

A High-Order Low-Order Algorithm with Exponentially-Convergent Monte Carlo for k -Eigenvalue Problems

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Outline

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A High-Order Low-Order Solution

Basic Idea

Build a low-order (LO) system that can be efficiently solved, with high-order (HO) correction from pure absorber MC simulations

- Useful when global solution of particle density needed, with low statistical noise
- The LO solution preserves the HO solution, upon convergence (consistent)
- The LO solver handles non-linearities and in-scattering source
 - Produces a linear-discontinuous (LD) representation of sources
 - Lower **dimensional** problem
- The HO system is a fixed-source transport problem
 - ECMC allows for the statistical noise to be reduced globally, giving accurate consistency terms

Eigenvalue Problem

- 1D transport equation

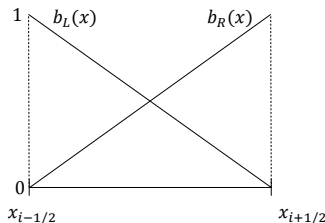
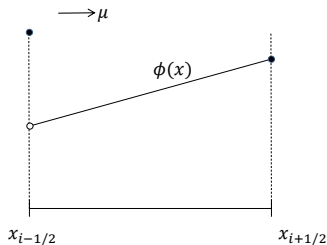
$$\mu \frac{\partial \psi}{\partial x} + \Sigma_t \psi(x, \mu) = \frac{1}{4\pi} \left(\Sigma_s + \frac{\nu \Sigma_f}{k_{\text{eff}}} \right) \phi(x)$$
$$\phi(x) = 2\pi \int_{-1}^1 \psi(x, \mu) d\mu$$

- MC solutions typically use power iteration
 - Accelerate source convergence with LO solution (e.g., CMFD)

We will use a low-order solution that handles the scattering and fission operators to determine k_{eff} and $\phi(x)$. The LO solution preserves MC transport solution

LO Discretization & Space-Angle Moments

- Linear Discontinuous (LD) finite elements in space and half range angular integrals



Spatial moments

$$\langle \cdot \rangle_{L,i} = \frac{2}{h_i} \int_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x)(\cdot) dx$$

Angular half-ranges

$$\phi^+(x) = 2\pi \int_0^1 I(x, \mu) d\mu$$

LO System

- Taking moments of TE yields **4 equations**, per cell i , e.g.

$$-2\mu_{i-1/2}^+ \phi_{i-1/2}^+ + \langle \mu \rangle_{L,i}^+ \langle \phi \rangle_{L,i}^+ + \langle \mu \rangle_{R,i}^+ \langle \phi \rangle_{R,i}^+ + \Sigma_t h_i \langle \phi \rangle_{L,i}^+ - \frac{\Sigma_s h_i}{4\pi} \left(\langle \phi \rangle_{L,i}^+ + \langle \phi \rangle_{L,i}^- \right) = \frac{1}{k_{\text{eff}}} \frac{\nu \Sigma_f h_i}{4\pi} \langle \phi \rangle_{L,i}^+$$

- Cell unknowns: $\langle \phi \rangle_{L,i}^+$, $\langle \phi \rangle_{R,i}^+$, $\langle \phi \rangle_{L,i}^-$, $\langle \phi \rangle_{R,i}^-$
- Need **angular** consistency terms and spatial closure (LD)

Computing LO Consistency terms from HO Solution

- For $\mu > 0$, L moment

$$\langle \mu \rangle_{L,i}^+ \simeq \frac{\frac{2}{h_i} \int_0^1 \int_{x_{i-1/2}}^{x_{i+1/2}} \mu b_{L,i}(x) \tilde{\psi}^{HO}(x, \mu) d\mu dx}{\frac{2}{h_i} \int_0^1 \int_{x_{i-1/2}}^{x_{i+1/2}} b_{L,i}(x) \tilde{\psi}^{HO}(x, \mu) d\mu dx} \quad (1)$$

- ECMC gives LDfE $\tilde{\psi}^{HO}(x, \mu)$ for computing terms directly
- To get S_2 , set $\langle \mu \rangle^\pm = \pm \frac{1}{\sqrt{3}}$

Solving LO System with Power Iteration

- Global System:
$$\mathbf{D}\Phi = \frac{1}{k_{\text{eff}}} \mathbf{F}\Phi$$

Algorithm

- 1 Guess $\Phi^{(0)}$ and $k_{\text{eff}}^{(0)}$

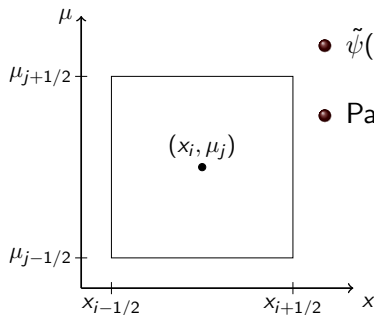
$$\Phi^{(l+1)} = \frac{1}{k_{\text{eff}}^{(l)}} \mathbf{D}^{-1} \mathbf{F} \Phi^{(l)}$$
$$k_{\text{eff}}^{(l+1)} = k_{\text{eff}}^{(l)} \frac{\int \nu \Sigma_f \phi^{(l+1)} dx}{\int \nu \Sigma_f \phi^{(l)} dx}.$$

