1. ACCELERATED ITERATIVE SOLUTION TO THE LO EQUATIONS

As described in Sec. **??**, the fully-discretized, S_2-like LO equations cannot be efficiently directly inverted in higher spatial dimensions. To demonstrate a possible path forward in higher dimensions, we have investigated the use of a standard source iteration scheme to solve the LO equations. As material properties become more diffusive (e.g., c_v is small and σ_a is large), the effective scattering cross sections becomes large. This results in a spectral radius of the source iterations that approaches unity [5]. These regimes are typical in TRT simulations, so an acceleration method for an iterative solution is critical. e will accelerate the source iterations with a nearly-consistent diffusion synthetic acceleration (DSA) method [8, 7].

We have also recast the DSA method as a preconditioner to an iterative Krylov solution [2] of the LO equations if acceleration degrades. Generally, Krylov methods mitigate acceleration losses due to inconsistencies in the acceleration method. In higher dimensions, the use of a Krylov method is necessary for effective acceleration for nearly-consistent acceleration methods such as WLA in problems with mixed optical thicknesses [2], e.g., typical radiative transfer problems. We would apply the preconditioned-Krylov approach to allow for the use of lumped DSA equations as a preconditioner, with the LO equations using one of the other fixup approaches detailed in the previous section. It is noted we are not interested in measuring the reduction of computational time because in 1D the LO equations can be directly solved efficiently. We are just interested in ensuring that DSA or a preconditioned-Krylov methods can reduce the number of scattering iterations sufficiently, including cases where inconsistencies in the LO equations are present.

1.1 Source Iteration Solution to the LO Equations

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The linearized LO equations can be solved with source iteration. In the source iteration process the scattering source $\sigma_{s,\text{eff}}\phi(x)$ is lagged, which uncouples unknowns between the two half-ranges. This produces a lower-triangular system where the spatial unknowns can be determined sequentially along the two directions of flow via a standard sweeping procedure [4, 5]. The newly computed half-range intensities can be used to compute the scattering source for the next iteration. This process is repeated until convergence.

In operator notation, we denote the source iteration process as

$$M \tag{1.1}$$

The scattering must be inverted within each Newton iteration.

1.2 Linear Diffusion Synthetic Acceleration

A form of DSA referred to as the WLA method is used to accelerate the source iterations [8, 7]. Between each source iteration, a residual equation is formed that provides the error in the current scattering iteration. The DSA method uses an approximate, lower-order operator to estimate the error in the zeroth angular moment of the intensity, i.e., the mean intensity ϕ. The DSA equations can be more efficiently solved than the S_2-like sweeps that are being accelerated, but will accurately resolve the slowly converging diffusive error modes. It is important for the spatial discretization of the DSA equations to be closely related to the discretization of the LO equations for the acceleration to be effective. The WLA method first solves a spatially-continuous discretization of the diffusion equation for the iterative error on

faces. The error on the faces is then mapped onto the volumetric moment unknowns via a LD discretization of diffusion equation [8]. The LD mapping resolves issues that would occur in optically-thick cells, while the continuous diffusion equation is accurate in the EDL where acceleration is important [1].

In the following section we will derived the WLA method. To simplify notation, we derive the general continuous diffusion equation necessary for the acceleration method. We then state the final equations for the acceleration method. Within a Newton iteration, the source iteration to be solved is simply a steady state transport equation with an isotropic source and scattering, i.e.,

$$\mu \frac{\partial I}{\partial x} + \sigma_t I = \frac{\sigma_s}{2}\phi(x) + q(x,\mu). \tag{1.2}$$

