1. ACCELERATED ITERATIVE SOLUTION TO THE LO EQUATIONS

As described in Sec. **??**, the fully-discrete, S_2-like LO equations are not easy to directly inverted efficiently in higher spatial dimensions. To demonstrate a possible path forward in higher dimensions, we have investigated the use of a standard source iteration scheme to invert the scattering terms in the LO equations. As material properties become more diffusive (e.g., c_v is small and σ_a is large), the effective scattering cross sections becomes large. This results in a spectral radius of the source iterations that approaches unity [7]. These regimes are typical in TRT simulations, so an acceleration method for an iterative solution is critical. We have accelerated the source iterations with a nearly-consistent diffusion synthetic acceleration (DSA) method known as WLA [10, 9].

REWRITE: STUFF ABOUT LUMPING ETC. We have also recast the DSA method as a preconditioner to an iterative Krylov solution [2] of the LO equations. Generally, Krylov methods mitigate acceleration losses due to inconsistencies in the acceleration method. In higher dimensions, the use of a Krylov method is necessary for effective acceleration for nearly-consistent acceleration methods in problems with mixed optical thicknesses [2], e.g., typical radiative transfer problems. Also, applying the preconditioned-Krylov approach allows for the use of spatially lumped DSA equations as a preconditioner, with the LO equations using alternative fix-up methods. We expect better acceleration performance (DID WE GET IT?) when the LDD discretization is introduced into the LO equations.

In the remainder of this chapter is structured as follows: The source iteration solution to the LO equations is detailed. Then, the equations for the WLA DSA method are derived and the acceleration algorithm is given. The DSA method is

then recast as a preconditoner to a GMRES solution to the scattering iteration equations. Finally, results are given for a modified test problem which requires the use of acceleration.

1.1 Source Iteration Solution to the Linearized LO Equations

The linearized LO equations can be solved with a source iteration method [5, 4, 6]. In the source iteration process the scattering source is lagged, which uncouples unknowns between the two half-ranges. This produces a lower-triangular system where the spatial unknowns can be solve for sequentially along the two directions of flow via a standard sweeping procedure [5, 7]. Beginning at the left boundary, the positive unknowns can be determined for each cell from $i = 1, \ldots, N_c$. Because the inflow to the i-th cell is defined from the previous cell or boundary condition, a local system of equations can be solved for the $\langle\phi\rangle^+_{L,i}$ and $\langle\phi\rangle^+_{R,i}$ unknowns. The negative direction unknowns are determined similarly, starting from the right boundary towards the left. The newly computed half-range intensities can then be used to estimate the scattering source for the next iteration. This process is repeated until convergence.

The source iteration process can be written in operator notation as

$$\mathbf{M}\psi^{l+1} = S\psi^l + Q, \tag{1.1}$$

where M is the discretized LO streaming plus removal operator (i.e., scattering is not included) and ψ is a vector of the half-range, FE moment unknowns. The vector Q contains the fixed source terms resulting from the linearized emission source and previous time step moments, for each equation; the terms for the i-th element and the L equation are

$$(\mathbf{Q})^{\pm}_{i,L} = \frac{\langle\phi\rangle^{\pm}_L}{c\Delta t} + \frac{1}{2} f_i \sigma_{a,i} ac \langle (T^n)^4 \rangle_{L,i} \tag{1.2}$$

The scattering operator terms for the i-th element and the L equations is

$$(\mathbf{S}\psi^l)^{\pm}_{i,L} = \frac{1}{2}\left(\sigma_{a,i}(1-f_i)+\sigma_{s,i}\right)\left(\langle\phi^l\rangle^{+}_{i,L}+\langle\phi^l\rangle^{-}_{i,L}\right). \tag{1.3}$$

Equivalent expressions are defined for the R moment equations and boundary conditions, forming a fully defined set of equations.

The scattering inversion must be performed within each Newton iteration. Thus, for the m-th Newton step, the source iteration process is defined as

1. Evaluate effective scattering and absorption cross sections with $\{T_i^m : \ i = 1, 2, \ldots, N_c\}$.
2. Compute new scattering source $\mathbf{S}\psi^l$.
3. Perform sweeps to calculate $\psi^{l+1} = \mathbf{M}^{-1}\mathbf{S}\psi^l + \mathbf{M}^{-1}Q$
4. If $\|\psi^{l+1} - \psi^l\| <$ tolerance, exit
5. Else: repeat steps 2–4.

