A High-Order Low-Order Algorithm with Exponentially-Convergent Monte Carlo for Thermal Radiative Transfer

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Outline

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Overview

- We are interested in modeling thermal radiation transport in the high-energy density physics regime
  - Temperatures on order of 10^6 K or more
  - Significant energy and momentum may be exchanged with material

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  - Temperatures on order of 10^6 K or more
  - Significant energy and momentum may be exchanged with material
- Radiative transfer simulations important in modeling:
  - Material under extreme conditions
  - Inertial confinement fusion
  - Supernovae and other astrophysical phenomena.

The Grey TRT Equations

- The 1D, grey equations

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial x} + \sigma_t I(x,\mu) = \frac{1}{2}\sigma_a acT^4(x),$$
$$C_v\frac{\partial T}{\partial t} = \sigma_a\left(\phi(x) - acT^4\right)$$
$$\phi(x) = \int_{-1}^{1} I(x,\mu)\mathrm{d}\mu.$$

- Fundamental unknowns are the radiation intensity $I(x,\mu)$ and material temperature $T(x)$

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- Fundamental unknowns are the radiation intensity $I(x,\mu)$ and material temperature $T(x)$
- Absorption cross section (σ_a) can be a strong function of T
- Equations are nonlinear and may be tightly coupled

The Implicit Monte Carlo Method

- Equations often solved with Monte Carlo (MC) via the Implicit Monte Carlo (IMC) method
- The material energy equation is linearized over a time step and eliminated from the system
  - Results in linear transport equation with a discrete emission and scattering terms
  - Continuous integration of time derivative

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  - Results in linear transport equation with a discrete emission and scattering terms
  - Continuous integration of time derivative
- Drawbacks
  - Effective scattering cross section can be very large
    - Local acceleration methods based on discrete diffusion approxmiations
  - Nonlinearities not converged
  - Reconstruction of linear source shape in cell

An alternative High-Order Low-Order approach

Basic Idea

Solve a fully non-linear, low-order (LO) system, that preserves a high-order (HO) solution from efficent MC simulations

- The LO system consists of space-angle moment equations, formed over a fixed finite-element (FE) spatial mesh

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  - Non-linear temperature dependence converged with Newton's method
  - **Output:** a linear-discontinuous (LD) FE representation of scattering and emission source

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  - **Output:** a linear-discontinuous (LD) FE representation of scattering and emission source
- The HO system is a fixed-source, pure-absorber transport problem
  - No effective scattering events
  - Solved with ECMC for efficient reduction of statisical noise
  - **Output:** angular consistency terms

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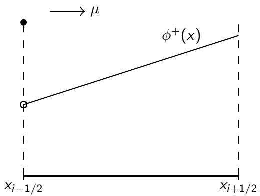
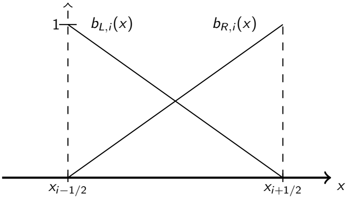
LO Discretization & Space-Angle Moments

- The LO system is "S_2-like" equations (similar to [Wolters 2013]), with a backward Euler discretization in time
- Linear discontinuous (LD) FE in space for ϕ, T, and $T^4(x)$





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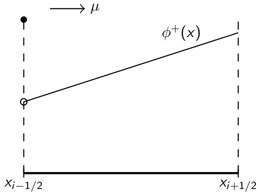

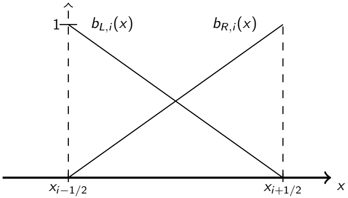

- Half-range integrals over $+$ and $-$ μ

- *Examples* of moments:

Spatial: left basis

$$\langle \cdot \rangle_{L,i} = \frac{2}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x)(\cdot)\mathrm{d}x$$

