

# The Uniform (Co-Irreducible) Dimension of Rings and Modules

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# Contents

<b>0 Generalities</b>	<b>1</b>
0.1 Semisimple modules . . . . .	1
0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings . . . . .	2
0.3 Finite length modules . . . . .	3
0.4 The Jacobson radical . . . . .	5
0.5 Semisimple rings . . . . .	6
<b>1 Essential submodules</b>	<b>7</b>
<b>2 Injective Modules</b>	<b>13</b>
2.1 Injective Module . . . . .	13
2.2 Injective envelopes . . . . .	20



# Chapter 0

## Generalities

### 0.1 Semisimple modules

**Definition 0.1.1.** A non-zero  $R$ -module  $S$  is called *simple* if its only submodules are  $0$  and  $S$ .

**Proposition 0.1.2.** Let  $S$  be an  $R$ -module. The following statements are equivalent:

1.  $S$  is simple;
2. for every non-zero element  $x \in S$  we have  $S = xR$ ;
3.  $S \cong R/I$ , where  $I$  is a maximal right ideal.

**Lemma 0.1.3** (Schur). Let  $S$  and  $S'$  be simple  $R$ -modules and let  $f: S \rightarrow S'$  be a homomorphism of  $R$ -modules. Then  $f = 0$  or  $f$  is an isomorphism. In particular  $\text{End}_R(S)$  is a division ring.

**Definition 0.1.4.** Let  $M$  be an  $R$ -module and let  $(S_i)_{i \in I}$  be the family of all simple submodules of  $M$ . If  $M = \sum_{i \in I} S_i$ , then  $M$  is called *semisimple*.

**Proposition 0.1.5.** Let  $M$  be a semisimple  $R$ -module and  $N$  a submodule of  $M$ . Then there exists a subset  $J \subseteq I$  such that

1. the family  $(S_j)_{j \in J}$  is independent;
2.  $M = N \oplus (\bigoplus_{j \in J} S_j)$ .

**Corollary 0.1.6.** With the above notation, for the semisimple module  $M$  there exists  $J \subseteq I$  such that the family  $(S_j)_{j \in J}$  is independent and

$$M = \bigoplus_{i \in I} M_i.$$

**Corollary 0.1.7.** If  $M$  is a semisimple  $R$ -module and  $N$  a submodule of  $M$ , then both  $N$  and  $M/N$  are semisimple.

**Corollary 0.1.8.** A direct sum of semisimple modules is a semisimple module.

**Theorem 0.1.9.** Let  $M$  be an  $R$ -module. The following statements are equivalent:

1.  $M$  is semisimple;
2.  $M$  is isomorphic to a direct sum of simple modules;
3. every submodule of  $M$  is a direct summand of  $M$ ;
4. every short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits.

**Definition 0.1.10.** The sum of all simple submodules of  $M$  is called the *socle* of  $M$  and is denoted by  $\text{soc}(M)$ . If  $M$  has no simple submodule we put  $\text{soc}(M) = 0$ .

**Proposition 0.1.11.** Let  $M$  and  $N$  be  $R$ -modules and  $f: M \rightarrow N$  a homomorphism. Then  $f(\text{soc}(M)) \subseteq \text{soc}(N)$ .

**Proposition 0.1.12.** Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . Then

$$\text{soc}(N) = \text{soc}(M) \cap N.$$

**Proposition 0.1.13.** If  $M = \bigoplus_{i \in I} M_i$ , then

$$\text{soc}(M) = \bigoplus_{i \in I} \text{soc}(M_i).$$

**Proposition 0.1.14.** Let  $R$  be a ring. Then the socle  $\text{soc}(R_R)$  is a two-sided ideal of  $R$ .

## 0.2 Noetherian (Artinian) modules and Noetherian (Artinian) rings

**Definition 0.2.1.** Let  $R$  be a ring and  $M$  a right  $R$ -module. We say that  $M$  satisfies the *maximal condition* (resp. the *minimal condition*) if every non-empty set of submodules of  $M$ , ordered by inclusion, has a maximal (resp. minimal) element.

We say that  $M$  satisfies the *ascending* (resp. *descending*) *chain condition* if every ascending chain of submodules of  $M$

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

(resp. every descending chain

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq \cdots$$

) is stationary, that is, there exists  $n \geq 1$  such that  $M_n = M_{n+1} = \cdots$ .

**Proposition 0.2.2.** Let  $M$  be an  $R$ -module. The following statements are equivalent:

1.  $M$  satisfies the maximal (minimal) condition;
2.  $M$  satisfies the ascending (descending) chain condition.

**Definition 0.2.3.** An  $R$ -module  $M$  is called *noetherian* (resp. *artinian*) if it satisfies the maximal (resp. minimal) condition. The ring  $R$  is called right noetherian (resp. right artinian) if the right module  $R_R$  is noetherian (resp. artinian).

**Example 0.2.4.**

1.  $\mathbb{Z}$  is a noetherian ring but not artinian.
2. Every finite group is a noetherian and artinian  $\mathbb{Z}$ -module.
3. Every finite ring is noetherian and artinian.
4. The ring  $\mathbb{Z}[X_1, X_2, \dots, X_n, \dots]$  is neither noetherian nor artinian:

$$(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \dots, X_n) \subsetneq \cdots$$

$$(X_1) \supsetneq (X_1^2) \supsetneq \cdots \supsetneq (X_1^k) \supsetneq \cdots$$

5. The Prüfer  $p$ -group  $\mathbb{Z}_{p^\infty}$  is an artinian but not noetherian  $\mathbb{Z}$ -module.

**Proposition 0.2.5.** Let  $N$  and  $P$  be submodules of  $M$  such that  $M = N + P$ . Then  $M$  is noetherian (artinian) if and only if  $N$  and  $P$  are noetherian (artinian).

**Proposition 0.2.6.** For an  $R$ -module  $M$  the following statements are equivalent:

1.  $M$  is noetherian;
2. every submodule of  $M$  is finitely generated.

**Proposition 0.2.7.** For an  $R$ -module  $M$  the following statements are equivalent:

1.  $M$  is artinian;
2. for every family  $(X_i)_{i \in I}$  of submodules of  $M$  there exists a finite subset  $J \subseteq I$  such that

$$\bigcap_{i \in I} X_i = \bigcap_{j \in J} X_j.$$

## 0.3 Finite length modules

**Definition 0.3.1.** Let  $M$  be a non-zero right  $R$ -module. A *composition series* or *Jordan–Hölder series* of  $M$  is a finite strictly ascending chain of submodules

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = M$$

such that  $X_{i+1}/X_i$  is a simple module for  $0 \leq i \leq n - 1$ . The integer  $n$  is called the *length* of the series, and the modules  $X_{i+1}/X_i$  are called its *factors*.

**Proposition 0.3.2.** Let  $M$  be an  $R$ -module. The following statements are equivalent:

1.  $M$  has a composition series;
2.  $M$  is noetherian and artinian.

**Proposition 0.3.3.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of right  $R$ -modules. Then  $M$  has a composition series if and only if both  $M'$  and  $M''$  have composition series.

If

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M$$

are two composition series of  $M$ , we say that they are *equivalent* if  $n = p$  and there exists a bijection  $\sigma : \{0, \dots, n-1\} \rightarrow \{0, \dots, n-1\}$  such that

$$M_{i+1}/M_i \cong M_{\sigma(i)+1}/M_{\sigma(i)} \quad (0 \leq i \leq n-1).$$

**Theorem 0.3.4** (Jordan–Hölder). *If an  $R$ -module  $M$  has two composition series*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M, \quad 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_p = M,$$

*then these two series are equivalent.*

**Definition 0.3.5.** An  $R$ -module  $M$  which admits a composition series is called a *module of finite length*. The length of its composition series is called the *length* of  $M$  and is denoted by  $l(M)$ . If  $M$  admits no composition series, we say that  $M$  has *infinite length* and we write  $l(M) = \infty$ .

