The Uniform (Co-Irreducible) Dimension of Rings and Modules

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Chapter 0

Generalities

0.1 Semisimple modules

Definition 0.1.1. A non-zero R-module S is called *simple* if its only submodules are 0 and S.

Proposition 0.1.2. *Let S be an R-module. The following statements are equivalent:*

1. *S is simple;*

2. *for every non-zero element $x \in S$ we have $S = xR$;*

3. *$S \cong R/I$, where I is a maximal right ideal.*

Lemma 0.1.3 (Schur)**.** *Let S and S' be simple R-modules and let $f \colon S \to S'$ be a homomorphism of R-modules. Then $f = 0$ or f is an isomorphism. In particular* $\mathrm{End}_R(S)$ *is a division ring.*

Definition 0.1.4. Let M be an R-module and let $(S_i)_{i\in I}$ be the family of all simple submodules of M. If $M = \sum_{i\in I} S_i$, then M is called *semisimple*.

Proposition 0.1.5. *Let M be a semisimple R-module and N a submodule of M. Then there exists a subset $J \subseteq I$ such that*

1. *the family $(S_j)_{j\in J}$ is independent;*

2. *$M = N \oplus \left(\bigoplus_{j\in J} S_j\right)$.*

Corollary 0.1.6. *With the above notation, for the semisimple module M there exists $J \subseteq I$ such that the family $(S_j)_{j\in J}$ is independent and*

$$M = \bigoplus_{i\in I} M_i.$$

Corollary 0.1.7. *If M is a semisimple R-module and N a submodule of M, then both N and M/N are semisimple.*

Corollary 0.1.8. *A direct sum of semisimple modules is a semisimple module.*

Theorem 0.1.9. *Let M be an R-module. The following statements are equivalent:*

1. *M is semisimple;*

2. *M is isomorphic to a direct sum of simple modules;*

3. *every submodule of M is a direct summand of M;*

4. *every short exact sequence*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits.

Definition 0.1.10. The sum of all simple submodules of M is called the *socle* of M and is denoted by $\operatorname{soc}(M)$. If M has no simple submodule we put $\operatorname{soc}(M) = 0$.

Proposition 0.1.11. *Let M and N be R-modules and $f\colon M \to N$ a homomorphism. Then $f(\operatorname{soc}(M)) \subseteq \operatorname{soc}(N)$.*

Proposition 0.1.12. *Let M be an R-module and N a submodule of M. Then*

$$\operatorname{soc}(N) = \operatorname{soc}(M) \cap N.$$

Proposition 0.1.13. *If $M = \bigoplus_{i \in I} M_i$, then*

$$\operatorname{soc}(M) = \bigoplus_{i \in I} \operatorname{soc}(M_i).$$

Proposition 0.1.14. *Let R be a ring. Then the socle $\operatorname{soc}(R_R)$ is a two-sided ideal of R.*

0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings

Definition 0.2.1. Let R be a ring and M a right R-module. We say that M satisfies the *maximal condition* (resp. the *minimal condition*) if every non-empty set of submodules of M, ordered by inclusion, has a maximal (resp. minimal) element.

We say that M satisfies the *ascending* (resp. *descending*) *chain condition* if every ascending chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

(resp. every descending chain

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots$$

) is stationary, that is, there exists $n \geq 1$ such that $M_n = M_{n+1} = \cdots$.

Proposition 0.2.2. *Let M be an R-module. The following statements are equivalent:*

1. *M satisfies the maximal (minimal) condition;*

2. *M satisfies the ascending (descending) chain condition.*

Definition 0.2.3. An R-module M is called *noetherian* (resp. *artinian*) if it satisfies the maximal (resp. minimal) condition. The ring R is called right noetherian (resp. right artinian) if the right module R_R is noetherian (resp. artinian).

Example 0.2.4.

1. $\mathbb{Z}$ is a noetherian ring but not artinian.

2. Every finite group is a noetherian and artinian $\mathbb{Z}$-module.

3. Every finite ring is noetherian and artinian.

4. The ring $\mathbb{Z}[X_1, X_2, \ldots, X_n, \ldots]$ is neither noetherian nor artinian:

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_n) \subsetneq \cdots$$

$$(X_1) \supsetneq (X_1^2) \supsetneq \cdots \supsetneq (X_1^k) \supsetneq \cdots$$

5. The Prüfer p-group $\mathbb{Z}_{p^\infty}$ is an artinian but not noetherian $\mathbb{Z}$-module.

Proposition 0.2.5. *Let N and P be submodules of M such that $M = N + P$. Then M is noetherian (artinian) if and only if N and P are noetherian (artinian).*

Proposition 0.2.6. *For an R-module M the following statements are equivalent:*

1. *M is noetherian;*

2. *every submodule of M is finitely generated.*

Proposition 0.2.7. *For an R-module M the following statements are equivalent:*

1. *M is artinian;*

2. *for every family $(X_i)_{i\in I}$ of submodules of M there exists a finite subset $J \subseteq I$ such that*
$$\bigcap_{i\in I} X_i = \bigcap_{j\in J} X_j.$$

0.3 Finite length modules

Definition 0.3.1. Let M be a non-zero right R-module. A *composition series* or *Jordan–Hölder series* of M is a finite strictly ascending chain of submodules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = M$$

such that X_{i+1}/X_i is a simple module for $0 \leq i \leq n-1$. The integer n is called the *length* of the series, and the modules X_{i+1}/X_i are called its *factors*.

Proposition 0.3.2. *Let M be an R-module. The following statements are equivalent:*

1. *M has a composition series;*

2. *M is noetherian and artinian.*

Proposition 0.3.3. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of right R-modules. Then M has a composition series if and only if both M' and M'' have composition series.

If

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \qquad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M$$

are two composition series of M, we say that they are *equivalent* if $n = p$ and there exists a bijection $\sigma : \{0, \dots, n-1\} \to \{0, \dots, n-1\}$ such that

$$M_{i+1}/M_i \cong M_{\sigma(i)+1}/M_{\sigma(i)} \quad (0 \leq i \leq n-1).$$

Theorem 0.3.4 (Jordan–Hölder)**.** *If an R-module M has two composition series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \qquad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M,$$

then these two series are equivalent.

Definition 0.3.5. An R-module M which admits a composition series is called a *module of finite length*. The length of its composition series is called the *length* of M and is denoted by $l(M)$. If M admits no composition series, we say that M has *infinite length* and we write $l(M) = \infty$.

Proposition 0.3.6. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of R-modules of finite length. Then

$$l(M) = l(M') + l(M'').$$

Corollary 0.3.7. *Let M be an R-module of finite length and let N, L be submodules of M. Then:*

1. *$l(M) = l(N) + l(M/N)$;*

2. *$l(N + L) + l(N \cap L) = l(N) + l(L)$.*

Corollary 0.3.8. *Let M be an R-module of finite length and let $M_1, M_2, \dots, M_n$ be submodules of M such that*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

Then

$$l(M) = \sum_{i=1}^{n} l(M_i).$$

0.4 The Jacobson radical

The Jacobson radical of a module

Definition 0.4.1. Let M be an R-module. The intersection of all maximal submodules of M is called the *Jacobson radical* of M and is denoted by $\operatorname{Rad}(M)$. If M has no maximal submodules, we adopt the convention $\operatorname{Rad}(M) = M$.

Remark 0.4.2. If M is a finitely generated R-module, then $\operatorname{Rad}(M) \neq M$.

Proposition 0.4.3. *Let M be an R-module. Then*

$$\operatorname{Rad}(M) = \bigcap_{\substack{f:M\to S\\ S \text{ simple}}} \ker(f) = \bigcap_{\substack{f:M\to X\\ X \text{ semisimple}}} \ker(f).$$

Proposition 0.4.4. *Let $f : M \to N$ be a homomorphism of R-modules. Then $f(\operatorname{Rad}(M)) \subseteq \operatorname{Rad}(N)$. If, in addition, f is an epimorphism and $\ker(f) \subseteq \operatorname{Rad}(M)$, then $f(\operatorname{Rad}(M)) = \operatorname{Rad}(N)$.*

Corollary 0.4.5. *For every R-module M we have $\operatorname{Rad}(M/\operatorname{Rad}(M)) = 0$.*

Corollary 0.4.6. *If M is a semisimple R-module, then $\operatorname{Rad}(M) = 0$.*

Corollary 0.4.7. *If $M = \bigoplus_{i\in I} M_i$, then*

$$\operatorname{Rad}(M) = \bigoplus_{i\in I} \operatorname{Rad}(M_i).$$

Proposition 0.4.8. *Let M be an R-module with $\operatorname{Rad}(M) \neq M$. Then*

$$\operatorname{Rad}(M) = \bigcap \{\, L \leq M \mid L \textit{ is a superfluous submodule} \,\}.$$

Proposition 0.4.9 (Nakayama's Lemma)**.** *Let M be a finitely generated R-module and N a submodule of M. If $N + \operatorname{Rad}(M) = M$, then $N = M$. (In other words, $\operatorname{Rad}(M)$ is the largest superfluous submodule of M.)*

The Jacobson radical of a ring

Let R be a ring. We consider the left ideal $\operatorname{Rad}({}_RR)$, the intersection of all maximal left ideals of R, and the right ideal $\operatorname{Rad}(R_R)$, the intersection of all maximal right ideals of R.

Proposition 0.4.10.

1. $\operatorname{Rad}(R_R)$ *is a two-sided ideal.*
2. $\operatorname{Rad}(R_R) = \{\, r \in R \mid 1 - ar \in U(R) \textit{ for all } a \in R \,\}$.
3. $\operatorname{Rad}(R_R) = \operatorname{Rad}({}_RR)$.

Definition 0.4.11. The two-sided ideal $\operatorname{Rad}(R_R) = \operatorname{Rad}({}_RR)$ is called the *Jacobson radical* of the ring R and is denoted by $\operatorname{Rad}(R)$.

Proposition 0.4.12.

1. *If J is a left (resp. right or two-sided) ideal such that $1 - x$ is invertible for every $x \in J$, then $J \subseteq \mathrm{Rad}(R)$.*

2. *If J is a nil left (resp. right or two-sided) ideal, then $J \subseteq \mathrm{Rad}(R)$.*

Proposition 0.4.13. *Let $\varphi : R \to S$ be a surjective ring homomorphism. Then $\varphi(\mathrm{Rad}(R)) \subseteq \mathrm{Rad}(S)$. If $\ker(\varphi) \subseteq \mathrm{Rad}(R)$, then $\varphi(\mathrm{Rad}(R)) = \mathrm{Rad}(S)$.*

Proposition 0.4.14. *If $(R_i)_{i\in I}$ is a family of rings, then*

$$\mathrm{Rad}\Big(\prod_{i\in I} R_i\Big) = \prod_{i\in I} \mathrm{Rad}(R_i).$$

Proposition 0.4.15. *Let M be an R-module. Then $M\,\mathrm{Rad}(R) \subseteq \mathrm{Rad}(M)$.*

Theorem 0.4.16. *If R is an artinian ring, then $\mathrm{Rad}(R)$ is nilpotent.*

0.5 Semisimple rings

Theorem 0.5.1. *For a ring R the following statements are equivalent:*

1. *every non-zero right R-module is semisimple;*

2. *R as a right R-module is semisimple;*

3. *R is artinian and $\mathrm{Rad}(R) = 0$.*

Definition 0.5.2. A ring R satisfying any (hence all) of the above conditions is called a *semisimple ring*.

