Comparison of Two Galerkin Quadrature Methods

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Abstract

We compare two methods for generating Galerkin quadratures. In Method 1, the standard S_n method is used to generate the moment-to-discrete matrix and the discrete-to-moment matrix is generated by inverting the moment-to-discrete matrix. This is a particular form of the original Galerkin quadrature method. In Method 2, which we introduce here, the standard S_n method is used to generate the discrete-to-moment matrix and the moment-to-discrete matrix is generated by inverting the discrete-to-moment matrix. Method 1 has the advantage that it preserves in a pointwise sense both N eigenvalues and N eigenvectors of the scattering operator with an N-point quadrature. Method 2 has the advantage that it generates consistent angular moment equations from the corresponding S_N equations while preserving N eigenvalues of the scattering operator with an N-point quadrature. Our computational results indicate that these two methods are quite comparable for the test problem considered.

1 Introduction

The Galerkin quadrature (GQ) method was introduced by Morel in 1990 [1] to better treat highly anisotropic scattering in S_n calculations. Every Galerkin quadrature set is completely defined by an invertible linear mapping between the discrete S_n angular fluxes and a set of spherical-harmonic flux moments. The matrix that maps the fluxes to the moments is called the discrete-to-moment matrix and the matrix that maps the moments to the fluxes is called the moment-to-discrete matrix. These matrices are inverses of each other, and they are used together with a diagonal matrix of Legendre cross section moments to construct the S_n scattering source. Distinct versions of the GQ method vary only with repect to how the invertible mapping is constructed. One can cast the standard S_n scattering source in terms of discrete-to-moment, moment-to-discrete, and cross section moment matrices, but one finds that the discrete-to-moment and moment-to-discrete matrices are generally not inverses of each other. This can be remedied by accepting either the standard S_n discrete-to-moment matrix and inverting it to obtain the moment-to-discrete matrix or by accepting the standard S_n moment-to-discrete matrix and inverting it to obtain the discrete-to-moment matrix. We refer to the former as Method 1 and to the latter as Method 2. We of course assume that the matrices will be invertible and well conditioned, but this is not necessarily so. In 1-D slabs and spheres, the choice of spherical-harmonics (Legendre polynomials) is trivial. With an N-point quadrature set, one maps to Legendre moments 0 through $N-1$. In multidimensions, the choice is not necessarily clear since the number of directions in a standard quadrature set is not equal the number of spherical harmonics of any given order. Nonetheless, prescriptions have been successfully made for triangular and product sets in various geometries [1,**?**,**?**].

Method 1 was introduced in the original GQ paper [1]. Method 2 is being introduced here, but the invertible mapping obtained by using a standard quadrature formula to map the discrete angular fluxes to the flux moments was recently used in two references. The first related to a 1-D Cartesian/spherical discretization for the Fokker-Planck operator [4], and the second related to the demonstration of an equivalence between the 1-D slab-geometry S_N equations with arbitrary quadrature and the P_{N-1} equations with a quadrature-dependent closure [5].

The purpose of the work described herein is to computationally compare Methods 1 and 2 for calculations with highly anisotropic scattering in 2-D Cartesian geometry. In particular we apply these methods to a problem with highly forward-peaked scattering that strongly approximates Fokker-Planck scattering. Fokker-Planck scattering represents a limit in which the scattering mean-free-path approaches zero while the mean cosine of the scattering angle approaches unity in such a way that the transport-corrected scattering cross section remains constant [6]. As such, it represents a severe test for schemes intended to treat anisotropic scattering.

Our computational results indicate that these two methods are very comparable for test problem considered. While the discrete-to-moment and moment-to-discrete matrices for each set differ significantly for sufficiently high order moments, the higher order moments in the problem we considered tend to be small, leading to similar angular flux solutions. Relatively small higher-order moments are expected because, as we later show, the Fokker-Planck operator preferentially attenuates higher-order moments.

This paper is organized as follows. First we describe Method 1. Then we describe Method 2. Our test problems are next described and computational results are given. Finally, we give conclusions and recommendations for future work.