**Proposition 0.3.6.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of  $R$ -modules of finite length. Then

$$l(M) = l(M') + l(M'').$$

**Corollary 0.3.7.** *Let  $M$  be an  $R$ -module of finite length and let  $N, L$  be submodules of  $M$ . Then:*

1.  $l(M) = l(N) + l(M/N);$
2.  $l(N + L) + l(N \cap L) = l(N) + l(L).$

**Corollary 0.3.8.** *Let  $M$  be an  $R$ -module of finite length and let  $M_1, M_2, \dots, M_n$  be submodules of  $M$  such that*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

*Then*

$$l(M) = \sum_{i=1}^n l(M_i).$$

## 0.4 The Jacobson radical

### The Jacobson radical of a module

**Definition 0.4.1.** Let  $M$  be an  $R$ -module. The intersection of all maximal submodules of  $M$  is called the *Jacobson radical* of  $M$  and is denoted by  $\text{Rad}(M)$ . If  $M$  has no maximal submodules, we adopt the convention  $\text{Rad}(M) = M$ .

*Remark 0.4.2.* If  $M$  is a finitely generated  $R$ -module, then  $\text{Rad}(M) \neq M$ .

**Proposition 0.4.3.** Let  $M$  be an  $R$ -module. Then

$$\text{Rad}(M) = \bigcap_{\substack{f:M \rightarrow S \\ S \text{ simple}}} \ker(f) = \bigcap_{\substack{f:M \rightarrow X \\ X \text{ semisimple}}} \ker(f).$$

**Proposition 0.4.4.** Let  $f : M \rightarrow N$  be a homomorphism of  $R$ -modules. Then  $f(\text{Rad}(M)) \subseteq \text{Rad}(N)$ . If, in addition,  $f$  is an epimorphism and  $\ker(f) \subseteq \text{Rad}(M)$ , then  $f(\text{Rad}(M)) = \text{Rad}(N)$ .

**Corollary 0.4.5.** For every  $R$ -module  $M$  we have  $\text{Rad}(M/\text{Rad}(M)) = 0$ .

**Corollary 0.4.6.** If  $M$  is a semisimple  $R$ -module, then  $\text{Rad}(M) = 0$ .

**Corollary 0.4.7.** If  $M = \bigoplus_{i \in I} M_i$ , then

$$\text{Rad}(M) = \bigoplus_{i \in I} \text{Rad}(M_i).$$

**Proposition 0.4.8.** Let  $M$  be an  $R$ -module with  $\text{Rad}(M) \neq M$ . Then

$$\text{Rad}(M) = \bigcap \{ L \leq M \mid L \text{ is a superfluous submodule} \}.$$

**Proposition 0.4.9** (Nakayama's Lemma). Let  $M$  be a finitely generated  $R$ -module and  $N$  a submodule of  $M$ . If  $N + \text{Rad}(M) = M$ , then  $N = M$ . (In other words,  $\text{Rad}(M)$  is the largest superfluous submodule of  $M$ .)

### The Jacobson radical of a ring

Let  $R$  be a ring. We consider the left ideal  $\text{Rad}(_R R)$ , the intersection of all maximal left ideals of  $R$ , and the right ideal  $\text{Rad}(R_R)$ , the intersection of all maximal right ideals of  $R$ .

**Proposition 0.4.10.**

1.  $\text{Rad}(R_R)$  is a two-sided ideal.
2.  $\text{Rad}(R_R) = \{ r \in R \mid 1 - ar \in U(R) \text{ for all } a \in R \}$ .
3.  $\text{Rad}(R_R) = \text{Rad}(_R R)$ .

**Definition 0.4.11.** The two-sided ideal  $\text{Rad}(R_R) = \text{Rad}(_R R)$  is called the *Jacobson radical* of the ring  $R$  and is denoted by  $\text{Rad}(R)$ .

**Proposition 0.4.12.**

1. If  $J$  is a left (resp. right or two-sided) ideal such that  $1 - x$  is invertible for every  $x \in J$ , then  $J \subseteq \text{Rad}(R)$ .
2. If  $J$  is a nil left (resp. right or two-sided) ideal, then  $J \subseteq \text{Rad}(R)$ .

**Proposition 0.4.13.** Let  $\varphi : R \rightarrow S$  be a surjective ring homomorphism. Then  $\varphi(\text{Rad}(R)) \subseteq \text{Rad}(S)$ . If  $\ker(\varphi) \subseteq \text{Rad}(R)$ , then  $\varphi(\text{Rad}(R)) = \text{Rad}(S)$ .

**Proposition 0.4.14.** If  $(R_i)_{i \in I}$  is a family of rings, then

$$\text{Rad}\left(\prod_{i \in I} R_i\right) = \prod_{i \in I} \text{Rad}(R_i).$$

**Proposition 0.4.15.** Let  $M$  be an  $R$ -module. Then  $M \text{ Rad}(R) \subseteq \text{Rad}(M)$ .

**Theorem 0.4.16.** If  $R$  is an artinian ring, then  $\text{Rad}(R)$  is nilpotent.

## 0.5 Semisimple rings

**Theorem 0.5.1.** For a ring  $R$  the following statements are equivalent:

1. every non-zero right  $R$ -module is semisimple;
2.  $R$  as a right  $R$ -module is semisimple;
3.  $R$  is artinian and  $\text{Rad}(R) = 0$ .

**Definition 0.5.2.** A ring  $R$  satisfying any (hence all) of the above conditions is called a *semisimple ring*.

**Proposition 0.5.3.** Let  $R$  be a semisimple ring and  $M$  a non-zero  $R$ -module. The following statements are equivalent:

1.  $M$  has finite length;
2.  $M$  is noetherian;
3.  $M$  is artinian.

**Theorem 0.5.4.** Let  $R$  be a right artinian ring and  $M$  a non-zero right  $R$ -module. The following statements are equivalent:

1.  $M$  has finite length;
2.  $M$  is noetherian;
3.  $M$  is artinian.

**Corollary 0.5.5** (Hopkins). A right artinian ring (respectively a left artinian ring) is right noetherian (respectively left noetherian).

# Chapter 1

## Essential submodules

**Definition 1.0.1.** Let  $M$  be a right  $R$ -module. A submodule  $N$  of  $M$  is called *essential* (or we say that  $M$  is an essential extension of  $N$ ) if  $N \cap N' \neq 0$  for every non-zero submodule  $N'$  of  $M$ . In this case we write  $N \trianglelefteq M_R$ .

A homomorphism of right  $R$ -modules  $f : M \rightarrow N$  is called *essential* if  $\text{Im } f$  is an essential submodule of  $N$  (i.e.  $\text{Im } f \trianglelefteq N_R$ ).

**Example.**

1.  $n\mathbb{Z} \trianglelefteq \mathbb{Z}_{\mathbb{Z}}$  for every  $n \geq 1$ .
2. Every submodule of the Prüfer group  $\mathbb{Z}_{p^\infty}$  is essential.

*Remark 1.0.2.* Let  $M$  be a right  $R$ -module and  $N$  a submodule of  $M$ . Then  $N \trianglelefteq M_R$  if and only if for every  $x \in M$ ,  $x \neq 0$ , there exists  $r \in R$  such that  $xr \in N \setminus \{0\}$ .

*Proof.* “ $\Rightarrow$ ” Let  $x \in M \setminus \{0\}$ . Since  $0 \neq xR \leq M_R$  and  $N \trianglelefteq M_R$ , we have  $xR \cap N \neq 0$ ; hence there exists  $r \in R$  with  $xr \in N \setminus \{0\}$ .

“ $\Leftarrow$ ” Let  $N' \leq M_R$ ,  $N' \neq 0$ . For every  $x \in N' \setminus \{0\}$  there exists  $r \in R$  such that  $xr \in N \setminus \{0\}$ , so  $N \cap N' \neq 0$ . Thus  $N \trianglelefteq M_R$ .  $\square$

**Definition 1.0.3.** A monomorphism of right  $R$ -modules  $f : N_R \rightarrow M_R$  is called *essential* if  $\text{Im } f \trianglelefteq M_R$ . It is immediate that, if  $N$  is a submodule of  $M$ , then the canonical inclusion  $i_N : N \rightarrow M$  is an essential monomorphism if and only if  $N \trianglelefteq M_R$ .