Proposition 0.5.3. *Let R be a semisimple ring and M a non-zero R-module. The following statements are equivalent:*

1. *M has finite length;*

2. *M is noetherian;*

3. *M is artinian.*

Theorem 0.5.4. *Let R be a right artinian ring and M a non-zero right R-module. The following statements are equivalent:*

1. *M has finite length;*

2. *M is noetherian;*

3. *M is artinian.*

Corollary 0.5.5 (Hopkins)**.** *A right artinian ring (respectively a left artinian ring) is right noetherian (respectively left noetherian).*

Chapter 1

Essential submodules

Definition 1.1. Let M be a right R-module. A submodule N of M is called *essential* (or we say that M is an essential extension of N) if $N \cap N' \neq 0$ for every non-zero submodule N' of M. In this case we write $N \trianglelefteq M_R$.

A homomorphism of right R-modules $f : M \to N$ is called *essential* if $\operatorname{Im} f$ is an essential submodule of N (i.e. $\operatorname{Im} f \trianglelefteq N_R$).

Example.

1. $n\mathbb{Z} \trianglelefteq \mathbb{Z}_{\mathbb{Z}}$ for every $n \geq 1$.

2. Every submodule of the Prüfer group $\mathbb{Z}_{p^\infty}$ is essential.

Remark 1.2. Let M be a right R-module and N a submodule of M. Then $N \trianglelefteq M_R$ if and only if for every $x \in M$, $x \neq 0$, there exists $r \in R$ such that $xr \in N \setminus \{0\}$.

Proof. "⇒" Let $x \in M \setminus \{0\}$. Since $0 \neq xR \leq M_R$ and $N \trianglelefteq M_R$, we have $xR \cap N \neq 0$; hence there exists $r \in R$ with $xr \in N \setminus \{0\}$.

"⇐" Let $N' \leq M_R$, $N' \neq 0$. For every $x \in N' \setminus \{0\}$ there exists $r \in R$ such that $xr \in N \setminus \{0\}$, so $N \cap N' \neq 0$. Thus $N \trianglelefteq M_R$. □

Definition 1.3. A monomorphism of right R-modules $f : N_R \to M_R$ is called *essential* if $\operatorname{Im} f \trianglelefteq M_R$. It is immediate that, if N is a submodule of M, then the canonical inclusion $i_N : N \to M$ is an essential monomorphism if and only if $N \trianglelefteq M_R$.

Proposition 1.4. *A monomorphism $f : N_R \to M_R$ is essential if and only if for every right R-module M' and every $g \in \operatorname{Hom}(M, M')$, the fact that $g \circ f$ is a monomorphism implies that g is a monomorphism.*

Proof. "⇒" Let g be as in the statement and suppose that $g \circ f$ is a monomorphism. Assume $\operatorname{Ker} g \neq 0$. Take $x \in \operatorname{Ker} g \cap \operatorname{Im} f \setminus \{0\}$. Then $x = f(x')$ for some $x' \in N$ and $g(x) = 0$, so $g(f(x')) = 0$. Since $g \circ f$ is a monomorphism, it follows that $x' = 0$ and hence $x = 0$, a contradiction.

"⇐" If f is not an essential monomorphism, then there exists $N' \leq M_R$, $N' \neq 0$, such that $N' \cap \operatorname{Im} f = 0$. Consider the canonical projection $\pi_{N'} : M \to M/N'$. If $x \in \operatorname{Ker}(\pi_{N'} \circ f)$, then $f(x) \in N'$, so $f(x) = 0$, hence $x = 0$. Thus $\pi_{N'} \circ f$ is injective. By the hypothesis in the statement, this implies that $\pi_{N'}$ is injective, so $N' = 0$, a contradiction. □

Corollary 1.5. *Let M be a right R-module and $N \leq M_R$. Then the following statements are equivalent:*

1. $N \trianglelefteq M_R$;
2. *the inclusion* $i_N : N \to M$ *is an essential monomorphism;*
3. *for every* $f \in \operatorname{Hom}(M, M')$, *with* M' *an arbitrary R-module, the fact that* $f \circ i_N$ *is a monomorphism implies that* f *is a monomorphism.*

Proposition 1.6. *Let $f : N_R \to M_R$ and $g : M_R \to P_R$ be monomorphisms. Then $g \circ f$ is essential if and only if both g and f are essential.*

Proof. "$\Leftarrow$" Let $z \in P \setminus \{0\}$. Since g is essential, there exists $r \in R$ such that $zr \in \operatorname{Im} g \setminus \{0\}$. Thus there exists $y \in M \setminus \{0\}$ with $zr = g(y)$. As f is essential, there exists $r' \in R$ such that $yr' \in \operatorname{Im} f \setminus \{0\}$. Hence there exists $x \in N \setminus \{0\}$ with $yr' = f(x)$. But $zr' = g(y)r' = g(yr') = g(f(x))$. If $zr' = 0$, then $g(f(x)) = 0$, hence $x = 0$, a contradiction. Therefore $zr' \in \operatorname{Im}(g \circ f)$ and $zr' \neq 0$, which shows that $g \circ f$ is essential.

"$\Rightarrow$" Let $y \in M \setminus \{0\}$. Since g is a monomorphism, $g(y) \neq 0$. Thus there exists $r \in R$ such that $g(yr) \in \operatorname{Im} g \setminus \{0\}$ and $g(yr) \neq 0$. Hence there exists $x \in N \setminus \{0\}$ such that $g(yr) = g(f(x))$, therefore $yr = f(x) \in \operatorname{Im} f$, which shows that f is an essential monomorphism.

If $z \in P \setminus \{0\}$, there exists $r \in R$ such that $zr \in \operatorname{Im}(g \circ f)$ and $zr \neq 0$. Since $\operatorname{Im}(g \circ f) \subseteq \operatorname{Im} g$, we obtain $zr \in \operatorname{Im} g$, so g is essential. $\square$

Proposition 1.7. *Let M be a right R-module and $L_1, L_2, \ldots, L_n$ submodules of M. Then:*

1) $\bigcap\limits_{i=1}^{n} L_i$ *is essential in M iff each L_i is essential in M.*

2) If $L_1 \subseteq L_2$ and L_1 is essential in M, then L_2 is essential in M.

The proof is obvious.

Proposition 1.8. *Let K and L be submodules of M.*

1) If $K \subseteq L \subseteq M$, then $K \trianglelefteq M$ iff $K \trianglelefteq L$ and $L \trianglelefteq M$.

2) If $h : K_R \to M_R$ is a module morphism and $L \trianglelefteq M$, then $h^{-1}(L) \trianglelefteq K$.

3) If $L_1, L_2 \leq M_R$ with $K_1 \trianglelefteq L_1$ and $K_2 \trianglelefteq L_2$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.

Proof.

1) Apply 1.5 and 1.6.

2) Let U be a non-zero submodule of K.

(i) If $h(U) = 0$, then $U \subseteq \ker h \subseteq h^{-1}(L)$, so $U \cap h^{-1}(L) \neq 0$.

(ii) If $h(U) \neq 0$, then $h(U) \cap L \neq 0$. Hence there exists $u \in U$ with $h(u) \in L$, $h(u) \neq 0$, so $u \in U \cap h^{-1}(L)$ and $u \neq 0$.

Thus $h^{-1}(L) \trianglelefteq K$.

3) Let $0 \neq X \leq L_1 \cap L_2$. Then $X \subseteq L_1$, so $0 \neq X \cap K_1 \leq L_1$. But $X \subseteq L_2$ implies $0 \neq (X \cap K_1) \cap L_2 = X \cap (K_1 \cap K_2)$. Hence $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.

Proposition 1.9. *Let $(K_\lambda)_{\lambda\in\Lambda}$, $(L_\lambda)_{\lambda\in\Lambda}$ be families of submodules of M. If (K_λ) is independent in M and $K_\lambda \trianglelefteq L_\lambda$ for all $\lambda \in \Lambda$, then (L_λ) is independent in M and*

$$\left(\bigoplus_{\lambda\in\Lambda} K_\lambda\right) \trianglelefteq \left(\bigoplus_{\lambda\in\Lambda} L_\lambda\right).$$

Proof.
Let $K_1 \trianglelefteq L_1$, $K_2 \trianglelefteq L_2$ with $K_1 \cap K_2 = 0$. By 1.8(3), $0 \trianglelefteq L_1 \cap L_2$, hence $L_1 \cap L_2 = 0$.

Let $\pi_1 : L_1 \oplus L_2 \to L_1$, $\pi_2 : L_1 \oplus L_2 \to L_2$ be the canonical projections. Since $K_1 \trianglelefteq L_1$ and $K_2 \trianglelefteq L_2$,

$$\pi_1^{-1}(K_1) = K_1 \oplus 0 \trianglelefteq L_1 \oplus L_2,$$

and

$$\pi_2^{-1}(K_2) = 0 \oplus K_2 \trianglelefteq L_1 \oplus L_2.$$

Hence

$$K_1 \oplus K_2 = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \trianglelefteq L_1 \oplus L_2.$$

Induction gives the finite case. For the general case, let $0 \neq m \in \oplus_{\lambda\in\Lambda} L_\lambda$. Then m lies in a finite direct sum $\oplus_{\lambda\in\Lambda_0} L_\lambda$ for some finite $\Lambda_0 \subseteq \Lambda$. Since $(\oplus_{\lambda\in\Lambda_0} K_\lambda) \trianglelefteq (\oplus_{\lambda\in\Lambda_0} L_\lambda)$, there exists $r \in R$ with $rm \in (\oplus_{\lambda\in\Lambda_0} K_\lambda) \setminus \{0\} \subseteq (\oplus_{\lambda\in\Lambda} K_\lambda)$. Thus

$$\left(\bigoplus_{\lambda\in\Lambda} K_\lambda\right) \trianglelefteq \left(\bigoplus_{\lambda\in\Lambda} L_\lambda\right).$$

Proposition 1.10. *Let N be a submodule of M. Then there exists a submodule Q with $N \subseteq Q \subseteq M$ such that Q is a maximal essential extension of N inside M.*

Proof.
Let

$$\mathfrak{S} = \{L \leq M \; ; \; N \subseteq L \subseteq M, \; N \trianglelefteq L\},$$

ordered by inclusion. Clearly $\mathfrak{S} \neq \varnothing$, since $N \in \mathfrak{S}$.

Let $(L_\lambda)_{\lambda\in\Lambda}$ be a totally ordered family of elements of $\mathfrak{S}$ and put

$$L := \bigcup_{\lambda\in\Lambda} L_\lambda.$$

Clearly $L \leq M_R$.

Let $x \in L \setminus \{0\}$. Then there exists $\lambda_0 \in \Lambda$ with $x \in L_{\lambda_0}$. Since N is essential in L_{λ_0}, there exists $r \in R$ such that $xr \in N$ and $xr \neq 0$, hence L is an essential extension of N. Thus $\mathfrak{S}$ is inductive and, by Zorn's lemma, $\mathfrak{S}$ admits a maximal element Q which satisfies the required conditions.

Definition 1.11. Let M be a right R-module and $N \leq M_R$. A submodule $K \leq M_R$ is called a *complement* of N in M if K is a maximal submodule of M with the property that $K \cap N = 0$. A submodule $K \leq M_R$ is called a *complement submodule* of M if there exists $N \leq M_R$ such that K is a complement of N in M.

