_k(a) = 0$  for every  $k \geq n + 1$ , that is,  $a \in M_k$  for all  $k \geq n + 1$ . Hence

$$L \subseteq M_{n+1} \subsetneq L_{n+1} \subseteq M_{n+2} \subsetneq L_{n+2} \subseteq \cdots \subseteq L,$$

a contradiction. Therefore  $R$  is right noetherian. □



# Chapter 3

## Direct Sums of Uniform (Co-irreducible) Modules

**Proposition 3.1.** *Let  $M$  be an  $R$ -module and  $E(M)$  its injective envelope. Then the following statements are equivalent:*

1.  $E(M)$  is indecomposable.
2. If  $L$  and  $K$  are non-zero submodules of  $M$ , then  $L \cap K \neq 0$ .
3. If  $x, y \in M \setminus \{0\}$ , then there exist  $a, b \in R$  such that  $0 \neq xa = yb$ .
4.  $M$  is an essential extension of each of its non-zero submodules.

*Proof.* (2)  $\Rightarrow$  (1). Assume that  $E(M) = L' \oplus K'$  with  $L'$  and  $K'$  non-zero submodules of  $E(M)$ . Since  $M \leq E(M)$ , we have  $L = L' \cap M \neq 0$  and  $K = K' \cap M \neq 0$ , hence

$$L \cap K = (L' \cap K') \cap M = 0,$$

which contradicts (2).

(1)  $\Rightarrow$  (2). Assume there exist non-zero submodules  $L, K \leq M$  such that  $L \cap K = 0$ . Then  $E(L) \leq E(M)$  and  $K \cap E(L) = 0$  (otherwise, since  $L \subseteq E(L)$ , we would have  $L \cap K \neq 0$ ). The short exact sequence

$$0 \longrightarrow E(L) \longrightarrow E(M) \longrightarrow E(M)/E(L) \longrightarrow 0$$

splits because  $E(L)$  is injective, hence  $E(M) \cong E(L) \oplus E(M)/E(L)$ . It follows that

$$0 = K \cap E(L) = K \cap E(M) = K,$$

a contradiction.

The equivalences (2)  $\Leftrightarrow$  (3) and (2)  $\Leftrightarrow$  (4) are straightforward.  $\square$

**Definition 3.2.** An  $R$ -module  $M$  satisfying any of the equivalent conditions above is called *uniform* (or *co-irreducible*). Let  $M$  be an  $R$ -module and  $N$  a proper submodule of  $M$ . If  $M/N$  is uniform, we say that  $N$  is *irreducible in  $M$* . Clearly,  $N$  is irreducible in  $M$  if and only if from an equality  $N = P \cap Q$  with  $P, Q \leq M$  it follows that  $N = P$  or  $N = Q$ .

**Example.**

1. Every simple module is clearly uniform.
2.  $\mathbb{Z}$  is a uniform  $\mathbb{Z}$ -module.
3. By Proposition 3.1, an injective module is indecomposable if and only if it is uniform. In particular, if  $M$  is a uniform  $R$ -module, then  $E(M)$  is uniform. Hence the  $\mathbb{Z}$ -modules  $\mathbb{Q}$  and  $\mathbb{Z}_{p^\infty}$  are uniform.
4. The  $\mathbb{Z}$ -module  $\mathbb{Z}_n$  is uniform if and only if  $n$  is a power of a prime. If  $n = p^k$  with  $p \geq 2$  and  $k \geq 1$ , then  $\langle p^m \rangle \subseteq \langle p^i \rangle \cap \langle p^j \rangle$  for all  $i, j \in \{1, \dots, k-1\}$ , where  $m = \min(i, j)$ , so  $\mathbb{Z}_{p^k}$  is uniform. Conversely, assume that  $\mathbb{Z}_n$  is uniform and write  $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$  as the factorisation of  $n$  into primes. If  $s \geq 2$ , then

$$\langle p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} \rangle + \mathbb{Z} \quad \text{and} \quad \langle p_s^{\alpha_s} \rangle + \mathbb{Z}$$

are non-zero submodules whose intersection is zero, a contradiction. Hence  $s = 1$  and  $n$  is a prime power.

**Proposition 3.3.** *Let  $M$  be an  $R$ -module and  $N$  a proper submodule of  $M$ . If  $x \in M \setminus N$ , then there exists an irreducible submodule  $P \leq M$  such that  $N \subseteq P$  and  $x \notin P$ .*

*Proof.* Set

$$\mathfrak{S} = \{ N' \leq M \mid N \subseteq N' \text{ and } x \notin N' \}.$$

The set  $\mathfrak{S}$ , ordered by inclusion, is inductive, so by Zorn's Lemma it has a maximal element  $P$ . We show that  $P$  is irreducible in  $M$ . Assume that  $P = U \cap V$  with  $U, V \leq M$  and  $P \subsetneq U$ ,  $P \subsetneq V$ . Since  $x \notin P$ , we have  $x \notin U$  or  $x \notin V$ ; suppose  $x \notin U$ . Then  $U \in \mathfrak{S}$  and  $P \subsetneq U$ , contradicting the maximality of  $P$ . Hence  $P$  is irreducible.  $\square$

**Corollary 3.4.** *Let  $M$  be an  $R$ -module and  $N$  a proper submodule of  $M$ . Then  $N$  is the intersection of irreducible submodules of  $M$ .*

*Proof.* This follows immediately from the previous proposition.  $\square$

**Proposition 3.5.** *Let  $M$  be a uniform  $R$ -module. Then:*

1. *Every non-zero submodule of  $M$  is uniform.*
2. *Every essential extension of  $M$  is uniform.*

*Proof.* 1) This follows immediately from 3.1.

2) Let  $M \trianglelefteq E$  be an essential extension of  $M$ . If  $E_1, E_2$  are non-zero submodules of  $E$ , then  $M \cap E_1 \neq 0$  and  $M \cap E_2 \neq 0$ , hence

$$(M \cap E_1) \cap (M \cap E_2) \neq 0,$$

and therefore  $E_1 \cap E_2 \neq 0$ . Thus  $E$  is uniform.  $\square$

**Lemma 3.6.** *Let  $M$  be an  $R$ -module and  $(M_\alpha)_{\alpha \in \Lambda}$  a family of independent submodules. If  $N$  is a submodule of  $M$  such that*

$$N \cap \left( \sum_{\alpha \in \Lambda} M_\alpha \right) \neq 0,$$

*then there exists a non-zero submodule of some  $M_\alpha$  that is isomorphic to a submodule of  $N$ .*

*Proof.* If  $\text{Card } \Lambda = 1$ , the statement is clear.