1.2 Linear Diffusion Synthetic Acceleration

A form of DSA referred to as the WLA method is used to accelerate the source iterations [10, 9]. Between each source iteration, a residual equation is formed that provides the error in the current scattering iteration. The DSA method uses an approximate, lower-order operator to solve the error equation for a correction to the zeroth angular moment of the intensity, i.e., the mean intensity $\phi(x)$. The DSA equations contain a standard finite-difference diffusion discretization that can be more efficiently solved than the S_2-like equations that are being accelerated, but will accurately resolve the slowly converging diffusive error modes. It is important for the spatial discretization of the DSA equations to be closely related to the discretization of the LO equations for the acceleration to be effective and stable [1]. The WLA

method first solves a spatially-continuous discretization of the diffusion equation for the iterative error on faces. The error on the faces is then mapped onto the volumetric moment unknowns via a LD discretization of diffusion equation [10]. The LD mapping resolves issues that would occur in optically-thick cells, while the continuous diffusion equation is accurate in the EDL where acceleration is important [1].

In the remained of this section we derive the discretized diffusion and update equations. To simplify notation, we derive the diffusion equations from a generic transport equation with isotropic scattering and source q_0, i.e.,

$$\mu \frac{\partial I}{\partial x} + \sigma_t I = \frac{\sigma_s}{2} \left(\phi(x) + q_0 \right) \tag{1.4}$$

The WLA-DSA algorithm is then detailed.

1.2.1 Forming a Continuous Diffusion Equation

First, a continuous spatial discretization of a diffusion equation is derived. The mean intensity ϕ will ultimately be assumed continuous at faces to produce a standard three-point finite-difference diffusion discretization. The zeroth and first μ moment of Eq. (1.4) produce the P_1 equations [5, 9], i.e.,

$$\frac{\partial J}{\partial x} + \sigma_a \phi = q_0 \tag{1.5}$$

$$\sigma_t J + \frac{1}{3} \frac{\partial \phi}{\partial x} = 0, \tag{1.6}$$

where $q_0 = \int_{-1}^{1} q(x, \mu) \mathrm{d}\mu$. The spatial finite element moments (defined by Eq. (**??**) and (**??**)) are taken of the above equations. The mean intensity is assumed linear on the interior of the cell, i.e., $\phi(x) = \phi_L b_L(x) + \phi_R b_R(x)$, for $x \in (x_{i-1/2}, x_{i+1/2})$.

Taking the left moment, evaluating integrals, and rearranging yields

$$J_i - J_{i-1/2} + \frac{\sigma_{a,i} h_i}{2} \left(\frac{2}{3} \phi_{L,i} + \frac{1}{3} \phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{L,i} \ , \tag{1.7}$$

where J_i is the average of the flux J over the cell. The moments of q are not simplified to be compatible with the error equations which are in terms of moments. For the R moment

$$J_{i+1/2} - J_i + \frac{\sigma_{a,i} h_i}{2} \left(\frac{2}{3} \phi_{L,i} + \frac{1}{3} \phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{R,i} \ . \tag{1.8}$$

The equation for the L moment is evaluated for cell $i+1$ and added to the R moment equation evaluated at i. The flux J is assumed continuous at $i + 1/2$ to eliminate the face fluxes from the equations. The sum of the two equations becomes

$$J_{i+1} - J_i + \frac{\sigma_{a,i+1} h_{i+1}}{2} \left(\frac{2}{3} \phi_{L,i+1} + \frac{1}{3} \phi_{R,i+1} \right) + \frac{\sigma_{a,i} h_i}{2} \left(\frac{1}{3} \phi_{L,i} + \frac{2}{3} \phi_{R,i} \right) = \frac{h}{2} \left(\langle q \rangle_{L,i+1} + \langle q \rangle_{R,i} \right). \tag{1.9}$$

The mean intensity is approximated as continuous at each face, i.e., $\phi_{L,i+1} = \phi_{R,i} \equiv \phi_{i+1/2}$. Adding the L and R moments of Eq. (1.6) together, with the continuous approximation for $\phi_{i+1/2}$, produces a discrete Fick's law equation [8]

$$J_i = -D_i \frac{\phi_{i+1/2} - \phi_{i-1/2}}{h_i}, \tag{1.10}$$

where $D_i = 1/(3\sigma_{t,i})$. Substitution of Eq. (1.10) into Eq. (1.9) and rearranging yields the following discrete diffusion equation:

$$\left(\frac{\sigma_{a,i+1} h_{i+1}}{6} - \frac{D_{i+1}}{h_{i+1}} \right) \phi_{i+3/2} + \left(\frac{D_{i+1}}{h_{i+1}} + \frac{D_i}{h_i} + \frac{\sigma_{a,i+1} h_{i+1}}{3} + \frac{\sigma_{a,i} h_i}{3} \right) \phi_{i+1/2}$$

$$+\left(\frac{\sigma_{a,i}h_i}{6}-\frac{D_i}{h_i}\right)\phi_{i-1/2}=\frac{h_{i+1}}{2}\langle q\rangle_{L,i+1}+\frac{h_i}{2}\langle q\rangle_{R,i}\ . \quad (1.11)$$