1.2.1 Forming a Continuous Diffusion Equation

Beginning with the P_1 equations for a steady-state problem [4, 7]

$$\frac{\partial J}{\partial x} + \sigma_a \phi = Q \tag{1.3}$$

$$\sigma_t J + \frac{1}{3}\frac{\partial \phi}{\partial x} = 0 \tag{1.4}$$

$$\tag{1.5}$$

spatial finite element moments are taken. The spatial moments are defined as

$$\langle \cdot \rangle_L = \frac{2}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{d}x \, b_{L,i}(x) \, (\cdot) \tag{1.6}$$

$$\langle \cdot \rangle_R = \frac{2}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathrm{d}x \, b_{R,i}(x) \, (\cdot) \, . \tag{1.7}$$

where $b_{L,i}(x) = (x_{i+1/2} - x)/h_i$ and $b_{R,i}(x) = (x - x_{i-1/2})/h_i$. The scalar flux ϕ will ultimately be assumed continuous at faces to produce a diffusion-like discretization. The mean intensity is assumed linear on the interior of the cell, i.e., $\phi(x) = \phi_L b_L(x) + \phi_R b_R(x)$, for $x \in (x_{i-1/2}, x_{i+1/2})$. Taking the left moment, evaluating integrals, and rearranging yields

$$J_i - J_{i-1/2} + \frac{\sigma_{a,i} h_i}{2} \left(\frac{2}{3}\phi_{L,i} + \frac{1}{3}\phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{L,i} \ , \tag{1.8}$$

where J_i is the average of the current over the cell. The moments of q are not simplified to be compatible with the LO moment equations. For the R moment

$$J_{i+1/2} - J_i + \frac{\sigma_{a,i} h_i}{2} \left(\frac{2}{3}\phi_{L,i} + \frac{1}{3}\phi_{R,i} \right) = \frac{h_i}{2} \langle q \rangle_{R,i} \ . \tag{1.9}$$

The equation for the L moment is evaluated for cell $i+1$ and added to the R moment equation evaluated at i. The current is assumed continuous at $i + 1/2$ to eliminate the face current from the system. The sum of the two equations becomes

$$J_{i+1} - J_i + \frac{\sigma_{a,i+1} h_{i+1}}{2} \left(\frac{2}{3}\phi_{L,i+1} + \frac{1}{3}\phi_{R,i+1} \right) + \frac{\sigma_{a,i} h_i}{2} \left(\frac{1}{3}\phi_{L,i} + \frac{2}{3}\phi_{R,i} \right) = \frac{h}{2} \left(\langle q \rangle_{L,i+1} + \langle q \rangle_{R,i} \right). \tag{1.10}$$

The scalar flux is assumed continuous at each face, i.e., $\phi_{L,i+1} = \phi_{R,i} \equiv \phi_{i+1/2}$. We then approximate the cell-averaged currents with Fick's law as

$$J_i = -D_i \frac{\phi_{i+1/2} - \phi_{i-1/2}}{h_i}. \tag{1.11}$$

Combining the definition and rearranging yields the following discrete diffusion equation:

$$\left(\frac{\sigma_{a,i+1}h_{i+1}}{6} - \frac{D_{i+1}}{h_{i+1}}\right)\phi_{i+3/2} + \left(\frac{D_{i+1}}{h_{i+1}} + \frac{D_i}{h_i} + \frac{\sigma_{a,i+1}h_{i+1}}{3} + \frac{\sigma_{a,i}h_i}{3}\right)\phi_{i+1/2} + \left(\frac{\sigma_{a,i}h_i}{6} - \frac{D_i}{h_i}\right)\phi_{i-1/2} = \frac{h_{i+1}}{2}\langle q\rangle_{L,i+1} + \frac{h_i}{2}\langle q\rangle_{R,i} \ . \quad (1.12)$$