If  $\text{Card } \Lambda = 2$ , set  $P = N \cap (M_1 + M_2)$ . If  $N \cap M_1 = 0$ , then  $P \cong (P + M_1)/M_1$  and

$$(P + M_1)/M_1 \leq (M_1 + M_2)/M_1 \cong M_2.$$

Hence  $P$  is isomorphic to a submodule of  $M_2$ . If  $\Lambda$  is finite, arguing by induction reduces the proof to this case.

If  $\Lambda$  is infinite, choose  $x \in N \cap \left( \sum_{\alpha \in \Lambda} M_\alpha \right)$ ,  $x \neq 0$ . Then there exist  $\alpha_1, \dots, \alpha_n \in \Lambda$  such that

$$xR \cap (M_{\alpha_1} + M_{\alpha_2} + \dots + M_{\alpha_n}) \neq 0,$$

and we are reduced to the case where  $\Lambda$  is finite. □

*Remark 3.7.* Let  $M$  be a right  $R$ -module and let  $\Omega$  be the set of uniform submodules of  $M$ . It may happen that  $\Omega = \emptyset$ . Set

$$\mathcal{S} = \{ \Omega' \subseteq \Omega \mid \sum_{N \in \Omega'} N \text{ is a direct sum} \}.$$

The partially ordered set  $(\mathcal{S}, \subseteq)$  is inductive and, by Zorn's lemma,  $\mathcal{S}$  has a maximal element  $\Omega_0$ . Put  $S = \bigoplus_{N \in \Omega_0} N$ . We say that  $S$  is a maximal direct sum of uniform submodules of  $M$ .

**Proposition 3.8.** *Let  $M$  be an  $R$ -module and  $S$  a maximal direct sum of uniform submodules of  $M$ . If  $N$  is a non-zero submodule of  $M$ , then the following statements are equivalent:*

1.  $S \cap N \neq 0$ ;
2.  $N$  contains a uniform submodule.

*Proof.* Implication (1)  $\Rightarrow$  (2) follows from Lemma 3.6.

Implication (2)  $\Rightarrow$  (1) follows from the maximality of  $S$ . □

**Theorem 3.9.** *Let  $M$  be an  $R$ -module and  $S$  a maximal direct sum of uniform submodules of  $M$ . There exists a submodule  $K$ , maximal among the submodules of  $M$  that contain no uniform submodule, with the properties:*

1.  $S + K$  is a direct sum;
2.  $(S \oplus K) \leq M$ .

*Proof.* 1. Let

$$\mathcal{S} = \{ L \leq M \mid \forall L' \leq L, L' \text{ is not uniform} \}.$$

Then  $\mathcal{S} \neq \emptyset$  because  $0 \in \mathcal{S}$  and  $\mathcal{S}$  is inductive. By Zorn's lemma,  $\mathcal{S}$  has a maximal element  $K$ . By Proposition 3.8 we have  $S \cap K = 0$ .

2. Suppose  $N \cap (S \oplus K) = 0$ , where  $N$  is a non-zero submodule of  $M$ . Then  $(N + K) \cap S = 0$ , and by Proposition 3.8,  $N + K$  contains no uniform submodule, which contradicts the maximality of  $K$ . Hence  $(S \oplus K) \trianglelefteq M$ . □

**Definition 3.10.** Let  $M$  be an  $R$ -module. A finite intersection of submodules

$$\bigcap_{i \in I} N_i \quad (I \text{ finite})$$

is called *reduced* if, for every  $i \in I$ , we have

$$\bigcap_{i \in I} N_i \neq \bigcap_{\substack{j \in I \\ j \neq i}} N_j.$$

It is clear that if we have a finite intersection  $N = \bigcap_{i \in I} N_i$  of irreducible submodules, then it can always be written as a reduced intersection of irreducible submodules.

**Theorem 3.11** (Kuroš–Ore Theorem). *Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . Assume that  $N$  has finite reduced intersections:*

$$N = N_1 \cap N_2 \cap \cdots \cap N_m = L_1 \cap L_2 \cap \cdots \cap L_n,$$

where each  $N_i$  and each  $L_j$  is an irreducible submodule of  $M$ . Then  $m = n$ .

*Proof.* Let  $N'_i = N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_m$ . Then  $N \subseteq N'_i$  and  $N = N_i \cap N'_i$ . For each  $j$ , set  $P_j = N_i \cap L_j$ . Then

$$N \subseteq P_j \subseteq N'_i \quad \text{and} \quad P_j \subseteq L_j.$$

Hence

$$N \subseteq \bigcap_{j=1}^n P_j \subseteq \bigcap_{j=1}^n L_j = N,$$

so the intersection of the  $P_j$  is exactly  $N$ .

Since  $N_i$  is irreducible in  $M$ , it is irreducible in  $N_i + N'_i$ , and therefore

$$\frac{N'_i}{N'_i \cap N_i} \cong \frac{N_i + N'_i}{N_i}$$

is uniform (co-irreducible), showing that  $N$  is irreducible in  $N'_i$ .

Thus from  $N = \bigcap_{j=1}^n P_j$ , there exists some  $j$  with  $N = P_j = N_i \cap L_j$ .