**Proposition 1.0.4.** *A monomorphism  $f : N_R \rightarrow M_R$  is essential if and only if for every right  $R$ -module  $M'$  and every  $g \in \text{Hom}(M, M')$ , the fact that  $g \circ f$  is a monomorphism implies that  $g$  is a monomorphism.*

*Proof.* “ $\Rightarrow$ ” Let  $g$  be as in the statement and suppose that  $g \circ f$  is a monomorphism. Assume  $\text{Ker } g \neq 0$ . Take  $x \in \text{Ker } g \cap \text{Im } f \setminus \{0\}$ . Then  $x = f(x')$  for some  $x' \in N$  and  $g(x) = 0$ , so  $g(f(x')) = 0$ . Since  $g \circ f$  is a monomorphism, it follows that  $x' = 0$  and hence  $x = 0$ , a contradiction.

“ $\Leftarrow$ ” If  $f$  is not an essential monomorphism, then there exists  $N' \leq M_R$ ,  $N' \neq 0$ , such that  $N' \cap \text{Im } f = 0$ . Consider the canonical projection  $\pi_{N'} : M \rightarrow M/N'$ . If  $x \in \text{Ker}(\pi_{N'} \circ f)$ , then  $f(x) \in N'$ , so  $f(x) = 0$ , hence  $x = 0$ . Thus  $\pi_{N'} \circ f$  is injective. By the hypothesis in the statement, this implies that  $\pi_{N'}$  is injective, so  $N' = 0$ , a contradiction.  $\square$

**Corollary 1.0.5.** *Let  $M$  be a right  $R$ -module and  $N \leq M_R$ . Then the following statements are equivalent:*

1.  $N \trianglelefteq M_R$ ;
2. the inclusion  $i_N : N \rightarrow M$  is an essential monomorphism;
3. for every  $f \in \text{Hom}(M, M')$ , with  $M'$  an arbitrary  $R$ -module, the fact that  $f \circ i_N$  is a monomorphism implies that  $f$  is a monomorphism.

**Proposition 1.0.6.** *Let  $f : N_R \rightarrow M_R$  and  $g : M_R \rightarrow P_R$  be monomorphisms. Then  $g \circ f$  is essential if and only if both  $g$  and  $f$  are essential.*

*Proof.* “ $\Leftarrow$ ” Let  $z \in P \setminus \{0\}$ . Since  $g$  is essential, there exists  $r \in R$  such that  $zr \in \text{Im } g \setminus \{0\}$ . Thus there exists  $y \in M \setminus \{0\}$  with  $zr = g(y)$ . As  $f$  is essential, there exists  $r' \in R$  such that  $yr' \in \text{Im } f \setminus \{0\}$ . Hence there exists  $x \in N \setminus \{0\}$  with  $yr' = f(x)$ . But  $zr' = g(y)r' = g(yr') = g(f(x))$ . If  $zr' = 0$ , then  $g(f(x)) = 0$ , hence  $x = 0$ , a contradiction. Therefore  $zr' \in \text{Im}(g \circ f)$  and  $zr' \neq 0$ , which shows that  $g \circ f$  is essential.

“ $\Rightarrow$ ” Let  $y \in M \setminus \{0\}$ . Since  $g$  is a monomorphism,  $g(y) \neq 0$ . Thus there exists  $r \in R$  such that  $g(yr) \in \text{Im } g \setminus \{0\}$  and  $g(yr) \neq 0$ . Hence there exists  $x \in N \setminus \{0\}$  such that  $g(yr) = g(f(x))$ , therefore  $yr = f(x) \in \text{Im } f$ , which shows that  $f$  is an essential monomorphism.

If  $z \in P \setminus \{0\}$ , there exists  $r \in R$  such that  $zr \in \text{Im}(g \circ f)$  and  $zr \neq 0$ . Since  $\text{Im}(g \circ f) \subseteq \text{Im } g$ , we obtain  $zr \in \text{Im } g$ , so  $g$  is essential.  $\square$

**Proposition 1.0.7.** *Let  $M$  be a right  $R$ -module and  $L_1, L_2, \dots, L_n$  submodules of  $M$ . Then:*

- 1)  $\bigcap_{i=1}^n L_i$  is essential in  $M$  iff each  $L_i$  is essential in  $M$ .
- 2) If  $L_1 \subseteq L_2$  and  $L_1$  is essential in  $M$ , then  $L_2$  is essential in  $M$ .

The proof is obvious.

**Proposition 1.0.8.** *Let  $K$  and  $L$  be submodules of  $M$ .*

- 1) If  $K \subseteq L \subseteq M$ , then  $K \trianglelefteq M$  iff  $K \trianglelefteq L$  and  $L \trianglelefteq M$ .
- 2) If  $h : K_R \rightarrow M_R$  is a module morphism and  $L \trianglelefteq M$ , then  $h^{-1}(L) \trianglelefteq K$ .
- 3) If  $L_1, L_2 \leq M_R$  with  $K_1 \trianglelefteq L_1$  and  $K_2 \trianglelefteq L_2$ , then  $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$ .

### Proof.

- 1) Apply 1.5 and 1.6.
- 2) Let  $U$  be a non-zero submodule of  $K$ .
  - (i) If  $h(U) = 0$ , then  $U \subseteq \ker h \subseteq h^{-1}(L)$ , so  $U \cap h^{-1}(L) \neq 0$ .
  - (ii) If  $h(U) \neq 0$ , then  $h(U) \cap L \neq 0$ . Hence there exists  $u \in U$  with  $h(u) \in L$ ,  $h(u) \neq 0$ , so  $u \in U \cap h^{-1}(L)$  and  $u \neq 0$ .
 Thus  $h^{-1}(L) \trianglelefteq K$ .
- 3) Let  $0 \neq X \leq L_1 \cap L_2$ . Then  $X \subseteq L_1$ , so  $0 \neq X \cap K_1 \leq L_1$ . But  $X \subseteq L_2$  implies  $0 \neq (X \cap K_1) \cap L_2 = X \cap (K_1 \cap K_2)$ . Hence  $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$ .

**Proposition 1.0.9.** Let  $(K_\lambda)_{\lambda \in \Lambda}$ ,  $(L_\lambda)_{\lambda \in \Lambda}$  be families of submodules of  $M$ . If  $(K_\lambda)$  is independent in  $M$  and  $K_\lambda \trianglelefteq L_\lambda$  for all  $\lambda \in \Lambda$ , then  $(L_\lambda)$  is independent in  $M$  and

$$\left( \bigoplus_{\lambda \in \Lambda} K_\lambda \right) \trianglelefteq \left( \bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

**Proof.**

Let  $K_1 \trianglelefteq L_1$ ,  $K_2 \trianglelefteq L_2$  with  $K_1 \cap K_2 = 0$ . By 1.8(3),  $0 \trianglelefteq L_1 \cap L_2$ , hence  $L_1 \cap L_2 = 0$ .

Let  $\pi_1 : L_1 \oplus L_2 \rightarrow L_1$ ,  $\pi_2 : L_1 \oplus L_2 \rightarrow L_2$  be the canonical projections. Since  $K_1 \trianglelefteq L_1$  and  $K_2 \trianglelefteq L_2$ ,

$$\pi_1^{-1}(K_1) = K_1 \oplus 0 \trianglelefteq L_1 \oplus L_2,$$

and

$$\pi_2^{-1}(K_2) = 0 \oplus K_2 \trianglelefteq L_1 \oplus L_2.$$

Hence

$$K_1 \oplus K_2 = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \trianglelefteq L_1 \oplus L_2.$$

Induction gives the finite case. For the general case, let  $0 \neq m \in \bigoplus_{\lambda \in \Lambda} L_\lambda$ . Then  $m$  lies in a finite direct sum  $\bigoplus_{\lambda \in \Lambda_0} L_\lambda$  for some finite  $\Lambda_0 \subseteq \Lambda$ . Since  $(\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \trianglelefteq (\bigoplus_{\lambda \in \Lambda_0} L_\lambda)$ , there exists  $r \in R$  with  $rm \in (\bigoplus_{\lambda \in \Lambda_0} K_\lambda) \setminus \{0\} \subseteq (\bigoplus_{\lambda \in \Lambda} K_\lambda)$ . Thus

$$\left( \bigoplus_{\lambda \in \Lambda} K_\lambda \right) \trianglelefteq \left( \bigoplus_{\lambda \in \Lambda} L_\lambda \right).$$

**Proposition 1.0.10.** Let  $N$  be a submodule of  $M$ . Then there exists a submodule  $Q$  with  $N \subseteq Q \subseteq M$  such that  $Q$  is a maximal essential extension of  $N$  inside  $M$ .