This system can be solved to get ϕ at each face. To allow for the use of lumped or standard LD in these equations, we introduce the factor θ, with $\theta = 1/3$ for standard LD, and $\theta = 1$ for lumped LD. The diffusion equation becomes

$$\left(\frac{\sigma_{a,i+1}h_{i+1}}{4}(1-\theta) - \frac{D_{i+1}}{h_{i+1}}\right)\phi_{i+3/2} + \left(\frac{D_{i+1}}{h_{i+1}} + \frac{D_i}{h_i} + \left(\frac{1+\theta}{2}\right)\left[\frac{\sigma_{a,i+1}h_{i+1}}{2} + \frac{\sigma_{a,i}h_i}{2}\right]\right)\phi_{i+1/2} + \left(\frac{\sigma_{a,i}h_i}{4}(1-\theta) - \frac{D_i}{h_i}\right)\phi_{i-1/2} = \frac{h_{i+1}}{2}\langle q\rangle_{L,i+1} + \frac{h_i}{2}\langle q\rangle_{R,i} \ . \quad (1.13)$$

1.2.1.1 Boundary Conditions

The LO system exactly satisfies the inflow boundary conditions, therefore we choose a vacuum boundary condition for the left-most cell. The equation for the left moment at the first cell is given by

$$J_1 - J_{1/2} + \frac{\sigma_{a,i}h_i}{2}\left(\frac{1+\theta}{2}\phi_{L,i} + \frac{1-\theta}{2}\phi_{R,i}\right) = \frac{h_i}{2}\langle q\rangle_{L,i} \ , \quad (1.14)$$

The Marshak boundary condition for the vacuum inflow at face $x_{1/2}$ is given as

$$J^+_{1/2} = 0 = \frac{\phi_{1/2}}{4} + \frac{J_{1/2}}{2}, \quad (1.15)$$

which can be solved for $J_{1/2}$. Substitution of the above equation and Eq. (1.11) into Eq. (1.14) gives

$$\left(\frac{1}{2}+\sigma_{a,1}h_1\frac{1+\theta}{4}-\frac{D_1}{h_1}\right)\phi_{1/2}+\left(\sigma_{a,1}h_1\frac{1-\theta}{4}-\frac{D_1}{h_1}\right)\phi_{3/2}=\frac{h_i}{2}\langle q\rangle_{L,1} \tag{1.16}$$

a similar expression can be derived for the last cell.

1.2.2 Mapping Solution onto LD Unknowns

Solution of the continuous diffusion equation in the previous section provides correction values for ϕ on the faces, denoted as $\phi^C_{i+1/2}$. We now need to determine the correction these results provide for the LD representation of ϕ. To do this, first we take the L and R finite element moments of the P_1 equations. A LDFE dependence is assumed on the interior of the cell for J and ϕ. Taking moments of Eq. (1.3) and simplifying yields

$$J_{i+1/2}-\frac{J_{L,i}+J_{R,i}}{2}+\frac{\sigma_{a,i}h_i}{2}\left(\frac{1}{3}\phi_{L,i}+\frac{2}{3}\phi_{R,i}\right)=\frac{h_i}{2}\langle q\rangle_{R,i} \tag{1.17}$$

$$\frac{J_{L,i}+J_{R,i}}{2}-J_{i-1/2}+\frac{\sigma_{a,i}h_i}{2}\left(\frac{2}{3}\phi_{L,i}+\frac{1}{3}\phi_{R,i}\right)=\frac{h_i}{2}\langle q\rangle_{L,i} \tag{1.18}$$

The moment equations for Eq. (1.4) are

$$\frac{1}{3}\left(\phi_{i+1/2}-\frac{\phi_{i,L}+\phi_{i,R}}{2}\right)+\frac{\sigma_{t,i}h_i}{2}\left(\frac{1}{3}J_{L,i}+\frac{2}{3}J_{R,i}\right)=0 \tag{1.19}$$

$$\frac{1}{3}\left(\frac{\phi_{i,L}+\phi_{i,R}}{2}-\phi_{i-1/2}\right)+\frac{\sigma_{t,i}h_i}{2}\left(\frac{2}{3}J_{L,i}+\frac{1}{3}J_{R,i}\right)=0 \tag{1.20}$$