To allow for the use of lumped or standard LD in these equations, we introduce the factor θ, with $\theta = 1/3$ for standard LD, and $\theta = 1$ for lumped LD. The diffusion equation becomes

$$\left(\frac{\sigma_{a,i+1}h_{i+1}}{4}(1-\theta)-\frac{D_{i+1}}{h_{i+1}}\right)\phi_{i+3/2}+\left(\frac{D_{i+1}}{h_{i+1}}+\frac{D_i}{h_i}+\left(\frac{1+\theta}{2}\right)\left[\frac{\sigma_{a,i+1}h_{i+1}}{2}+\frac{\sigma_{a,i}h_i}{2}\right]\right)\phi_{i+1/2}$$
$$+\left(\frac{\sigma_{a,i}h_i}{4}(1-\theta)-\frac{D_i}{h_i}\right)\phi_{i-1/2}=\frac{h_{i+1}}{2}\langle q\rangle_{L,i+1}+\frac{h_i}{2}\langle q\rangle_{R,i}\ . \quad (1.12)$$

Summation over all cells forms a system of equations for ϕ at each face.

1.2.1.1 Diffusion Boundary Conditions

The upwinding in the LO system exactly satisfies the inflow boundary conditions, therefore a vacuum boundary condition is applied to the diffusion error equations. The equation for the left moment at the first cell is given by

$$J_1-J_{1/2}+\frac{\sigma_{a,i}h_i}{2}\left(\frac{1+\theta}{2}\phi_{L,i}+\frac{1-\theta}{2}\phi_{R,i}\right)=\frac{h_i}{2}\langle q\rangle_{L,i}\ , \quad (1.13)$$

The Marshak boundary condition for the vacuum inflow at face $x_{1/2}$ is given as

$$J^+_{1/2}=0=\frac{\phi_{1/2}}{4}+\frac{J_{1/2}}{2}, \quad (1.14)$$

which can be solved for $J_{1/2}$. Substitution of the above equation and Eq. (1.10) into Eq. (1.13) gives

$$\left(\frac{1}{2}+\sigma_{a,1}h_1\frac{1+\theta}{4}-\frac{D_1}{h_1}\right)\phi_{1/2}+\left(\sigma_{a,1}h_1\frac{1-\theta}{4}-\frac{D_1}{h_1}\right)\phi_{3/2}=\frac{h_i}{2}\langle q\rangle_{L,1} \quad (1.15)$$

A similar expression can be derived for the right-most cell.

1.2.2 Mapping Solution onto LD Unknowns

Solution of the continuous diffusion equation will provide an approximation to the error of ϕ on faces, denoted as $\phi^C_{i+1/2}$. We now need to determine the correction these results provide for the LD representation of ϕ. To do this, first we take the L and R finite element moments of the P_1 equations. A LDFE dependence is assumed on the interior of the cell for J and ϕ. Taking moments of Eq. (1.5) and simplifying yields

$$J_{i+1/2} - \frac{J_{L,i} + J_{R,i}}{2} + \frac{\sigma_{a,i} h_i}{2} \left(\frac{1}{3}\phi_{L,i} + \frac{2}{3}\phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{R,i} \tag{1.16}$$

$$\frac{J_{L,i} + J_{R,i}}{2} - J_{i-1/2} + \frac{\sigma_{a,i} h_i}{2} \left(\frac{2}{3}\phi_{L,i} + \frac{1}{3}\phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{L,i} \tag{1.17}$$

The moment equations for Eq. (1.6) are

$$\frac{1}{3} \left(\phi_{i+1/2} - \frac{\phi_{i,L} + \phi_{i,R}}{2} \right) + \frac{\sigma_{t,i} h_i}{2} \left(\frac{1}{3} J_{L,i} + \frac{2}{3} J_{R,i} \right) = 0 \tag{1.18}$$

$$\frac{1}{3} \left(\frac{\phi_{i,L} + \phi_{i,R}}{2} - \phi_{i-1/2} \right) + \frac{\sigma_{t,i} h_i}{2} \left(\frac{2}{3} J_{L,i} + \frac{1}{3} J_{R,i} \right) = 0 \tag{1.19}$$

The face terms $J_{i\pm1/2}$ and $\phi_{i\pm1/2}$ need to be eliminated from the system. The scalar flux is assumed to be the value provided by the continuous diffusion solution at each face, i.e., $\phi_{i\pm1/2} = \phi^C_{i\pm1/2}$. The fluxes are decomposed into half-range values to decouple the equations between cells. At $x_{i+1/2}$, the current is composed as $J_{i+1/2} = J^+_{i+1/2} - J^-_{i+1/2}$, where $+$ and $-$ denote the positive and negative half ranges of μ, respectively. We approximate the incoming fluxes, e.g., $J^-_{i+1/2}$, based on $\phi^C_{i+1/2}$. The