Set  $I = \{1, \dots, m\}$ ,  $J = \{1, \dots, n\}$ , and choose  $i_1 \in I$ . Then for some  $j_1 \in J$ ,

$$N = L_{j_1} \cap \left( \bigcap_{i \neq i_1} N_i \right).$$

Since the intersection is finite we may reduce it step by step, maintaining at each step at least one  $L_{j_k}$  that does not disappear. We obtain a decreasing chain

$$J \supseteq J_1 \supseteq \dots \supseteq J_r,$$

with  $r \leq m$ , such that for the final set  $J' \subseteq J$ ,

$$N = \bigcap_{j \in J'} L_j.$$

Because the intersection  $L_1 \cap \dots \cap L_n$  is reduced, we must have  $J' = J$ . Since  $J'$  has at most  $r$  elements,  $n \leq r \leq m$ . A symmetric argument gives  $m \leq n$ .  $\square$

**Theorem 3.12.** *Let  $M$  be an  $R$ -module and let  $(N_i)_{i=1}^m$  and  $(L_j)_{j=1}^n$  be two independent families of uniform (co-irreducible) submodules such that*

$$\bigoplus_{i=1}^m N_i \subseteq M \quad \text{and} \quad \bigoplus_{j=1}^n L_j \subseteq M$$

*are essential in  $M$ . Then  $m = n$ .*

*Proof.* For each  $k$ , set  $N'_k = \bigoplus_{i \neq k} N_i$ . Then  $N_k \cap N'_k = 0$ .

Consider

$$\mathfrak{S}_k = \{P \leq M \mid N_k \cap P = 0 \text{ and } N_k \subseteq P\}.$$

This set is inductive, so by Zorn's lemma it has a maximal element  $P_k$ . If  $Q \subseteq Q' \subseteq M$  and  $P_k \subseteq Q \subseteq Q'$ , then the irreducibility of  $N_k$  forces  $P_k = Q = Q'$ . Thus each  $P_k$  is co-irreducible.

For any  $k$ ,

$$N = \bigoplus_{k=1}^m N_k = N_k \oplus N'_k.$$

By the modular law,

$$N \cap P_k = (N_k \oplus N'_k) \cap P_k = N_k + (N'_k \cap P_k),$$

so  $N \cap P_k = N_k$ . Hence

$$N \cap \bigcap_{k=1}^m P_k = \bigcap_{k=1}^m (N \cap P_k) = \bigcap_{k=1}^m N_k = 0.$$

But since  $N$  is essential in  $M$ , this implies  $\bigcap_{k=1}^m P_k = 0$ .

Moreover,  $N_i = \bigcap_{k \neq i} P_k$ , so each such intersection is reduced.

Repeating the argument for  $(L_j)$ , we obtain irreducible submodules  $(Q_j)$  with  $\bigcap_j Q_j = 0$ .

Applying the Kuroš–Ore theorem gives  $m = n$ .  $\square$

**Definition 3.13.** An  $R$ -module  $M$  is said to have *finite uniform (co-irreducible) dimension* if there exists a finite independent family  $(N_i)_{i=1}^n$  of uniform submodules of  $M$  such that

$$\bigoplus_{i=1}^n N_i \leq M.$$

In this case the integer  $n$  is called the *uniform (co-irreducible) dimension* of  $M$  and we write  $\dim M = n$ .

**Theorem 3.14.** *Let  $M$  be an  $R$ -module. The following statements are equivalent:*

1.  $M$  has finite uniform (co-irreducible) dimension.
2.  $M$  satisfies the ascending chain condition for direct sums.

*Proof.* (2)  $\Rightarrow$  (1). Let  $\mathcal{S}'$  and  $\mathcal{S}$  be two finite families of independent submodules of  $M$ . We say that  $\mathcal{S}'$  is a *refinement* of  $\mathcal{S}$  if every submodule in  $\mathcal{S}'$  is contained in some submodule from  $\mathcal{S}$ . It is easy to check that refinement defines a partial order on the set of families of submodules of  $M$ . Since  $M$  satisfies (2), there exists a family  $\mathcal{S}_0$  of independent submodules of  $M$  which is maximal with respect to this order.

Take  $N \in \mathcal{S}_0$ . If  $N$  is not uniform, then there exist non-zero submodules  $P, Q \leq N$  with  $P \cap Q = 0$ . From  $\mathcal{S}_0$  we can then construct a proper refinement of  $\mathcal{S}_0$ , a contradiction. Hence every submodule in  $\mathcal{S}_0$  is uniform, so  $M$  contains a uniform submodule. If  $L$  is a non-zero submodule of  $M$ , then  $L$  also satisfies (2), and the same argument shows that  $L$  contains a uniform submodule.

Now let

$$S = \bigoplus_{i \in I} N_i$$

be a maximal direct sum of uniform submodules of  $M$  and let  $N \leq M$  be non-zero. Since  $M$  satisfies the ascending chain condition on direct sums, the index set  $I$  is finite. By Proposition 3.8 we have  $S \cap N \neq 0$ , so  $S \leq M$ . By definition,  $M$  has finite uniform dimension.

(1)  $\Rightarrow$  (2). Let

$$S = \bigoplus_{i=1}^n N_i$$

be a direct sum of uniform submodules with  $S \leq M$ . If  $N$  is a non-zero submodule of  $M$ , then  $S \cap N \neq 0$ , and by Lemma 3.6 the submodule  $N$  contains a uniform submodule.

Let  $(L_j)_{j=1}^m$  be an independent family of submodules of  $M$ . From the previous paragraph, each  $L_j$  contains a uniform submodule  $P_j$ . Clearly, the family  $(P_j)_{j=1}^m$  is independent. By Theorem 3.12 we deduce  $m \leq n$ . Thus every independent family of submodules has bounded finite length, which is equivalent to the ascending chain condition for direct sums.  $\square$

**Remark 3.15.** Let  $M$  be an  $R$ -module. Then:

1. If  $N \leq M$ , then  $M$  has finite uniform (co-irreducible) dimension if and only if  $N$  has finite uniform dimension, and in this case  $\dim M = \dim N$ .

2. If  $M = \bigoplus_{i=1}^n N_i$ , then  $M$  has finite uniform dimension if and only if each  $N_i$  has finite uniform dimension for all  $i = 1, \dots, n$ . Moreover,

$$\dim M = \sum_{i=1}^n \dim N_i.$$



# Chapter 4

## Ore domains and Goldie rings

**Definition 4.1.** An integral domain  $D$  is called a *right Ore domain* (respectively a left Ore domain) if  $aD \cap bD \neq 0$  (respectively  $Da \cap Db \neq 0$ ) for all  $a, b \in D \setminus \{0\}$ .

**Proposition 4.2.** *Let  $D$  be an integral domain. Then  $D$  is a right Ore domain if and only if  $D$  contains a uniform right ideal. In particular, this condition holds whenever  $D$  satisfies the ascending chain condition on direct sums of right ideals.*

*Proof.* “ $\Rightarrow$ ” If  $D$  is a right Ore domain, then the right module  $D_D$  is uniform.

