Kinematic Notations

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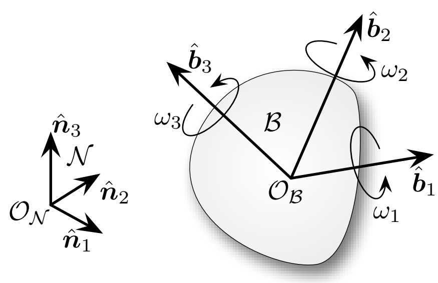

Figure 1: Coordinate Frame Illustration

1 Vectors and Matrices

The coordinate frame

$$\mathcal{B} : \{\mathcal{O}, \hat{b}_1, \hat{b}_2, \hat{b}_3\}$$

illustrated in Figure 1 is defined through its origin $\mathcal{O}$ and the three mutually orthogonal unit direction vectors

$$\{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$$

A vector with a ˆ symbol denotes a unit length direction vector. A right-handed coordinate frame satisfies $\hat{b}_1 \times \boldsymbol{b}_2 = \hat{b}_3$. If this coordinate frame $\mathcal{B}$ is attached to a rigid body, then describing the orientation of the rigid body is equivalent to studying the orientation of $\mathcal{B}$. For the study of attitude dynamics, the translation of the rigid body is not of interest. As a result coordinate frames are thus often defined through their unit direction vectors only.

A position vector $\boldsymbol{r}$ can be written in vector form for example as

$$\boldsymbol{r} = x\hat{\boldsymbol{n}}_1 + y\hat{\boldsymbol{n}}_2 + z\hat{\boldsymbol{n}}_3 \tag{1}$$

Each vector is a magnitude times a direct vector. Note that the vector components are taken here with respect to the frame $\mathcal{N}$. To represent the vector in a matrix form the following left-superscript notation is used:

$${}^{\mathcal{N}}\boldsymbol{r} = {}^{\mathcal{N}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

The left superscript $\mathcal{N}$ denotes that the (x, y, z) vector components are taken with respect to the $\mathcal{N}$ frame.

As with positions, orientations can only be described relative to a reference orientation. Let $\mathcal{N}$ be an inertial (non-accelerating) coordinate frame. By defining the $\hat{\boldsymbol{b}}_i$ direction vectors with respect to $\mathcal{N}$, we are able to describe the orientation of the body $\mathcal{B}$ with respect to $\mathcal{N}$.

The angular motion of $\mathcal{B}$ relative to $\mathcal{N}$ is described through the angular velocity vector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$

$$\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \omega_1\hat{\boldsymbol{b}}_1 + \omega_2\hat{\boldsymbol{b}}_2 + \omega_3\hat{\boldsymbol{b}}_3 \tag{3}$$

This vector is the instantaneous angular rotation vector of body $\mathcal{B}$ relative to $\mathcal{N}$, and is typically expressed in body frame vector components. If only the $\mathcal{B}$ and $\mathcal{N}$ frames are considered, the $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ vector is often written simply as $\boldsymbol{\omega}$.

While we will see that attitude coordinates sets are not vectors, and do not abide by vector addition laws, amazingly the angular velocity vector is truly a vector. Thus, if three frames $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{N}$ are present, their angular velocities relate through

$$\boldsymbol{\omega}_{\mathcal{A}/\mathcal{N}} = \boldsymbol{\omega}_{\mathcal{A}/\mathcal{B}} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \tag{4}$$

If we wish to express a vector $\boldsymbol{\omega}$ as a 3×1 matrix, we must specify with respect to which frame the vector components have been taken. If $\mathcal{B}$ frame components

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are used as in Eq. (3), then the left super-script notation is used again.

$$ {}^{\mathcal{B}}\boldsymbol{\omega} = {}^{\mathcal{B}}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{5} $$

These frame declarations can become cumbersome after a while if only 2 frames are present. If no label is made on a matrix representation of a vector, then body frame components are implied.

The tilde matrix notation is

$$ [\tilde{\boldsymbol{\omega}}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{6} $$

is a matrix representation of the vector cross product through $[\tilde{\boldsymbol{\omega}}]\boldsymbol{a} \equiv \boldsymbol{\omega} \times \boldsymbol{a}$.

2 Vector Differentation

Having defined a rotating frame $\mathcal{B}$, let us briefly discuss how to differentiate a vector $\boldsymbol{r}$ expressed in $\mathcal{B}$ vector components. To discuss the time evolution of $\boldsymbol{r}$, an observer frame must be specified. For example, if $\boldsymbol{r}$ points from the Space Shuttle cockpit to tail, then this vector appears stationary as seen by a Shuttle fixed observer. However, this same $\boldsymbol{r}$ is rotating as seen by an earth-fixed observer. A left super-script label is used on the time differential operator to denote the observer frame. The transport theorem is used to map a time derivative seen by a frame $\mathcal{B}$ into the equivalent derivative seen by another frame $\mathcal{N}$:[1,2]

$$ \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{r} \tag{7} $$

The vector $\boldsymbol{r}$ and frames $\mathcal{B}$ and $\mathcal{N}$ are only placeholders in Eq. (7). The transport theorem applies equally if any other vector or frames are substituted into this expression.

If no frame label is provided, then the time derivative is assumed to be taken with respect to an inertial frame.

$$ \dot{\boldsymbol{x}} \equiv \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}) \tag{8} $$

This is by far the most common time derivative of a vector as Newton's and Euler's laws require inertial derivatives to be taken. Further, let $\boldsymbol{x}$ be expressed in $\mathcal{B}$ frame components as ${}^{\mathcal{B}}\boldsymbol{x} = (x_1, x_2, x_3)^T$ Notice that

$$ {}^{\mathcal{B}}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \Rightarrow \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{x}) \quad \not\Rightarrow \quad \dot{\boldsymbol{x}} \tag{9} $$

The time derivative of the angular velocity vector $\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}$ has a special property worth noting:

$$ \dot{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}) + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}}) \tag{10} $$

In other words, if $\boldsymbol{\omega}$ is the angular velocity vector between two particular frames, then the time derivative of $\boldsymbol{\omega}$ as seen by either of these frames is the same.

References

[1] P. W. Likins. *Elements of Engineering Mechanics*. McGraw-Hill, New York, 1973.

[2] Hanspeter Schaub and John L. Junkins. *Analytical Mechanics of Space Systems*. AIAA Education Series, Reston, VA, 3rd edition, 2014.