P_1 approximation provides the following relation [9]

$$\phi = 2(J^+ + J^-). \tag{1.20}$$

At $x_{i+1/2}$, the above expression is solved for the incoming current $J^-_{i+1/2}$. The total current becomes, with $\phi_{i+1/2} = \phi^C_{i+1/2}$,

$$J_{i+1/2} = J^+_{i+1/2} - J^-_{i+1/2} = 2J^+_{i+1/2} - \frac{\phi^C_{i+1/2}}{2}, \tag{1.21}$$

In the positive direction, at the right face, the values of ϕ and J are based on the LD representation within the cell at that face, i.e., $\phi_{R,i}$ and $J_{R,i}$. The standard P_1 approximation for the half-range fluxes is used[8], i.e.,

$$J^\pm = \frac{\gamma\phi}{2} \pm \frac{J}{2}, \tag{1.22}$$

where γ accounts for the difference between the LO parameters and the true P_1 approximation. Thus, for the right face and positive half-range,

$$J^+_{i+1/2} = \frac{\gamma}{2}\phi_{i,R} + \frac{J_{i,R}}{2} \tag{1.23}$$

A similar expression can be derived for $x_{i-1/2}$. The total fluxes at each face are thus

$$J_{i+1/2} = \gamma\phi_{i,R} + J_{i,R} - \frac{\phi^C_{i+1/2}}{2} \tag{1.24}$$

$$J_{i-1/2} = \frac{\phi^C_{i-1/2}}{2} - \gamma\phi_{i,L} + J_{i,L} \tag{1.25}$$

Substitution of these results back into the LD balance equations and introduction of the lumping notation yields the final equations

$$\left(\gamma\phi_{i,R} + J_{i,R} - \frac{\phi^C_{i+1/2}}{2}\right) - \frac{J_{L,i} + J_{R,i}}{2} + \frac{\sigma_{a,i}h_i}{2}\left(\frac{(1-\theta)}{2}\phi_{L,i} + \frac{(1+\theta)}{2}\phi_{R,i}\right) = \frac{h_i}{2}\langle q\rangle_{R,i} \tag{1.26}$$

$$\frac{J_{L,i} + J_{R,i}}{2} - \left(\frac{\phi^C_{i-1/2}}{2} - \gamma\phi_{i,L} + J_{i,L}\right) + \frac{\sigma_{a,i}h_i}{2}\left(\frac{(1+\theta)}{2}\phi_{L,i} + \frac{(1-\theta)}{2}\phi_{R,i}\right) = \frac{h_i}{2}\langle q\rangle_{L,i} \tag{1.27}$$

$$\frac{1}{3}\left(\phi^C_{i+1/2} - \frac{\phi_{i,L} + \phi_{i,R}}{2}\right) + \frac{\sigma_{t,i}h_i}{2}\left(\frac{(1-\theta)}{2}J_{L,i} + \frac{(1+\theta)}{2}J_{R,i}\right) = 0 \tag{1.28}$$

$$\frac{1}{3}\left(\frac{\phi_{i,L} + \phi_{i,R}}{2} - \phi^C_{i-1/2}\right) + \frac{\sigma_{t,i}h_i}{2}\left(\frac{(1+\theta)}{2}J_{L,i} + \frac{(1-\theta)}{2}J_{R,i}\right) = 0. \tag{1.29}$$

The above equations are completely local to each cell and fully defined, including for boundary cells. The system can be solved for the desired unknowns $\phi_{i,L}$, $\phi_{i,R}$, $J_{i,L}$, and $J_{i,R}$, which represent the mapping of $\phi^C_{i+1/2}$ onto the LD representation for $\phi(x)$.

1.2.3 DSA Source Definition

The above discretization procedure is used to determine the error in the scalar flux. The sources $\langle q\rangle_{L/R}$ for the error equations remain to be defined. In operator notation, the following iteration equations can be defined

$$\mathbf{L}\psi^{l+1/2} = \mathbf{S}\psi^{l+1} + Q \tag{1.30}$$

$$\mathbf{D}\delta\phi^{l+1/2} = \mathbf{S}(\phi^{l+1/2} - \phi^l) \tag{1.31}$$

$$\psi^{l+1/2} = \psi^{l+1/2} + \delta\phi^{l+1/2} \tag{1.32}$$

It is noted that unlike fully consistent DSA equations, the WLA-DSA algorithm does not preserve particle balance to round off. This is because the mapping procedure uses an approximate inflow to each cell, that is inconsistent with the partial outflow into the next cell.