- 2 Converge $\phi(x)$ and k_{eff}
- 3 **Accelerate** $\Phi^{(l)}$ and $k_{\text{eff}}^{(l+1)}$ after each power iteration with Nonlinear Krylov Acceleration (NKA)

Exponentially Convergent Monte Carlo

- An iterative form of residual Monte Carlo
- Requires a functional form of the angular flux in space and angle
 - Use finite element representation
- Can reduce statistical error globally $\propto e^{-\alpha N}$
 - Does not make difficult problems easier
- Adaptive mesh refinement allows the error to continue to be reduced
 - Increase N per batch by factor of new cells added

Space-Angle Mesh and MC Implementation Details



- $\tilde{\psi}(x, \mu) = \psi_{a,i} + \psi_{x,i} \frac{2}{h_x}(x - x_i) + \psi_{\mu,i} \frac{2}{h_\mu}(\mu - \mu_i)$

- Path-length estimators of moments, e.g.

$$\psi_{x,i} = \frac{6}{h_x^2 h_\mu} \iint_{\mathcal{D}} (x_{c,j} - x_i) \psi(x, \mu) dx d\mu$$

- Particles are allowed to stream, $w(s) = w_0 e^{-\Sigma_t s}$
- LDFE and upwinding **eliminates** surface tallies
- Cell-wise, global representation allows for **stratified** sampling
 - $N_{i,j} \propto |r_i(x, \mu)|$
 - Force $N_i \geq N_{\min}$ and adjust particle weights

High Order System and ECMC Algorithm

- **Pure absorber** transport problem

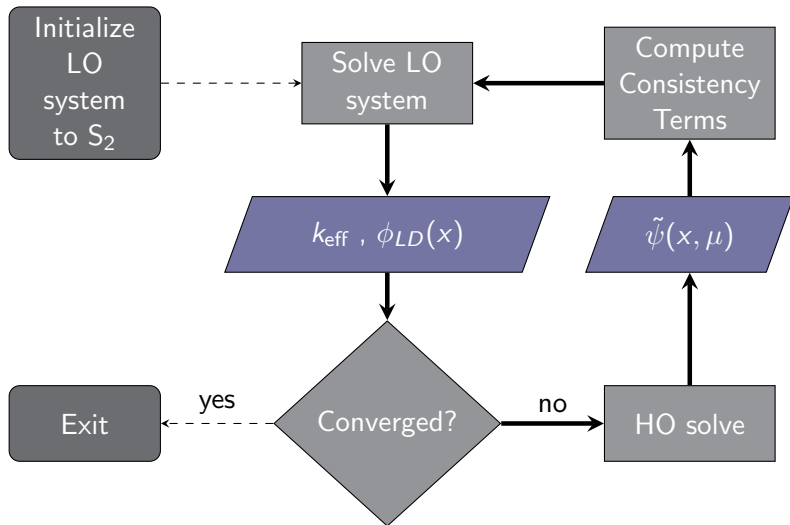
$$\left[\mu \frac{\partial}{\partial x} + \Sigma_t \right] \psi(x, \mu) = \frac{1}{4\pi} \left(\Sigma_s + \frac{1}{k_{\text{eff}}^{LO}} \right) \phi^{LO}(x)$$

$$\mathbf{L}\psi = q^{LO}$$

ECMC Algorithm

- Residual Equation: $\mathbf{L}\tilde{\epsilon}^{(m)} = \tilde{r}^{(m)} = q^{LO} - \mathbf{L}\tilde{\psi}^{(m)}$
- Compute $\tilde{\epsilon}^{(m)} = \mathbf{L}^{-1}\tilde{r}^{(m)}$ with MC, **projecting** the solution
- Update: $\tilde{\psi}^{(m)} = \tilde{\psi}^{(m-1)} + \tilde{\epsilon}^{(m)}$
 - If $\tilde{\epsilon}$ is reduced each batch, **exponential convergence achieved**
 - h -refine when $\epsilon(x, \mu)$ not represented sufficiently
 - Must sufficiently reduce noise in ϵ tallies, each batch
- Repeat until $\|\tilde{\epsilon}\|_2 < \text{tol} \times \|\psi\|_2$

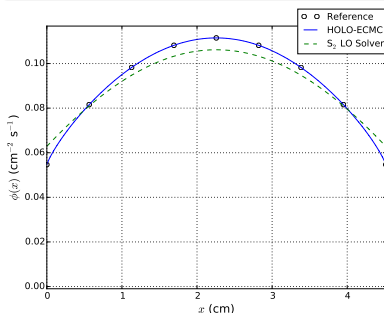
High-Order Low-Order Algorithm



Critical slab benchmark

Problem Parameters

- $k_{\infty} = 2.29$, $\Sigma_t = 0.326 \text{ cm}^{-1}$
- 2.4×10^5 histories per batch, $100 \times$ & 20μ cells
- Adaptive HO convergence