Remark 1.12. The set
$$\widetilde{\mathfrak{S}} = \{ L \leq M_R \mid N \cap L = 0 \}$$
is inductive and, by applying Zorn's lemma, it follows that there exists a complement of N in M. In particular, 0 and M are complement submodules of M.

Proposition 1.13. *Let M_R, $N \leq M_R$ and $K \leq M_R$, with K a complement of N in M. There exists a complement Q of K in M such that $N \subseteq Q$. Moreover, Q is a maximal essential extension of N in M.*

Proof. It is easy to see that the set
$$\widetilde{\mathfrak{S}} = \{ L \leq M_R \mid K \cap L = 0,\ N \subseteq L \}$$
is inductive, and Zorn's lemma guarantees the existence of Q.

Let L be a non-zero submodule of Q such that $L \cap N = 0$. Put $K_1 = L + K$. Clearly $K \subseteq K_1$. If $x \in N \cap (L + K)$, then $x = y + z$ with $y \in L$, $z \in K$. But $z = x - y \in Q$. Since $Q \cap K = 0$, we obtain $z = 0$ and hence $x = y$. From $L \cap N = 0$ it follows that $x = y = 0$, and therefore $N \cap (L + K) = 0$, which contradicts the fact that K is a complement of N in M. Thus $L \cap N \neq 0$ for every $0 \neq L \leq Q$, so Q is an essential extension of N.

Suppose that there exists $Q' \leq M_R$ with $N \trianglelefteq Q'$ and $Q \subsetneq Q'$. Since Q' is a complement of K, we have $Q' \cap K \neq 0$. But $N \cap (Q' \cap K) = 0$ and $0 \neq Q' \cap K \leq Q'$, contradicting $N \trianglelefteq Q'$. Hence Q is a maximal essential extension of N in M. □

Definition 1.14. A submodule N of M_R is called *closed* if N has no proper (meaning different from N) essential extension in M.

Corollary 1.15. *Let M_R be a right R-module. The complement submodules of M coincide with the closed submodules of M.*

Proof. From 1.13 it follows immediately that every closed submodule of M is a complement submodule of M.

Conversely, let K be a complement submodule of M_R. Then there exists $N \leq M_R$ such that K is a complement of N in M. Assume that K has a proper essential extension in M; that is, there exists $K' \leq M_R$ with $K \trianglelefteq K'$ and $K \subsetneq K'$. By the maximality of K we have $K' \cap N \neq 0$, and since $K \trianglelefteq K'$, it follows that
$$K \cap K' \cap N \neq 0,$$
contradiction. □

Corollary 1.16. *Let N be a submodule of M_R. If K is a complement of N in M, then:*

1. $(N + K) \trianglelefteq M_R$.

2. *The canonical morphism $\pi_K \circ i_N : N \to M/K$ is an essential monomorphism.*

Proof. (1) Let $x \in M \setminus \{0\}$. If $x \notin K$, then $K + Rx \neq K$ and hence $N \cap (K + Rx) \neq 0$. Let $y \in N \cap (K + Rx)$, $y \neq 0$. There exist $z \in K$ and $r \in R$ such that $y = z + rx$. If $rx = 0$, then $y = z$ and, since $N \cap K = 0$, we obtain $y = 0$, a contradiction. Thus $rx \neq 0$ and, because $rx = y - z$, we have $rx \in N + K$, which shows that $(N + K) \trianglelefteq M_R$.

(2) We have $\mathrm{Im}(\pi_K \circ i_N) = (N + K)/K$. Let L/K be a non-zero submodule of M/K. Then

$$\frac{N+K}{K} \cap \frac{L}{K} = \frac{(N+K) \cap L}{K} = \frac{N \cap L + K}{K}.$$

Since K is a complement of N, we have $N \cap L \neq 0$, and hence

$$\frac{N \cap L + K}{K} \neq 0,$$

which shows that $\pi_K \circ i_N$ is an essential monomorphism. □

Chapter 2

Injective Modules

2.1 Injective Module

Let Q and M be two right R-modules. We say that Q is *M-injective* if for every monomorphism $u : M' \to M$ and every morphism $f : M' \to Q$, there exists $g : M \to Q$ such that $g \circ u = f$; that is, the diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[dl, dashed, "g"] \\ & Q & \end{tikzcd}$$

is commutative.

This property is equivalent to the condition that the map

$$\mathrm{Hom}(u, Q) : \mathrm{Hom}(M, Q) \longrightarrow \mathrm{Hom}(M', Q)$$

is surjective for every monomorphism $u : M' \to M$. Since the functor $\mathrm{Hom}(-, Q)$ is left exact, it follows that Q is M-injective if and only if $\mathrm{Hom}(-, Q)$ is exact with respect to every short exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

The R-module Q is called *quasi-injective* (or *self-injective*) if it is Q-injective. If Q is M-injective for every R-module M, then Q is called *injective*.

2.1.1 Proposition

Proposition 2.1.1. *Let Q and M be two R-modules. The following statements are equivalent:*

1. *Q is M-injective.*
2. *For every submodule N of M and every morphism $f : N \to Q$, there exists $g : M \to Q$ such that $g_{|N} = f$.*
3. *For every essential submodule N of M and every morphism $f : N \to Q$, there exists $g : M \to Q$ such that $g_{|N} = f$.*

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1). Let M'_R, $0 \longrightarrow M' \xrightarrow{u} M$ and $f : M' \to Q$. Then $u(M') \leq M$. Consider $i : u(M') \to M$ the canonical injection and $\bar{u} : M' \to u(M')$ the isomorphism induced by u. There exists $g : M \to Q$ such that $g \circ i = f \circ \bar{u}^{-1}$. Hence

$$g \circ i \circ \bar{u} = f \quad \text{and therefore} \quad g \circ u = f.$$

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "\bar{u}"] \arrow[d, "f"'] & u(M') \arrow[r, hook, "i"] & M \arrow[dll, dashed, "g"] \\
& Q & &
\end{tikzcd}$$

(3) $\Rightarrow$ (2). Let N be a submodule of M and K a complement of N in M. Then $(N \oplus K) \trianglelefteq M$. Define $h : N \oplus K \to Q$ by $h(n + k) = f(n)$ for all $n \in N$, $k \in K$. Since $N \cap K = 0$, the map h is well-defined. There exists $g : M \to Q$ such that $g_{|N \oplus K} = h$, and hence $g_{|N} = h_{|N} = f$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook] \arrow[d, "f"'] & N \oplus K \arrow[r, hook] \arrow[dl, "h"] & M \arrow[dll, dashed, bend left, "g"] \\
& Q & &
\end{tikzcd}$$

$\square$

Proposition 2.1.2. *Let $(M_\alpha)_{\alpha \in \Lambda}$ be a family of R-modules and M an R-module. Then $\prod_{\alpha \in \Lambda} M_\alpha$ is M-injective if and only if each M_α is M-injective.*

Proof. Let N be a submodule of M. Put $P = \prod_{\alpha \in \Lambda} M_\alpha$ and let $\pi_\alpha : P \to M_\alpha$ be the canonical projections for all $\alpha \in \Lambda$.

" $\Leftarrow$ " Given a morphism $f : N \to P$, the morphisms $\pi_\alpha \circ f : N \to M_\alpha$ can be extended to $g_\alpha : M \to M_\alpha$. There exists $g : M \to P$ such that $g_{|N} = f$.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[dl, "g"'] \arrow[ddl, "g_\alpha"] \\
& P \arrow[d, "\pi_\alpha"'] & \\
& M_\alpha &
\end{tikzcd}$$

" $\Rightarrow$ " Let $\forall \alpha \in \Lambda$ and $f : N \to M_\alpha$. Considering the canonical inclusion $\varepsilon_\alpha : M_\alpha \to P$, since P is M-injective, there exists $g : M \to P$ which extends $\varepsilon_\alpha \circ f : N \to P$. Then $\varepsilon_\alpha : M_\alpha \to P$ extends f and hence M_α is M-injective.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook] \arrow[d, "f"'] & M \arrow[ddl, "g"'] \arrow[dddl, bend left, "\pi_\alpha g"] \\
& M_\alpha \arrow[d, "\varepsilon_\alpha"'] & \\
& P \arrow[d, "\pi_\alpha"'] & \\
& M_\alpha &
\end{tikzcd}$$

□

Corollary 2.1.3.

1. *Let $(Q_\alpha)_{\alpha\in\Lambda}$ be a family of R-modules. Then $\prod_{\alpha\in\Lambda} Q_\alpha$ is injective if and only if each Q_α is injective for every $\alpha \in \Lambda$.*

2. *The module $Q_1 \oplus Q_2$ is an injective R-module if and only if each Q_i is injective for $i = 1, 2$. In particular, a direct summand of an injective module is injective.*

Proposition 2.1.4. *Let Q be an R-module.*

1. *If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules and Q is M-injective, then Q is M'-injective and M''-injective.*

2. *If $(M_\alpha)_{\alpha\in\Lambda}$ is a family of submodules of M such that $M = \sum_{\alpha\in\Lambda} M_\alpha$ and Q is M_α-injective for every α, then Q is M-injective.*

3. *Let $(N_\alpha)_{\alpha\in\Lambda}$ be a family of R-modules. Then Q is $\bigoplus_{\alpha\in\Lambda} N_\alpha$-injective if and only if Q is N_α-injective for every $\alpha \in \Lambda$.*

Proof. To show that Q is M'-injective, we consider N a submodule of M' and $\varphi : N \to Q$ a morphism of R-modules. Since Q is M-injective, there exists $\phi : M \to Q$ such that $g \circ f\,|_N = \varphi$, and hence $\psi \circ f : M' \to Q$ is a morphism which extends φ.

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\varphi"'] & M' \arrow[r, "f"] \arrow[dl, "\psi"] & M \arrow[dll, bend left] \\
& Q & &
\end{tikzcd}$$

Let $h : L \to M''$ be a monomorphism. We may assume, without loss of generality, that $M' \le M$ and $M'' = M/M'$. Since $L \cong h(L) \le M''$, there exist $P \le M$, $M' \subseteq P$ such that $h(L) = P/M'$ and hence $L \cong P/M'$. We obtain the commutative diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Since Q is M-injective, applying the functor $\mathrm{Hom}(-,Q)$ we obtain the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M'',Q) \arrow[r] \arrow[d, "h^*"'] & \mathrm{Hom}(M,Q) \arrow[r] \arrow[d] & \mathrm{Hom}(M',Q) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Hom}(L,Q) \arrow[r] & \mathrm{Hom}(P,Q) \arrow[r] \arrow[d] & \mathrm{Hom}(M',Q) & \\
 & & 0 & &
\end{tikzcd}$$

We obtain that $h^* = \mathrm{Hom}(h,Q)$ is an epimorphism, which shows that Q is M''-injective.