**Proof.**

Let

$$\mathfrak{S} = \{L \leq M ; N \subseteq L \subseteq M, N \trianglelefteq L\},$$

ordered by inclusion. Clearly  $\mathfrak{S} \neq \emptyset$ , since  $N \in \mathfrak{S}$ .

Let  $(L_\lambda)_{\lambda \in \Lambda}$  be a totally ordered family of elements of  $\mathfrak{S}$  and put

$$L := \bigcup_{\lambda \in \Lambda} L_\lambda.$$

Clearly  $L \leq M_R$ .

Let  $x \in L \setminus \{0\}$ . Then there exists  $\lambda_0 \in \Lambda$  with  $x \in L_{\lambda_0}$ . Since  $N$  is essential in  $L_{\lambda_0}$ , there exists  $r \in R$  such that  $rx \in N$  and  $rx \neq 0$ , hence  $L$  is an essential extension of  $N$ . Thus  $\mathfrak{S}$  is inductive and, by Zorn's lemma,  $\mathfrak{S}$  admits a maximal element  $Q$  which satisfies the required conditions.

**Definition 1.0.11.** Let  $M$  be a right  $R$ -module and  $N \leq M_R$ . A submodule  $K \leq M_R$  is called a *complement* of  $N$  in  $M$  if  $K$  is a maximal submodule of  $M$  with the property that  $K \cap N = 0$ . A submodule  $K \leq M_R$  is called a *complement submodule* of  $M$  if there exists  $N \leq M_R$  such that  $K$  is a complement of  $N$  in  $M$ .

*Remark 1.0.12.* The set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid N \cap L = 0 \}$$

is inductive and, by applying Zorn's lemma, it follows that there exists a complement of  $N$  in  $M$ . In particular, 0 and  $M$  are complement submodules of  $M$ .

**Proposition 1.0.13.** *Let  $M_R$ ,  $N \leq M_R$  and  $K \leq M_R$ , with  $K$  a complement of  $N$  in  $M$ . There exists a complement  $Q$  of  $K$  in  $M$  such that  $N \subseteq Q$ . Moreover,  $Q$  is a maximal essential extension of  $N$  in  $M$ .*

*Proof.* It is easy to see that the set

$$\tilde{\mathfrak{S}} = \{ L \leq M_R \mid K \cap L = 0, N \subseteq L \}$$

is inductive, and Zorn's lemma guarantees the existence of  $Q$ .

Let  $L$  be a non-zero submodule of  $Q$  such that  $L \cap N = 0$ . Put  $K_1 = L + K$ . Clearly  $K \subseteq K_1$ . If  $x \in N \cap (L + K)$ , then  $x = y + z$  with  $y \in L$ ,  $z \in K$ . But  $z = x - y \in Q$ . Since  $Q \cap K = 0$ , we obtain  $z = 0$  and hence  $x = y$ . From  $L \cap N = 0$  it follows that  $x = y = 0$ , and therefore  $N \cap (L + K) = 0$ , which contradicts the fact that  $K$  is a complement of  $N$  in  $M$ . Thus  $L \cap N \neq 0$  for every  $0 \neq L \leq Q$ , so  $Q$  is an essential extension of  $N$ .

Suppose that there exists  $Q' \leq M_R$  with  $N \trianglelefteq Q'$  and  $Q \subsetneq Q'$ . Since  $Q'$  is a complement of  $K$ , we have  $Q' \cap K \neq 0$ . But  $N \cap (Q' \cap K) = 0$  and  $0 \neq Q' \cap K \leq Q'$ , contradicting  $N \trianglelefteq Q'$ . Hence  $Q$  is a maximal essential extension of  $N$  in  $M$ .  $\square$

**Definition 1.0.14.** A submodule  $N$  of  $M_R$  is called *closed* if  $N$  has no proper (meaning different from  $N$ ) essential extension in  $M$ .

**Corollary 1.0.15.** *Let  $M_R$  be a right  $R$ -module. The complement submodules of  $M$  coincide with the closed submodules of  $M$ .*

*Proof.* From 1.13 it follows immediately that every closed submodule of  $M$  is a complement submodule of  $M$ .

Conversely, let  $K$  be a complement submodule of  $M_R$ . Then there exists  $N \leq M_R$  such that  $K$  is a complement of  $N$  in  $M$ . Assume that  $K$  has a proper essential extension in  $M$ ; that is, there exists  $K' \leq M_R$  with  $K \trianglelefteq K'$  and  $K \subsetneq K'$ . By the maximality of  $K$  we have  $K' \cap N \neq 0$ , and since  $K \trianglelefteq K'$ , it follows that

$$K \cap K' \cap N \neq 0,$$

contradiction.  $\square$

**Corollary 1.0.16.** *Let  $N$  be a submodule of  $M_R$ . If  $K$  is a complement of  $N$  in  $M$ , then:*

$$1. (N + K) \trianglelefteq M_R.$$

$$2. \text{The canonical morphism } \pi_K \circ i_N : N \rightarrow M/K \text{ is an essential monomorphism.}$$

*Proof.* (1) Let  $x \in M \setminus \{0\}$ . If  $x \notin K$ , then  $K + Rx \neq K$  and hence  $N \cap (K + Rx) \neq 0$ . Let  $y \in N \cap (K + Rx)$ ,  $y \neq 0$ . There exist  $z \in K$  and  $r \in R$  such that  $y = z + rx$ . If  $rx = 0$ , then  $y = z$  and, since  $N \cap K = 0$ , we obtain  $y = 0$ , a contradiction. Thus  $rx \neq 0$  and, because  $rx = y - z$ , we have  $rx \in N + K$ , which shows that  $(N + K) \trianglelefteq M_R$ .

(2) We have  $\text{Im}(\pi_K \circ i_N) = (N + K)/K$ . Let  $L/K$  be a non-zero submodule of  $M/K$ . Then

$$\frac{N + K}{K} \cap \frac{L}{K} = \frac{(N + K) \cap L}{K} = \frac{N \cap L + K}{K}.$$

Since  $K$  is a complement of  $N$ , we have  $N \cap L \neq 0$ , and hence

$$\frac{N \cap L + K}{K} \neq 0,$$

which shows that  $\pi_K \circ i_N$  is an essential monomorphism.  $\square$



# Chapter 2

## Injective Modules

### 2.1 Injective Module

Let  $Q$  and  $M$  be two right  $R$ -modules. We say that  $Q$  is  $M$ -injective if for every monomorphism  $u : M' \rightarrow M$  and every morphism  $f : M' \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g \circ u = f$ ; that is, the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M \\ & & f \downarrow & \nearrow g & \\ & & Q & & \end{array}$$

is commutative.

This property is equivalent to the condition that the map

$$\text{Hom}(u, Q) : \text{Hom}(M, Q) \longrightarrow \text{Hom}(M', Q)$$

is surjective for every monomorphism  $u : M' \rightarrow M$ . Since the functor  $\text{Hom}(-, Q)$  is left exact, it follows that  $Q$  is  $M$ -injective if and only if  $\text{Hom}(-, Q)$  is exact with respect to every short exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

The  $R$ -module  $Q$  is called *quasi-injective* (or *self-injective*) if it is  $Q$ -injective. If  $Q$  is  $M$ -injective for every  $R$ -module  $M$ , then  $Q$  is called *injective*.