The source in Eq. (1.31), i.e., q in Eqs. (1.12) and (1.15), is the residual for a given scattering iteration, given by [4, 5]

$$q = \sigma_s \left(\phi^{l+1/2} - \phi^l \right). \tag{1.33}$$

The spatial moments are straight forward:

$$\langle q \rangle_{L,i} = \sigma_{s,i} \left(\langle \phi^{l+1/2} \rangle_{L,i} - \langle \phi^l \rangle_{L,i} \right) \tag{1.34}$$

The above equation is valid for lumping or standard LD. This is because the LO moments are defined differently for LLD or LD, resulting in equations that are consistent. For instance, for lumped LD, we desire the right equation to have $\langle q \rangle_{R,i} = \sigma_s(\phi_{R,i}^{l+1/2} - \phi_{R,i}^l)$. Th LO system uses a spatial closure and linear relation where the edge values are defined as the moments, i.e., $\langle \phi \rangle_{R,i} \equiv \phi_{R,i}$. Substituting the lumped closure into the right hand side of this equation gives back the original equation, i.e., $\langle q \rangle_{R,i} = \sigma_{s,i} \left(\langle \phi^{l+1/2} \rangle_{R,i} - \langle \phi^l \rangle_{R,i} \right)$. The same is true for standard LD.

Solution of the diffusion equation provides a correction to J and ϕ for the volumetric finite element unknowns. Because we are interested in the time-dependent solution, we need to accelerate the solution for the half-range fluxes, rather than just the scalar flux. We only accelerate the zeroth moment of the angular intensity, as the solution for J is inaccurate due to the approximations introduced. The error in

the scalar intensities are defined as, for the P-1 approximation,

$$\delta\phi^{\pm} = \frac{\delta\phi}{2} \pm \frac{3\delta J}{4} \tag{1.35}$$

Spatial moments are taken of $\delta\phi^{\pm}$, using the lumping notation of LD on the interior

$$\langle\delta\phi^{\pm}\rangle_L = \frac{1+\theta}{2}\delta\phi_L^{\pm} + \frac{1-\theta}{2}\delta\phi_R^{\pm} \tag{1.36}$$

$$\langle\delta\phi^{\pm}\rangle_R = \frac{1-\theta}{2}\delta\phi_L^{\pm} + \frac{1+\theta}{2}\delta\phi_R^{\pm}, \tag{1.37}$$

where Eq. (1.35) can be used to get in terms of $\delta\phi_{L/R}$, noting that we do not add on the first moment correction.

1.2.4 The WLA-DSA Accelerated Source Iteration Algorithm

We define the process of solving the diffusion like equations and mapping the error uknowns back onto the moment equations as the operator D^{-1}.

The source iteration with linear DSA procedure, for the m-th Newton iteration, is defined as

1. Evaluate effective scattering and absorption cross sections with $\{T_i^m : \ i = 1, 2, \ldots, N_c\}$.
2. Compute new scattering source $\mathbf{S}\psi^l$.
3. Perform sweeps to calculate $\psi^{l+1/2} = \mathbf{M}^{-1}\mathbf{S}\psi^l + \mathbf{M}^{-1}Q$
4. Compute DSA residual source $\sigma_s(\phi^{l+1/2} - \phi^l)$
5. Solve continuous DSA equations (i.e., Eq. (1.12)) for $\{\delta\phi_{i+1/2}^{l+1/2} : \ i = 0, 1, \ldots, N_c\}$
6. Map the continuous error onto the moment unknowns.
7. Add correction to the moment uknowns, $\phi^{\pm,l+1} = \phi^{\pm,l+1/2} + \delta\phi^{l+1/2}/2$.

8. If $\|\psi^{l+1} - \psi^{l}\| <$ tolerance, exit

9. Else: repeat steps 2–7.

1.3 GMRES Solution to the LO Equations

The source iteration procedure can be recast as an iterative solution to a linear maxtrix system. GMRES is an effective iterative solution method for asymmetric, sparse matrix equations [**?**]. Using operator notation, we manipulate the moment equations to form a matrix equation:

$$\left(\mathbf{I} - \mathbf{M}^{-1}\mathbf{S}\right)\psi = \mathbf{M}^{-1}Q, \tag{1.38}$$

where I is an identity matrix. The GMRES method is used to iteratively solve the above linear system. Krylov solutions to a linear system repeatedly apply the matrix operator to vectors, projecting the system onto a Krylov subspace [**?**]. Rather than forming the full matrix system, we can apply the operation of $\mathbf{S}$ and $\mathbf{M}^{-1}$ operators as defined in the previous sections to apply $(\mathbf{I} - \mathbf{M}^{-1}\mathbf{S})$ to the Krylov vectors.