2 Method 1

Method 1 is most easily illustrated in terms of 1-D quadrature. Consider a set of N discrete cosines:$\{\mu_m\}_{m=1}^{N}$, and an associated set of interpolatory basis functions, $\{B_m(\mu)\}_{m=1}^{N}$. These basis functions can span any space in principle, but they must satisfy

$$B_i(\mu_j) = \delta_{ij} \quad , \quad \text{for all } i \text{ and } j, \tag{1}$$

where δ_{ij} is the Dirac delta. This implies that the GQ angular flux solution can be represented as follows:

$$\tilde{\psi}(\mu) = \sum_{m=1}^{N} \psi_m B_m(\mu) \quad , \quad \text{for all } m. \tag{2}$$

where

$$\psi_m = \psi(\mu_m) \quad . \tag{3}$$

Thus the angular flux unknowns associated with the GQ method are standard S_N discrete angular fluxes. The discrete angular flux values are mapped to a set of N Legendre polynomial moments as follows:

$$\phi_k = \frac{1}{2}\int_{-1}^{+1} P_k(\mu)\tilde{\psi}(\mu)\, d\mu \quad , \quad k = 0, N-1. \tag{4}$$

where $\mathrm{P}_k(\mu)$ is the Legendre polynomial of degree k. We can express Eq. (4) in matrix form as follows:

$$\overrightarrow{\phi} = \mathbf{D}\overrightarrow{\psi} \quad , \tag{5}$$

where

$$\overrightarrow{\phi} = (\phi_0, \phi_1, \phi_2, \ldots, \phi_{N-1})^T \quad , \tag{6}$$

$$\overrightarrow{\psi} = (\psi_1, \psi_2, \psi_3, \ldots, \psi_N)^T \quad , \tag{7}$$

and

$$D_{i,j} = \frac{1}{2} \int_{-1}^{+1} P_i(\mu) B_j(\mu) \, d\mu \quad , \quad i = 0, N-1, \; j = 1, N. \tag{8}$$

The matrix $\mathbf{D}$ is called the discrete-to-moment matrix since it maps the discrete angular flux values to the Legendre moments of the angular flux. In order to proceed, we must assume that $\mathbf{D}$ is invertible. Making this assumption, we generate the discrete S_N scattering source via a similarity transformation from the Legendre moment basis to the discrete angular flux basis. In particular, the discrete scattering source values can be represented as follows:

$$\overrightarrow{S} = \mathbf{S}\overrightarrow{\psi} = \mathbf{MXD}\overrightarrow{\psi} \quad , \tag{9}$$

where

$$\mathbf{M} = \mathbf{D}^{-1} \quad , \tag{10}$$

$$\overrightarrow{S} = (S_1, S_2, S_3, \ldots, S_N)^T \quad , \tag{11}$$

and

$$\mathbf{X} = diag(\sigma_0, \sigma_1, \ldots, \sigma_{N-1}) \quad . \tag{12}$$

where σ_k is the Legendre scattering cross section moment of degree k:

$$\sigma_k = \frac{1}{2} \int_{-1}^{+1} P_k(\mu_0) \sigma_s(\mu_0) \, d\mu_0 \quad . \tag{13}$$

The matrix $\mathbf{M}$ is called the moment-to-discrete matrix because it maps Legendre flux moments to discrete angular flux values:

$$\overrightarrow{\psi} = \mathbf{M}\overrightarrow{\phi} \quad . \tag{14}$$

The analytic Boltzmann scattering operator, $\mathbf{S}_B$ can be represented as follows:

$$\mathbf{S}_B \psi = \sum_{k=0}^{\infty} (2k+1)\sigma_k \phi_k P_k(\mu) \quad , \tag{15}$$

where ϕ_k is the k'th Legendre moment of the angular flux:

$$\phi_k = \frac{1}{2} \int_{-1}^{+1} \psi(\mu') P_k(\mu')\, d\mu' \quad . \tag{16}$$

It is important to recognize that $\mathbf{X}$ represents $\mathbf{S}_B$ in the Legendre polynomial or P_{N-1} basis, i.e., if you write the P_n equations in matrix form, the scattering operator is given by $\mathbf{X}$. Thus the S_N scattering matrix given by Eq. (9) is equivalent to the P_{N-1} scattering matrix via a similarity transformation. This property offers several advantages, which we will now describe.