“ $\Leftarrow$ ” Let  $X$  be a uniform right ideal of  $D$  and  $x \in X \setminus \{0\}$ . Consider the map

$$\varphi_x : D_D \longrightarrow X, \quad \varphi_x(a) = xa.$$

This is an injective homomorphism of right  $D$ -modules. Hence  $D_D$  is uniform and therefore  $D$  is a right Ore domain.  $\square$

**Example 4.3.** Let  $K$  be a commutative field and  $y$  an indeterminate. Let  $K(y)$  be the field of fractions of the polynomial ring  $K[y]$ . If  $x$  is another indeterminate, we consider  $K(y)[x]$ , the set of polynomials in the indeterminate  $x$  with coefficients in  $K(y)$ . On this set we define two operations: addition is the usual addition of polynomials, and multiplication in  $K(y)[x]$  is defined by the commutation rule

$$xf(y) = f(y^2)x, \quad f \in K(y).$$

It is straightforward to check that  $K(y)[x]$  with these two operations is a unital ring. We write  $A = K(y)[x]$ . Then  $A$  is a non-commutative integral domain.

Every element of  $A$  can be written uniquely in the form

$$P(x) = f_n(y)x^n + f_{n-1}(y)x^{n-1} + \cdots + f_1(y)x + f_0(y),$$

where  $n$  is a natural number,  $f_0(y), f_1(y), \dots, f_n(y) \in K(y)$  and  $f_n(y) \neq 0$ . We set  $\deg P(x) = n$ .

If  $P(x), Q(x) \in A$  with  $Q(x) \neq 0$ , then there exist  $S(x), R(x) \in A$  such that

$$P(x) = S(x)Q(x) + R(x) \tag{*}$$

with either  $R(x) = 0$  or  $\deg R(x) < \deg Q(x)$ . From (\*) it follows that  $A$  is a left principal ideal ring. Hence  $A$  satisfies the ascending chain condition on direct sums of left ideals, and therefore  $A$  is a left Ore domain.

We now show that  $xA \cap yA = 0$ . Let  $\alpha \in xA \cap yA$ . Then

$$\alpha = x(f_n(y)x^n + \cdots + f_0(y)) = yx(g_m(y)x^m + \cdots + g_0(y)),$$

so that

$$f_n(y^2)x^{n+1} + \cdots + f_0(y^2)x = y(g_m(y^2)x^{m+1} + \cdots + g_0(y^2)x),$$

and hence  $m = n$ . Thus

$$f_n(y^2) = yg_m(y^2), \dots, f_0(y^2) = yg_0(y^2),$$

which holds if and only if  $f_n(y) = \cdots = f_1(y) = f_0(y) = 0$ , and therefore  $\alpha = 0$ . Consequently,  $A$  is not a right Ore domain.

## GOLDIE RINGS

**Definition 4.4.** A right (respectively left) ideal  $I$  is called a *right annihilator ideal* (respectively a *left annihilator ideal*) if there exists a non-empty subset  $X$  of  $R$  such that  $I = \text{ann}_r(X)$  (respectively  $I = \text{ann}_l(X)$ ).

A ring  $R$  is called a *right Goldie ring* if  $R$  satisfies the ascending chain condition on right annihilator ideals and there is no infinite direct sum of non-zero right ideals in  $R$ .

### Example 4.5.

1. If  $R$  is right noetherian, then  $R$  is a right Goldie ring. In particular,  $\mathbb{Z}$  is a Goldie ring.
2. If  $R$  is an integral domain, then  $R$  is a right Goldie ring if and only if  $R$  is a right Ore domain.
3. From Example 4.3 it follows that there exist rings which are Goldie on the left but not on the right.

**Proposition 4.6.** *Let  $R$  be a ring. Then  $R$  satisfies the ascending chain condition on right (respectively left) annihilator ideals if and only if  $R$  satisfies the descending chain condition for left (respectively right) annihilator ideals.*

*Proof.* “ $\Rightarrow$ ” Let  $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$  be a descending sequence of left annihilator ideals. For each  $j \geq 1$  there exists a non-empty subset  $X_j \subseteq R$  such that  $I_j = \text{ann}_l(X_j)$ .

We thus obtain an ascending sequence of right annihilator ideals

$$\text{ann}_r(I_1) \subseteq \text{ann}_r(I_2) \subseteq \cdots \subseteq \text{ann}_r(I_n) \subseteq \cdots$$

Hence there exists  $m \geq 1$  such that  $\text{ann}_r(I_m) = \text{ann}_r(I_{m+k})$  for all  $k \geq 1$ . It follows that

$$\text{ann}_l(\text{ann}_r(I_m)) = \text{ann}_l(\text{ann}_r(I_{m+k})) \quad \text{for all } k \geq 1,$$

that is,  $I_m = I_{m+k}$  for all  $k \geq 1$ .

Using the relations  $\text{ann}_l(\text{ann}_r(\text{ann}_l(X))) = \text{ann}_l(X)$  and  $\text{ann}_r(Y) \subseteq \text{ann}_r(X)$  whenever  $X \subseteq Y$ , the converse implication “ $\Leftarrow$ ” is proved in the same way.  $\square$

**Proposition 4.7.** *Let  $R$  be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If  $I$  and  $J$  are right ideals with  $I \subseteq J$ , then there exists an element  $b \in R$  such that  $bJ \neq 0$  and  $bJ \cap I = 0$ .*

*Proof.* By the preceding lemma there exists a minimal element  $a$  in the set of left annihilator ideals such that  $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(I)$ . Then  $aJ \neq 0$ ; since  $R$  is semiprime we have  $aJ \cdot aJ \neq 0$ , so there exist  $x \in J$  and  $a \in aJ$  with  $b = xa$  such that  $xaJ \neq 0$ . Clearly  $bJ \neq 0$ .