#### 2.1.1 Proposition

**Proposition 2.1.1.** *Let  $Q$  and  $M$  be two  $R$ -modules. The following statements are equivalent:*

1.  *$Q$  is  $M$ -injective.*
2. *For every submodule  $N$  of  $M$  and every morphism  $f : N \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g|_N = f$ .*
3. *For every essential submodule  $N$  of  $M$  and every morphism  $f : N \rightarrow Q$ , there exists  $g : M \rightarrow Q$  such that  $g|_N = f$ .*

*Proof.* Implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are obvious.

(2)  $\Rightarrow$  (1). Let  $M'_R$ ,  $0 \longrightarrow M' \xrightarrow{u} M$  and  $f : M' \rightarrow Q$ . Then  $u(M') \leq M$ . Consider  $i : u(M') \rightarrow M$  the canonical injection and  $\bar{u} : M' \rightarrow u(M')$  the isomorphism induced by  $u$ . There exists  $g : M \rightarrow Q$  such that  $g \circ i = f \circ \bar{u}^{-1}$ . Hence

$$g \circ i \circ \bar{u} = f \quad \text{and therefore} \quad g \circ u = f.$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \xrightarrow{\bar{u}} & u(M') & \xhookrightarrow{i} & M \\ & & f \downarrow & & & & \\ & & Q & & \xleftarrow{g} & & \end{array}$$

(3)  $\Rightarrow$  (2). Let  $N$  be a submodule of  $M$  and  $K$  a complement of  $N$  in  $M$ . Then  $(N \oplus K) \trianglelefteq M$ . Define  $h : N \oplus K \rightarrow Q$  by  $h(n+k) = f(n)$  for all  $n \in N, k \in K$ . Since  $N \cap K = 0$ , the map  $h$  is well-defined. There exists  $g : M \rightarrow Q$  such that  $g|_{N \oplus K} = h$ , and hence  $g|_N = h|_N = f$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \hookrightarrow & N \oplus K & \hookrightarrow & M \\ & & f \downarrow & & \nearrow h & & \\ & & Q & & \xleftarrow{g} & & \end{array}$$

□

**Proposition 2.1.2.** *Let  $(M_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules and  $M$  an  $R$ -module. Then  $\prod_{\alpha \in \Lambda} M_\alpha$  is  $M$ -injective if and only if each  $M_\alpha$  is  $M$ -injective.*

*Proof.* Let  $N$  be a submodule of  $M$ . Put  $P = \prod_{\alpha \in \Lambda} M_\alpha$  and let  $\pi_\alpha : P \rightarrow M_\alpha$  be the canonical projections for all  $\alpha \in \Lambda$ .

"  $\Leftarrow$  " Given a morphism  $f : N \rightarrow P$ , the morphisms  $\pi_\alpha \circ f : N \rightarrow M_\alpha$  can be extended to  $g_\alpha : M \rightarrow M_\alpha$ . There exists  $g : M \rightarrow P$  such that  $g|_N = f$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \hookrightarrow & M \\ & & f \downarrow & \nearrow g & \\ & & P & \xleftarrow{\pi_\alpha} & M_\alpha \\ & & \searrow g_\alpha & & \end{array}$$

"  $\Rightarrow$  " Let  $\forall \alpha \in \Lambda$  and  $f : N \rightarrow M_\alpha$ . Considering the canonical inclusion  $\varepsilon_\alpha : M_\alpha \rightarrow P$ , since  $P$  is  $M$ -injective, there exists  $g : M \rightarrow P$  which extends  $\varepsilon_\alpha \circ f : N \rightarrow P$ . Then  $\varepsilon_\alpha : M_\alpha \rightarrow P$  extends  $f$  and hence  $M_\alpha$  is  $M$ -injective.

$$\begin{array}{ccccc}
0 & \longrightarrow & N & \hookrightarrow & M \\
& & f \downarrow & & \swarrow \\
& & M_\alpha & \xrightarrow{g} & \\
& & \varepsilon_\alpha \downarrow & & \pi_\alpha g \\
& & P & & \\
& & \pi_\alpha \downarrow & & \\
& & M_\alpha & &
\end{array}$$

□

**Corollary 2.1.3.**

1. Let  $(Q_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules. Then  $\prod_{\alpha \in \Lambda} Q_\alpha$  is injective if and only if each  $Q_\alpha$  is injective for every  $\alpha \in \Lambda$ .
2. The module  $Q_1 \oplus Q_2$  is an injective  $R$ -module if and only if each  $Q_i$  is injective for  $i = 1, 2$ . In particular, a direct summand of an injective module is injective.

**Proposition 2.1.4.** Let  $Q$  be an  $R$ -module.

1. If  $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$  is an exact sequence of  $R$ -modules and  $Q$  is  $M$ -injective, then  $Q$  is  $M'$ -injective and  $M''$ -injective.
2. If  $(M_\alpha)_{\alpha \in \Lambda}$  is a family of submodules of  $M$  such that  $M = \sum_{\alpha \in \Lambda} M_\alpha$  and  $Q$  is  $M_\alpha$ -injective for every  $\alpha$ , then  $Q$  is  $M$ -injective.
3. Let  $(N_\alpha)_{\alpha \in \Lambda}$  be a family of  $R$ -modules. Then  $Q$  is  $\bigoplus_{\alpha \in \Lambda} N_\alpha$ -injective if and only if  $Q$  is  $N_\alpha$ -injective for every  $\alpha \in \Lambda$ .

*Proof.* To show that  $Q$  is  $M'$ -injective, we consider  $N$  a submodule of  $M'$  and  $\varphi : N \rightarrow Q$  a morphism of  $R$ -modules. Since  $Q$  is  $M$ -injective, there exists  $\phi : M \rightarrow Q$  such that  $g \circ f|_N = \varphi$ , and hence  $\psi \circ f : M' \rightarrow Q$  is a morphism which extends  $\varphi$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \longrightarrow & M' & \xrightarrow{f} & M \\
& & \varphi \downarrow & & \swarrow \psi & & \\
& & Q & & & &
\end{array}$$

Let  $h : L \rightarrow M''$  be a monomorphism. We may assume, without loss of generality, that  $M' \leq M$  and  $M'' = M/M'$ . Since  $L \cong h(L) \leq M''$ , there exist  $P \leq M$ ,  $M' \subseteq P$  such that  $h(L) = P/M'$  and hence  $L \cong P/M'$ . We obtain the commutative diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & L \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow h \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0
\end{array}$$

Since  $Q$  is  $M$ -injective, applying the functor  $\text{Hom}(-, Q)$  we obtain the commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(M'', Q) & \longrightarrow & \text{Hom}(M, Q) & \longrightarrow & \text{Hom}(M', Q) \longrightarrow 0 \\
 & & h^* \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \text{Hom}(L, Q) & \longrightarrow & \text{Hom}(P, Q) & \longrightarrow & \text{Hom}(M', Q) \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

We obtain that  $h^* = \text{Hom}(h, Q)$  is an epimorphism, which shows that  $Q$  is  $M''$ -injective.

2) Let  $N$  be a submodule of  $M$  and  $f : N \rightarrow Q$  a morphism of  $R$ -modules. Consider the set

$$\mathfrak{S} = \{(L, h) \mid N \leq L \leq M, h : L \rightarrow Q, h|_N = f\}.$$

Since  $(N, f) \in \mathfrak{S}$ , we have  $\mathfrak{S} \neq \emptyset$ . Define on  $\mathfrak{S}$  the order relation  $(L_1, h_1) \preccurlyeq (L_2, h_2)$  if and only if  $L_1 \leq L_2$  and  $h_2|_{L_1} = h_1$ . One checks that  $\mathfrak{S}$  is inductive and, by Zorn's lemma, there exists a maximal element  $(L_0, g_0)$  of  $\mathfrak{S}$ . To show that  $L_0 = M$  it is enough to prove that  $M_\alpha \leq L_0$  for every  $\alpha \in \Lambda$ .