It is well-known that the eigenvalues of $\mathbf{S}_B$ are the Legendre scattering cross section moments and the eigenfunctions are the Legendre polynomials:

$$\begin{aligned} \mathbf{S}_B P_j(\mu) &= \sum_{k=0}^{\infty} (2k+1)\sigma_k \left[\frac{1}{2} \int_{-1}^{+1} P_j(\mu') P_k(\mu')\, d\mu' \right] P_k(\mu) \quad , \\ &= \sigma_j P_j(\mu) \quad . \end{aligned} \tag{17}$$

Note from Eqs. (12) and (17) that $\mathbf{X}$ preserves the first N eigenvalues of $\mathbf{S}_B$. Since $\mathbf{S}$ is equivalent to $\mathbf{X}$ via a similarity transformation, it follows that $\mathbf{S}$ also preserves the first N

eigenvalues of $\mathbf{S}_B$.

Another advantage of the equivalence to $\mathbf{X}$ is that forward-peaked delta-function scattering is exactly treated. More specifically, when

$$\sigma_s(\mu_0) = 2\delta(\mu_0 - 1) \quad , \tag{18}$$

one finds that

$$\sigma_k = 1 \quad , \quad \text{for all } k. \tag{19}$$

Substituting from Eq. (19) into Eq. (15), we find that the exact scattering source corresponding to the interpolated representation for the angular flux is given by

$$\mathbf{S}_B\tilde{\psi} = \sum_{k=0}^{\infty}(2k+1)\phi_k P_k(\mu) = \psi(\mu) \quad . \tag{20}$$

Substituting from Eq. (19) into Eq. (9) via Eq. (12), we find that the discrete scattering source is in pointwise agreement with Eq. (20):

$$\overrightarrow{S} = \mathbf{MID}\overrightarrow{\psi} = \mathbf{D}^{-1}\mathbf{D}\overrightarrow{\psi} = \overrightarrow{\psi} \quad . \tag{21}$$

In Method 1 we choose to interpolate the angular flux with global polynomials. We do so for several reasons, but the first is that the GQ matrix $\mathbf{M}$ is easily generated without forming the interpolatory basis functions. In particular, the matrix $\mathbf{M}$ is given by

$$M_{i,j} = (2j+1)P_j(\mu_i) \quad , \tag{22}$$

which is just the usual Legendre polynomial expansion of degree $N-1$ for the flux evaluated at the quadrature points:

$$\phi_i = \sum_{j=1}^{N-1} (2j+1)\sigma_k \phi_k P_k(\mu_i) \qquad , \tag{23}$$

The matrix $\mathbf{D}$ can be obtained simply by inverting $\mathbf{M}$.

The equivalence of $\mathbf{S}$ and $\mathbf{X}$ coupled with polynomial interpolation of the discrete angular flux values results in a discrete scattering source that is actually exact in a certain pointwise sense. It is not the discrete scattering source corresponding to the exact angular flux solution, but rather it is the discrete scattering source obtained by applying the exact scattering operator to the assumed interpolated polynomial angular flux solution, and evaluating that source at the quadrature points. To demonstrate this we first note that the scattering source for any arbitrary angular flux, ψ can be exactly represented as follows:

$$\mathbf{S}_B \psi = \sum_{k=0}^{\infty} (2k+1)\sigma_k \phi_k P_k(\mu) \qquad . \tag{24}$$

However, if $\tilde{\psi}$ is a polynomial of degree $N-1$, then $\phi_k = 0$, for $k > N-1$, and Eq. (24) reduces to a polynomial of degree $N-1$:

$$\mathbf{S}_B \tilde{\psi} = \sum_{k=0}^{N-1} (2k+1)\sigma_k \phi_k P_k(\mu) \qquad . \tag{25}$$

Thus the discrete scattering source for $\tilde{\psi}$ is given by

$$S_m = \sum_{k=0}^{N-1} (2k+1)\sigma_k \phi_k P_k(\mu_m) \qquad , \quad m = 1, N, \tag{26}$$

which, given Eq. (25), is clearly pointwise exact.