It remains to show that  $bJ \cap I = 0$ . Take  $\lambda \in bJ \cap I$ ; then  $\lambda = b\mu = xa\mu \in I$  for some  $\mu \in J$ . Since  $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\lambda)$ , we obtain

$$\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\lambda) \subseteq \text{ann}_\ell(\mu).$$

Because  $a \in aJ \subseteq aR$ , we have

$$xa \in a \quad \text{and hence} \quad xaJ \subseteq aJ. \quad (4.1)$$

On the other hand,  $xaJ \subseteq I$ , while  $\lambda \in I$  and  $a \subseteq \text{ann}_r(I)$ ; therefore

$$a \subseteq \text{ann}_\ell(\mu). \quad (4.2)$$

Since  $xaJ \neq 0$ , we also have

$$a \not\subseteq \text{ann}_\ell(J). \quad (4.3)$$

From (4.1), (4.2) and (4.3) we obtain  $\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\mu) \cap a$ . If  $\lambda \neq 0$ , then  $xa\mu \notin \text{ann}_\ell(\mu)$ , and, since  $xa \in a$ , it follows that  $xa\mu \neq 0$ , contradicting the minimality of  $a$ . Thus  $\lambda = 0$ , and hence  $bJ \cap I = 0$ .  $\square$

**Corollary 4.8.** *If  $R$  is a right Goldie semiprime ring, then  $R$  satisfies the descending chain condition on right annihilator ideals.*

*Proof.* Assume that there exists a strictly descending sequence of right annihilator ideals

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$$

Then for every  $n \geq 1$  we have  $\text{ann}_\ell(I_n) \neq \text{ann}_\ell(I_{n+1})$ . By Proposition 4.7, for each  $n \geq 1$  there exists a right ideal  $K_n$  such that  $K_n \subseteq I_n$  and  $K_n \cap I_{n+1} \neq 0$ . Hence the sum  $\sum_{n \geq 1} K_n$  is an infinite direct sum of non-zero right ideals, which contradicts the right Goldie property of  $R$ .  $\square$

**Proposition 4.9.** *Let  $R$  be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If  $x, y \in R$  are such that  $xR$  and  $yR$  are essential in  $R_R$ , then  $xyR$  is essential.*

*Proof.* Let  $I \leq R_R$  with  $I \neq 0$ . Define

$$J = \{I : x\} = \{a \in R \mid xa \in I\}.$$

Then  $J$  is a right ideal of  $R$ , and  $xJ = xR \cap I \neq 0$  because  $xR \leq R_R$ . Since  $\text{ann}_r(x) \subseteq J$ ,  $xJ \neq 0$  and  $\text{ann}_\ell(x) = 0$ , we obtain

$$\text{ann}_\ell(J) \subseteq \text{ann}_\ell(\text{ann}_r(x)).$$

By Proposition 4.7 there exists a right ideal  $K \neq 0$  such that  $K \subseteq J$  and  $K \cap \text{ann}_r(x) = 0$ .

Now set

$$L = \{K : y\} = \{a \in R \mid ya \in K\}.$$

Since  $yR \trianglelefteq R_R$ , we have  $yL = yR \cap K \neq 0$ . If  $xyL = 0$ , then  $yL \subseteq \text{ann}_r(x)$  and hence  $yL \subseteq \text{ann}_r(x) \cap K = 0$ , a contradiction. Thus  $xyL \neq 0$ .

Because  $xyL \subseteq xK \subseteq xJ \subseteq xR \cap I$ , it follows that  $0 \neq xyL \subseteq xyR \cap I$ , showing that  $xyR \trianglelefteq R_R$ .  $\square$

**Proposition 4.10.** *Let  $R$  be a ring which satisfies the ascending chain condition on right annihilator ideals. Then, for every  $a \in R$  there exists an integer  $k \geq 0$  such that*

$$\text{ann}_r(a^n) = \text{ann}_r(a^m) \quad \text{for all } n \geq 0 \text{ and } m \geq k.$$

*Proof.* We have the ascending chain of right annihilator ideals

$$\text{ann}_r(a) \subseteq \text{ann}_r(a^2) \subseteq \cdots \subseteq \text{ann}_r(a^n) \subseteq \cdots.$$

Let  $k \geq 0$  be such that  $\text{ann}_r(a^k) = \text{ann}_r(a^m)$  for every  $m \geq k$ . If  $\lambda \in \text{ann}_r(a^n) \cap a^m R$ , then  $a^n \lambda = 0$  and  $\lambda = a^m \mu$ , hence  $a^{m+n} \mu = 0$ . Since  $m + n \geq k$ , it follows that  $\mu \in \text{ann}_r(a^k)$  and thus  $a^k \mu = 0$ . As  $m \geq k$ , we may write  $\lambda = a^{m-k} a^k \mu = 0$ , and therefore

$$\text{ann}_r(a^n) \cap a^m R = 0.$$

$\square$

**Corollary 4.11.** *Assume the hypotheses of Proposition 4.10. If  $xR \trianglelefteq R_R$ , then  $x$  is a regular element of  $R$ .*

*Proof.* If  $\text{ann}_r(x) \neq 0$ , then  $0 = \text{ann}_r(R) \neq \text{ann}_r(xR)$ , so there exists a right ideal  $I \neq 0$  such that  $I \cap xR = 0$ , a contradiction. Hence  $\text{ann}_r(x) = 0$ .

By Proposition 4.9 we have  $x^n R \trianglelefteq R_R$  for every  $n \geq 1$ . Applying Proposition 4.10, we obtain  $k \geq 0$  such that  $\text{ann}_r(x^n) \cap x^m R = 0$  for all  $m \geq k$ . In particular  $\text{ann}_r(x) = 0$ , so  $x$  is a regular element of  $R$ .  $\square$

**Corollary 4.12.** *Let  $R$  be a right Goldie and semiprime ring. If  $\text{ann}_r(x) = 0$  for some  $x \in R$ , then  $xR \trianglelefteq R_R$  and  $x$  is regular.*

*Proof.* Let  $I$  be a non-zero right ideal of  $R$  and suppose  $I \cap xR = 0$ . Then the sum

$$\sum_{n \geq 1} x^n I$$

is direct. Indeed, for  $p = 1$  we have  $x^p I \cap \sum_{n \neq p} x^n I = xI \cap x^p I$ . If  $y \in xI \cap x^p I$ , then  $y = x\lambda = x^2\mu$  for some  $\lambda, \mu \in I$ . Hence  $\lambda - x\mu \in \text{ann}_r(x)$  and so  $\lambda = x\mu$ , which implies  $\lambda \in I \cap xR = 0$ . Thus  $\lambda = 0$  and consequently  $y = 0$ . Therefore  $xI \cap x^2 I = 0$ , and a fortiori  $xI \cap x^p I = 0$  for  $p \neq 1$ . Hence  $x^p I \cap \sum_{n \neq p} x^n I = 0$ , so the sum  $\sum_{n \geq 1} x^n I$  is direct, a contradiction.