2) Let N be a submodule of M and $f : N \to Q$ a morphism of R-modules. Consider the set

$$\mathfrak{S} = \{(L,h) \mid N \leq L \leq M,\ h : L \to Q,\ h|_N = f\}.$$

Since $(N,f) \in \mathfrak{S}$, we have $\mathfrak{S} \neq \varnothing$. Define on $\mathfrak{S}$ the order relation $(L_1,h_1) \preccurlyeq (L_2,h_2)$ if and only if $L_1 \leq L_2$ and $h_2|_{L_1} = h_1$. One checks that $\mathfrak{S}$ is inductive and, by Zorn's lemma, there exists a maximal element (L_0,g_0) of $\mathfrak{S}$. To show that $L_0 = M$ it is enough to prove that $M_\alpha \leq L_0$ for every $\alpha \in \Lambda$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & L_0 \cap M_\alpha \arrow[r, "i_\alpha"] \arrow[d, "i_0"'] & M_\alpha \arrow[ldd, "h_\alpha"] \\
 & L_0 \arrow[d, "g_0"'] & \\
 & Q &
\end{tikzcd}$$

from which it follows that there exists $h_\alpha : M_\alpha \to Q$ such that $h_\alpha \circ i_\alpha = g_0 \circ i_\alpha$. Define $h^* : L_0 + M_\alpha \to Q$ by $h^*(l + m_\alpha) = g_0(l) + h_\alpha(m_\alpha)$, for all $l \in L_0$, $m_\alpha \in M_\alpha$. If $l + m_\alpha = 0$, then $l = -m_\alpha \in L_0 \cap M_\alpha$ and hence $h^*(l + m_\alpha) = g_0(l) + h_\alpha(l)$, which shows that h^* is well defined. Thus $(L_0 + M_\alpha, h^*) \in \mathfrak{S}$, and since $(L_0,g_0) \preccurlyeq (L_0 + M_\alpha, h^*)$, by the maximality of (L_0,g_0) we obtain $L_0 = L_0 + M_\alpha$, i.e. $M_\alpha \leq L_0$ for every $\alpha \in \Lambda$.

3) "$\Rightarrow$" Since $N_\alpha \leq N$ and Q is N-injective, it follows that Q is N_α-injective for every $\alpha \in \Lambda$.

"$\Leftarrow$" Let $N'_\alpha = i_\alpha(N_\alpha)$. Since Q is N_α-injective and $N'_\alpha \cong N_\alpha$, we see that Q is N'_α-injective. Now apply (2). □

Corollary 2.1.5.

1. *The module $Q_1 \oplus Q_2$ is a quasi-injective R-module if and only if each Q_i is Q_j-injective for all $i,j = 1,2$. In particular, a direct summand of a quasi-injective module is quasi-injective.*

2. *The module Q^n is quasi-injective over R if and only if Q is quasi-injective.*

Corollary 2.1.6. *Let Q and M be two R-modules. Then Q is M-injective if and only if Q is mR-injective for every $m \in M$.*

Proof. The implication "$\Rightarrow$" is clear.

For "$\Leftarrow$", since $M = \sum_{m \in M} mR$, it follows from 2.1.4(2) that Q is M-injective. $\square$

Theorem 2.1.7 (Baer's criterion)**.** *For an R-module Q the following statements are equivalent:*

1. *Q is injective.*

2. *Q is R-injective.*

3. *For every right ideal I of R and every morphism $f : I \to Q$ there exists $x \in Q$ such that $f(a) = xa$ for all $a \in I$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Let M be an R-module and $x \in M$. Since $\varphi_x : R \to xR$, $\varphi_x(a) = xa$ for all $a \in R$, is a surjective morphism of R-modules, it follows that $R/\operatorname{Ker}\varphi_x \cong xR$. As Q is R-injective, 2.1.4(1) implies that Q is $R/\operatorname{Ker}\varphi_x$-injective, and hence Q is xR-injective for every $x \in M$. Therefore, by 2.1.6 we obtain that Q is M-injective.

(2) $\Rightarrow$ (3). Let I be a right ideal of R and $f : I \to Q$. There exists $g : R \to Q$ such that $g|_I = f$. Put $x = g(1) \in Q$. Then $f(a) = g(a) = ag(1) = xa$ for all $a \in I$.

(3) $\Rightarrow$ (2). Suppose that for a morphism $f : I \to Q$ there exists $x \in Q$ with $f(a) = xa$ for all $a \in I$. Define $g : R \to Q$ by $g(r) = xr$ for all $r \in R$. Then clearly $g|_I = f$. $\square$

Definition 2.1.8.

1. An R-module Q is called *divisible* if for every $y \in Q$ and every $a \in R$ which is not a zero divisor, there exists $x \in Q$ such that $ax = y$. It is easily seen that any factor module of a divisible module is divisible.

2. A commutative integral domain is called a *PID ring* (principal ideal domain) if every ideal of it is principal.

Proposition 2.1.9.

1. *Every injective module is divisible.*

2. *Let R be a PID.*

   (i) If Q is an R-module, then Q is injective if and only if it is divisible.

   (ii) If I is a non-zero ideal of R, then R/I is a quasi-injective R-module. In particular, $\mathbb{Z}_n$ is a quasi-injective $\mathbb{Z}$-module for every $n \geq 1$.

Proof.
1) Let Q be a divisible R-module, $y \in Q$ and $a \in R$ a non–zero divisor. Define $f : aR \to Q$ by $f(ax) = yx$ for all $x \in R$. Since a is not a zero divisor, f is well defined. By Baer's criterion there exists $x \in Q$ such that $f(\lambda) = x\lambda$ for all $\lambda \in aR$. In particular $y = f(a) = f(a \cdot 1) = xa$; hence Q is divisible.

2)(i) The implication "$\Rightarrow$" follows immediately from (1).

"$\Leftarrow$" Let Q be a divisible R-module and I a right ideal of R. Then there exists $a \in R$ with $I = aR$. Given a morphism of R-modules $f : I \to Q$, choose $x \in Q$ with $f(a) = xa$. For any $r \in R$, $f(ar) = f(a)r = xar$, so by Baer's criterion Q is injective.

2)(ii) Let $I = aR$ and $J = bR$ be non-zero ideals of R with $I \subseteq J$, and let $f : J/I \to R/I$ be a morphism of R-modules. There exists $c \in R$ such that $a = bc$. Write $f(\widehat{b}) = \widehat{x} \in R/I$. Then $xc \in I = aR$, so there is $a_1 \in R$ with $xc = aa_1$, and hence $x = ba_1$.

Define $g : R/I \to R/I$ by $g(\widehat{r}) = \widehat{a_1 r}$ for all $r \in R$. Then g is an R-module homomorphism and $g(\widehat{b}) = \widehat{x}$, so $g|_{J/I} = f$. Thus R/I is quasi-injective. In particular, for $R = \mathbb{Z}$ and $I = n\mathbb{Z}$ we obtain that $\mathbb{Z}_n$ is a quasi-injective $\mathbb{Z}$-module for every $n \geq 1$. □

Corollary 2.1.10. *An abelian group G is an injective $\mathbb{Z}$-module if and only if G is divisible.*

Corollary 2.1.11.

1. *$\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ are injective $\mathbb{Z}$-modules.*
2. *Any direct sum of injective $\mathbb{Z}$-modules is an injective $\mathbb{Z}$-module.*
3. *Any factor group of an injective $\mathbb{Z}$-module is injective.*

Lemma 2.1.12. *Let A, S, T be rings and let ${}_AM_S$, ${}_AN_T$ be bimodules. Then $\mathrm{Hom}_A(M, N)$ has a structure of left S-module and right T-module given by*

$$(s \cdot f)(x) = f(xs), \qquad (f \cdot t)(x) = f(x)t,$$

for $s \in S$, $t \in T$, $x \in M$, $f \in \mathrm{Hom}_A(M, N)$.

Proof. Let $a, b \in A$ and $x, y \in M$. Then

$$(s\cdot f)(ax+by) = f((ax+by)s) = f((ax)s)+f((by)s) = af(xs)+bf(ys) = a(s\cdot f)(x)+b(s\cdot f)(y),$$

so $s \cdot f \in \mathrm{Hom}_A(M, N)$. Similarly $f \cdot t \in \mathrm{Hom}_A(M, N)$.

For $s, s' \in S$ and $f, g \in \mathrm{Hom}_A(M, N)$ we have

$$(s \cdot (f + g))(x) = (f + g)(xs) = f(xs) + g(xs) = (s \cdot f)(x) + (s \cdot g)(x), \tag{1}$$
$$((s + s') \cdot f)(x) = f(x(s + s')) = f(xs + xs') = f(xs) + f(xs') = (s \cdot f)(x) + (s' \cdot f)(x), \tag{2}$$
$$((ss') \cdot f)(x) = f(xss') = f((xs)s') = (s' \cdot f)(xs) = (s \cdot (s' \cdot f))(x), \tag{3}$$
$$(1_S \cdot f)(x) = f(x1_S) = f(x). \tag{4}$$

From (1)–(4) we see that $\mathrm{Hom}_A(M, N)$ is a left S-module. A similar computation shows that $\mathrm{Hom}_A(M, N)$ is a right T-module. Moreover, $((s \cdot f) \cdot t)(x) = (s \cdot f)(xt) = f(xts)t$ and $(s \cdot (f \cdot t))(x) = f(xts)t$, so the two actions commute and $\mathrm{Hom}_A(M, N)$ is an S–T bimodule. □

Proposition 2.1.13 (Eckmann–Schopf). *Let Q be a divisible abelian group. Then the left R-module $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is injective.*

Proof. By the previous lemma, $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ carries a structure of left R–module given by

$$(r \cdot f)(a) = f(ar), \qquad \forall a, r \in R,\ f \in \mathrm{Hom}_{\mathbb{Z}}(R, Q).$$

Let I be a left ideal of R and $h : I \to \mathrm{Hom}_{\mathbb{Z}}(R, Q)$ a morphism of left R–modules. Define

$$\gamma : {}_{\mathbb{Z}}I \longrightarrow {}_{\mathbb{Z}}Q, \qquad \gamma(a) = h(a)(1).$$

Then γ is a morphism of $\mathbb{Z}$–modules. Since Q is $\mathbb{Z}$–injective, there exists $\widetilde{\gamma} : {}_{\mathbb{Z}}R \to {}_{\mathbb{Z}}Q$ such that $\widetilde{\gamma}_{|I} = \gamma$. For $a \in I$ and $r \in R$ we have

$$(a \cdot \widetilde{\gamma})(r) = \widetilde{\gamma}(ra) = h(ra)(1) = (r \cdot h(a))(1) = h(a)(r),$$

hence $h(a) = a \cdot \widetilde{\gamma}$ for all $a \in I$. By Baer's criterion, $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective left R–module. □

Proposition 2.1.14. *Every left R–module M can be embedded into an injective left R–module.*

Proof. There exists a free abelian group $\mathbb{Z}^{(A)}$ and a surjective $\mathbb{Z}$–morphism $f : \mathbb{Z}^{(A)} \to M$. Hence

$${}_{\mathbb{Z}}M \cong \mathbb{Z}^{(A)}/\ker f \subseteq \mathbb{Q}^{(A)}/\ker f,$$

and therefore there is a divisible abelian group G with ${}_{\mathbb{Z}}M \subseteq {}_{\mathbb{Z}}G$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ we obtain a monomorphism

$${}_{R}M \cong \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, G).$$

Since G is divisible, Proposition 2.1.13 implies that $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is an injective left R–module. Thus M embeds into an injective left R–module. □

Proposition 2.1.15. *Let Q be an R–module. Then Q is injective if and only if every short exact sequence*

$$0 \longrightarrow Q \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

splits.