If we set the discrete values of the angular flux to the Legendre polynomial of degree j, where $j \leq N-1$, we get

$$\psi_m = P_j(\mu_m) \quad , \quad m = 1, N. \tag{27}$$

Computing the moments of the angular flux, we get:

$$\mathbf{D}\overrightarrow{\psi} = \overrightarrow{e}_j \quad , \tag{28}$$

where $\overrightarrow{e}_j$ is the cardinal vector having a j'th component of unity with all other components zero. Next applying the cross section matrix, we obtain the legendre moment of the scattering source:

$$\mathbf{XD}\overrightarrow{\psi} = \sigma_j \overrightarrow{e}_j \quad . \tag{29}$$

Finally, we compute the discrete scattering source:

$$\mathbf{MDD}\overrightarrow{\psi} = \sigma_j \mathbf{M}\overrightarrow{e}_j = \sigma_j P_j(\mu_m) \quad , \quad m = 1, N, \tag{30}$$

It is clear from Eqs. (17) and (30) that the first N eigenfunctions as well as the first N eigenvalues of the analytic scattering operator are preserved in a pointwise sense with Method 1.

In multiple dimensions, the advantages for global polynomial interpolation generalize to analogous advantages for spherical-harmonic interpolation. The choice of harmonic moments to which to map the directions is complicated by the fact that the number of harmonics of a given degree does not match the number of directions, but prescriptions for the mapping have been defined for triangular and product sets in R-Z and Cartesian geometries [1–3].

3 METHOD 2

In a recent paper, it was shown that in 1-D slab-geometry with isotropic scattering, the standard S_N equations are equivalent to P_{N-1} equations with a quadrature-dependent closure if the discrete-to-moment matrix defined by Eq. (23) is invertible [5]. This is clearly a highly desirable property, but anisotropic scattering was not considered in Reference 5. We show in Section (3.1) that this property also applies with anisotropic scattering if one uses a certain variation of the GQ method to represent the scattering source. In particular, one must define $\mathbf{D}$ in accordance with the standard application of an S_N quadrature formula:

$$D_{i,j} = P_i(\mu_j) w_j \quad , \tag{31}$$

or equivalently,

$$\phi_i = \sum_{m=1}^{N} \psi_m P_i(\mu_m) w_m \quad , \quad i = 0, N-1. \tag{32}$$

Then $\mathbf{M}$ is obtained simply by inverting $\mathbf{D}$. This approach, which we refer to as Method 2 is similar to Method 1 in that it ensures that the first N eigenvalues of the scattering source are exactly preserved and that forward-peaked delta-function scattering is treated exactly. However, the scattering source is generally not pointwise exact in any sense, and the first N eigenvectors of the scattering operator are generally not pointwise preserved. Thus both Method 1 and Method 2 offer certain advantages, but it is not clear that either will be superior in general. Warsa and Prinja [4] recently proposed a moment-preserving discretization for the 1-D Fokker-Planck scattering operator that is actually equivalent to Method 2 using the Fokker-Planck cross section moments defined by Morel [7], although it is not obvious from their derivation.

3.1 Angular Moment Equations

Here we demonstrate that Method 2 yields Legendre moment equations through degree $N-1$ with a quadrature-dependent closure. We begin with the S_N equations assuming use of a Method 2 Galerkin quadrature:

$$\mu_m \frac{\partial \psi}{\partial x} + \sigma_t \psi = \sum_{k=0}^{N-1} M_{m,k}(\sigma_k \phi_k + q_k) \quad , \quad m = 1, N, \tag{33}$$

where $M_{m,k}$ is the column m and row k element of $\mathbf{M} = \mathbf{D}^{-1}$, the k'th Legendre moment of the homogeneous source is denoted by q_k, and

$$\phi_k = \sum_{m=1}^{N} \psi_m P_k(\mu_m) w_m \quad , \quad k = 0, N. \tag{34}$$