Angular: positive flow

$$\phi^+(x) = \int_0^1 I(x,\mu)\mathrm{d}\mu$$

Forming the LO System

1. Applying moments to the TRT equations yields 4 radiation equations, and 2 material energy equations, per cell

2. The resulting radiation equations are manipulated to produce weighted averages, which we refer to as consistency terms, e.g.,

$$\{\mu\}_{L,i}^{+} := \frac{\int\limits_{0}^{1}\int\limits_{x_{i-1/2}}^{x_{i+1/2}} \mu\, b_{L,i}(x) I^{n+1}(x,\mu)\, \mathrm{d}\mu \mathrm{d}x}{\int\limits_{0}^{1}\int\limits_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x) I^{n+1}(x,\mu)\, \mathrm{d}\mu \mathrm{d}x}$$

3. Use HO $\tilde{I}^{n+1}(x,\mu)$ and the LD spatial closure to close the system

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- Cell unknowns: $\langle\phi\rangle_{L,i}^{n+1,+}$, $\langle\phi\rangle_{R,i}^{n+1,+}$, $\langle\phi\rangle_{L,i}^{n+1,-}$, $\langle\phi\rangle_{R,i}^{n+1,-}$, $T_{L,i}$, $T_{R,i}$

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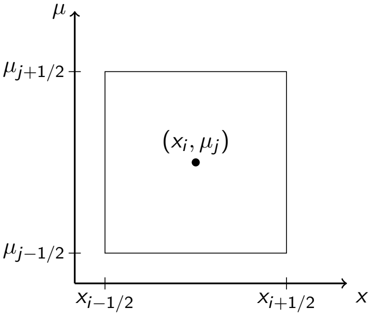
- Non-linear, fully discrete system, solved with approximate Newton's method

Outline

Overview of Exponentially Convergent Monte Carlo

- Each batch tallies the error in current estimate of solution, which is a transport problem with a reduced source
    - Can reduce statistical error globally $\propto e^{-\alpha N}$
    - In TRT problems, old angular intensity provides a *very good* guess of new solution, significantly reducing the required number of histories
- Requires a discretized form of the angular intensity $\tilde{I}(x, \mu)$
    - Use projection of the solution onto a space-angle FE mesh
    - LD FE $\tilde{I}^{HO}(x, \mu)$ allows for direct computation of consistency terms



High Order System and ECMC Algorithm

- Pure absorber transport problem because we have LD representation of T^4 from LO solution

$$\left[\mu\frac{\partial}{\partial x}+\left(\sigma_t+\frac{1}{c\Delta t}\right)\right]I^{n+1}(x,\mu)=\boxed{\frac{I^n}{c\Delta t}+\frac{1}{2}\sigma_a acT_{LO}^{n+1,4}}$$

$$\mathbf{L}I^{n+1}=q_{LO}$$

For each batch:

- Residual Equation: $\mathbf{L}\tilde{\epsilon}^{(m)}=\tilde{r}^{(m)}=q-\mathbf{L}\tilde{I}^{n+1,(m)}$
- Compute $\tilde{\epsilon}^{(m)}=\mathbf{L}^{-1}\tilde{r}^{(m)}$ with MC
  - Particles are allowed to stream along path s, $w(s)=w_0e^{-\sigma_t s}$
  - Cell-wise, global representation allows for easy stratified sampling
- Update: $\tilde{I}^{n+1,(m)}=\tilde{I}^{n+1,(m-1)}+\tilde{\epsilon}^{(m)}$
  - If $\tilde{\epsilon}$ is reduced each batch, exponential convergence achieved

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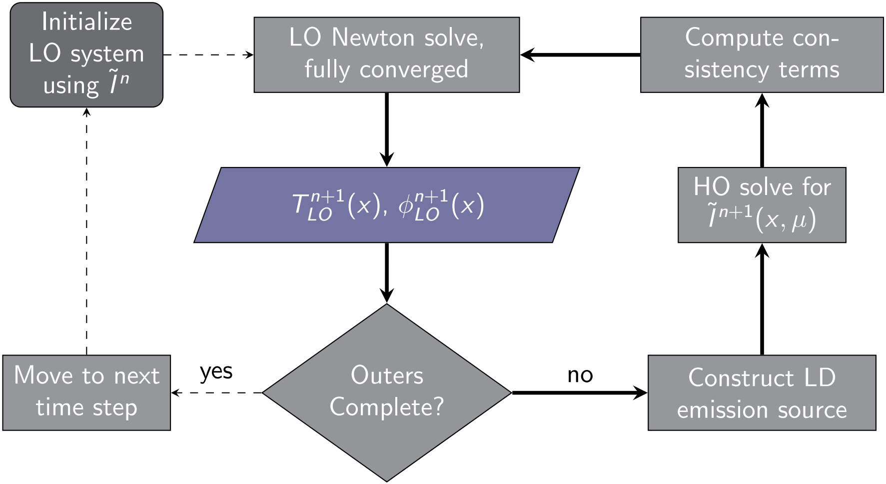
High-Order Low-Order Algorithm