Using similar equation for all the inflow currents, the balance equations for ϕ become The face terms $J_{i\pm1/2}$ and $\phi_{i\pm1/2}$ need to be eliminated from the system. The scalar flux is assumed to be the value provided by the continuous diffusion solution

at each face, i.e., $\phi_{i\pm1/2} = \phi^C_{i\pm1/2}$. The currents are decomposed into half-range values to decouple the equations between cells. At $x_{i+1/2}$, the current is composed as $J_{i+1/2} = J^+_{i+1/2} - J^-_{i+1/2}$, where $+$ and $-$ denote the positive and negative half ranges of μ, respectively. Typically, the incoming current $J^-_{i+1/2}$ is upwinded from cell $i+1$. However, we approximate the incoming current based on $\phi^C_{i+1/2}$. The P_1 approximation provides the following relation

$$\phi = 2(J^+ + J^-). \tag{1.21}$$

At $x_{i+1/2}$, the above expression is solved for the incoming current $J^-_{i+1/2}$. The total current becomes, with $\phi_{i+1/2} = \phi^C_{i+1/2}$,

$$J_{i+1/2} = J^+_{i+1/2} - J^-_{i+1/2} = 2J^+_{i+1/2} - \frac{\phi^C_{i+1/2}}{2}, \tag{1.22}$$

In the positive direction, at the right face, the values of ϕ and J are based on the LD representation within the cell at that face, i.e., $\phi_{R,i}$ and $J_{R,i}$. The standard P_1 approximation for the half-range currents and fluxes are used[6], i.e.,

$$J^\pm = \frac{\gamma\phi}{2} \pm \frac{J}{2}, \tag{1.23}$$

where γ accounts for the difference between the LO parameters and the true P_1 approximation. Thus, for the right face and positive half-range,

$$J^+_{i+1/2} = \frac{\gamma}{2}\phi_{i,R} + \frac{J_{i,R}}{2} \tag{1.24}$$

A similar expression can be derived for $x_{i-1/2}$. The total currents at each face are

thus

$$J_{i+1/2} = \gamma\phi_{i,R} + J_{i,R} - \frac{\phi^C_{i+1/2}}{2} \tag{1.25}$$

$$J_{i-1/2} = \frac{\phi^C_{i-1/2}}{2} - \gamma\phi_{i,L} + J_{i,L} \tag{1.26}$$

Substitution of these results back into the LD balance equations and introduction of the lumping notation yields the final equations

$$\left(\gamma\phi_{i,R} + J_{i,R} - \frac{\phi^C_{i+1/2}}{2}\right) - \frac{J_{L,i} + J_{R,i}}{2} + \frac{\sigma_{a,i}h_i}{2}\left(\frac{(1-\theta)}{2}\phi_{L,i} + \frac{(1+\theta)}{2}\phi_{R,i}\right) = \frac{h_i}{2}\langle q\rangle_{R,i} \tag{1.27}$$

$$\frac{J_{L,i} + J_{R,i}}{2} - \left(\frac{\phi^C_{i-1/2}}{2} - \gamma\phi_{i,L} + J_{i,L}\right) + \frac{\sigma_{a,i}h_i}{2}\left(\frac{(1+\theta)}{2}\phi_{L,i} + \frac{(1-\theta)}{2}\phi_{R,i}\right) = \frac{h_i}{2}\langle q\rangle_{L,i} \tag{1.28}$$

$$\frac{1}{3}\left(\phi^C_{i+1/2} - \frac{\phi_{i,L} + \phi_{i,R}}{2}\right) + \frac{\sigma_{t,i}h_i}{2}\left(\frac{(1-\theta)}{2}J_{L,i} + \frac{(1+\theta)}{2}J_{R,i}\right) = 0 \tag{1.29}$$

$$\frac{1}{3}\left(\frac{\phi_{i,L} + \phi_{i,R}}{2} - \phi^C_{i-1/2}\right) + \frac{\sigma_{t,i}h_i}{2}\left(\frac{(1+\theta)}{2}J_{L,i} + \frac{(1-\theta)}{2}J_{R,i}\right) = 0. \tag{1.30}$$

The above equations are completely local to each cell and fully defined. The system can be solved for the the desired unknowns $\phi_{i,L}$, $\phi_{i,R}$, $J_{i,L}$, and $J_{i,R}$.