Note that Eq. (34) is consistent with the Method 2 definition for $\mathbf{D}$ for $k = 0, N-1$, and defines ϕ_N as well. Next we multiply Eq. (33) by $P_k(\mu_m)w_m$ where $k = 0, N-1$, and sum over all directions. Recognizing that

$$\mu P_k(\mu) = \frac{k+1}{2k+1} P_{k+1}(\mu) + \frac{k}{2k+1} P_{k-1}(\mu) \quad , \tag{35}$$

and using the fact that $P_k(\mu_m)w_m$ is the column k and row m element of $\mathbf{D}$, we obtain the following system of moment equations:

$$\frac{d\phi_1}{dx} + \sigma_t \phi_0 = \sigma_0 \phi_0 + q_0 \varrho \quad , \tag{36}$$

$$\frac{k+1}{2k+1}\frac{d\phi_{k+1}}{dx} + \frac{k}{2k+1}\frac{d\phi_{k-1}}{dx} + \sigma_t \phi_k = \sigma_k \phi_k + q_k \quad , \quad k = 1, N-2, \tag{37}$$

$$\frac{N}{2N-1}\frac{d\phi_N}{dx} + \frac{N-1}{2N-1}\frac{d\phi_{N-2}}{dx} + \sigma_t \phi_{N-1} = \sigma_0 \phi_{N-1} + q_{N-1} \quad . \tag{38}$$

These moment equations are in fact the exact Legendre moment equations. The system appears to be open because there are $N+1$ unknowns and N equations. However, since ϕ_N is given by Eq. (34) and there is an mapping between the first N Legendre flux moments and the angular fluxes via $\mathbf{M} = \mathbf{D}^{-1}$, it follows that ϕ_N is a function of these moments. In particular, using $\mathbf{M}$ to express the angular fluxes in Eq. (34) in terms of the first N Legendre flux moments, we obtain

$$\begin{aligned} \phi_N &= \sum_{m=1}^{N} \psi_m P_N(\mu_m) w_m \quad , \\ &= \sum_{m=1}^{N} \left[\sum_{k=0}^{N-1} M_{m,k} \phi_k \right] \quad , \\ &= \sum_{k=0}^{N-1} \left[\sum_{m=1}^{N} M_{m,k} P_N(\mu_m) w_m \right] \phi_k \quad . \end{aligned} \tag{39}$$

Equation (39) represents the closure for a set of exact Legendre moment equations of degree $N-1$ that are completely equivalent to the S_N equations with Galerkin quadrature as defined by Method 2. It has not been demonstrated that this property holds in any geometry other than 1-D slab geometry, but we suspect that it does in all Cartesian geometries. This property does not apply in general even in 1-D slab geometry with Method 1.

4 Standard S_N Scattering Source

While the GQ method may seem quite different from the standard S_N method for representing the scattering source, the generation of the source requires the same algebraic steps. For instance, if we assume a scattering cross section expansion order of degree $N-1$, the standard S_N scattering source can be represented by Eq. (9) with **M** defined by

$$M_{i,j} = (2j+1)P_j(\mu_i) \quad , \tag{40}$$

and **D** defined by

$$D_{i,j} = P_i(\mu_j)w_j \quad , \tag{41}$$

where w_j is the quadrature weight for direction j, and with **X** defined by Eq. (12). The most obvious problem with the S_N method is that **M** and **D** are in general not inverses of each other. The only exception is the case of Gauss quadrature. In this case, the standard S_N method, the GQ Method 1, and the GQ Method 2 are all *equivalent.*

We note that if the scattering is weakly anisotropic, i.e., if the cross section expansion converges with K moments where K is less than the number of quadrature directions, one need only store K columns of the moment-to-discrete matrix and K rows of the discrete-to-moment matrix to compute the scattering source. Thus the GQ method can be used with weakly anisotropic scattering at the same computational cost as the standard S_N method.