The GMRES method will generally converge faster than the source iteration procedure [4]. However, as the system becomes scattering dominated, convergence will degrade. To improve the convergence rate, we precondition the GMRES system with the WLA-DSA method. The goal of preconditioning is to efficiently apply an operator to the equation that will approximate the inverse of the matrix operator. Left preconditioning [**?**] was applied to the above system. In matrix form, we write the preconditioned GMRES equations as

$$\left(\mathbf{I} + \mathbf{D}^{-1}\mathbf{S}\right)\left(\mathbf{I} - \mathbf{L}^{-1}\mathbf{S}\right)\psi = \left(\mathbf{I} + \mathbf{D}^{-1}\mathbf{S}\right)\mathbf{L}^{-1}Q. \tag{1.39}$$

The operation of $\left(\mathbf{I} + \mathbf{D}^{-1}\mathbf{S}\right)^{-1}$ is equivalent to the DSA procedure, adding the cor-

rection to a Krylov vector rather than applying to the scattering residual.

The opensource library `mgmres` was modified to implement the matrix-free version of the GMRES procedure. The infrastructure from the source iteration with DSA procedure is reused to provide the operation of $(\mathbf{I} + \mathbf{D}^{-1}\mathbf{S})(\mathbf{I} - \mathbf{M}^{-1}\mathbf{S})$ applied to the Krylov vectors returned from the GMRES solve.

Some modifications need to be made to solve these equations explicilty. Thus, the total algorithm for the GMRES solution is

1. Evaluate effective scattering and absorption cross sections with $\{T_i^m : \ i = 1, 2, \ldots, N_c\}$.
2. Compute new scattering source $\mathbf{S}\psi^l$.
3. Perform sweeps to calculate $\psi^{l+1/2} = \mathbf{M}^{-1}\mathbf{S}\psi^l + \mathbf{M}^{-1}Q$
4. Compute DSA residual source $\sigma_s(\phi^{l+1/2} - \phi^l)$
5. Solve continuous DSA equations (i.e., Eq. (1.12)) for $\{\delta\phi_{i+1/2}^{l+1/2} : \ i = 0, 1, \ldots, N_c\}$
6. Map the continuous error onto the moment unknowns.
7. If $\|\psi^{l+1} - \psi^l\| <$ tolerance, exit
8. Else: repeat steps 2–7.

Without preconditioning, the diffusion solve is simply removed.

1.4 Computational Results

It is noted we are not interested in measuring the reduction of computational time in this section because in 1D the LO equations can be directly solved efficiently. We are just interested in ensuring that the acceleration methods can reduce the number of scattering iterations sufficiently, including cases where inconsistencies in the LO equations are present.

Method	Iters./Time Step	Iters./Newton Step
SI		
SI-DSA		
GMRES		
GMRES-DSA		

We test our acceleration methods with three test problems. The first is the two material problem in Sec **??**. The second problem is a modification of the two material problem described in Sec. **??**. The problem specifications are the same as before, except for modifications to the material properties for $x > 0.5$ cm. In the right region, the parameters are $\sigma_a = 20,000$ cm^{-1}, $\sigma_s = 500$ cm^{-1} c_v =??? Jk g^{-1} keV^{-1}. The third test problem is the diffusion limit problem described in Sec. **??**.

1.4.1 Results for LD Spatial Discretization

We first test this problem with the source iteration using DSA to accelerate and compare to an unaccelerated SI solution. 3 batches of 10,000 particles are ran for each HO solve, one HO solver per time step. We tally the total number of source iterations per time step, over the two solves. We initialize the solution for the scattering iterations to zero at the beginning of each LO solve. We reduce the solve time for each to 1.0 sh. The time step begins at 0.001 sh and linearly increases by 15 % each time step to a maximum time step size of 0.01 sh. The larger time step size increases the amount of diffusive behavior.

REFERENCES

[1] Marvin Adams and William Martin. Diffusion synthetic acceleration of discontinuous finite element transport iterations. *Nuclear Science and Engineering*, 111(2):145–167, 1992.

[2] Edward W. Larsen and Jim E. Morel. Advances in discrete-ordinates methodology. *Nuclear Computational Science. Springer Netherlands*, pages 1–84, 2010.

[3] J. A. Fleck, Jr. and J. D. Cummings, Jr. An implicit monte carlo scheme for calculating time and frequency dependent nonlinear radiation transport. *J. Comput. Phys.*, 8(3):313–342, December 1971.

[4] Edward W Larsen and Jim E Morel. Advances in discrete-ordinates methodology. In *Nuclear Computational Science*, pages 1–84. Springer, 2010.

[5] Elmer Eugene Lewis and Warren F Miller. *Computational methods of neutron transport*. John Wiley and Sons, Inc., New York, NY, 1984.

[6] Ryan McClarren. Numerical methods in reactor analysis. Class Lecture Notes, 2015.