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Relevant Implementation Specifics

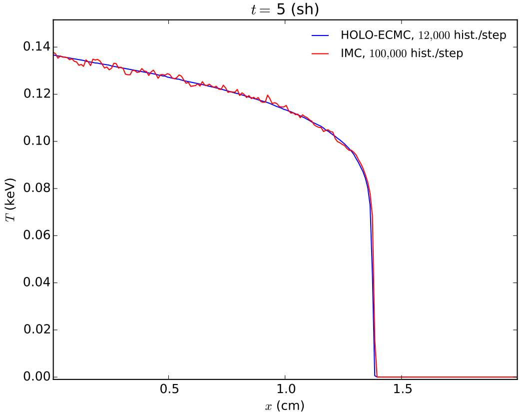
- LO Newton iterations are is fully converged each solve
- Difficulties in resolving the solution near the wave-front
  - The LD representation for *I* results in negativite values within a cell
  - In these results, no correction is applied to the HO solution, and the LO solution uses lumped LD and S_2 equivalent terms in bad cells
- For all results, one HO solve per time step

Marshak Wave Test Problem

- From equilibrium, a radiation source is applied at the left boundary at $t = 0$. With $\sigma_a \propto T^{-3}$, energy slowly moves across the system.

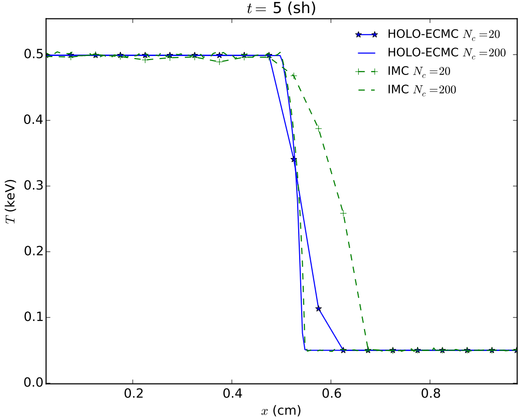
Marshak Wave Test Problem

- From equilibrium, a radiation source is applied at the left boundary at $t = 0$. With $\sigma_a \propto T^{-3}$, energy slowly moves across the system.
- Transient solution after 5 shakes plotted as $T_r = \sqrt[4]{\phi/ac}$, with 200 x cells



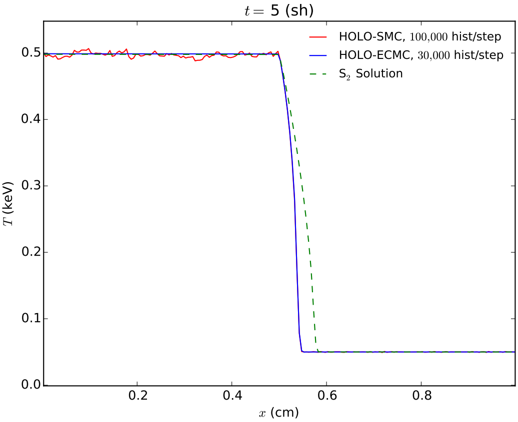
Two Material Problem, Comparison of Spatial Convergence

- Problem the same as Marshak Wave, but with constant opacities and an optically thin (left) and optically thick (right) region



Comparison of statistical noise for standard and ECMC HO solvers

- One HO solve, with a *fixed number of histories* per time step, for two different HO solvers: a comparison of ECMC with 3 batches and standard MC (SMC), as well as S_2 solution



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Current & Future Development

- Can accurately reproduce IMC results with HOLO method
  - ECMC requires significantly less particles
  - LO solver determines non-linear Material temperature distribution
  - Linear shape within a cell mitigates teleportation error
  - Very efficient for diffusive problems
- Currently developing strategies for dealing with unresolvable solutions
- Next step will be to implement in 2 spatial dimensions

Questions?

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