5 COMPUTATIONAL RESULTS

The standard S_N scattering source treatment in 1-D slab-geometry with Gauss quadrature is equivalent to *both* the Method 1 and Method 2 Galerkin treatments. There is no quadrature set in multidimensions that yields an equivalence between the standard S_N scattering source treatment and any Galerkin treatment. Thus we choose to compare Methods 1 and 2 in 2-D Cartesian geometry rather than 1-D slab geometry. To provide a stringent test of the quadratures, we assumed anisotropic scattering in the test problems that approximates Fokker-Planck scattering. Such scattering is described in Sec. (1). To obtain a standard for comparison we performed Monte Carlo calculations.

5.1 Problem Definition

There is one test problem. The directional coordinate system for the test problem is oriented such that $\xi = \cos(\theta)$ is the z-axis cosine, $\mu = \sin(\theta)\cos(\phi)$ is the x-axis cosine, and $\eta = \sin(\theta)\sin(\phi)$ is the y-axis cosine. The test problem is defined in 2-D x-y geometry. The spatial domain is a 10 cm $\times$ 10 cm square with $x \in [0, 10]$ and $y \in [0, 10]$. There are four faces, x^-, x^+, y^- and y^+. On the x^- face the value of x is everywhere equal to zero, and on the x^+ face the value of x is everywhere equal to 10. The definitions for $y^\mp$ are analogous. The boundary conditions are vacuum on both y-faces and the x^+ face. There is an angular flux uniformly incident on the x^- face that is constant for $\xi \in [0, 1]$ and $\phi \in [0, \pi/2]$, and otherwise zero. The flux is normalized to yield a unit incident half-range current (p/s). The problem geometry is illustrated in Fig. 1. A 65 $\times$ 65 uniform rectangular spatial mesh was used in the calculations with each rectangle cut into four triangles. A linear-discontinuous approximation was used on each triangle.

The quantities of interest are the leakages (p/s) out of each of the four faces. To aid us in understanding the results we obtained for these leakages, we also computed the angular fluxes and flux moments moments at the center of each face using the interior face fluxes as illustrated in Fig. 2. Due to upwinding, only the interior outgoing fluxes on each face are true face fluxes, but the interior incoming fluxes nonetheless represent a second-order accurate approximation to the true incoming face fluxes. We did not calculate these angular fluxes and flux moments by Monte Carlo. For reasons that will later be clear, it is not necessary for our purposes to know the accuracy of these moments.

Fokker-Planck scattering can be represented by Legendre scattering cross section moments because the spherical-harmonics are eigenfunctions of both the Boltzmann and Fokker-Planck scattering operators [7]. In particular, for a cross section expansion of degree K,

$$\sigma_k = \alpha \left[K(K+1) - k(k+1) \right] \quad , \quad k = 0, K. \tag{42}$$

where α (cm^{-1}) is the transport-corrected scattering cross section. However, we chose not to use these coefficients because Monte Carlo codes cannot use the Legendre coefficients defined in Eq. (42) even if a discrete scattering angle formulation based upon Legendre cross section expansions [8] is used. This is so because the expansion coefficients defined in Eq. (42) approach the Fokker-Planck limit as $N \to \infty$ from a non-physical region in moment space. Furthermore, we know of no Monte Carlo algorithm to exactly simulate Fokker-Planck scattering. Thus we assume assume a forward-peaked scattering cross section that approximates Fokker-Planck scattering as described in Reference 8. In particular, the following cross section is assumed:

$$\sigma_s(\mu_0) = 500\ \delta(\mu_0 - 0.999) \quad . \tag{43}$$

Note that the transport-corrected scattering cross section corresponding to Eq. (43) is equal to 0.25 cm^{-1}:

$$\sigma_{s,tr} = \frac{1}{2} \int_{-1}^{+1} \sigma_s(\mu_0)\,(1 - \mu_0)\ d\mu_0 \quad . \tag{44}$$

We assume an absorption cross section of 0.1 cm^{-1} in all calculations.