Thus  $xR \trianglelefteq R_R$ . By Corollary 4.11 we conclude that  $x$  is a regular element of  $R$ .  $\square$



**Proposition 4.13.** *Let  $R$  be a right Goldie and semiprime ring. Then:*

- (a) *Every two-sided right annihilator ideal of  $R$  contains a minimal two-sided right annihilator ideal.*
- (b) *There exists a finite direct sum of non-zero minimal two-sided right annihilator ideals of  $R$  which is essential in  $R$ .*

*Proof.* (a) This follows from Corollary 4.8.

(b) Let  $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$  be a maximal direct sum where  $I_1, \dots, I_n$  are non-zero minimal two-sided right annihilator ideals of  $R$ . Let  $K \leq R_R$ ,  $K \neq 0$  with  $I \cap K = 0$ . Since  $KI \subseteq I \cap K = 0$ , we have  $K \subseteq \text{ann}_r(I)$ . As  $R$  is semiprime,  $I \cap \text{ann}_r(I) = 0$ , hence  $I \text{ann}_r(I) = 0$  and thus  $\text{ann}_l(I) \subseteq \text{ann}_r(I)$ , so in particular  $\text{ann}_r(I) \neq 0$ . Again,  $R$  being semiprime implies  $I \cap \text{ann}_l(I) = 0$ , where  $\text{ann}_l(I)$  is a two-sided ideal. By (a) there exists a non-zero two-sided right annihilator ideal  $J$ , minimal with respect to  $J \subseteq \text{ann}_r(I)$  and  $J \cap I = 0$ , which contradicts the maximality of  $I$  as a direct sum. Hence  $I \trianglelefteq R$ .  $\square$

**Proposition 4.14.** *Let  $R$  be a right Goldie and prime ring. If  $I$  is an essential right ideal of  $R$ , then  $I$  contains a regular element of  $R$ .*

*Proof.* Choose  $a \in I$  such that  $\text{ann}_r(a)$  is minimal in the set  $\{\text{ann}_r(x) \mid x \in I\}$ . Let  $J \leq R_R$ ,  $J \neq 0$  with  $aR \cap J = 0$ . Since  $I \trianglelefteq R_R$ , we have  $I \cap J \neq 0$ ; we may assume that  $J$  is a non-zero right ideal of  $R$  contained in  $I$  and still satisfying  $aR \cap J = 0$ .

Let  $x \in J$ . If  $\lambda \in \text{ann}_r(a + x)$ , then  $a\lambda + x\lambda = 0$ , so  $x\lambda = -a\lambda \in aR \cap J$ , hence  $x\lambda = 0$  and consequently  $a\lambda = 0$ . Thus  $\lambda \in \text{ann}_r(a) \cap \text{ann}_r(x)$ . Since the inclusion

$$\text{ann}_r(a) \cap \text{ann}_r(x) \subseteq \text{ann}_r(a + x)$$

is obvious, we obtain  $\text{ann}_r(a + x) = \text{ann}_r(a) \cap \text{ann}_r(x)$ . By the minimality of  $\text{ann}_r(a)$  it follows that  $\text{ann}_r(a) \subseteq \text{ann}_r(x)$  and hence  $x \text{ann}_r(a) = 0$ , so  $J \text{ann}_r(a) = 0$ . As  $R$  is prime, we must have  $\text{ann}_r(a) = 0$ .

By Corollary 4.12 we deduce  $J = 0$ , a contradiction. Therefore  $aR \trianglelefteq R_R$ , and Corollary 4.11 implies that  $a$  is regular in  $R$ .  $\square$



# Chapter 5

## The Osofsky–Smith Theorem

**Definition 5.1.** An  $R$ -module  $M$  is called a *CS-module* if every submodule which is a complement of  $M$  is a direct summand of  $M$ . The module  $M$  is called *CS-complete* if  $M/N$  is a CS-module for every  $N \leq M$ .

*Remark 5.2.* From 2.2.2 it follows that every injective module is a CS-module.

**Proposition 5.3.** For an  $R$ -module  $M$  the following statements are equivalent:

1.  $M$  is a CS-module.
2. Every maximal essential extension of a submodule of  $M$  is a direct summand of  $M$ .
3. Every submodule of  $M$  is essential in a direct summand of  $M$ .

*Proof.* (1)  $\Rightarrow$  (2) Let  $N$  be a submodule of  $M$  and  $Q$  a maximal essential extension of  $N$  in  $M$  (Proposition 1.13). Then  $Q$  is a closed submodule, and by 1.15 it follows that  $Q$  is a direct summand of  $M$ .

(2)  $\Rightarrow$  (3) is immediate from Proposition 1.13.

(3)  $\Rightarrow$  (1) Let  $K$  be a complement submodule of  $M$ . There exists a direct summand  $Q$  of  $M$  such that  $K \leq Q$ . Since  $K$  is a closed submodule, we obtain  $K = Q$ , and therefore  $K$  is a direct summand of  $M$ .  $\square$

**Proposition 5.4.** Every quasi-injective module is a CS-module.

*Proof.* Let  $M$  be a quasi-injective  $R$ -module and  $N$  a submodule of  $M$ . Then there exists  $E_2 \leq E(M)$  such that  $E(M) = E_1 \oplus E_2$ , where  $E_1 = E(N)$ . Let  $\pi_i : E(M) \rightarrow E_i$ ,  $i = 1, 2$ , be the canonical projections. Since  $\pi_i(M) \leq M$  for all  $i = 1, 2$  (Corollary 2.2.7), it follows that

$$M = (M \cap E_1) \oplus (M \cap E_2).$$

Clearly  $N \leq (M \cap E_1)$ , so  $N$  is essential in a direct summand of  $M$ . By the previous proposition we conclude that  $M$  is a CS-module.  $\square$

**Lemma 5.5.** Let  $X$  be a cyclic CS-module with the property that  $S = \text{soc}(X) \leq X_R$ , where  $S$  is not finitely generated, but every finitely generated submodule of  $S$  is a direct summand of  $X$ . If every cyclic submodule of  $X$  is a CS-module, then  $X/S$  is not a CS-module.