1.2.3 DSA Source Definition

The above discretization procedure is used to determine the error in the scalar flux. The sources $\langle q\rangle_{L/R}$ thus need to be defined. They are simply the residual in

the scattering iterations, given by

$$q = \sigma_s \left(\phi^{l+1/2} - \phi^l \right). \tag{1.31}$$

The spatial moments are straight forward:

$$\langle q \rangle_{L,i} = \sigma_{s,i} \left(\langle \phi^{l+1/2} \rangle_{L,i} - \langle \phi^l \rangle_{L,i} \right) \tag{1.32}$$

The above equation is valid for lumping or standard LD. This is because the LO moments are defined differently for LLD or LD, resulting in equations that are consistent. For instance, for lumped LD, the LO system uses the spatial closure that the edge value is defined as the moment, i.e., $\langle \phi \rangle_{R,i} \equiv \phi_{R,i}$. For a standard lumped source, we desire the right equation to have $\langle q \rangle_{R,i} = \sigma_s(\phi_{R,i}^{l+1/2} - \phi_{R,i}^l)$. Substituting the lumped closure into the right hand side of this equation gives back the original equation, i.e., $\langle q \rangle_{R,i} = \sigma_{s,i} \left(\langle \phi^{l+1/2} \rangle_{R,i} - \langle \phi^l \rangle_{R,i} \right)$. The same is true for standard LD.

1.2.4 Updating the LO Unkowns

We now have a correction to J and ϕ for the volumetric finite element unknowns. Because we are interested in the time-dependent solution, we need to accelerate the solution for the half-range fluxes, rather than just the scalar flux. We only accelerate the zeroth moment of the angular intensity. The error in the scalar intensities are defined as

$$\delta\phi^{\pm} = \frac{\delta\phi}{2} \pm \frac{3\delta J}{4} \tag{1.33}$$

Spatial moments are taken of $\delta\phi^{\pm}$, using the lumping notation of LD on the interior

$$\langle \delta\phi^{\pm} \rangle_L = \frac{1+\theta}{2} \delta\phi_L^{\pm} + \frac{1-\theta}{2} \delta\phi_R^{\pm} \tag{1.34}$$

$$\langle \delta\phi^{\pm} \rangle_R = \frac{1-\theta}{2}\delta\phi_L^{\pm} + \frac{1+\theta}{2}\delta\phi_R^{\pm}, \tag{1.35}$$

where Eq. (1.33) can be used to get in terms of $\delta\phi_{L/R}$ and $\delta J_{L/R}$. It is noted that for consistency, the updates to the moments depend on the lumping notation, even though the sources are defined the same in both cases.

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APPENDIX A

FIRST APPENDIX

A.1 Useful Moment Relations for LO Equations

There are several relations between various moment definitions that are useful in derivation and manipulation of the LO equations. The following are derived for $\phi(x)$, but can be applied to general moments of functions. The volumetric average terms can be eliminated in terms of the L and R moments from the relation $b_{L,i}(x) + b_{R,i}(x) = 1$.

$$\phi_i = \frac{1}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} 1 \, \phi(x) \mathrm{d}x \tag{A.1}$$

$$= \frac{1}{h_i} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x)\phi(x)\mathrm{d}x + \int_{x_{i-1/2}}^{x_{i+1/2}} b_{R,i}(x)\phi(x)\mathrm{d}x \right) \tag{A.2}$$

$$= \frac{1}{2} \left(\langle \phi \rangle_{L,i} + \langle \phi \rangle_{R,i} \right) \tag{A.3}$$