Consider the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & L_0 \cap M_\alpha & \xrightarrow{i_\alpha} & M_\alpha \\
 & & i_0 \downarrow & & \swarrow h_\alpha \\
 & & L_0 & & \\
 & & g_0 \downarrow & & \\
 & & Q & &
 \end{array}$$

from which it follows that there exists  $h_\alpha : M_\alpha \rightarrow Q$  such that  $h_\alpha \circ i_\alpha = g_0 \circ i_\alpha$ . Define  $h^* : L_0 + M_\alpha \rightarrow Q$  by  $h^*(l + m_\alpha) = g_0(l) + h_\alpha(m_\alpha)$ , for all  $l \in L_0, m_\alpha \in M_\alpha$ . If  $l + m_\alpha = 0$ , then  $l = -m_\alpha \in L_0 \cap M_\alpha$  and hence  $h^*(l + m_\alpha) = g_0(l) + h_\alpha(l)$ , which shows that  $h^*$  is well defined. Thus  $(L_0 + M_\alpha, h^*) \in \mathfrak{S}$ , and since  $(L_0, g_0) \preccurlyeq (L_0 + M_\alpha, h^*)$ , by the maximality of  $(L_0, g_0)$  we obtain  $L_0 = L_0 + M_\alpha$ , i.e.  $M_\alpha \leq L_0$  for every  $\alpha \in \Lambda$ .

3) “ $\Rightarrow$ ” Since  $N_\alpha \leq N$  and  $Q$  is  $N$ -injective, it follows that  $Q$  is  $N_\alpha$ -injective for every  $\alpha \in \Lambda$ .

“ $\Leftarrow$ ” Let  $N'_\alpha = i_\alpha(N_\alpha)$ . Since  $Q$  is  $N_\alpha$ -injective and  $N'_\alpha \cong N_\alpha$ , we see that  $Q$  is  $N'_\alpha$ -injective. Now apply (2).  $\square$

### Corollary 2.1.5.

1. The module  $Q_1 \oplus Q_2$  is a quasi-injective  $R$ -module if and only if each  $Q_i$  is  $Q_j$ -injective for all  $i, j = 1, 2$ . In particular, a direct summand of a quasi-injective module is quasi-injective.

2. The module  $Q^n$  is quasi-injective over  $R$  if and only if  $Q$  is quasi-injective.

**Corollary 2.1.6.** Let  $Q$  and  $M$  be two  $R$ -modules. Then  $Q$  is  $M$ -injective if and only if  $Q$  is  $mR$ -injective for every  $m \in M$ .

*Proof.* The implication “ $\Rightarrow$ ” is clear.

For “ $\Leftarrow$ ”, since  $M = \sum_{m \in M} mR$ , it follows from 2.1.4(2) that  $Q$  is  $M$ -injective.  $\square$

**Theorem 2.1.7** (Baer’s criterion). For an  $R$ -module  $Q$  the following statements are equivalent:

1.  $Q$  is injective.
2.  $Q$  is  $R$ -injective.
3. For every right ideal  $I$  of  $R$  and every morphism  $f : I \rightarrow Q$  there exists  $x \in Q$  such that  $f(a) = xa$  for all  $a \in I$ .

*Proof.* The implication (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (1). Let  $M$  be an  $R$ -module and  $x \in M$ . Since  $\varphi_x : R \rightarrow xR$ ,  $\varphi_x(a) = xa$  for all  $a \in R$ , is a surjective morphism of  $R$ -modules, it follows that  $R/\text{Ker } \varphi_x \cong xR$ . As  $Q$  is  $R$ -injective, 2.1.4(1) implies that  $Q$  is  $R/\text{Ker } \varphi_x$ -injective, and hence  $Q$  is  $xR$ -injective for every  $x \in M$ . Therefore, by 2.1.6 we obtain that  $Q$  is  $M$ -injective.

(2)  $\Rightarrow$  (3). Let  $I$  be a right ideal of  $R$  and  $f : I \rightarrow Q$ . There exists  $g : R \rightarrow Q$  such that  $g|_I = f$ . Put  $x = g(1) \in Q$ . Then  $f(a) = g(a) = ag(1) = xa$  for all  $a \in I$ .

(3)  $\Rightarrow$  (2). Suppose that for a morphism  $f : I \rightarrow Q$  there exists  $x \in Q$  with  $f(a) = xa$  for all  $a \in I$ . Define  $g : R \rightarrow Q$  by  $g(r) = xr$  for all  $r \in R$ . Then clearly  $g|_I = f$ .  $\square$

### Definition 2.1.8.

1. An  $R$ -module  $Q$  is called *divisible* if for every  $y \in Q$  and every  $a \in R$  which is not a zero divisor, there exists  $x \in Q$  such that  $ax = y$ . It is easily seen that any factor module of a divisible module is divisible.
2. A commutative integral domain is called a *PID ring* (principal ideal domain) if every ideal of it is principal.

### Proposition 2.1.9.

1. Every injective module is divisible.
2. Let  $R$  be a PID.
  - (i) If  $Q$  is an  $R$ -module, then  $Q$  is injective if and only if it is divisible.
  - (ii) If  $I$  is a non-zero ideal of  $R$ , then  $R/I$  is a quasi-injective  $R$ -module. In particular,  $\mathbb{Z}_n$  is a quasi-injective  $\mathbb{Z}$ -module for every  $n \geq 1$ .

*Proof.*

1) Let  $Q$  be a divisible  $R$ -module,  $y \in Q$  and  $a \in R$  a non-zero divisor. Define  $f : aR \rightarrow Q$  by  $f(ax) = yx$  for all  $x \in R$ . Since  $a$  is not a zero divisor,  $f$  is well defined. By Baer's criterion there exists  $x \in Q$  such that  $f(\lambda) = x\lambda$  for all  $\lambda \in aR$ . In particular  $y = f(a) = f(a \cdot 1) = xa$ ; hence  $Q$  is divisible.

2)(i) The implication " $\Rightarrow$ " follows immediately from (1).

" $\Leftarrow$ " Let  $Q$  be a divisible  $R$ -module and  $I$  a right ideal of  $R$ . Then there exists  $a \in R$  with  $I = aR$ . Given a morphism of  $R$ -modules  $f : I \rightarrow Q$ , choose  $x \in Q$  with  $f(a) = xa$ . For any  $r \in R$ ,  $f(ar) = f(a)r = xar$ , so by Baer's criterion  $Q$  is injective.

2)(ii) Let  $I = aR$  and  $J = bR$  be non-zero ideals of  $R$  with  $I \subseteq J$ , and let  $f : J/I \rightarrow R/I$  be a morphism of  $R$ -modules. There exists  $c \in R$  such that  $a = bc$ . Write  $f(\widehat{b}) = \widehat{x} \in R/I$ . Then  $xc \in I = aR$ , so there is  $a_1 \in R$  with  $xc = aa_1$ , and hence  $x = ba_1$ .

Define  $g : R/I \rightarrow R/I$  by  $g(\widehat{r}) = \widehat{a_1 r}$  for all  $r \in R$ . Then  $g$  is an  $R$ -module homomorphism and  $g(\widehat{b}) = \widehat{x}$ , so  $g|_{J/I} = f$ . Thus  $R/I$  is quasi-injective. In particular, for  $R = \mathbb{Z}$  and  $I = n\mathbb{Z}$  we obtain that  $\mathbb{Z}_n$  is a quasi-injective  $\mathbb{Z}$ -module for every  $n \geq 1$ .  $\square$

**Corollary 2.1.10.** *An abelian group  $G$  is an injective  $\mathbb{Z}$ -module if and only if  $G$  is divisible.*

**Corollary 2.1.11.**

1.  $\mathbb{Q}$  and  $\mathbb{Z}_{p^\infty}$  are injective  $\mathbb{Z}$ -modules.
2. Any direct sum of injective  $\mathbb{Z}$ -modules is an injective  $\mathbb{Z}$ -module.
3. Any factor group of an injective  $\mathbb{Z}$ -module is injective.