*Proof.* Assume that  $X/S$  is a CS-module. Since  $S$  is not finitely generated, we can write

$$S = \bigoplus_{i \geq 1} S_i,$$

with  $S_i \leq X$  and  $S_i$  not finitely generated for every  $i \geq 1$ .

For every  $i \geq 1$  there exists a complement submodule  $D_i$  of  $X$  such that  $S_i \leq D_i$ . Then  $D_i$  is a direct summand of  $X$ , hence  $D_i$  is cyclic for all  $i \geq 1$ , and therefore  $S_i \subsetneq D_i$ .

Since  $X/S$  is a CS-module, there exists a direct summand  $\overline{E}$  of  $X$  such that

$$\sum_{i \geq 1} \frac{D_i + S}{S} = \frac{D}{S} \leq \overline{E},$$

where  $D := \bigoplus_{i \geq 1} D_i$ . Let  $E$  be a cyclic submodule of  $X$  such that  $\overline{E} = (E + S)/S$ . We obtain

$$\frac{D}{S} \leq \frac{E + S}{S}.$$

Since  $E \cap S \leq S$  and  $S$  is semisimple, there exists  $T \leq S$  with

$$S = (E \cap S) \oplus T,$$

and hence

$$E + S = (E \cap S) \oplus T.$$

Suppose there exists  $i \geq 1$  such that  $D_i \cap E = 0$ . Considering the canonical projection  $\pi : E \oplus T \rightarrow T$ , we have  $D_i \cap \ker \pi = 0$ , which shows that  $\pi|_{D_i}$  is a monomorphism and therefore  $D_i$  is semisimple. But  $S_i \leq D_i$ , which implies  $S_i = D_i$ , a contradiction. Hence  $D_i \cap E \neq 0$  for all  $i \geq 1$ , and consequently  $S_i \cap E \neq 0$  for all  $i \geq 1$ .

Since  $S \leq X_R$ , there exist simple modules  $V_i$  with  $V_i \leq S_i \cap E$  for all  $i \geq 1$ . Let  $V := \bigoplus_{i \geq 1} V_i$ . The module  $E$  is a CS-module because  $E$  is cyclic. There exists a direct summand  $L$  of  $X$  such that  $V \leq L$ . Note that  $L$  is cyclic and therefore  $V \subsetneq L$ . For  $n \geq 1$  set

$$P_n := \left( \bigoplus_{i=1}^n D_i \right) \cap L.$$

Then

$$P_n \cap S = L \cap \left( \bigoplus_{i=1}^n S_i \right) = V \cap \left( \bigoplus_{i=1}^n S_i \right) = \bigoplus_{i=1}^n V_i,$$

and hence  $P_n \cap S$  is a finitely generated submodule of  $S$ .

It follows that  $P_n \cap S$  is a direct summand of  $X$ . Since  $S \leq X_R$ , we have

$$P_n := \left( \bigoplus_{i=1}^n D_i \right) \cap L \leq S \quad \text{for all } n \geq 1,$$

which shows that  $D \cap L \leq S$ .

Using the modular distributive law we obtain

$$\frac{D}{S} \cap \frac{L+S}{S} = \frac{D \cap (L+S)}{S} = \frac{S + D \cap L}{S} = 0.$$

Since  $\frac{D}{S} \trianglelefteq \frac{E+S}{S}$ , it follows that  $L+S = S$ , hence  $L \leq S$ . Thus  $L$  is semisimple and, as  $V \trianglelefteq L$ , we get  $V = L$ , a contradiction. Therefore  $X/S$  is not a CS-module.  $\square$

**Theorem 5.6** (Osofsky–Smith Theorem). *Let  $R$  be a ring and  $M$  a cyclic  $R$ -module such that  $mR$  is a completely CS-module for every  $m \in M$ . Then  $M$  is a finite direct sum of uniform modules.*

*Proof.* Every indecomposable CS-module is uniform, so it is enough to show that  $M$  is a finite direct sum of indecomposable modules.

Assume that  $M$  does not admit such a decomposition. Then there exist submodules

$$M = A_1 \oplus B_1, \quad 0 \neq A_1, B_1 \leq M,$$

with  $B_1$  not a direct sum of indecomposable modules. Hence we can write

$$B_1 = A_2 \oplus B_2, \quad 0 \neq A_2, B_2 \leq B_1,$$

where  $B_2$  is again not a direct sum of indecomposable modules.

Iterating this construction we obtain

$$0 \neq A_i, B_i \leq M \quad (i \geq 1)$$

with

$$M = A_1 \oplus \cdots \oplus A_n \oplus B_n \quad (n \geq 1)$$

and

$$A_{n+1} \oplus A_{n+2} \oplus \cdots \leq B_n \quad (n \geq 1).$$

Since each  $A_i$  is cyclic, choose for every  $i \geq 1$  a maximal submodule  $C_i \leq A_i$ . Set

$$C := \bigoplus_{i \geq 1} C_i, \quad P := M/C, \quad S := \bigoplus_{i \geq 1} \frac{A_i}{C_i}.$$

Then  $S$  is a countably generated semisimple submodule of  $P$ .

For every  $n \geq 1$ , by the modular distributive law we have

$$(A_1 \oplus \cdots \oplus A_n \oplus C) \cap (B_n + C) = C + (A_1 \oplus \cdots \oplus A_n) \cap B_n = C + \bigoplus_{k \geq n+1} C_k = C.$$

Hence

$$P = \frac{M}{C} = \frac{A_1 \oplus \cdots \oplus A_n \oplus C}{C} \oplus \frac{B_n + C}{C} \quad \text{for all } n \geq 1.$$

If  $T \leq S$  is a finitely generated submodule, then for some  $n \geq 1$  we have

$$T = \frac{A_1 \oplus \cdots \oplus A_n + C}{C},$$

so every finitely generated submodule of  $S$  is a direct summand of  $P$ .