5.2 S$_N$ Calculations

S$_N$ calculations are performed for the test problem using Method 1 and Method 2 Galerkin quadrature in conjunction with the discrete directions for Gauss-Chebychev triangular quadrature. The problem is performed for several different orders of quadrature to gauge convergence rate in addition to comparing the two methods. The Gauss-Chebychev directions are given by the 1-D Gauss directions in the ξ-cosine and the 1-D Chebychev (equally-spaced) directions in the azimuthal angle. The spherical-harmonics to which the directions of an S_N set are mapped are defined as follows:

$$\begin{aligned} Y_k^m &= \sqrt{C_k^m} P_k^m(\xi) cos(m\omega) \quad , \quad \text{for } k \geq 0 \text{ and } k \geq m \geq 0, && \text{(45a)} \\ &= \sqrt{C_k^m} P_k^m(\xi) sin(|m|\omega) \quad , \quad \text{for } k \geq 0 \text{ and } 0 > m \geq -k, && \text{(45b)} \end{aligned}$$

where $P_k^m(x)$ is the associated Legendre function, and

$$C_k^m = (2 - \delta_{m,0}) \frac{(k - |m|!)}{(k + |m|!)} \quad , \tag{46}$$

The specific set of harmonics mapped to triangular S_N quadrature in 2-D Cartesian geometry are as follows. The harmonic Y_k^m is a member of the set if and only if:

- $k + |m|$ is even, and
- $k < N$, and $k \geq m \geq -k$, or
- $k = N$, and $0 > m \geq -k$.

We suspect that this is the only acceptable set of harmonics with $k \leq N$, but we cannot prove it. An unacceptable set of harmonics leads to a singular mapping between $\overrightarrow{\psi}$ and $\overrightarrow{\phi}$. Discovering the "rule" for choosing the harmonics is basically a trial and error process guided by geometric and quadrature symmetries. We are fairly certain that there are other sets containing harmonics with $k > N$ that are acceptable, but we would expect them to yield mappings that are more ill-conditioned than those associated with our choice of harmonics. Ideally, one would like to obtain mappings that are as well-conditioned as possible.

It is important to note that in applying Method 2, we did not use the standard Gauss-Chebychev weights. Rather, we used the companion quadrature weights associated with Method 1. The companion weights correspond to the first row of the discrete-to-moment matrix, which is used to generate the first flux moment, i.e., the scalar flux. The correspondence between these matrix elements and quadrature weights is clear from Eq. (5);

$$\phi = \sum_{m=1}^{N} D_{0,m} \psi_m \quad , \tag{47}$$

where ϕ is the scalar flux. Although Eq. (47) is taken from a 1-D expression, it applies in general where N is the total number of directions. The first angular moment will always correspond to the scalar flux since the interpolatory space must always be able to represent an isotropic angular flux. One need not generate the discrete-to-moment matrix to obtain the companion weights. Rather, given the Gauss-Chebychev directions, one can obtain these weights simply by requiring that the quadrature formula exactly integrate the spherical harmonics used to interpolate the angular fluxes in Method 1. This results in a linear system of equations for the weights.

We use the companion weights in Method 2 because use of the standard Gauss-Chebychev weights results in an anisotropic scattering source with isotropic scattering. This is clearly unacceptable. In general, one should always apply Method 2 with a quadrature formula that exactly integrates the harmonics used to interpolate the angular fluxes in Method 1. In some instances, standard quadratures can do this. For instance, standard product (square) Gauss-Chebychev quadrature do integrate the interpolatory harmonics.

We also attempted to perform S_n calculations with the standard quadrature treatment for comparison with our Galerkin quadrature calculations. The standard Gauss-Chebychev directions and weights were used in conjunction with a scattering cross section expansion of degree $N-1$. All of the harmonics of degree less than or equal to $N-1$ can be exactly integrated with the standard triangular Gauss-Chebychev quadrature, so particle conservation was maintained (this is not the case for expansion degrees greater than $N-1$). However, we found that the source iteration matrices that had eigenvalues with magnitudes greater than one, i.e., source iteration was unstable. Thus we could not provide computational results for the standard S_n quadrature treatment.