[7] J.E. Morel, T.A. Wareing, and K. Smith. Linear-Discontinuous Spatial Differencing Scheme for S_n Radiative Transfer Calculations. *Journal of Computational Physics*, 128:445–462, 1996.

[8] "Weston M. Stacey". *"Nuclear Reactor Physics"*. Wiley, 2007.

[9] T.A. Wareing. *Asymptotic diffusion accelerated discontinuous finite element methods for transport problems*. PhD thesis, Michigan, 1991.

[10] T.A. Wareing, E.W. Larsen, and M.L. Adams. Asymptotic Diffusion Accelerated Discontinuous Finite Element Schemes for the S_N Equations in Slab and X-Y

Geometries. In *International Topical Meeting on Advances in Mathematics, Computations, Reactor Physics*, volume 3, Pittsburgh, PA, 1991.

APPENDIX A

FIRST APPENDIX

A.1 Useful Moment Relations for LO Equations

There are several relations between various moment definitions that are useful in derivation and manipulation of the LO equations. The following are derived for $\phi(x)$, but can be applied to general moments of functions. The volumetric average terms can be eliminated in terms of the L and R moments from the relation $b_{L,i}(x) + b_{R,i}(x) = 1$.

$$\phi_i = \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} 1 \, \phi(x) \mathrm{d}x \tag{A.1}$$

$$= \frac{1}{h_i} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x)\phi(x)\mathrm{d}x + \int_{x_{i-1/2}}^{x_{i+1/2}} b_{R,i}(x)\phi(x)\mathrm{d}x \right) \tag{A.2}$$

$$= \frac{1}{2} \left(\langle \phi \rangle_{L,i} + \langle \phi \rangle_{R,i} \right) \tag{A.3}$$

A similar relation can be derived for the first moment in space as

$$\phi_{x,i} = \frac{3}{2} \left(\langle \phi \rangle_{L,i} + \langle \phi \rangle_{R,i} \right) \tag{A.4}$$

The above relations can be inverted to derived a relation for the L and R moments in terms of the slope and average moments. These moment expressions are defined purely in terms of integrals, and are independent of the chosen spatial representation

Once a linear relation on the interior has been assumed, there are other useful closures that can be derived. The standard linear interpolatory expansion, for the

positive half-range, is restated here:

$$\phi^+(x) = \phi^+_{L,i} b_{L,i}(x) + \phi^+_{R,i} b_{R,i}(x) \tag{A.5}$$

Using this expansion, one can derive a relation between the outflow from a cell and the hat function moments that is equivalent to the standard LDFE Galerkin method:

$$\phi^+_{i,R} = 2\langle\phi\rangle^+_{R,i} - \langle\phi\rangle^+_{L,i} \tag{A.6}$$

this linear relation also defines the value for $\phi^+_{i,L}$.

To eliminate the LO unknowns in a manner that is equivalent to lumping the discrete system, the following expression can be used for the outflow from a cell

$$\phi^+_{i+1/2} = \phi^+_i + \frac{\phi^+_x}{3}, \tag{A.7}$$

which in terms of the hat function moments is equivalent to $\phi^+_{i+1/2} = \langle\phi\rangle^+_{R,i}$. Inserting this expression into Eq. (**??**), and using the same definition for the linear representation over the interior of $\phi^+_{i+1/2}(x) = \phi_{L,i} b_{L,i}(x) + \phi_{R,i} b_{R,i}(x)$, will produce an equivalent set of unknowns as a linear discontinuous method with a lumped representation for the radiation. The temperature equation must be independently lumped. This relation preserves the average within a cell but modifies the first moment.

A similar expression produces a lumped-equivalent representation on the interior of the cell:

$$\phi^+_{i,R} = \phi^+_i + \frac{\phi^+_x}{3}, \tag{A.8}$$

The moment equations are not modified by using this expression, however the interpretation of the moments as a linear representation over the cell has been altered.

This allows for us to ensure a lumped representation on the interior while still using the HO solution to eliminate the outflow from the equations.

A.2 Newtons Method for the LO Equations

Because we have only considered problems with constant densities and heat capacities, the linearization described below is in terms of temperature T rather than material internal energy, for simplicity. However, the linearization can be formed in terms of internal energy to apply this method to a general equation of state.