Now let  $X$  be a submodule of  $P$  with  $S \trianglelefteq X \leq P$  such that  $X$  is CS and every finitely generated submodule of  $S$  is a direct summand of  $X$ . If  $\text{soc}(X)$  denotes the socle of  $X$ , then  $S \trianglelefteq \text{soc}(X)$ , and since  $S \trianglelefteq X$  we get  $S = \text{soc}(X)$ .

Thus the hypotheses of the previous lemma are satisfied, so  $X/S$  is not a CS-module, a contradiction.  $\square$

**Corollary 5.7.** *Let  $R$  be a ring such that every cyclic  $R$ -module is a CS-module. Then every cyclic  $R$ -module is a finite direct sum of uniform modules.*

*Proof.* Let  $M_R$  be a cyclic module,  $m \in M$  and  $N \leq mR$ . Then the factor module  $mR/N$  is cyclic and, by hypothesis, a CS-module. By the Osofsky–Smith Theorem it follows that  $mR$ , hence  $M$ , is a finite direct sum of uniform modules.  $\square$

**Corollary 5.8.** *Let  $N$  be an  $R$ -module such that every factor module of every cyclic submodule of  $N$  is injective. Then  $N$  is semisimple.*

*Proof.* Take  $0 \neq x \in N$  and set  $C = xR \leq N$ . By hypothesis, every factor module of a cyclic submodule of  $C$  is injective; hence every cyclic submodule of  $C$  is completely CS. By the Osofsky–Smith Theorem we have

$$C = xR = \bigoplus_{i=1}^n U_i$$

for certain uniform submodules  $U_i$ .

Since

$$U_i \cong C / \bigoplus_{j \neq i} U_j,$$

each  $U_i$  is injective and therefore indecomposable for all  $i = 1, \dots, n$  (see Proposition 3.1).

Let  $y \in U_i$ . Then  $yR \cong yR/0$  is injective, hence the exact sequence

$$0 \longrightarrow yR \longrightarrow U_i \longrightarrow U_i/yR \longrightarrow 0$$

splits. By 2.1.15 it follows that  $yR$  is a direct summand of  $U_i$ , and thus  $yR = U_i$  or  $yR = 0$  for every  $y \in U_i$ . We obtain that  $U_i$  is simple and therefore  $C = xR$  is semisimple for every  $x \in N$ . Hence  $N$  is semisimple.  $\square$

**Corollary 5.9** (Osofsky). *If  $R$  is a ring such that every cyclic  $R$ -module is injective, then  $R$  is semisimple Artinian.*

*Proof.* Observing that we can apply the previous corollary with  $N = R$ , we obtain that  $R$  is semisimple.  $\square$

**Corollary 5.10.** *Let  $R$  be a ring such that every singular cyclic  $R$ -module is injective. Then every singular  $R$ -module is semisimple.*

*Proof.* Let  $M$  be a singular  $R$ -module. Submodules and factor modules of a singular module are singular. In particular, if  $x \in M \setminus \{0\}$ , every factor module of any cyclic submodule of  $xR$  is injective. By Corollary 5.8 it follows that  $xR$  is semisimple for every  $x \in M \setminus \{0\}$ , which shows that  $M$  is semisimple.  $\square$

**Lemma 5.11.** *Let  $S$  be a simple  $R$ -module. Then  $S$  is singular if and only if  $S \cdot \text{soc}(R) = 0$ .*

*Proof.* Let  $M$  be a maximal right ideal of  $R$ . We show that  $M \trianglelefteq R_R$  if and only if  $\text{soc}(R) \subseteq M$ .

The necessity is clear if we take into account that

$$\text{soc}(R) = \bigcap_{L \trianglelefteq R} L,$$

where the intersection runs over all maximal right ideals  $L$  of  $R_R$ .

Conversely, if  $M$  is not essential in  $R$ , there exists  $0 \neq a \in R$  such that  $aR \cap M = 0$ . Using the maximality of  $M$  we get  $aR \oplus M = R$ , hence  $aR \cong R/M$ , which shows that  $aR$  is simple and therefore  $aR \subseteq \text{soc}(R)$ , so  $aR \subseteq M$ , a contradiction.

Since  $S$  is a simple singular  $R$ -module, for every  $x \in S$ ,  $\text{ann}_R(x)$  is an essential maximal ideal of  $R$ ; from the above observation it follows that  $\text{soc}(R) \subseteq \text{ann}_R(x)$  for all  $x \in S$ , which is equivalent to  $S \cdot \text{soc}(R) = 0$ .  $\square$

**Corollary 5.12.** *The following statements are equivalent:*

1. *Every singular cyclic  $R$ -module is injective.*
2. *Every singular  $R$ -module is injective.*

*Proof.* (1)  $\Rightarrow$  (2). Assume that  $Z(R_R) \neq 0$  and let  $a \in Z(R_R) \setminus \{0\}$ . Then  $aR$  is a singular cyclic  $R$ -module, hence injective. It follows that  $aR$  is a direct summand of  $R$ , so there exists  $e = e^2 \in R \setminus \{0\}$  such that  $aR = eR$  and hence  $e \in Z(R_R)$ , a contradiction since  $\text{ann}_r(e) \cap eR = 0$ . Thus  $R$  is nonsingular, and the injective envelope of a singular module is singular (see A. W. Goldie, *Torsion free modules and rings*, J. Algebra (1964), 268–287).

Let  $M$  be a singular  $R$ -module. Since  $E(M)$  is a singular  $R$ -module, Corollary 5.10 yields that  $E(M)$  is semisimple. Hence  $M$  is a direct summand of  $E(M)$  and therefore  $M = E(M)$ , so  $M$  is injective.

(2)  $\Rightarrow$  (1). This implication is obvious.  $\square$

**Corollary 5.13** (Goodearl). *If every singular cyclic  $R$ -module is injective, then:*

1. *every simple  $R/\text{soc}(R)$ -module is injective;*
2.  *$R/\text{soc}(R)$  is noetherian.*

*Proof.* (1) Let  $S$  be a simple  $R/\text{soc}(R)$ -module. Since  $S \cdot \text{soc}(R) = 0$ , Lemma 5.11 shows that  $S$  is a simple singular  $R$ -module and hence a simple injective  $R$ -module. It follows that  $S$  is a simple injective  $R/\text{soc}(R)$ -module.  $\square$