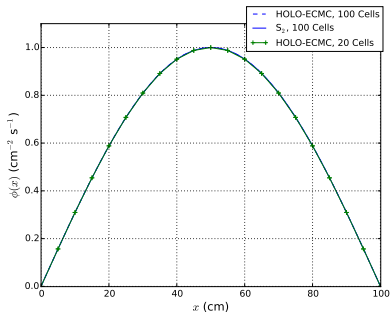


- $|\Delta\Phi| < 10^{-4}$ in 4 outer iterations, using $\sim 2.4 \times 10^7$ histories
- For 10 independent simulations:
 - $\overline{k_{\text{eff}}} = 0.999998$,
 $\sigma(k_{\text{eff}}) = 4.1 \times 10^{-6}$,
 $\text{max dev.} = 1.1 \times 10^{-5}$
 - $\overline{\sigma_{\text{rel}}(\phi_i)} = 1.4 \times 10^{-5}$

Optically thick, near-critical slab

Problem Parameters

- $k_{\infty} = 1$, $\Sigma_t = 30.0 \text{ cm}^{-1}$, $\Sigma_s = 29.5 \text{ cm}^{-1}$, $\text{DR} \approx 0.999$
- Relative Tolerance of $1.0\text{E-}05$ for all solvers

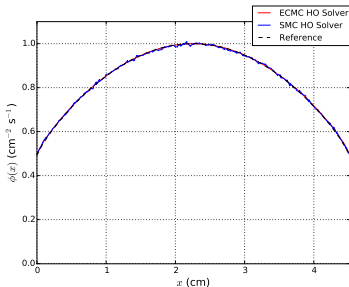


- Fission source convergence:
 - Power iteration not converged after 10,000 iterations
 - NKA converged in 246 iterations
- 1 outer iteration, 4.8×10^6 total histories
- $k_{\text{eff}} = 0.99997$

Comparison of statistical noise for standard and ECMC HO solvers

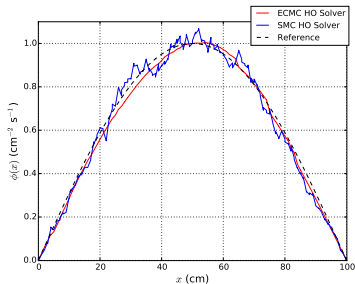
- One HO solve, with a fixed number of histories N .
Comparison of **ECMC** with 5 batches or standard MC (**SMC**)

Critical Slab



$$N = 1.5 \times 10^5$$

Diffuse Slab, $k_{\infty} = 1$



$$N = 10^6$$

Current & Future Development

- Can Solve for k_{eff} and fission source with HOLO method
 - Pure absorber histories are cheaper than standard MC simulations
 - ECMC efficiently reduces noise globally
 - LO solver handles scattering, fission, and
- Stratified source sampling more efficient at reducing statistical variance than a constant number of particles per cell
- Need to use the estimated statistical error in tallies for $\tilde{\epsilon}(x, \mu)$

Questions?

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Backup Slides



Algorithm

- ❶ $\tilde{\psi}^{(0)} = \tilde{\psi}$ or from last batch this time step
- ❷ Using Monte Carlo, solve $\epsilon^{(m)} = \mathbf{L}^{-1} \tilde{r}^{(m)}$
 - Use volumetric tallies, weighted with x and μ basis moments time ψ to construct LD $\tilde{\epsilon}^{(m)}(x, \mu)$ over the current space-angle mesh
- ❸ $\tilde{\psi}^{(j+1)} = \tilde{\psi}^{(m)} + \tilde{\epsilon}^{(m)}$
- ❹ **IF** error stagnation:
 - Refine mesh based on relative jump error in $\tilde{\psi}(x, \mu)$
- ❺ Repeat 2-4 until $\|\tilde{\epsilon}\|_2 < \text{tol} \times \|\psi\|_2$

Algorithm

- ➊ Initialize $\langle \mu \rangle^\pm$ parameters to S_2
- ➋ Solve LO system using power iteration
- ➌ Build q^{LD} for HO solver, and set $\tilde{\psi}$ to latest HO estimate on coarsest $x-\mu$ mesh
- ➍ Solve $\tilde{\psi}(x, \mu) = \mathbf{L}^{-1} q^{LO}$ using ECMC
- ➎ Compute new $\langle \mu \rangle^\pm$ parameters using $\tilde{\psi}^{HO}$ over LO mesh
- ➏ Repeat 2-5 until $\underline{\phi}^{LO}$ is converged
 - Use adaptive convergence criteria

Exponential Convergence Plot