Proof. "⇒". Assume Q is injective. Exactness implies that f is a monomorphism. By injectivity of Q there exists $h : M \to Q$ with $hf = \mathrm{id}_Q$, hence the sequence splits.

"⇐". By Proposition 2.1.14 there is an injective R–module Q' and a monomorphism $i : Q \to Q'$. Consider the exact sequence

$$0 \longrightarrow Q \xrightarrow{i} Q' \longrightarrow Q'/i(Q) \longrightarrow 0.$$

By the hypothesis this sequence splits, so Q is a direct summand of Q'. A direct summand of an injective module is injective, therefore Q is injective. □

2.2 Injective envelopes

Definition 2.2.1. Let M be an R–module. A pair (E, i) is called an *injective envelope* of M if E is injective and $i : M \to E$ is an essential monomorphism.

Proposition 2.2.2. *Let Q be an injective R–module. Then every complement submodule of Q is a direct summand of Q.*

Proof. Let K be a submodule of Q and N a complement of K in Q, that is, $K \cap N = 0$ and $K + N$ is an essential submodule of Q. Then $(K + N)/N \cong Q/N$. Define $g : (K + N)/N \to Q$ by

$$g((x + y) + N) = x, \qquad x \in K,\ y \in N.$$

Since $K \cap N = 0$, g is well defined and injective. As Q is injective, there exists $h : Q/N \to Q$ with $h_{|(K+N)/N} = g$. Because $(K + N)/N \cong Q/N$ and g is a monomorphism, h is also a monomorphism. We have $K = \operatorname{Im} g = h((K + N)/N) \subseteq h(Q/N)$. As K is a closed submodule, it follows that $K = h(Q/N)$. Since h is a monomorphism, $(K + N)/N = Q/N$, hence $K + N = Q$. Thus K is a direct summand of Q. □

Theorem 2.2.3 (Eckmann–Schopf)**.** *Every R–module M has an injective envelope, unique up to isomorphism.*

Proof. By Proposition 2.1.14 there exists an injective R–module Q with $M \leq Q$. Let E be a maximal essential extension of M in Q. Then E is a complement submodule of Q, and by the previous proposition E is injective. Hence (E, i), where $i : M \hookrightarrow E$ is the inclusion, is an injective envelope of M.

For uniqueness, let (E_1, i_1) and (E_2, i_2) be two injective envelopes of M. Since E_2 is injective, there exists $f : E_1 \to E_2$ with $fi_1 = i_2$. The map i_2 is a monomorphism and i_1 is an essential monomorphism, so (using 1.4) f is a monomorphism. Thus $E_1 \cong f(E_1)$ and $E_2 = f(E_1) \oplus E_3$ for some submodule E_3. But $i_2(M) \subseteq f(E_1)$, hence $i_2(M) \cap E_3 = 0$. Since i_2 is essential, we must have $E_3 = 0$, so $E_2 = f(E_1)$ and f is an isomorphism. □

In practice we fix one representative of this isomorphism class and denote it by $E(M)$, with $M \trianglelefteq E(M)$.

Proposition 2.2.4. *Let M be an R–module and $i : M \to Q$ a monomorphism with Q_R injective. The following statements are equivalent:*

1. *(Q, i) is an injective envelope of M;*
2. *for every monomorphism $f : M \to Q'$ with Q' injective, there exists a monomorphism $g : Q \to Q'$ such that $gi = f$.*

Proof. (1) ⇒ (2). Let $f : M \to Q'$ be a monomorphism with Q' injective. By injectivity of Q' there is $u : Q \to Q'$ with $ui = f$. Because i is an essential monomorphism and Q' is injective, the image $u(Q)$ is a complement of $f(M)$; by the definition of injective envelope this forces u to be a monomorphism. Set $g = u$.

(2) $\Rightarrow$ (1). Let $(E(M), j)$ be an injective envelope of M. Applying (2) to $f = j$ we obtain a monomorphism $g : Q \to E(M)$ with $gi = j$. Since j is an essential monomorphism, it follows that i is also essential; hence (Q, i) is an injective envelope of M. $\square$

Proposition 2.2.5. *For any family of right R–modules $M_1, M_2, \ldots, M_n$ we have*

$$E\left(\bigoplus_{i=1}^{n} M_i\right) \cong \bigoplus_{i=1}^{n} E(M_i).$$

Proof. By 1.9, $\bigoplus_{i=1}^{n} E(M_i)$ is an essential extension of $\bigoplus_{i=1}^{n} M_i$. Moreover,

$$\bigoplus_{i=1}^{n} E(M_i) \cong \prod_{i=1}^{n} E(M_i),$$

so by 2.1.3 the module $\bigoplus_{i=1}^{n} E(M_i)$ is injective. By uniqueness of injective envelopes we obtain

$$E\left(\bigoplus_{i=1}^{n} M_i\right) \cong \bigoplus_{i=1}^{n} E(M_i).$$

$\square$

Theorem 2.2.6. *Let Q and M be two R-modules. Then Q is M-injective if and only if $f(M) \subseteq Q$ for every $f \in \operatorname{Hom}(E(M), E(Q))$.*

Proof. "$\Rightarrow$" Let $f \in \operatorname{Hom}(E(M), E(Q))$ and set

$$K := \{\, m \in M \mid f(m) \in Q \,\}.$$

Since Q is M-injective, there exists a morphism $\bar{f} : M \to Q$ such that $\bar{f}\,|_K = f\,|_K$. We claim that

$$Q \cap (\bar{f} - f)(M) = 0.$$

Take $x \in Q$ and $m \in M$ with $x = (\bar{f} - f)(m)$. Then

$$f(m) = \bar{f}(m) - x \in Q,$$

so $m \in K$. Hence

$$x = \bar{f}(m) - f(m) = f(m) - f(m) = 0,$$

and therefore $Q \cap (\bar{f} - f)(M) = 0$. Since Q is an essential submodule of $E(Q)$, it follows that $(\bar{f} - f)(M) = 0$. Thus $f(M) = \bar{f}(M) \subseteq Q$.

"$\Leftarrow$" As $E(Q)$ is injective, it is enough to work with $f \in \operatorname{Hom}(M, E(Q))$. Let N be a submodule of M and $g : N \to Q$ a morphism of R-modules. Because $E(Q)$ is injective, there exists $\tilde{g} : M \to E(Q)$ such that $\tilde{g}\,|_N = i \circ g$, where $i : Q \to E(Q)$ is the canonical injection. By hypothesis, $\tilde{g}(M) \subseteq Q$, so identifying $\tilde{g}$ with its corestriction to Q we obtain a morphism $h : M \to Q$ with $h\,|_N = g$. Therefore Q is M-injective. $\square$

Corollary 2.2.7. *An R-module Q is quasi-injective if and only if $f(Q) \subseteq Q$ for every $f \in \operatorname{End}(E(Q))$.*

Theorem 2.2.8 (Matlis–Bass)**.** *Let R be a ring. Then R is right noetherian if and only if, for every simple right R-module S_i $(i \geq 1)$,*

$$Q := \bigoplus_{i=1}^{\infty} E(S_i)$$

is an injective right R-module.

Proof. "$\Rightarrow$" Let L be a right ideal of R,

$$Q = \bigoplus_{i=1}^{\infty} E(S_i)$$

and let $f : L \to Q$ be a morphism of right R-modules. There exist elements $a_1, \ldots, a_n \in L$ such that

$$L = a_1R + a_2R + \cdots + a_nR.$$

Clearly, there is $m \geq 1$ with $f(a_k) \in \bigoplus_{j=1}^{m} E(S_j)$ for all $k = 1, \ldots, n$, hence

$$\Im f \subseteq \bigoplus_{j=1}^{m} E(S_j).$$

Since $\bigoplus_{j=1}^{m} E(S_j)$ is injective, there exists $g : R \to \bigoplus_{j=1}^{m} E(S_j)$ such that $g\,|_L = f$. Let $\bar{f} = i \circ g$, where $i : \bigoplus_{j=1}^{m} E(S_j) \to Q$ is the canonical injection. Then $\bar{f}\,|_L = f$, so Q is injective.

"$\Leftarrow$" Suppose that R is not right noetherian. Then there exists a strictly ascending chain of finitely generated right ideals:

$$L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq \cdots .$$

By Krull's lemma, for every $n \geq 1$ there exists a maximal submodule $M_n \subsetneq L_n$ such that

$$L_{n-1} \subseteq M_n \quad \text{for all } n \geq 2.$$

Set

$$L := \bigcup_{k=1}^{\infty} L_k, \qquad \pi_k : L_k \longrightarrow L_k/M_k$$

for the canonical projections, and put

$$E_k := E(L_k/M_k) \quad \text{for each } k \geq 1.$$

Then

$$E := \bigoplus_{k=1}^{\infty} E_k$$

is injective, and

$$f : L \longrightarrow E, \qquad f(a) = \sum_{k=1}^{\infty} \pi_k(a)$$

is well defined. There exists an element $x \in E_1 \oplus \cdots \oplus E_n$ such that $f(a) = xa$ for all $a \in L$. It follows that $\pi_k(a) = 0$ for every $k \geq n+1$, that is, $a \in M_k$ for all $k \geq n+1$. Hence

$$L \subseteq M_{n+1} \subsetneq L_{n+1} \subseteq M_{n+2} \subsetneq L_{n+2} \subseteq \cdots \subseteq L,$$

a contradiction. Therefore R is right noetherian. $\square$

Chapter 3

Direct Sums of Uniform (Co-irreducible) Modules

Proposition 3.1. *Let M be an R-module and $E(M)$ its injective envelope. Then the following statements are equivalent:*

1. *$E(M)$ is indecomposable.*
2. *If L and K are non–zero submodules of M, then $L \cap K \neq 0$.*
3. *If $x, y \in M \setminus \{0\}$, then there exist $a, b \in R$ such that $0 \neq xa = yb$.*
4. *M is an essential extension of each of its non–zero submodules.*

Proof. (2) $\Rightarrow$ (1). Assume that $E(M) = L' \oplus K'$ with L' and K' non–zero submodules of $E(M)$. Since $M \trianglelefteq E(M)$, we have $L = L' \cap M \neq 0$ and $K = K' \cap M \neq 0$, hence

$$L \cap K = (L' \cap K') \cap M = 0,$$

which contradicts (2).

(1) $\Rightarrow$ (2). Assume there exist non–zero submodules $L, K \leq M$ such that $L \cap K = 0$. Then $E(L) \leq E(M)$ and $K \cap E(L) = 0$ (otherwise, since $L \subseteq E(L)$, we would have $L \cap K \neq 0$). The short exact sequence

$$0 \longrightarrow E(L) \longrightarrow E(M) \longrightarrow E(M)/E(L) \longrightarrow 0$$

splits because $E(L)$ is injective, hence $E(M) \cong E(L) \oplus E(M)/E(L)$. It follows that

$$0 = K \cap E(L) = K \cap E(M) = K,$$

a contradiction.

The equivalences (2) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) are straightforward. $\square$

Definition 3.2. An R-module M satisfying any of the equivalent conditions above is called *uniform* (or *co-irreducible*). Let M be an R-module and N a proper submodule of M. If M/N is uniform, we say that N is *irreducible in M*. Clearly, N is irreducible in M if and only if from an equality $N = P \cap Q$ with $P, Q \leq M$ it follows that $N = P$ or $N = Q$.

Example.