A similar relation can be derived for the first moment in space as

$$\phi_{x,i} = \frac{3}{2} \left(\langle \phi \rangle_{L,i} + \langle \phi \rangle_{R,i} \right) \tag{A.4}$$

The above relations can be inverted to derived a relation for the L and R moments in terms of the slope and average moments. These moment expressions are defined purely in terms of integrals, and are independent of the chosen spatial representation

Once a linear relation on the interior has been assumed, there are other useful closures that can be derived. The standard linear interpolatory expansion, for the

positive half-range, is restated here:

$$\phi^+(x) = \phi^+_{L,i} b_{L,i}(x) + \phi^+_{R,i} b_{R,i}(x) \tag{A.5}$$

Using this expansion, one can derive a relation between the outflow from a cell and the hat function moments that is equivalent to the standard LDFE Galerkin method:

$$\phi^+_{i,R} = 2\langle\phi\rangle^+_{R,i} - \langle\phi\rangle^+_{L,i} \tag{A.6}$$

this linear relation also defines the value for $\phi^+_{i,L}$.

To eliminate the LO unknowns in a manner that is equivalent to lumping the discrete system, the following expression can be used for the outflow from a cell

$$\phi^+_{i+1/2} = \phi^+_i + \frac{\phi^+_x}{3}, \tag{A.7}$$

which in terms of the hat function moments is equivalent to $\phi^+_{i+1/2} = \langle\phi\rangle^+_{R,i}$. Inserting this expression into Eq. (**??**), and using the same definition for the linear representation over the interior of $\phi^+_{i+1/2}(x) = \phi_{L,i} b_{L,i}(x) + \phi_{R,i} b_{R,i}(x)$, will produce an equivalent set of unknowns as a linear discontinuous method with a lumped representation for the radiation. The temperature equation must be independently lumped. This relation preserves the average within a cell but modifies the first moment.

A similar expression produces a lumped-equivalent representation on the interior of the cell:

$$\phi^+_{i,R} = \phi^+_i + \frac{\phi^+_x}{3}, \tag{A.8}$$

The moment equations are not modified by using this expression, however the interpretation of the moments as a linear representation over the cell has been altered.

This allows for us to ensure a lumped representation on the interior while still using the HO solution to eliminate the outflow from the equations.

A.2 Newtons Method for the LO Equations

Because we have only considered problems with constant densities and heat capacities, the linearization described below is in terms of temperature T rather than material internal energy, for simplicity. However, the linearization can be formed in terms of internal energy to apply this method to a general equation of state.

To formulate the Newton iterations, the Planckian source is linearized in the material and radiation equations (Eq. (**??**) & Eq. (**??**)). Application of the first order Taylor expansion in time to the implicit emission source $B(T^{n+1})$, about some temperature T^* at some time $t^* \in [t^n, t^{n+1}]$, yields

$$\sigma_a^{n+1} acT^{4,n+1} \simeq \sigma_a^* ac \left[T^{*4} + (T^{n+1} - T^*)4T^{*3}\right] \tag{A.9}$$

where $\sigma_a^* \equiv \sigma_a(T^*)$. Substitution of this expression into Eq. (**??**) yields

$$\rho c_v \left(\frac{T^{n+1} - T^n}{\Delta t}\right) = \sigma_a^* \phi^{n+1} - \sigma_a^* ac \left[T^{*4} + (T^{n+1} - T^*)4T^{*3}\right]. \tag{A.10}$$

Algebraic manipulation of this equation yields an expression for $T^{n+1} - T^*$:

$$\left(T^{n+1} - T^*\right) = \frac{\frac{\sigma_a^* \Delta t}{\rho c_v}\left[\phi^{n+1} - acT^{*4}\right] + (T^n - T^*)}{1 + \sigma_a^* ac\Delta t \frac{4T^{*3}}{\rho c_v}}.$$

This expression is substituted back into Eq. (A.9) to form an explicit approximation for the emission source at t^{n+1} as

$$\sigma_a acT^{4,n+1} \simeq \sigma_a^*(1 - f^*)\phi^{n+1} + f^* \sigma_a^* acT^{4,n} + \rho c_v \frac{1 - f^*}{\Delta t}(T^n - T^*) \tag{A.11}$$

where $f^* = [1 + \sigma_a^* c\Delta t 4aT^{*3}/(\rho c_v)]^{-1}$ is often referred to as the Fleck factor [3].

