• Balance eq:

$$\sum_m w_m \left(\frac{\mu_m}{\Delta x_j} \left(F_{m,j+1/2} - F_{m,j-1/2} \right) + \sigma_t F_{m,j} \right) = Q$$

$$\frac{f_{1,j+1/2} - f_{1,j-1/2}}{\Delta x} + \sigma_t f_{0,j} = Q_0$$

• First moment

$$\sum_m w_m \mu_m \left(\frac{\mu_m}{\Delta x} \left(F_{m,j+1/2} - F_{m,j-1/2} \right) + \sigma_t F_{m,j} \right) = Q$$

$$\frac{1}{3} \left(f_{0,j+1/2} - f_{0,j-1/2} \right) + \sigma_t F_1 = 0$$

• closure eq:

$$F_{m,j+1/2} = \overrightarrow{F}_{m,j} + \hat{F}_{m,j}, \quad \mu_m > 0$$

$$\sum_{\mu_m > 0} w_m (\cdot) \left(\left(\frac{f_{0,j+1/2} + 3 f_{1,j+1/2} \mu_m}{2} \right) = \left(\frac{f_{0,j} + 3\mu_m f_{1,j}}{2} \right) + \left(\frac{\hat{f}_{0,j} + 3\mu_m \hat{f}_{1,j}}{2} \right) \right) \quad (\mu_m > 0)$$

$$\frac{1}{2} \left(f_{0,j+1/2} + 3\delta \hat{f}_{1,j+1/2} \right) = \frac{1}{2} \left(f_{0,j} + 3\delta \hat{f}_{1,j} \right) + \frac{1}{2} \left(\hat{f}_{0,j} + 3\delta f_{1,j} \right) \quad ?$$

$\mu_m < 0$:

$$\sum_{\mu_m < 0} \left(F_{m,j+1/2} \right) = \sum_{\mu_m < 0} \left(f_0 + 3 f \mu_m \right)$$

$$= \frac{f_{0,j+1/2} - 3 f_{1,j+1/2} \delta}{2}$$

• I want $\sum_{\mu_m} \left(F_{m,j+1/2} \right)$

$$f_{0,j+1/2} = \frac{1}{2} \left(f_{0,j} + 3\delta f_{1,j} \right) + \frac{1}{2} \left(\hat{f}_{0,j} + 3\delta \hat{f}_{1,j} \right) + \frac{f_{0,j+1/2} - 3 f_{1,j+1/2} \delta}{2}$$

$\sum w_m = 2$

$f_0 = \delta\phi, \quad f_1 = \delta J$

$$\sum w_m \mu_m^2 = \int_{-1}^{1} \mu^2 d\mu = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\sum w_m = \int_{-1}^{1} d\mu = 2$$

$$\sum w_m \mu_m^2 = \int_{-1}^{1} \mu^2 d\mu = \frac{2}{3}$$

$$2 \sum_{\mu_m > 0} w_m \mu_m^2 = 2 \int_0^1 \mu^2 d\mu = \frac{2}{3}$$

$$f_{0,j+1/2} = \frac{1}{2}(f_{0,j} + 3\delta f_{1,j}) + \frac{1}{2}(\hat{f}_{0,j} + 3\delta \hat{f}_{1,j}) + \frac{f_{0,j+1/2} - 3f_{1,j+1/2}\delta}{2}$$

- First Moment

$$f_{1,j+1/2} = \sum_{\mu_m} w_m \mu_m \left(F_{m+1/2}\right) = \frac{1}{2}\left(\delta f_{0,j} + f_{1,j} + \delta \hat{f}_{0,j} + \hat{f}_{1,j} - \delta f_{0,j+1/2} + f_{1,j+1/2}\right)$$

$$\sum_{\mu_m > 0} w_m \mu_m \left(F_{m+1/2}\right) = \sum_{\mu_m > 0} w_m \mu_m \left(\frac{f_{0,j} + 3\mu_m f_{1,j}}{2}\right) + \sum_{\mu_m < 0} \mu_m \left(\frac{\hat{f}_{0,j} + 3\mu_m \hat{f}_{1,j}}{2}\right)$$

$$= \frac{\delta f_{0,j} + f_{1,j}}{2} + \frac{\delta \hat{f}_{0,j} + \hat{f}_{1,j}}{2}$$

$$\sum_{\mu_m < 0} w_m \mu_m \left(F_{m+1/2}\right) = \sum_{\mu_m < 0} w_m \mu_m \left(\frac{f_{0,j+1/2} + 3\mu_m f_{1,j+1/2}}{2}\right) = -\frac{\delta f_{0,j+1/2}}{2} + \frac{f_{1,j+1/2}}{2}$$

