Techniques for solving differential equations :

- Inspection
- $\frac{d^n y}{dt^n} = f(t) \rightarrow$ direct integration
- $\frac{dy}{dt} = f(y)\,g(t) \rightarrow$ separation of variables
- Linear other $\rightarrow$ previously learnt methods

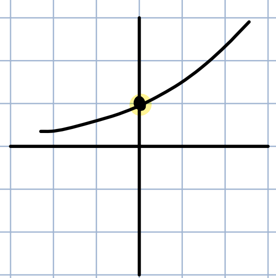
We need methods for non-linear differential equations and systems of DEs

Euler's Method :

- Given initial conditions, we estimate x value at another value of t, using $x_1 = x_0 + hf(x_0, t_0)$

e.g $\frac{dx}{dt} = x$ $\quad x(0) = 1 \quad$ find $x(1) \quad h = 1$

$x(t) = e^t \quad x(1) = e \approx 2.7 \quad$ (using exact method)



subbing x_0, t_0 into rhs of ode

$x_1 = x_0 + hf(x_0, t_0)$

$= 1 + 1 \times 1 = 2$ (estimation with Euler)

2.7 and 2 not too close $\therefore$ needed to use smaller step size

- Instead, multiple steps are used

$\rightarrow$ for form $\frac{dx}{dt} = f(x, t) \quad x(t_0) = x_0$

$t_{n+1} = t_n + h \quad \& \quad x_{n+1} = x_n + hf(x_n, t_n)$ $\quad$ analogous to distance = current + time x speed

- the smaller the step, h, the more accurate the values

Works as $\frac{x(t+h) - x(t)}{h} \approx \frac{dx}{dt}$ and $\frac{dx}{dt} = f(x,t)$

$\therefore$ rearranging $\quad x(t+h) = x(t) + hf(x,t)$

$x_{n+1} = x_n + hf(x_n, t_n)$

Example : Susceptible - Infected - Recovered (SIR) model of infection spread :

$$\frac{dS}{dt} = -\lambda S I \qquad \frac{dI}{dt} = \lambda S I - \gamma I \qquad \frac{dR}{dt} = -\gamma I$$

$$\downarrow$$

$$S(t+h) = S(t) + h \times \lambda S(t) I(t)$$

$$I(t+h) = I(t) + h \times (\lambda S(t) I(t) - \gamma I(t))$$

lower h = more accurate result

Changing to 1st order form :

- Introduce new variables to change system to 1st order

e.g $\frac{d^2\theta}{dt^2} = -\frac{g}{e} \sin\theta$

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -\frac{g}{e}\sin\theta$$

$\rightarrow$ create iteration from each

Root Finding :

- Refers to finding the roots of functions ie. to find x such that

$$f(x) = 0$$

Fixed - Point Iteration Method :

- Rearrange $f(x) = 0$ to $x = g(x)$
- Make a sequence of values : x_0, x_1, x_2 , guessing x_0.

using $\boxed{x_{n+1} = g(x_n)}$

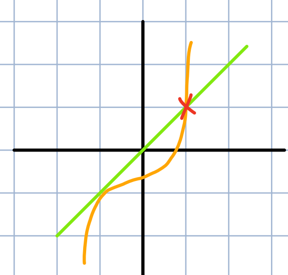
$\rightarrow$ if this sequence converges, then it converges to a root of f.

What can go wrong ?

- Divergence

e.g $x^3 - x - 1 = 0$ $\rightarrow$ $x = x^3 - 1$

using $x_0 = 0$
$x_1 = -1$
$x_2 = -2$



$x_3 = -9$ ... numbers get large (-ve)
although solution exists

- Chaos

Theory of Fixed Point Iteration :

- Consider $x_{n+1} = g(x_n)$
- let root be x^* & $E_n := x_n - x^*$ denote the error of n^{th} iterate

Using Taylor Series for small E_n

$$x^* + E_{n+1} = g(x_n) = g(x^* + E_n) \approx g(x^*) + g'(x^*)E_n = x^* + g'(L)E_n$$

Cancelling x^* from both sides :

$$x^* \approx g'(x^*)E_n$$

- Let $r = g'(x^*)$ be the (theoretical) rate of convergence

→ we want $|r| < 1$ $\therefore$ $\boxed{|g'(x^*)| < 1}$ is necessary for convergence

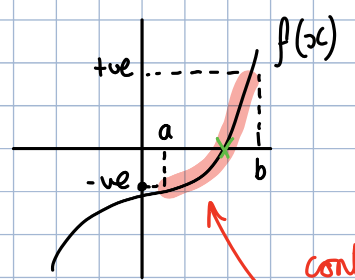
if $0 < g'(x^*) < 1$ then convergence is monotonic (error same sign, just reduces)

if $-1 < g'(L) < 0$ then convergence is oscillatory (step is overestimate, under, over etc.
→ error +ve, -ve, +ve..)

Intermediate Value Theorem :

For a continuous function $f(x)$, and x values of a & b where $f(a)$ and $f(b)$ are non-zero and opposite signs, there exists a root x^* with $a < x^* < b$ such that $f(x^*) = 0$

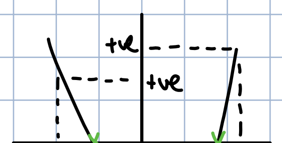
ie.



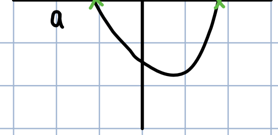
(if function continuous and goes from -ve to +ve, it crosses zero between)

- is a sufficient condition, NOT a necessary condition

e.g

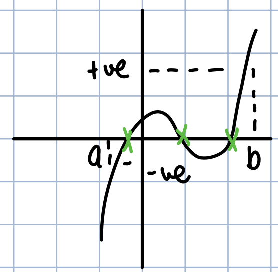


$f(a)$ & $f(b)$ not opposite signs so doesn't satisfy condition, but clearly does have roots



→ also it shows there is one root but there could be multiple

e.g



IMVT applies which means at least one root, but doesn't tell you there's 3.

To summarise, a function that doesn't satisfy the IMVT CAN have roots in interval, but if a function does satisfy IMVT you know it has at least one root in interval.

Bisection Method :

- Using IMVT repeatedly

→ let $f(x)$ continuous with $f(a_1)f(b_1) < 0$

- half interval

→ let $c_n = \frac{(a_n + b_n)}{2}$ for $n = 1, 2 \ldots$

- If $f(c_n) = 0$, stop
- If $f(c_n)f(a_n) < 0$, let $a_{n+1} = a_n$ & $b_{n+1} = c_n$
- If $f(c_n)f(b_n) < 0$, let $a_{n+1} = a_n$ & $b_{n+1} = b_n$

→ Keep halving interval, use midpoint of interval as root guess.
→ Stop when half interval less than tolerance

- This method is robust (guaranteed to work, unlike fixed point) but converges very slowly. Rate of convergence = 0.5
- Has linear convergence : for 2x correct digits, need 2 x iterations

Newton-Raphson Method :

- Uses tangent to generate sequence of approximations:

$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$ so $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$

- Requires $f(x)$ to be continuous and that $f'(x)$ exists and is continuous.

- Converges much faster than bisection, but things can go wrong:

→ may jump way outside of interval before converging

→ may diverge

$\boxed{\text{Newton-Raphson faster convergence, but not as robust as bisection.}}$