Next, the above equation must be spatially discretized. Application of the L spatial moment yields

$$\langle \sigma_a^* acT^{4,n+1} \rangle_{L,i} = \sigma_{ai}^*(1 - f_i^*)\langle \phi^{n+1} \rangle_{L,i} + f_i^* \sigma_{ai}^* ac \left(\frac{2}{3} T_{L,i}^{4,n} + \frac{1}{3} T_{R,i}^{4,n} \right) \\ \rho_i c_{vi} \frac{1 - f_i^*}{\Delta t} \left[\frac{2}{3} \left(T_{L,i}^n - T_{L,i}^* \right) + \frac{1}{3} \left(T_{R,i}^n - T_{R,i}^* \right) \right], \quad \text{(A.12)}$$

where $T^{4,n}$ and T^n have been assumed LD and f^* is assumed constant over a cell, i.e., $f_i^* \equiv \sigma_a(T_i^*)$. The error introduced by a constant f^* approaches zero as the non-linearity is converged because T^* approaches T^{n+1}. Based on an estimate for T^*, Eq. (A.12) is an expression for the Planckian emission source in the radiation moment equations with an additional effective scattering source. A similar expression can be derived for $\langle \sigma_{a,i} acT^4 \rangle_R$ and the right moment equations. The expressions for the emissions source is substituted into the radiation moment equations (Eq. (**??**)–(**??**)) to produce a linear system of equations for the new radiation intensity moments.

Once the linear equations have been solved for new radiation moments, new temperature unknowns can be estimated. To conserve energy, the same linearization and discretizations used to solve the radiation equation must be used in the material energy equation. Substitution of Eq. (A.12) into the material energy L moment equation ultimately yields

$$\frac{2}{3} T_{L,i}^{n+1} + \frac{1}{3} T_{R,i}^{n+1} = \frac{f_i^* \sigma_{ai}^* \Delta t}{\rho c_v} \left[\langle \phi^{n+1} \rangle_{L,i} - ac \left(\frac{2}{3} T_{L,i}^{4,n} + \frac{1}{3} T_{R,i}^{4,n} \right) \right] + \\ (1 - f_i^*) \left(\frac{2}{3} T_{L,i}^* + \frac{1}{3} T_{R,i}^* \right) + f \left(\frac{2}{3} T_{L,i}^n + \frac{1}{3} T_{R,i}^n \right) \quad \text{(A.13)}$$

A similar expression is produced for the R moment equation. This produces a local

matrix equation to solve for new T unknowns. If both the radiation and temperature unknowns are lumped, this matrix becomes diagonalized.

Based on these equations, the algorithm for solving the LO equations, with iteration index l, is defined as

1. Initialize T unknowns using T^n or the last estimate of T^{n+1} from previous LO solve

2. Build the LO system based on the effective scattering $(1 - f^l)$ and emission terms evaluated using T^l.

3. Solve the linearized LO system to produce an estimate for $\phi^{n+1,l}$.

4. Evaluate a new estimate of T^{n+1} unknowns using Eq. (A.13).

5. $T^* \leftarrow \tilde{T}^{n+1}$.

6. Repeat 2-5 until $(T^{n+1,k})^4$ and $\phi^{n+1,k}$ are converged.

Figure A.1: TAMU figure

APPENDIX B

SECOND APPENDIX WITH A LONGER TITLE - MUCH LONGER IN FACT

Text for the Appendix follows.

Figure B.1: TAMU figure

B.1 Appendix Section