Second-Order Discretization in Space and Time for Grey S_2 Radiation-Hydrodynamics

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Introduction

- We have developed a second-order accurate in space and time non-relativistic radiation-hydrodynamics method based upon MUSCL-Hancock hydrodynamics and the S_2 grey radiation transport equations with trapezoidal-BDF2 (TBDF2) temporal discretization and linear-discontinuous Galerkin (LDG) spatial discretization.
- This method preserves total mass, total momentum, and total energy; yields to $O(v/c)$ the correct equilibrium solutions for the energy-integrated radiation intensity, the energy-integrated radiation flux, and the energy-integrated radiation pressure; and is correct to $O(v/c)$ in the equilibrium-diffusion limit (EDL).
- It is an IMEX method, using a combination of both explicit and implicit time integration.
- The method is asymptotic preserving with respect to the EDL, which requires particular attention.
- The method is designed so that the standard MUSCL-Hancock solution is obtained if the radiation momentum and energy deposition is insignificant, and the standard LDG coupled S_2-material temperature solution is obtained if the material velocities are insignificant.

Asymptotic Preservation of the Equilibrium-Diffusion Limit

- It is well known that use of an implicit MUSCL Hancock-type scheme for the S_n equations yields poor behavior in the EDL.
- In general an DG scheme of at least linear order is required in 1-D for good behavior. In 2-D a DG scheme of at least bilinear order is required on rectangles, and in 3-D a DG scheme of at least trilinear order is required on bricks.
- Because of the coupling between the material internal energy (temperature) and the radiation intensity, proper behavior in the diffusion limit requires a DG treatment for the material internal energy as well as the radiation intensity.
- This results in two different slope definitions for the material internal energy from the MUSCL-Hancock method and the DG method.
- We are able to achieve the desired properties for our method by using each set of slopes in different parts of the calculation.

The Equations

- The equations that we solve are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,, \tag{1a}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial x}(p) = \frac{\sigma_t}{c}F_{r,0}\,, \tag{1b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[(E+p)\,u\right] = -\sigma_a c\left(aT^4 - E_r\right) + \frac{\sigma_t u}{c}F_{r,0}\,, \tag{1c}$$

$$\frac{1}{c}\frac{\partial \psi^+}{\partial t} + \frac{1}{\sqrt{3}}\frac{\partial \psi^+}{\partial x} + \sigma_t\psi^+ = \frac{\sigma_s}{4\pi}cE_r + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t u}{4\pi c}F_{r,0} + \frac{\sigma_t}{\sqrt{3}\pi}E_r u\,, \tag{1d}$$

$$\frac{1}{c}\frac{\partial \psi^-}{\partial t} - \frac{1}{\sqrt{3}}\frac{\partial \psi^-}{\partial x} + \sigma_t\psi^- = \frac{\sigma_s}{4\pi}cE_r + \frac{\sigma_a}{4\pi}acT^4 - \frac{\sigma_t u}{4\pi c}F_{r,0} - \frac{\sigma_t}{\sqrt{3}\pi}E_r u\,, \tag{1e}$$

where ρ is the density, u is the velocity, $E = \frac{\rho u^2}{2} + \rho e$ is the total material energy density, e is the specific internal energy density, T is the material temperature, E_r is the radiation energy density,

$$E_r = \frac{2\pi}{c}\left(\psi^+ + \psi^-\right)\,, \tag{2}$$

F_r is the radiation energy flux,

$$F_r = \frac{2\pi}{\sqrt{3}}\left(\psi^+ - \psi^-\right)\,, \tag{3}$$

and $F_{r,0}$ is an approximation to the comoving-frame flux,

$$F_{r,0} = F_r - \frac{4}{3}E_r u\,. \tag{4}$$

- Equations (1a) through (1e) are closed in our calculations by assuming an ideal equation of state (EOS):

$$p = \rho e(\gamma - 1)\,, \tag{5a}$$

$$T = \frac{e}{C_v}\,, \tag{5b}$$

where γ is the adiabatic index, and C_v is the specific heat. However, our method is compatible with any valid EOS.