**Lemma 2.1.12.** *Let  $A, S, T$  be rings and let  ${}_A M_S, {}_A N_T$  be bimodules. Then  $\text{Hom}_A(M, N)$  has a structure of left  $S$ -module and right  $T$ -module given by*

$$(s \cdot f)(x) = f(xs), \quad (f \cdot t)(x) = f(x)t,$$

for  $s \in S$ ,  $t \in T$ ,  $x \in M$ ,  $f \in \text{Hom}_A(M, N)$ .

*Proof.* Let  $a, b \in A$  and  $x, y \in M$ . Then

$$(s \cdot f)(ax+by) = f((ax+by)s) = f((ax)s) + f((by)s) = af(xs) + bf(ys) = a(s \cdot f)(x) + b(s \cdot f)(y),$$

so  $s \cdot f \in \text{Hom}_A(M, N)$ . Similarly  $f \cdot t \in \text{Hom}_A(M, N)$ .

For  $s, s' \in S$  and  $f, g \in \text{Hom}_A(M, N)$  we have

$$(s \cdot (f + g))(x) = (f + g)(xs) = f(xs) + g(xs) = (s \cdot f)(x) + (s \cdot g)(x), \quad (1)$$

$$((s + s') \cdot f)(x) = f(x(s + s')) = f(xs + xs') = f(xs) + f(xs') = (s \cdot f)(x) + (s' \cdot f)(x), \quad (2)$$

$$((ss') \cdot f)(x) = f(xss') = f((xs)s') = (s' \cdot f)(xs) = (s \cdot (s' \cdot f))(x), \quad (3)$$

$$(1_S \cdot f)(x) = f(x1_S) = f(x). \quad (4)$$

From (1)–(4) we see that  $\text{Hom}_A(M, N)$  is a left  $S$ -module. A similar computation shows that  $\text{Hom}_A(M, N)$  is a right  $T$ -module. Moreover,  $((s \cdot f) \cdot t)(x) = (s \cdot f)(xt) = f(xts)t$  and  $(s \cdot (f \cdot t))(x) = f(xts)t$ , so the two actions commute and  $\text{Hom}_A(M, N)$  is an  $S$ – $T$  bimodule.  $\square$

**Proposition 2.1.13** (Eckmann–Schopf). *Let  $Q$  be a divisible abelian group. Then the left  $R$ -module  $\text{Hom}_{\mathbb{Z}}(R, Q)$  is injective.*

*Proof.* By the previous lemma,  $\text{Hom}_{\mathbb{Z}}(R, Q)$  carries a structure of left  $R$ -module given by

$$(r \cdot f)(a) = f(ar), \quad \forall a, r \in R, f \in \text{Hom}_{\mathbb{Z}}(R, Q).$$

Let  $I$  be a left ideal of  $R$  and  $h : I \rightarrow \text{Hom}_{\mathbb{Z}}(R, Q)$  a morphism of left  $R$ -modules. Define

$$\gamma : \mathbb{Z}I \longrightarrow \mathbb{Z}Q, \quad \gamma(a) = h(a)(1).$$

Then  $\gamma$  is a morphism of  $\mathbb{Z}$ -modules. Since  $Q$  is  $\mathbb{Z}$ -injective, there exists  $\tilde{\gamma} : \mathbb{Z}R \rightarrow \mathbb{Z}Q$  such that  $\tilde{\gamma}|_I = \gamma$ . For  $a \in I$  and  $r \in R$  we have

$$(a \cdot \tilde{\gamma})(r) = \tilde{\gamma}(ra) = h(ra)(1) = (r \cdot h(a))(1) = h(a)(r),$$

hence  $h(a) = a \cdot \tilde{\gamma}$  for all  $a \in I$ . By Baer's criterion,  $\text{Hom}_{\mathbb{Z}}(R, Q)$  is an injective left  $R$ -module.  $\square$

**Proposition 2.1.14.** *Every left  $R$ -module  $M$  can be embedded into an injective left  $R$ -module.*

*Proof.* There exists a free abelian group  $\mathbb{Z}^{(A)}$  and a surjective  $\mathbb{Z}$ -morphism  $f : \mathbb{Z}^{(A)} \rightarrow M$ . Hence

$$\mathbb{Z}M \cong \mathbb{Z}^{(A)} / \ker f \subseteq \mathbb{Q}^{(A)} / \ker f,$$

and therefore there is a divisible abelian group  $G$  with  $\mathbb{Z}M \subseteq \mathbb{Z}G$ . Applying the functor  $\text{Hom}_{\mathbb{Z}}(R, -)$  we obtain a monomorphism

$${}_R M \cong \text{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \text{Hom}_{\mathbb{Z}}(R, G).$$

Since  $G$  is divisible, Proposition 2.1.13 implies that  $\text{Hom}_{\mathbb{Z}}(R, G)$  is an injective left  $R$ -module. Thus  $M$  embeds into an injective left  $R$ -module.  $\square$

**Proposition 2.1.15.** *Let  $Q$  be an  $R$ -module. Then  $Q$  is injective if and only if every short exact sequence*

$$0 \longrightarrow Q \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0$$

splits.

*Proof.* “ $\Rightarrow$ ”. Assume  $Q$  is injective. Exactness implies that  $f$  is a monomorphism. By injectivity of  $Q$  there exists  $h : M \rightarrow Q$  with  $hf = \text{id}_Q$ , hence the sequence splits.

“ $\Leftarrow$ ”. By Proposition 2.1.14 there is an injective  $R$ -module  $Q'$  and a monomorphism  $i : Q \rightarrow Q'$ . Consider the exact sequence

$$0 \longrightarrow Q \xrightarrow{i} Q' \longrightarrow Q'/i(Q) \longrightarrow 0.$$

By the hypothesis this sequence splits, so  $Q$  is a direct summand of  $Q'$ . A direct summand of an injective module is injective, therefore  $Q$  is injective.  $\square$

## 2.2 Injective envelopes

**Definition 2.2.1.** Let  $M$  be an  $R$ -module. A pair  $(E, i)$  is called an *injective envelope* of  $M$  if  $E$  is injective and  $i : M \rightarrow E$  is an essential monomorphism.

**Proposition 2.2.2.** Let  $Q$  be an injective  $R$ -module. Then every complement submodule of  $Q$  is a direct summand of  $Q$ .

*Proof.* Let  $K$  be a submodule of  $Q$  and  $N$  a complement of  $K$  in  $Q$ , that is,  $K \cap N = 0$  and  $K + N$  is an essential submodule of  $Q$ . Then  $(K + N)/N \cong Q/N$ . Define  $g : (K + N)/N \rightarrow Q$  by

$$g((x + y) + N) = x, \quad x \in K, y \in N.$$

Since  $K \cap N = 0$ ,  $g$  is well defined and injective. As  $Q$  is injective, there exists  $h : Q/N \rightarrow Q$  with  $h|_{(K+N)/N} = g$ . Because  $(K + N)/N \cong Q/N$  and  $g$  is a monomorphism,  $h$  is also a monomorphism. We have  $K = \text{Im } g = h((K + N)/N) \subseteq h(Q/N)$ . As  $K$  is a closed submodule, it follows that  $K = h(Q/N)$ . Since  $h$  is a monomorphism,  $(K + N)/N = Q/N$ , hence  $K + N = Q$ . Thus  $K$  is a direct summand of  $Q$ .  $\square$

**Theorem 2.2.3** (Eckmann–Schopf). Every  $R$ -module  $M$  has an injective envelope, unique up to isomorphism.

*Proof.* By Proposition 2.1.14 there exists an injective  $R$ -module  $Q$  with  $M \leq Q$ . Let  $E$  be a maximal essential extension of  $M$  in  $Q$ . Then  $E$  is a complement submodule of  $Q$ , and by the previous proposition  $E$  is injective. Hence  $(E, i)$ , where  $i : M \hookrightarrow E$  is the inclusion, is an injective envelope of  $M$ .