- Left side

$$f_{0,j-1/2} = \frac{1}{2}(f_{0,j} - 3\delta f_{1,j}) - \frac{1}{2}\left(\hat{f}_{0,j} - 3\delta \hat{f}_{1,j}\right) + \frac{f_{0,j-1/2} + 3\delta f_{j-1/2,1}}{2}$$

$$f_{1,j-1/2} = \frac{1}{2}\left(-\delta f_{0,j} + f_{1,j} - \hat{f}_{0,j} + \hat{f}_{1,j}\right) + \frac{1}{2}\left(\delta f_{0,j-1/2} + f_{1,j-1/2}\right)$$

- My $\delta = \frac{1}{2}\delta^{\text{wavedisp}}$, so in next step δ is divided by 2

$$f_{0,j+1/2} = \frac{1}{2}\left(f_{0,j} + \frac{3\delta}{2} f_{1,j}\right) + \frac{1}{2}\left(\hat{f}_{0,j} + \frac{3\delta}{2}\hat{f}_{1,j}\right) + \frac{f_{0,j+1/2}}{2} - \frac{3f_{1,j+1/2}}{4}\delta$$

$$f_{1,j+1/2} = \frac{1}{2}\left(\frac{\delta}{2} f_{0,j} + f_{1,j} + \frac{\delta}{2}\hat{f}_{0,j} + \hat{f}_{1,j} - \frac{\delta}{2} f_{0,j+1/2} + f_{1,j+1/2}\right)$$

$$f_{0,j-1/2} = \frac{1}{2}\left(f_{0,j} - \frac{3\delta}{8} f_{1,j}\right) - \frac{1}{2}\left(\hat{f}_{0} - \frac{3}{2}\delta\hat{f}_{1,j}\right) + \frac{1}{2}\left(f_{0,j-1/2} + \frac{3\delta}{4} f_{j-1/2,1}\right)$$

$$f_{1,j-1/2} = \frac{1}{2}\left(-\frac{\delta}{2} f_{0,j} + f_{1,j} + \frac{\delta}{2}\hat{f}_{0,j} - \hat{f}_{1,j}\right) + \frac{1}{2}\left(\frac{\delta}{2} f_{0,j-1/2} + f_{1,j-1/2}\right)$$

- Add $\left(f_{0,j+1/2} + f_{0,j-1/2}\right)$

$$f_{0,j+1/2} + f_{0,j-1/2} = \hat{f}_{0,j} + \frac{3\delta}{2}\hat{f}_{1,j} + \left(\frac{f_{0,j+1/2} + f_{0,j-1/2}}{2}\right) - \frac{3\delta}{4}\left(f_{1,j+1/2} - f_{1,j-1/2}\right)$$

$$\Rightarrow \boxed{f_{0,j} + \frac{3\delta}{2}\hat{f}_{1,j} = \frac{1}{2}\left(f_{0,j+1/2} + f_{0,j-1/2}\right) + \frac{3}{4}\delta\left(f_{1,j+1/2} - f_{1,j-1/2}\right)} \quad \text{(Eq. 4.129)}$$

- Add $f_{1,j+1/2} + f_{1,j-1/2}$

$$f_{1,j+1/2} + f_{1,j-1/2} = f_{1,j} + \frac{\delta}{2}\hat{f}_{0,j} + \frac{1}{2}\left(\frac{\delta}{2} f_{0,j-1/2} - \frac{\delta}{2} f_{0,j+1/2}\right) + \frac{1}{2}\left(f_{1,j+1/2} + f_{1,j-1/2}\right)$$

$$\boxed{f_{1,j} + \frac{\delta}{2}\hat{f}_{0,j} = \frac{\delta}{4}\left(f_{0,j+1/2} - f_{0,j-1/2}\right) + \frac{1}{2}\left(f_{1,j+1/2} + f_{1,j-1/2}\right)} \quad (4.130)$$