1. Every simple module is clearly uniform.
2. $\mathbb{Z}$ is a uniform $\mathbb{Z}$-module.
3. By Proposition 3.1, an injective module is indecomposable if and only if it is uniform. In particular, if M is a uniform R-module, then $E(M)$ is uniform. Hence the $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ are uniform.
4. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is uniform if and only if n is a power of a prime. If $n = p^k$ with $p \geq 2$ and $k \geq 1$, then $\langle p^m \rangle \subseteq \langle p^i \rangle \cap \langle p^j \rangle$ for all $i, j \in \{1, \ldots, k-1\}$, where $m = \min(i, j)$, so $\mathbb{Z}_{p^k}$ is uniform. Conversely, assume that $\mathbb{Z}_n$ is uniform and write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ as the factorisation of n into primes. If $s \geq 2$, then
$$\langle p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} \rangle + \mathbb{Z} \quad \text{and} \quad \langle p_s^{\alpha_s} \rangle + \mathbb{Z}$$
are non–zero submodules whose intersection is zero, a contradiction. Hence $s = 1$ and n is a prime power.

Proposition 3.3. *Let M be an R-module and N a proper submodule of M. If $x \in M \setminus N$, then there exists an irreducible submodule $P \leq M$ such that $N \subseteq P$ and $x \notin P$.*

Proof. Set
$$\mathfrak{S} = \{\, N' \leq M \mid N \subseteq N' \text{ and } x \notin N' \,\}.$$
The set $\mathfrak{S}$, ordered by inclusion, is inductive, so by Zorn's Lemma it has a maximal element P. We show that P is irreducible in M. Assume that $P = U \cap V$ with $U, V \leq M$ and $P \subsetneq U$, $P \subsetneq V$. Since $x \notin P$, we have $x \notin U$ or $x \notin V$; suppose $x \notin U$. Then $U \in \mathfrak{S}$ and $P \subsetneq U$, contradicting the maximality of P. Hence P is irreducible. □

Corollary 3.4. *Let M be an R-module and N a proper submodule of M. Then N is the intersection of irreducible submodules of M.*

Proof. This follows immediately from the previous proposition. □

Proposition 3.5. *Let M be a uniform R-module. Then:*

1. *Every non–zero submodule of M is uniform.*
2. *Every essential extension of M is uniform.*

Proof. 1) This follows immediately from 3.1.

2) Let $M \trianglelefteq E$ be an essential extension of M. If E_1, E_2 are non-zero submodules of E, then $M \cap E_1 \neq 0$ and $M \cap E_2 \neq 0$, hence
$$(M \cap E_1) \cap (M \cap E_2) \neq 0,$$
and therefore $E_1 \cap E_2 \neq 0$. Thus E is uniform. □

Lemma 3.6. *Let M be an R-module and $(M_\alpha)_{\alpha\in\Lambda}$ a family of independent submodules. If N is a submodule of M such that*

$$N \cap \Big(\sum_{\alpha\in\Lambda} M_\alpha\Big) \neq 0,$$

then there exists a non-zero submodule of some M_α that is isomorphic to a submodule of N.

Proof. If $\operatorname{Card}\Lambda = 1$, the statement is clear.

If $\operatorname{Card}\Lambda = 2$, set $P = N \cap (M_1 + M_2)$. If $N \cap M_1 = 0$, then $P \cong (P + M_1)/M_1$ and

$$(P + M_1)/M_1 \leq (M_1 + M_2)/M_1 \cong M_2.$$

Hence P is isomorphic to a submodule of M_2. If Λ is finite, arguing by induction reduces the proof to this case.

If Λ is infinite, choose $x \in N \cap \big(\sum_{\alpha\in\Lambda} M_\alpha\big)$, $x \neq 0$. Then there exist $\alpha_1, \dots, \alpha_n \in \Lambda$ such that

$$xR \cap (M_{\alpha_1} + M_{\alpha_2} + \cdots + M_{\alpha_n}) \neq 0,$$

and we are reduced to the case where Λ is finite. □

Remark 3.7. Let M be a right R-module and let Ω be the set of uniform submodules of M. It may happen that $\Omega = \emptyset$. Set

$$\mathcal{S} = \Big\{\Omega' \subseteq \Omega \;\Big|\; \sum_{N\in\Omega'} N \text{ is a direct sum}\Big\}.$$

The partially ordered set $(\mathcal{S}, \subseteq)$ is inductive and, by Zorn's lemma, $\mathcal{S}$ has a maximal element Ω_0. Put $S = \bigoplus_{N\in\Omega_0} N$. We say that S is a maximal direct sum of uniform submodules of M.

Proposition 3.8. *Let M be an R-module and S a maximal direct sum of uniform submodules of M. If N is a non-zero submodule of M, then the following statements are equivalent:*

1. *$S \cap N \neq 0$;*

2. *N contains a uniform submodule.*

Proof. Implication (1) ⇒ (2) follows from Lemma 3.6.

Implication (2) ⇒ (1) follows from the maximality of S. □

Theorem 3.9. *Let M be an R-module and S a maximal direct sum of uniform submodules of M. There exists a submodule K, maximal among the submodules of M that contain no uniform submodule, with the properties:*

1. *$S + K$ is a direct sum;*

2. *$(S \oplus K) \trianglelefteq M$.*

Proof. 1. Let
$$\mathcal{S} = \{ L \leq M \mid \forall L' \leq L,\ L' \text{ is not uniform} \}.$$
Then $\mathcal{S} \neq \emptyset$ because $0 \in \mathcal{S}$ and $\mathcal{S}$ is inductive. By Zorn's lemma, $\mathcal{S}$ has a maximal element K. By Proposition 3.8 we have $S \cap K = 0$.

2. Suppose $N \cap (S \oplus K) = 0$, where N is a non-zero submodule of M. Then $(N+K) \cap S = 0$, and by Proposition 3.8, $N+K$ contains no uniform submodule, which contradicts the maximality of K. Hence $(S \oplus K) \trianglelefteq M$.

□

Definition 3.10. Let M be an R-module. A finite intersection of submodules
$$\bigcap_{i \in I} N_i \qquad (I \text{ finite})$$
is called *reduced* if, for every $i \in I$, we have
$$\bigcap_{i \in I} N_i \neq \bigcap_{\substack{j \in I \\ j \neq i}} N_j.$$

It is clear that if we have a finite intersection $N = \bigcap_{i \in I} N_i$ of irreducible submodules, then it can always be written as a reduced intersection of irreducible submodules.

Theorem 3.11 (Kuros–Ore Theorem)**.** *Let M be an R-module and N a submodule of M. Assume that N has finite reduced intersections:*
$$N = N_1 \cap N_2 \cap \cdots \cap N_m = L_1 \cap L_2 \cap \cdots \cap L_n,$$
where each N_i and each L_j is an irreducible submodule of M. Then $m = n$.

Proof. Let $N_i' = N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_m$. Then $N \subseteq N_i'$ and $N = N_i \cap N_i'$. For each j, set $P_j = N_i \cap L_j$. Then
$$N \subseteq P_j \subseteq N_i' \quad \text{and} \quad P_j \subseteq L_j.$$
Hence
$$N \subseteq \bigcap_{j=1}^{n} P_j \subseteq \bigcap_{j=1}^{n} L_j = N,$$
so the intersection of the P_j is exactly N.

Since N_i is irreducible in M, it is irreducible in $N_i + N_i'$, and therefore
$$\frac{N_i'}{N_i' \cap N_i} \cong \frac{N_i + N_i'}{N_i}$$
is uniform (co-irreducible), showing that N is irreducible in N_i'.

Thus from $N = \bigcap_{j=1}^{n} P_j$, there exists some j with $N = P_j = N_i \cap L_j$.

Set $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$, and choose $i_1 \in I$. Then for some $j_1 \in J$,

$$N = L_{j_1} \cap \Big(\bigcap_{i \neq i_1} N_i \Big).$$

Since the intersection is finite we may reduce it step by step, maintaining at each step at least one L_{j_k} that does not disappear. We obtain a decreasing chain

$$J \supseteq J_1 \supseteq \cdots \supseteq J_r,$$

with $r \leq m$, such that for the final set $J' \subseteq J$,

$$N = \bigcap_{j \in J'} L_j.$$

Because the intersection $L_1 \cap \cdots \cap L_n$ is reduced, we must have $J' = J$. Since J' has at most r elements, $n \leq r \leq m$. A symmetric argument gives $m \leq n$. □

Theorem 3.12. *Let M be an R-module and let $(N_i)_{i=1}^m$ and $(L_j)_{j=1}^n$ be two independent families of uniform (co-irreducible) submodules such that*

$$\bigoplus_{i=1}^{m} N_i \subseteq M \quad \textit{and} \quad \bigoplus_{j=1}^{n} L_j \subseteq M$$

are essential in M. Then $m = n$.

Proof. For each k, set $N'_k = \bigoplus_{i \neq k} N_i$. Then $N_k \cap N'_k = 0$.

Consider

$$\mathfrak{S}_k = \{ P \leq M \mid N_k \cap P = 0 \text{ and } N_k \subseteq P \}.$$

This set is inductive, so by Zorn's lemma it has a maximal element P_k. If $Q \subseteq Q' \subseteq M$ and $P_k \subseteq Q \subseteq Q'$, then the irreducibility of N_k forces $P_k = Q = Q'$. Thus each P_k is co-irreducible.

For any k,

$$N = \bigoplus_{k=1}^{m} N_k = N_k \oplus N'_k.$$

By the modular law,

$$N \cap P_k = (N_k \oplus N'_k) \cap P_k = N_k + (N'_k \cap P_k),$$

so $N \cap P_k = N_k$. Hence

$$N \cap \bigcap_{k=1}^{m} P_k = \bigcap_{k=1}^{m} (N \cap P_k) = \bigcap_{k=1}^{m} N_k = 0.$$

But since N is essential in M, this implies $\bigcap_{k=1}^m P_k = 0$.

Moreover, $N_i = \bigcap_{k \neq i} P_k$, so each such intersection is reduced.

Repeating the argument for (L_j), we obtain irreducible submodules (Q_j) with $\bigcap_j Q_j = 0$.

Applying the Kuros–Ore theorem gives $m = n$. □

Definition 3.13. An R-module M is said to have *finite uniform (co-irreducible) dimension* if there exists a finite independent family $(N_i)_{i=1}^n$ of uniform submodules of M such that

$$\bigoplus_{i=1}^{n} N_i \trianglelefteq M.$$

In this case the integer n is called the *uniform (co-irreducible) dimension* of M and we write $\dim M = n$.

Theorem 3.14. *Let M be an R-module. The following statements are equivalent:*

1. *M has finite uniform (co-irreducible) dimension.*
2. *M satisfies the ascending chain condition for direct sums.*

Proof. (2) $\Rightarrow$ (1). Let $\mathcal{S}'$ and $\mathcal{S}$ be two finite families of independent submodules of M. We say that $\mathcal{S}'$ is a *refinement* of $\mathcal{S}$ if every submodule in $\mathcal{S}'$ is contained in some submodule from $\mathcal{S}$. It is easy to check that refinement defines a partial order on the set of families of submodules of M. Since M satisfies (2), there exists a family $\mathcal{S}_0$ of independent submodules of M which is maximal with respect to this order.

