$$\psi_i^{LD} = \frac{\frac{\sigma h}{3}+2}{\frac{4}{3}\sigma h + 2 + \frac{\sigma^2 h^2}{3}} = \frac{6+\sigma h}{6+4\sigma h+\sigma^2 h^2} \approx 1 - \frac{\tau}{2} + \frac{\tau^2}{6} - \frac{\tau^3}{36} - \frac{\tau^4}{108} + O(\tau^5)$$

$$\psi_i^{\alpha} = \frac{1}{\sigma h}\left(1-e^{-\sigma h}\right) \approx 1 - \frac{\tau}{2} + \frac{\tau^2}{6} - \frac{\tau^3}{24}$$

- Balance and 1st moment eq.'s, exact: $\psi_{i-1/2} = 1$

$$\psi_{i+1/2} - \psi_{i-1/2} + \sigma \psi_i h = 0 \quad (1)$$

$$\frac{3}{h}\left(\psi_{i+1/2} + \psi_{i-1/2}\right) - \frac{6}{h}\psi_i + \sigma \psi_x = 0 \quad (2)$$

- Eliminate outflow as $\psi_{i+1/2} = \psi_i \alpha + \psi_x$, solve for ψ_i, $\tau = \sigma h$

$$(1) \rightarrow \psi_x + \psi_i \alpha - \psi_{i-1/2} + \sigma \psi_i h = 0 \Rightarrow \psi_x \equiv \underbrace{\psi_{i-1/2}}_{1} - \psi_i(\tau+\alpha)$$

$$(2) \rightarrow 3\left(\psi_i \alpha + \psi_x + 1\right) - 6\psi_i + \tau\left(1 - \psi_i(\tau+\alpha)\right) = 0$$

$$3\left(\psi_i \alpha + 1 - \psi_i(\tau+\alpha) + 1\right) - 6\psi_i + \tau\left(1-\psi_i(\tau+\alpha)\right) = 0$$

$$\psi_i\left(3\alpha - 3\tau - 3\alpha - 6 - \tau^2 - \alpha\tau\right) + (6+\tau) = 0$$

$$\boxed{\psi_i = \frac{6+\tau}{6+(3+\alpha)\tau+\tau^2}}$$

- For LD, $\alpha = 1$:

$$\psi_i = \frac{6+\tau}{6+4\tau+\tau^2}, \quad \text{error } O(\tau^3)?$$

- If we look @ $\psi_i \tau$:

$$(\psi_i \tau)^{LD} = 1 - \frac{\tau^2}{2} + \frac{\tau^3}{6} - \frac{\tau^4}{36}$$

$$(\psi_i^{\alpha}\tau) = \left(1 - e^{-\tau}\right) = 1 - \frac{\tau}{2} + \frac{\tau^3}{6} - \frac{\tau^4}{24}, \quad \text{error } O(\tau^4)$$

- for α based on ψ^{ex}:

$$\alpha^{ex} = \frac{\psi^{ex}_{i+1/2} - \psi^{ex}_{x}}{\psi^{ex}_{i}} = \frac{e^{-\tau} - \frac{(6-3\tau)+(6+3\tau)e^{-\tau}}{\tau^2}}{\frac{1}{\tau}(1-e^{-\tau})} = \frac{\tau^2 e^{-\tau} - (6-3\tau) + (6+3\tau)e^{-\tau}}{\tau(1-e^{-\tau})}$$

- $\psi_i = \dfrac{6+\tau}{6+(3+\alpha^{ex})\tau+\tau^2}$ $\xrightarrow{\text{Wolfram}}$ $\psi_i \approx 1 - \frac{\tau}{2} + \frac{\tau^2}{6} - \frac{\tau^3}{24} + \frac{\tau^4}{120} - \frac{\tau^5}{720} + \Theta(\tau^6)$

$$\psi^{ex}_i = 1 - \frac{\tau}{2} + \frac{\tau^2}{6} - \frac{\tau^3}{24} + \frac{\tau^4}{120} - \frac{\tau^5}{720} + \Theta(\tau^6)$$

$\boxed{\psi_i = \psi_i^{ex}}$ ✓

$$\psi_i = \frac{6+\tau}{6+3\tau+\left(\frac{\tau^2 e^{-\tau} - (6-3\tau)+(6+3\tau)e^{-\tau}}{1-e^{-\tau}}\right)+\tau^2}$$

$$\psi_i = \frac{(6+\tau)(1-e^{-\tau})}{(6+3\tau)(1-e^{-\tau}) + \tau^2 e^{-\tau} - (6-3\tau) + \cancel{(6+3\tau)e^{-\tau}} + \tau^2 - e^{-\tau}\tau^2}$$

$$\psi_i = \frac{(6+\tau)(1-e^{-\tau})}{(6+3\tau) - (6-3\tau) + \tau^2} = \frac{(6+\tau)(1-e^{-\tau})}{6\tau+\tau^2} = \boxed{\frac{1-e^{-\tau}}{\tau}}$$ ✓

- Any value of ψ_i^{ex}, ψ_x^{ex} will reproduce moments exactly, as long as they satisfy the balance eq. locally (and higher mom. eq. in case of other closure forms that use a higher mom.)

$$\psi^{ex}_{i+1/2} - \underbrace{\psi^{ex}_{i-1/2}}_{\text{assume known, ex.}} + \sigma\psi^{ex}_i = 0$$

&

$$\tilde{\psi}_{i+1/2} - \tilde{\psi}_{i-1/2} + \sigma\tilde{\psi}_i = 0$$

$$\psi^{ex}_{i+1/2} - \tilde{\psi}_{i+1/2} - 0 + \sigma(\psi^{ex}_i - \tilde{\psi}_i) = 0$$