TBDF2 Form

- We have found the following non-standard form of the TBDF2 method to facilitate determining the form of the radiation coupling terms in the hydrodynamics equations:

$$\frac{2\left(f^{n+1/2} - f^n\right)}{\Delta t} = \frac{1}{2}(\mathbf{A}f)^{n+1/2} + \frac{1}{2}(\mathbf{A}f)^n\,, \tag{6}$$

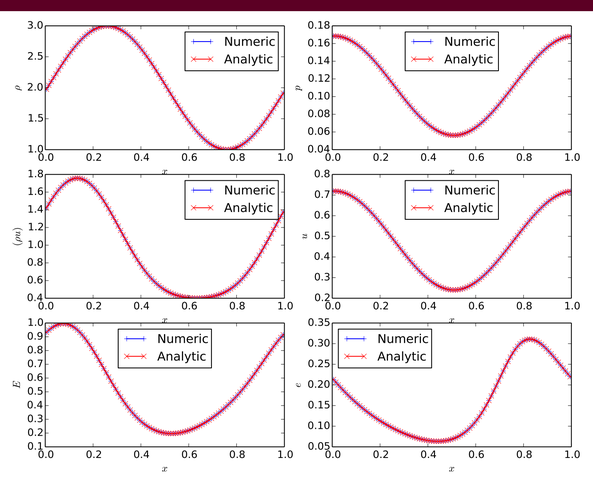
$$\frac{2\left(f^{n+1} - f^{n+1/2}\right)}{\Delta t} = \frac{2}{3}(\mathbf{A}f)^{n+1} + \frac{1}{6}(\mathbf{A}f)^{n+1/2} + \frac{1}{6}(\mathbf{A}f)^n\,. \tag{7}$$

- Note that each of these expression represents a conservation statement over each half time step. The usual expression that Eq. (7) replaces does not have this property.

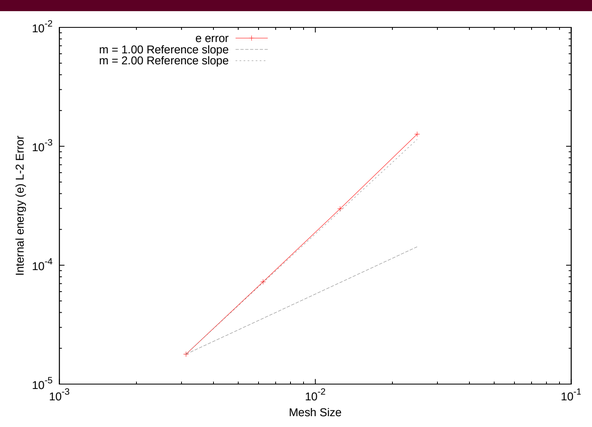
Algorithm

- Each time step is broken into two half time steps.
- During the first half time step:
  1. Perform standard explicit MUSCL-Hancock predictor step.
  2. Enter iteration loop
  - 2a. Update material momentum change due to radiation momentum deposition.
  - 2b. Perform coupled solve of total material energy and S_2 equations with Crank-Nicholson temporal differencing.
  3. Perform standard explicit MUSCL-Hancock corrector step.
  4. Enter iteration loop
  - 4a. Compute new material momentum due to radiation momentum deposition.
  - 4b. Perform simultaneous solve for new total material energy due to radiation energy deposition and the radiation intensities with Crank-Nicholson temporal differencing.
- The second half time step is the same as the first except that TBDF2 temporal diffencing is used in the corrector step in the simultaneous solve for the material total energy and the radiation intensities.
- The standard MUSCL-Hancock algorithm does not need internal energy slopes, but one can nonetheless consistently compute them. These slopes are not used to initialize the LDG representation for the internal energy in the coupled radiation total marterial energy solve. Rather, the LDG slopes computed at previous time steps are used. This is necessary to preserve the diffusion limit.

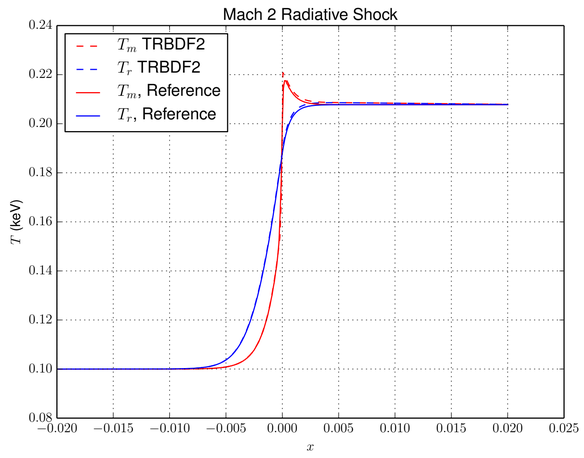
Results



(a) MMS Solutions



(b) MMS convergence ($\Delta x/\Delta t$ constant)



(c) Mach 2 Radiative Shock Solution

Conclusions

- We are able to produce accurate radiative shock solutions.
- We demonstrated second-order accuracy for a smooth manufactured solution.