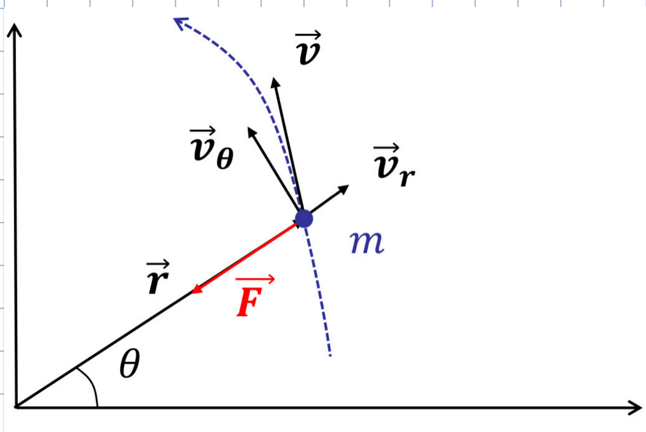
Newton & Kepler needed analytical solution to describe the relative motion of one body to another when mass of one body is insignificant when compared with another.

The Kepler Problem: to find the radial and angular coordinates of an object in orbit as a function of time.

↳ given masses, initial positions & velocities



We use conservation of angular momentum and conservation of energy to derive an expression for the motion of massless particle m.

$m << M$ so
$m \approx$ massless

Consider single body acted on by central force.

- Using polar coordinates.

Law of conservation of momentum: when the net external torque acting on a system about a given axis is zero, the total angular momentum of the system about that axis is constant.

$L = I\omega$

angular momentum

2nd moment of inertia

angular velocity

analagous to $p = mv$

linear momentum

for large r and small m :

$$\boxed{\vec{L} = \vec{r} \times m\dot{\vec{r}}}$$

units = $kg\,m^2/s$

also denoted as 'H'

$\dot{r} = \vec{v}$ (vector)

We can prove $\vec{L}$ constant if we differentiate (using chain rule) : $\left(\cancelto{0}{\dot{r} \times m\dot{r}} + r \times m\ddot{r}\right)$

$$\dot{L} = r \times m\ddot{r} = r \times F = 0$$

r & F act parallel (directed to origin ∴ cross-product = 0)

if $\dot{L} = 0$, $L =$ constant

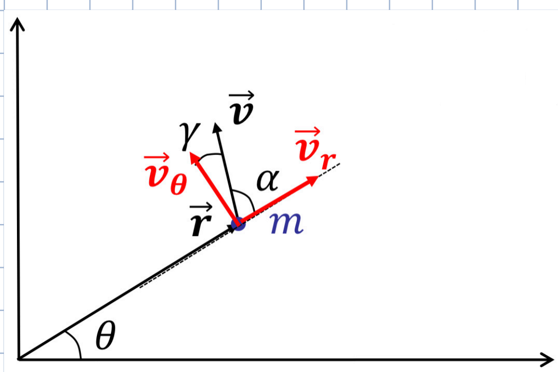
- Because L is constant, and $\vec{r} \times \vec{v}$ defines L as perpendicular to both vectors, m must have planar motion.

Specific Angular Momentum : h (all books use 'h' for this quantity)

$$h = r \times v \quad \text{(all vectors)}$$

Bold letters usually denote vectors and regular size denotes magnitude.

$$r = ||\mathbf{r}||$$



$$h = |r|\,|v| \sin\alpha$$

no $\hat{n}$ as we're looking for magnitude

$$\boxed{h = |r|\,|v| \cos\gamma}$$

flight path angle

$$\boxed{\tan\gamma = \vec{v}_r / \vec{v}_\theta}\ *$$

for peri/apo $\gamma = 0$

∴

$$\boxed{r_a v_a = r_p v_p}$$

$$\boxed{v = \sqrt{\vec{v}_\theta^{\,2} + \vec{v}_r^{\,2}}}$$

due to conservation of momentum :

$$r_1 v_1 \cos\gamma_1 = r_2 v_2 \cos\gamma_2$$

from circular motion : $v_\theta = r\omega = r\dfrac{d\theta}{dt}$

using $v_\theta = v\cos\gamma$: $h = rv\cos\gamma = rv_\theta = r\left(r\dfrac{d\theta}{dt}\right) = r^2\dot{\theta}$

$$\rightarrow \boxed{h = r^2\dot{\theta}} \quad \& \quad \boxed{h = rv_\theta}$$

* Circular orbits → $v_r = 0$ at all times, any other orbit has radial comp.

Following non-examinable derivation found in slides:

$$\ddot{\mathbf{r}} = \left[\ddot{r} - r\dot{\theta}^2\right]\hat{r} + \left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\theta} = -\frac{\mu}{r^2}\hat{r}$$

from orbits 3

due to acceleration along r
Centripetal acceleration
Coriolis acceleration due to change in radius
Euler acceleration due to angular acceleration

no $\hat{\theta}$ component on rhs
$\therefore$ lhs $\hat{\theta}$ component equated to 0

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

Following further algebra, we find:

$$\boxed{r = \frac{h^2/\mu}{1 + e\cos(\theta - \omega)}}$$

Orbit Equation

θ is true anomaly measured from periapsis.

Solution 1 = θ, Solution 2 = $2\pi - \theta$

for an ellipse: $$\boxed{r = \frac{a(1-e^2)}{1 + e\cos(\theta-\omega)}}$$

Specific Energy $$\boxed{\varepsilon = (e^2 - 1)\frac{\mu^2}{2h^2}}$$

can define specific energy of different shaped orbits by plugging different e values in.

$$\boxed{\frac{dA}{dT} = \frac{h}{2} = \text{constant}} \longrightarrow K2$$

$$\boxed{T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}} \longrightarrow K3$$