Take $N \in \mathcal{S}_0$. If N is not uniform, then there exist non-zero submodules $P, Q \leq N$ with $P \cap Q = 0$. From $\mathcal{S}_0$ we can then construct a proper refinement of $\mathcal{S}_0$, a contradiction. Hence every submodule in $\mathcal{S}_0$ is uniform, so M contains a uniform submodule. If L is a non-zero submodule of M, then L also satisfies (2), and the same argument shows that L contains a uniform submodule.

Now let

$$S = \bigoplus_{i \in I} N_i$$

be a maximal direct sum of uniform submodules of M and let $N \leq M$ be non-zero. Since M satisfies the ascending chain condition on direct sums, the index set I is finite. By Proposition 3.8 we have $S \cap N \neq 0$, so $S \trianglelefteq M$. By definition, M has finite uniform dimension.

(1) $\Rightarrow$ (2). Let

$$S = \bigoplus_{i=1}^{n} N_i$$

be a direct sum of uniform submodules with $S \trianglelefteq M$. If N is a non-zero submodule of M, then $S \cap N \neq 0$, and by Lemma 3.6 the submodule N contains a uniform submodule.

Let $(L_j)_{j=1}^m$ be an independent family of submodules of M. From the previous paragraph, each L_j contains a uniform submodule P_j. Clearly, the family $(P_j)_{j=1}^m$ is independent. By Theorem 3.12 we deduce $m \leq n$. Thus every independent family of submodules has bounded finite length, which is equivalent to the ascending chain condition for direct sums. $\square$

Remark 3.15. Let M be an R-module. Then:

1. If $N \trianglelefteq M$, then M has finite uniform (co-irreducible) dimension if and only if N has finite uniform dimension, and in this case $\dim M = \dim N$.

2. If $M = \bigoplus_{i=1}^{n} N_i$, then M has finite uniform dimension if and only if each N_i has finite uniform dimension for all $i = 1, \ldots, n$. Moreover,

$$\dim M = \sum_{i=1}^{n} \dim N_i.$$

Chapter 4

Ore domains and Goldie rings

Definition 4.1. An integral domain D is called a *right Ore domain* (respectively a left Ore domain) if $aD \cap bD \neq 0$ (respectively $Da \cap Db \neq 0$) for all $a, b \in D \setminus \{0\}$.

Proposition 4.2. *Let D be an integral domain. Then D is a right Ore domain if and only if D contains a uniform right ideal. In particular, this condition holds whenever D satisfies the ascending chain condition on direct sums of right ideals.*

Proof. "$\Rightarrow$" If D is a right Ore domain, then the right module D_D is uniform.

"$\Leftarrow$" Let X be a uniform right ideal of D and $x \in X \setminus \{0\}$. Consider the map

$$\varphi_x : D_D \longrightarrow X, \qquad \varphi_x(a) = xa.$$

This is an injective homomorphism of right D-modules. Hence D_D is uniform and therefore D is a right Ore domain. $\square$

Example 4.3. Let K be a commutative field and y an indeterminate. Let $K(y)$ be the field of fractions of the polynomial ring $K[y]$. If x is another indeterminate, we consider $K(y)[x]$, the set of polynomials in the indeterminate x with coefficients in $K(y)$. On this set we define two operations: addition is the usual addition of polynomials, and multiplication in $K(y)[x]$ is defined by the commutation rule

$$xf(y) = f(y^2)x, \qquad f \in K(y).$$

It is straightforward to check that $K(y)[x]$ with these two operations is a unital ring. We write $A = K(y)[x]$. Then A is a non-commutative integral domain.

Every element of A can be written uniquely in the form

$$P(x) = f_n(y)x^n + f_{n-1}(y)x^{n-1} + \cdots + f_1(y)x + f_0(y),$$

where n is a natural number, $f_0(y), f_1(y), \ldots, f_n(y) \in K(y)$ and $f_n(y) \neq 0$. We set $\deg P(x) = n$.

If $P(x), Q(x) \in A$ with $Q(x) \neq 0$, then there exist $S(x), R(x) \in A$ such that

$$P(x) = S(x)Q(x) + R(x) \tag{$*$}$$

with either $R(x) = 0$ or $\deg R(x) < \deg Q(x)$. From $(*)$ it follows that A is a left principal ideal ring. Hence A satisfies the ascending chain condition on direct sums of left ideals, and therefore A is a left Ore domain.

We now show that $xA \cap yA = 0$. Let $\alpha \in xA \cap yA$. Then

$$\alpha = x\big(f_n(y)x^n + \cdots + f_0(y)\big) = yx\big(g_m(y)x^m + \cdots + g_0(y)\big),$$

so that

$$f_n(y^2)x^{n+1} + \cdots + f_0(y^2)x = y\big(g_m(y^2)x^{m+1} + \cdots + g_0(y^2)x\big),$$

and hence $m = n$. Thus

$$f_n(y^2) = yg_m(y^2), \ldots, f_0(y^2) = yg_0(y^2),$$

which holds if and only if $f_n(y) = \cdots = f_1(y) = f_0(y) = 0$, and therefore $\alpha = 0$. Consequently, A is not a right Ore domain.

GOLDIE RINGS

Definition 4.4. A right (respectively left) ideal I is called a *right annihilator ideal* (respectively a left annihilator ideal) if there exists a non-empty subset X of R such that $I = \mathrm{ann}_r(X)$ (respectively $I = \mathrm{ann}_l(X)$).

A ring R is called a *right Goldie ring* if R satisfies the ascending chain condition on right annihilator ideals and there is no infinite direct sum of non-zero right ideals in R.

Example 4.5.

1. If R is right noetherian, then R is a right Goldie ring. In particular, $\mathbb{Z}$ is a Goldie ring.

2. If R is an integral domain, then R is a right Goldie ring if and only if R is a right Ore domain.

3. From Example 4.3 it follows that there exist rings which are Goldie on the left but not on the right.

Proposition 4.6. *Let R be a ring. Then R satisfies the ascending chain condition on right (respectively left) annihilator ideals if and only if R satisfies the descending chain condition for left (respectively right) annihilator ideals.*

Proof. "$\Rightarrow$" Let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \ldots$ be a descending sequence of left annihilator ideals. For each $j \geq 1$ there exists a non-empty subset $X_j \subseteq R$ such that $I_j = \mathrm{ann}_\ell(X_j)$.

We thus obtain an ascending sequence of right annihilator ideals

$$\mathrm{ann}_r(I_1) \subseteq \mathrm{ann}_r(I_2) \subseteq \cdots \subseteq \mathrm{ann}_r(I_n) \subseteq \ldots.$$

Hence there exists $m \geq 1$ such that $\mathrm{ann}_r(I_m) = \mathrm{ann}_r(I_{m+k})$ for all $k \geq 1$. It follows that

$$\mathrm{ann}_\ell(\mathrm{ann}_r(I_m)) = \mathrm{ann}_\ell(\mathrm{ann}_r(I_{m+k})) \quad \text{for all } k \geq 1,$$

that is, $I_m = I_{m+k}$ for all $k \geq 1$.

Using the relations $\mathrm{ann}_\ell(\mathrm{ann}_r(\mathrm{ann}_\ell(X))) = \mathrm{ann}_\ell(X)$ and $\mathrm{ann}_r(Y) \subseteq \mathrm{ann}_r(X)$ whenever $X \subseteq Y$, the converse implication "$\Leftarrow$" is proved in the same way. $\square$

Proposition 4.7. *Let R be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If I and J are right ideals with $I \subsetneq J$, then there exists an element $b \in R$ such that $bJ \neq 0$ and $bJ \cap I = 0$.*

Proof. By the preceding lemma there exists a minimal element $\mathfrak{a}$ in the set of left annihilator ideals such that $\operatorname{ann}_\ell(J) \subsetneq \operatorname{ann}_\ell(I)$. Then $\mathfrak{a}J \neq 0$; since R is semiprime we have $\mathfrak{a}J \cdot \mathfrak{a}J \neq 0$, so there exist $x \in J$ and $a \in \mathfrak{a}J$ with $b = xa$ such that $xaJ \neq 0$. Clearly $bJ \neq 0$.

It remains to show that $bJ \cap I = 0$. Take $\lambda \in bJ \cap I$; then $\lambda = b\mu = xa\mu \in I$ for some $\mu \in J$. Since $\operatorname{ann}_\ell(J) \subseteq \operatorname{ann}_\ell(\lambda)$, we obtain

$$\operatorname{ann}_\ell(J) \subseteq \operatorname{ann}_\ell(\lambda) \subseteq \operatorname{ann}_\ell(\mu).$$

Because $a \in \mathfrak{a}J \subseteq \mathfrak{a}R$, we have

$$xa \in \mathfrak{a} \quad \text{and hence} \quad xaJ \subseteq \mathfrak{a}J. \tag{4.1}$$

On the other hand, $xaJ \subseteq I$, while $\lambda \in I$ and $\mathfrak{a} \subseteq \operatorname{ann}_r(I)$; therefore

$$\mathfrak{a} \subseteq \operatorname{ann}_\ell(\mu). \tag{4.2}$$

Since $xaJ \neq 0$, we also have

$$\mathfrak{a} \nsubseteq \operatorname{ann}_\ell(J). \tag{4.3}$$

From (4.1), (4.2) and (4.3) we obtain $\operatorname{ann}_\ell(J) \subsetneq \operatorname{ann}_\ell(\mu) \cap \mathfrak{a}$. If $\lambda \neq 0$, then $xa\mu \notin \operatorname{ann}_\ell(\mu)$, and, since $xa \in \mathfrak{a}$, it follows that $xa\mu \neq 0$, contradicting the minimality of $\mathfrak{a}$. Thus $\lambda = 0$, and hence $bJ \cap I = 0$. ☐

Corollary 4.8. *If R is a right Goldie semiprime ring, then R satisfies the descending chain condition on right annihilator ideals.*

Proof. Assume that there exists a strictly descending sequence of right annihilator ideals

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots.$$

Then for every $n \geq 1$ we have $\operatorname{ann}_\ell(I_n) \neq \operatorname{ann}_\ell(I_{n+1})$. By Proposition 4.7, for each $n \geq 1$ there exists a right ideal K_n such that $K_n \subseteq I_n$ and $K_n \cap I_{n+1} \neq 0$. Hence the sum $\sum_{n\geq 1} K_n$ is an infinite direct sum of non-zero right ideals, which contradicts the right Goldie property of R. ☐

Proposition 4.9. *Let R be a semiprime ring which satisfies the ascending chain condition on right annihilator ideals. If $x, y \in R$ are such that xR and yR are essential in R_R, then xyR is essential.*

Proof. Let $I \leq R_R$ with $I \neq 0$. Define

$$J = \{I : x\} = \{a \in R \mid xa \in I\}.$$

Then J is a right ideal of R, and $xJ = xR \cap I \neq 0$ because $xR \trianglelefteq R_R$. Since $\operatorname{ann}_r(x) \subseteq J$, $xJ \neq 0$ and $\operatorname{ann}_\ell(x) = 0$, we obtain

$$\operatorname{ann}_\ell(J) \subseteq \operatorname{ann}_\ell(\operatorname{ann}_r(x)).$$

By Proposition 4.7 there exists a right ideal $K \neq 0$ such that $K \subseteq J$ and $K \cap \mathrm{ann}_r(x) = 0$.