To formulate the Newton iterations, the Planckian source is linearized in the material and radiation equations (Eq. (**??**) & Eq. (**??**)). Application of the first order Taylor expansion in time to the implicit emission source $B(T^{n+1})$, about some temperature T^* at some time $t^* \in [t^n, t^{n+1}]$, yields

$$\sigma_a^{n+1} acT^{4,n+1} \simeq \sigma_a^* ac \left[T^{*4} + (T^{n+1} - T^*)4T^{*3}\right] \tag{A.9}$$

where $\sigma_a^* \equiv \sigma_a(T^*)$. Substitution of this expression into Eq. (**??**) yields

$$\rho c_v \left(\frac{T^{n+1} - T^n}{\Delta t}\right) = \sigma_a^* \phi^{n+1} - \sigma_a^* ac \left[T^{*4} + (T^{n+1} - T^*)4T^{*3}\right]. \tag{A.10}$$

Algebraic manipulation of this equation yields an expression for $T^{n+1} - T^*$:

$$\left(T^{n+1} - T^*\right) = \frac{\frac{\sigma_a^* \Delta t}{\rho c_v}\left[\phi^{n+1} - acT^{*4}\right] + (T^n - T^*)}{1 + \sigma_a^* ac\Delta t \frac{4T^{*3}}{\rho c_v}}.$$

This expression is substituted back into Eq. (A.9) to form an explicit approximation for the emission source at t^{n+1} as

$$\sigma_a acT^{4,n+1} \simeq \sigma_a^*(1 - f^*)\phi^{n+1} + f^* \sigma_a^* acT^{4,n} + \rho c_v \frac{1 - f^*}{\Delta t}(T^n - T^*) \tag{A.11}$$

where $f^* = [1 + \sigma_a^* c\Delta t 4aT^{*3}/(\rho c_v)]^{-1}$ is often referred to as the Fleck factor [3].

Next, the above equation must be spatially discretized. Application of the L spatial moment yields

$$\langle \sigma_a^* acT^{4,n+1} \rangle_{L,i} = \sigma_{ai}^*(1 - f_i^*)\langle \phi^{n+1} \rangle_{L,i} + f_i^* \sigma_{ai}^* ac \left(\frac{2}{3} T_{L,i}^{4,n} + \frac{1}{3} T_{R,i}^{4,n} \right) \\ \rho_i c_{vi} \frac{1 - f_i^*}{\Delta t} \left[\frac{2}{3} \left(T_{L,i}^n - T_{L,i}^* \right) + \frac{1}{3} \left(T_{R,i}^n - T_{R,i}^* \right) \right], \quad \text{(A.12)}$$

where $T^{4,n}$ and T^n have been assumed LD and f^* is assumed constant over a cell, i.e., $f_i^* \equiv \sigma_a(T_i^*)$. The error introduced by a constant f^* approaches zero as the non-linearity is converged because T^* approaches T^{n+1}. Based on an estimate for T^*, Eq. (A.12) is an expression for the Planckian emission source in the radiation moment equations with an additional effective scattering source. A similar expression can be derived for $\langle \sigma_{a,i} acT^4 \rangle_R$ and the right moment equations. The expressions for the emissions source is substituted into the radiation moment equations (Eq. (**??**)–(**??**)) to produce a linear system of equations for the new radiation intensity moments.

Once the linear equations have been solved for new radiation moments, new temperature unknowns can be estimated. To conserve energy, the same linearization and discretizations used to solve the radiation equation must be used in the material energy equation. Substitution of Eq. (A.12) into the material energy L moment equation ultimately yields

$$\frac{2}{3} T_{L,i}^{n+1} + \frac{1}{3} T_{R,i}^{n+1} = \frac{f_i^* \sigma_{ai}^* \Delta t}{\rho c_v} \left[\langle \phi^{n+1} \rangle_{L,i} - ac \left(\frac{2}{3} T_{L,i}^{4,n} + \frac{1}{3} T_{R,i}^{4,n} \right) \right] + \\ (1 - f_i^*) \left(\frac{2}{3} T_{L,i}^* + \frac{1}{3} T_{R,i}^* \right) + f \left(\frac{2}{3} T_{L,i}^n + \frac{1}{3} T_{R,i}^n \right) \quad \text{(A.13)}$$

A similar expression is produced for the R moment equation. This produces a local

matrix equation to solve for new T unknowns. If both the radiation and temperature unknowns are lumped, this matrix becomes diagonalized.

Based on these equations, the algorithm for solving the LO equations, with iteration index l, is defined as

1. Initialize T unknowns using T^n or the last estimate of T^{n+1} from previous LO solve

2. Build the LO system based on the effective scattering $(1 - f^l)$ and emission terms evaluated using T^l.

3. Solve the linearized LO system to produce an estimate for $\phi^{n+1,l}$.

4. Evaluate a new estimate of T^{n+1} unknowns using Eq. (A.13).

5. $T^* \leftarrow \tilde{T}^{n+1}$.

6. Repeat 2-5 until $(T^{n+1,k})^4$ and $\phi^{n+1,k}$ are converged.

Figure A.1: TAMU figure

APPENDIX B

SECOND APPENDIX WITH A LONGER TITLE - MUCH LONGER IN FACT

Text for the Appendix follows.

Figure B.1: TAMU figure

B.1 Appendix Section