6 Monte Carlo Calculation

To obtain benchmark solutions for comparison with the discrete-ordinates solutions, we performed Monte Carlo calculations for each of the two problems. Each calculation was performed with 10^9 total histories in 100 batches.

6.1 Comparison of Methods

The results of our tests calculations are given in Table 1. As expected from the shape of the incident flux distribution, the leakage through the y^- face is the most difficult to compute. Thus we focus on this leakage. The GQ Method 1 formulation is significantly more accurate than the GQ Method 2 formulation at the S_4 level, but the two approaches yield very similar results for higher order quadratures and the differences in the results decrease with increasing quadrature order. The question arises as to why the GQ methods yield such similar results when their respective **D** and **M** matrices differ significantly for the higher-order moments. One possible explanation is that the higher order moments are relatively small, rendering higher-order differences in the **D** and **M** matrices moot. We computed the angular flux moments at the face centers to investigate this possiblity. The S_8 flux moments on the y^- face are given for both the GQ Method 1 and the GQ Method 2 in Table 2. It can be ween from Table 2 that the higher-order moments are indeed relatively small for both methods. This result is not surprising given the nature of the attenuation coefficients associated with highly forward peaked scattering. For instance, it can be determined from Eq. (42) that for Fokker-Planck scattering the k'th total attenuation coefficient is given by $\sigma_0 - \sigma_k = \alpha k(k+1)$. This coefficient rapidly increases with increasing k. Thus the higher-order flux moments are strongly attenuated relative to the lower-order moments. This is in contrast to isotropic scattering for which all moments other than the scalar flux have the same attenuation coefficient. The two quadrature sets yield identical solutions with isotropic scattering and with no scattering at all. Thus making the scattering more isotropic or making the problem more optically thin does not change the similarity of results from the two methods. We have been unable to construct a problem for which the GQ1 and GQ2 sets yield significantly different results for any other than the very lowest quadrature orders.

7 CONCLUSIONS

Our results indicate that the two types of triangular Gauss-Chebychev Galerkin quadrature sets are quite comparable for the highly forward-peaked scattering problem considered. Indeed, we were unable to construct any problem for which the two methods yielded significantly different results. In contrast, the standard triangular Gauss-Chebychev sets are not suitable for the problem considered due to instability of the source iteration process.

We intend to investigate the application of the two Galerkin quadrature methods to other types of quadrature sets in the future.

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Figure 1: Problem geometry with incident flux. The arrows are intended to indicate the quarter-range isotropic nature of the incident flux, which is constant on the x^- face. Each face is 10 cm in length.

Figure 2: Illustration of face flux calculation. The two interior corner fluxes about each face center are averaged to obtain a face-centered value.

Table 1: **Comparison of leakages** (p/s) **for GQ Method 1 (GQ1), GQ Method 2 (GQ2), and Monte Carlo (MC).**

Method		Face			
		x^-	x^+	y^-	y^+
MC		$0.199278 \pm 1.4 \cdot 10^{-5}$	$0.019233 \pm 4.6 \cdot 10^{-6}$	$0.072106 \pm 8.7 \cdot 10^{-6}$	$0.190231 \pm 1.4 \cdot 10^{-5}$
S_4	GQ1	2.220842e-01	1.821048e-02	6.744504e-02	1.930308e-01
	GQ2	2.220842e-01	1.821048e-02	6.532314e-02	1.951527e-01
S_8	GQ1	2.091496e-01	1.894772e-02	7.002274e-02	1.899175e-01
	GQ2	2.091496e-01	1.894771e-02	6.970301e-02	1.902372e-01
S_{16}	GQ1	2.024318e-01	1.914877e-02	7.141571e-02	1.900609e-01
	GQ2	2.024318e-01	1.914877e-02	7.138919e-02	1.900875e-01
S_{32}	GQ1	2.000532e-01	1.920919e-02	7.192883e-02	1.902319e-01
	GQ2	2.000532e-01	1.920917e-02	7.192763e-02	1.902331e-01

Table 2: **Comparison of the S_8 GQ Method 1 and the GQ Method flux moments at the center of the y^- face.**