Now set

$$L = \{K : y\} = \{a \in R \mid ya \in K\}.$$

Since $yR \trianglelefteq R_R$, we have $yL = yR \cap K \neq 0$. If $xyL = 0$, then $yL \subseteq \mathrm{ann}_r(x)$ and hence $yL \subseteq \mathrm{ann}_r(x) \cap K = 0$, a contradiction. Thus $xyL \neq 0$.

Because $xyL \subseteq xK \subseteq xJ \subseteq xR \cap I$, it follows that $0 \neq xyL \subseteq xyR \cap I$, showing that $xyR \trianglelefteq R_R$. □

Proposition 4.10. *Let R be a ring which satisfies the ascending chain condition on right annihilator ideals. Then, for every $a \in R$ there exists an integer $k \geq 0$ such that*

$$\mathrm{ann}_r(a^n) = \mathrm{ann}_r(a^m) \quad \textit{for all } n \geq 0 \textit{ and } m \geq k.$$

Proof. We have the ascending chain of right annihilator ideals

$$\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(a^2) \subseteq \cdots \subseteq \mathrm{ann}_r(a^n) \subseteq \cdots.$$

Let $k \geq 0$ be such that $\mathrm{ann}_r(a^k) = \mathrm{ann}_r(a^m)$ for every $m \geq k$. If $\lambda \in \mathrm{ann}_r(a^n) \cap a^m R$, then $a^n\lambda = 0$ and $\lambda = a^m\mu$, hence $a^{m+n}\mu = 0$. Since $m + n \geq k$, it follows that $\mu \in \mathrm{ann}_r(a^k)$ and thus $a^k\mu = 0$. As $m \geq k$, we may write $\lambda = a^{m-k}a^k\mu = 0$, and therefore

$$\mathrm{ann}_r(a^n) \cap a^m R = 0.$$

□

Corollary 4.11. *Assume the hypotheses of Proposition 4.10. If $xR \trianglelefteq R_R$, then x is a regular element of R.*

Proof. If $\mathrm{ann}_r(x) \neq 0$, then $0 = \mathrm{ann}_r(R) \neq \mathrm{ann}_r(xR)$, so there exists a right ideal $I \neq 0$ such that $I \cap xR = 0$, a contradiction. Hence $\mathrm{ann}_r(x) = 0$.

By Proposition 4.9 we have $x^n R \trianglelefteq R_R$ for every $n \geq 1$. Applying Proposition 4.10, we obtain $k \geq 0$ such that $\mathrm{ann}_r(x^n) \cap x^m R = 0$ for all $m \geq k$. In particular $\mathrm{ann}_r(x) = 0$, so x is a regular element of R. □

Corollary 4.12. *Let R be a right Goldie and semiprime ring. If $\mathrm{ann}_r(x) = 0$ for some $x \in R$, then $xR \trianglelefteq R_R$ and x is regular.*

Proof. Let I be a non-zero right ideal of R and suppose $I \cap xR = 0$. Then the sum

$$\sum_{n \geq 1} x^n I$$

is direct. Indeed, for $p = 1$ we have $x^p I \cap \sum_{n \neq p} x^n I = xI \cap x^p I$. If $y \in xI \cap x^p I$, then $y = x\lambda = x^2\mu$ for some $\lambda, \mu \in I$. Hence $\lambda - x\mu \in \mathrm{ann}_r(x)$ and so $\lambda = x\mu$, which implies $\lambda \in I \cap xR = 0$. Thus $\lambda = 0$ and consequently $y = 0$. Therefore $xI \cap x^2 I = 0$, and a fortiori $xI \cap x^p I = 0$ for $p \neq 1$. Hence $x^p I \cap \sum_{n \neq p} x^n I = 0$, so the sum $\sum_{n \geq 1} x^n I$ is direct, a contradiction.

Thus $xR \trianglelefteq R_R$. By Corollary 4.11 we conclude that x is a regular element of R. □

Proposition 4.13. *Let R be a right Goldie and semiprime ring. Then:*

(a) Every two–sided right annihilator ideal of R contains a minimal two–sided right annihilator ideal.

(b) There exists a finite direct sum of non-zero minimal two–sided right annihilator ideals of R which is essential in R.

Proof. (a) This follows from Corollary 4.8.

(b) Let $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ be a maximal direct sum where $I_1, \dots, I_n$ are non-zero minimal two–sided right annihilator ideals of R. Let $K \leq R_R$, $K \neq 0$ with $I \cap K = 0$. Since $KI \subseteq I \cap K = 0$, we have $K \subseteq \operatorname{ann}_r(I)$. As R is semiprime, $I \cap \operatorname{ann}_r(I) = 0$, hence $I \operatorname{ann}_r(I) = 0$ and thus $\operatorname{ann}_l(I) \subseteq \operatorname{ann}_r(I)$, so in particular $\operatorname{ann}_r(I) \neq 0$. Again, R being semiprime implies $I \cap \operatorname{ann}_l(I) = 0$, where $\operatorname{ann}_l(I)$ is a two–sided ideal. By (a) there exists a non-zero two–sided right annihilator ideal J, minimal with respect to $J \subseteq \operatorname{ann}_r(I)$ and $J \cap I = 0$, which contradicts the maximality of I as a direct sum. Hence $I \trianglelefteq R$. □

Proposition 4.14. *Let R be a right Goldie and prime ring. If I is an essential right ideal of R, then I contains a regular element of R.*

Proof. Choose $a \in I$ such that $\operatorname{ann}_r(a)$ is minimal in the set $\{\operatorname{ann}_r(x) \mid x \in I\}$. Let $J \leq R_R$, $J \neq 0$ with $aR \cap J = 0$. Since $I \trianglelefteq R_R$, we have $I \cap J \neq 0$; we may assume that J is a non-zero right ideal of R contained in I and still satisfying $aR \cap J = 0$.

Let $x \in J$. If $\lambda \in \operatorname{ann}_r(a + x)$, then $a\lambda + x\lambda = 0$, so $x\lambda = -a\lambda \in aR \cap J$, hence $x\lambda = 0$ and consequently $a\lambda = 0$. Thus $\lambda \in \operatorname{ann}_r(a) \cap \operatorname{ann}_r(x)$. Since the inclusion

$$\operatorname{ann}_r(a) \cap \operatorname{ann}_r(x) \subseteq \operatorname{ann}_r(a + x)$$

is obvious, we obtain $\operatorname{ann}_r(a + x) = \operatorname{ann}_r(a) \cap \operatorname{ann}_r(x)$. By the minimality of $\operatorname{ann}_r(a)$ it follows that $\operatorname{ann}_r(a) \subseteq \operatorname{ann}_r(x)$ and hence $x \operatorname{ann}_r(a) = 0$, so $J \operatorname{ann}_r(a) = 0$. As R is prime, we must have $\operatorname{ann}_r(a) = 0$.

By Corollary 4.12 we deduce $J = 0$, a contradiction. Therefore $aR \trianglelefteq R_R$, and Corollary 4.11 implies that a is regular in R. □

Chapter 5

The Osofsky–Smith Theorem

Definition 5.1. An R-module M is called a *CS-module* if every submodule which is a complement of M is a direct summand of M. The module M is called *CS-complete* if M/N is a CS-module for every $N \leq M$.

Remark 5.2. From 2.2.2 it follows that every injective module is a CS-module.

Proposition 5.3. *For an R-module M the following statements are equivalent:*

1. *M is a CS-module.*
2. *Every maximal essential extension of a submodule of M is a direct summand of M.*
3. *Every submodule of M is essential in a direct summand of M.*

Proof. (1) $\Rightarrow$ (2) Let N be a submodule of M and Q a maximal essential extension of N in M (Proposition 1.13). Then Q is a closed submodule, and by 1.15 it follows that Q is a direct summand of M.

(2) $\Rightarrow$ (3) is immediate from Proposition 1.13.

(3) $\Rightarrow$ (1) Let K be a complement submodule of M. There exists a direct summand Q of M such that $K \trianglelefteq Q$. Since K is a closed submodule, we obtain $K = Q$, and therefore K is a direct summand of M. $\square$

Proposition 5.4. *Every quasi-injective module is a CS-module.*

Proof. Let M be a quasi-injective R-module and N a submodule of M. Then there exists $E_2 \leq E(M)$ such that $E(M) = E_1 \oplus E_2$, where $E_1 = E(N)$. Let $\pi_i : E(M) \to E_i$, $i = 1, 2$, be the canonical projections. Since $\pi_i(M) \leq M$ for all $i = 1, 2$ (Corollary 2.2.7), it follows that

$$M = (M \cap E_1) \oplus (M \cap E_2).$$

Clearly $N \trianglelefteq (M \cap E_1)$, so N is essential in a direct summand of M. By the previous proposition we conclude that M is a CS-module. $\square$

Lemma 5.5. *Let X be a cyclic CS-module with the property that $S = \mathrm{soc}(X) \trianglelefteq X_R$, where S is not finitely generated, but every finitely generated submodule of S is a direct summand of X. If every cyclic submodule of X is a CS-module, then X/S is not a CS-module.*

Proof. Assume that X/S is a CS-module. Since S is not finitely generated, we can write

$$S = \bigoplus_{i \geq 1} S_i,$$

with $S_i \leq X$ and S_i not finitely generated for every $i \geq 1$.

For every $i \geq 1$ there exists a complement submodule D_i of X such that $S_i \trianglelefteq D_i$. Then D_i is a direct summand of X, hence D_i is cyclic for all $i \geq 1$, and therefore $S_i \subsetneq D_i$.

Since X/S is a CS-module, there exists a direct summand $\overline{E}$ of X such that

$$\sum_{i \geq 1} \frac{D_i + S}{S} = \frac{D}{S} \trianglelefteq \overline{E},$$

where $D := \bigoplus_{i \geq 1} D_i$. Let E be a cyclic submodule of X such that $\overline{E} = (E + S)/S$. We obtain

$$\frac{D}{S} \leq \frac{E + S}{S}.$$

Since $E \cap S \leq S$ and S is semisimple, there exists $T \leq S$ with

$$S = (E \cap S) \oplus T,$$

and hence

$$E + S = (E \cap S) \oplus T.$$

Suppose there exists $i \geq 1$ such that $D_i \cap E = 0$. Considering the canonical projection $\pi : E \oplus T \to T$, we have $D_i \cap \ker \pi = 0$, which shows that $\pi_{|D_i}$ is a monomorphism and therefore D_i is semisimple. But $S_i \trianglelefteq D_i$, which implies $S_i = D_i$, a contradiction. Hence $D_i \cap E \neq 0$ for all $i \geq 1$, and consequently $S_i \cap E \neq 0$ for all $i \geq 1$.

Since $S \trianglelefteq X_R$, there exist simple modules V_i with $V_i \leq S_i \cap E$ for all $i \geq 1$. Let $V := \bigoplus_{i \geq 1} V_i$. The module E is a CS-module because E is cyclic. There exists a direct summand L of X such that $V \trianglelefteq L$. Note that L is cyclic and therefore $V \subsetneq L$. For $n \geq 1$ set

$$P_n := \left(\bigoplus_{i=1}^{n} D_i \right) \cap L.$$

Then

$$P_n \cap S = L \cap \left(\bigoplus_{i=1}^{n} S_i \right) = V \cap \left(\bigoplus_{i=1}^{n} S_i \right) = \bigoplus_{i=1}^{n} V_i,$$

and hence $P_n \cap S = \bigoplus_{i=1}^{n} V_i$. □