For uniqueness, let  $(E_1, i_1)$  and  $(E_2, i_2)$  be two injective envelopes of  $M$ . Since  $E_2$  is injective, there exists  $f : E_1 \rightarrow E_2$  with  $fi_1 = i_2$ . The map  $i_2$  is a monomorphism and  $i_1$  is an essential monomorphism, so (using 1.4)  $f$  is a monomorphism. Thus  $E_1 \cong f(E_1)$  and  $E_2 = f(E_1) \oplus E_3$  for some submodule  $E_3$ . But  $i_2(M) \subseteq f(E_1)$ , hence  $i_2(M) \cap E_3 = 0$ . Since  $i_2$  is essential, we must have  $E_3 = 0$ , so  $E_2 = f(E_1)$  and  $f$  is an isomorphism.  $\square$

In practice we fix one representative of this isomorphism class and denote it by  $E(M)$ , with  $M \trianglelefteq E(M)$ .

**Proposition 2.2.4.** Let  $M$  be an  $R$ -module and  $i : M \rightarrow Q$  a monomorphism with  $Q_R$  injective. The following statements are equivalent:

1.  $(Q, i)$  is an injective envelope of  $M$ ;
2. for every monomorphism  $f : M \rightarrow Q'$  with  $Q'$  injective, there exists a monomorphism  $g : Q \rightarrow Q'$  such that  $gi = f$ .

*Proof.* (1)  $\Rightarrow$  (2). Let  $f : M \rightarrow Q'$  be a monomorphism with  $Q'$  injective. By injectivity of  $Q'$  there is  $u : Q \rightarrow Q'$  with  $ui = f$ . Because  $i$  is an essential monomorphism and  $Q'$  is injective, the image  $u(Q)$  is a complement of  $f(M)$ ; by the definition of injective envelope this forces  $u$  to be a monomorphism. Set  $g = u$ .

(2)  $\Rightarrow$  (1). Let  $(E(M), j)$  be an injective envelope of  $M$ . Applying (2) to  $f = j$  we obtain a monomorphism  $g : Q \rightarrow E(M)$  with  $gi = j$ . Since  $j$  is an essential monomorphism, it follows that  $i$  is also essential; hence  $(Q, i)$  is an injective envelope of  $M$ .  $\square$

**Proposition 2.2.5.** *For any family of right  $R$ -modules  $M_1, M_2, \dots, M_n$  we have*

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

*Proof.* By 1.9,  $\bigoplus_{i=1}^n E(M_i)$  is an essential extension of  $\bigoplus_{i=1}^n M_i$ . Moreover,

$$\bigoplus_{i=1}^n E(M_i) \cong \prod_{i=1}^n E(M_i),$$

so by 2.1.3 the module  $\bigoplus_{i=1}^n E(M_i)$  is injective. By uniqueness of injective envelopes we obtain

$$E\left(\bigoplus_{i=1}^n M_i\right) \cong \bigoplus_{i=1}^n E(M_i).$$

$\square$

**Theorem 2.2.6.** *Let  $Q$  and  $M$  be two  $R$ -modules. Then  $Q$  is  $M$ -injective if and only if  $f(M) \subseteq Q$  for every  $f \in \text{Hom}(E(M), E(Q))$ .*

*Proof.* “ $\Rightarrow$ ” Let  $f \in \text{Hom}(E(M), E(Q))$  and set

$$K := \{m \in M \mid f(m) \in Q\}.$$

Since  $Q$  is  $M$ -injective, there exists a morphism  $\bar{f} : M \rightarrow Q$  such that  $\bar{f}|_K = f|_K$ . We claim that

$$Q \cap (\bar{f} - f)(M) = 0.$$

Take  $x \in Q$  and  $m \in M$  with  $x = (\bar{f} - f)(m)$ . Then

$$f(m) = \bar{f}(m) - x \in Q,$$

so  $m \in K$ . Hence

$$x = \bar{f}(m) - f(m) = f(m) - f(m) = 0,$$

and therefore  $Q \cap (\bar{f} - f)(M) = 0$ . Since  $Q$  is an essential submodule of  $E(Q)$ , it follows that  $(\bar{f} - f)(M) = 0$ . Thus  $f(M) = \bar{f}(M) \subseteq Q$ .

“ $\Leftarrow$ ” As  $E(Q)$  is injective, it is enough to work with  $f \in \text{Hom}(M, E(Q))$ . Let  $N$  be a submodule of  $M$  and  $g : N \rightarrow Q$  a morphism of  $R$ -modules. Because  $E(Q)$  is injective, there exists  $\tilde{g} : M \rightarrow E(Q)$  such that  $\tilde{g}|_N = i \circ g$ , where  $i : Q \rightarrow E(Q)$  is the canonical injection. By hypothesis,  $\tilde{g}(M) \subseteq Q$ , so identifying  $\tilde{g}$  with its corestriction to  $Q$  we obtain a morphism  $h : M \rightarrow Q$  with  $h|_N = g$ . Therefore  $Q$  is  $M$ -injective.  $\square$

**Corollary 2.2.7.** *An  $R$ -module  $Q$  is quasi-injective if and only if  $f(Q) \subseteq Q$  for every  $f \in \text{End}(E(Q))$ .*

**Theorem 2.2.8** (Matlis–Bass). *Let  $R$  be a ring. Then  $R$  is right noetherian if and only if, for every simple right  $R$ -module  $S_i$  ( $i \geq 1$ ),*

$$Q := \bigoplus_{i=1}^{\infty} E(S_i)$$

*is an injective right  $R$ -module.*

*Proof.* “ $\Rightarrow$ ” Let  $L$  be a right ideal of  $R$ ,

$$Q = \bigoplus_{i=1}^{\infty} E(S_i)$$

and let  $f : L \rightarrow Q$  be a morphism of right  $R$ -modules. There exist elements  $a_1, \dots, a_n \in L$  such that

$$L = a_1R + a_2R + \cdots + a_nR.$$

Clearly, there is  $m \geq 1$  with  $f(a_k) \in \bigoplus_{j=1}^m E(S_j)$  for all  $k = 1, \dots, n$ , hence

$$\Im f \subseteq \bigoplus_{j=1}^m E(S_j).$$

Since  $\bigoplus_{j=1}^m E(S_j)$  is injective, there exists  $g : R \rightarrow \bigoplus_{j=1}^m E(S_j)$  such that  $g|_L = f$ . Let  $\bar{f} = i \circ g$ , where  $i : \bigoplus_{j=1}^m E(S_j) \rightarrow Q$  is the canonical injection. Then  $\bar{f}|_L = f$ , so  $Q$  is injective.

“ $\Leftarrow$ ” Suppose that  $R$  is not right noetherian. Then there exists a strictly ascending chain of finitely generated right ideals:

$$L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq \cdots.$$

By Krull’s lemma, for every  $n \geq 1$  there exists a maximal submodule  $M_n \subsetneq L_n$  such that

$$L_{n-1} \subseteq M_n \quad \text{for all } n \geq 2.$$

Set

$$L := \bigcup_{k=1}^{\infty} L_k, \quad \pi_k : L_k \longrightarrow L_k/M_k$$

for the canonical projections, and put

$$E_k := E(L_k/M_k) \quad \text{for each } k \geq 1.$$

Then

$$E := \bigoplus_{k=1}^{\infty} E_k$$

is injective, and

$$f : L \longrightarrow E, \quad f(a) = \sum_{k=1}^{\infty} \pi_k(a)$$

is well defined. There exists an element  $x \in E_1 \oplus \cdots \oplus E_n$  such that  $f(a) = xa$  for all  $a \in L$ . It follows that  $\pi_k(a) = 0$  for every  $k \geq n + 1$ , that is,  $a \in M_k$  for all  $k \geq n + 1$ . Hence

$$L \subseteq M_{n+1} \subsetneq L_{n+1} \subseteq M_{n+2} \subsetneq L_{n+2} \subseteq \cdots \subseteq L,$$

a contradiction. Therefore  $R$  is right noetherian.  $\square$

