Homework 1

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Problem 1:

Henyey and Greenstein (1941) introduced a function which, by the variation of one parameter, $1 \leq h \leq 1$, ranges from backscattering through isotropic scattering to forward scattering. In our notation we can write this as

$$K(\mu_0, v' \to v) = \frac{1}{2} \frac{1 - h^2}{(1 + h^2 - 2h\mu_0)^{3/2}} \delta(v' - v). \tag{1}$$

Verify that this is a properly normalized $f(\mu_0)$ and compute $K_l(v' \to v)$ for $l = 0, 1, 2$ as a function of h.

Solution:

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- First, expand scattering kernel in Legendre Polynomials.
- Begin w/ Legendre generating fn:

$$\frac{1}{\sqrt{1-2xh+h^2}} = \sum_{n=0}^{\infty} P_n(x) h^n \qquad (1) \text{ [wiki]}$$

- Take a derivative w.r.t. x

$$\frac{h}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) h^n \qquad (2)$$

- Multiply both sides by $\frac{(1-h^2)}{2h}$:

$$K(x) = \frac{1}{2} \frac{(1-h^2)}{(1-2xh+h^2)^{3/2}} = \frac{1}{2}\sum_{n=0}^{\infty} P_n'(x) h^{n-1}(1-h^2) \qquad (3)$$

- (3) is our scattering kernel. Now we need to eliminate the derivative in terms of $P_n(x)$. Rewrite sum by letting $n \to n+1$

$$\frac{1}{2}\sum_{n=0}^{\infty} P_n'(x) h^{n-1}(1-h^2) = \frac{1}{2}\sum_{n=0}^{\infty} P_{n+1}'(x) h^{n}(1-h^2) \qquad (4)$$

- Note that the sum still starts at zero because $P_0'(x)=0$, so we can trivially add this term. From Abromowitz:

$$P_{n+1}'(x) = \sum_{m \text{ even}}^{n} (2m+1) P_m(x) \qquad (5)$$

- (5)→(4) and write as two sums:

$$K(x) = \frac{1}{2}\sum_{n=0}^{\infty} \sum_{\substack{m \\ \text{even}}}^{n} (2m+1) P_m(x) h^n - \frac{1}{2}\sum_{n=0}^{\infty} \sum_{\substack{m \\ \text{even}}}^{n} (2m+1) P_m(x) h^{n+2} \qquad (6)$$

* Note: Here orthogonality is defined as $\int P_m P_n = \frac{\delta_{mn}\, 2}{2n+1}$ *

- Shift the second infinite sum by $n'=n+2$

$$K=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{m\\ \text{even}}}^{n}(2m+1)P_m(x)h^n-\frac{1}{2}\sum_{n'=2}^{\infty}\sum_{\substack{m\\ \text{even}}}^{n'-2}(2m+1)P_m(x)h^{n'}$$

- Combine the two series, writing out $n=0,1$ & letting $n'\to n$

$$K=\frac{1}{2}\sum_{n=0}^{1}(2n+1)P_n(x)h^n+\frac{1}{2}\sum_{n=2}^{\infty}\left(\sum_{\substack{m\\ \text{even}}}^{n}(2m+1)P_m(x)-\sum_{\substack{m\\ \text{even}}}^{n-2}(2m+1)P_m(x)\right)h^n$$

- For the inner sums all terms but n cancel, adding back energy:

$$\boxed{K(\mu_0,\nu'\to\nu)=\frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\mu_0)h^n\delta(\nu'-\nu)} \qquad (7)$$

- Taking legendre moments:

$$K_\ell(\nu'\to\nu)=\frac{\delta(\nu'-\nu)}{2}\sum_{n=0}^{\infty}(2n+1)h^n\int_{-1}^{1}d\mu_0\,P_n(\mu_0)P_\ell(\mu_0)$$

$$K_\ell(\nu'\to\nu)=\delta(\nu'-\nu)\sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)h^n\frac{\delta_{n\ell}2}{(2\ell+1)}$$

$$\boxed{K_\ell(\nu'\to\nu)=h^\ell}$$

- Noting $\int_0^\infty d\nu\int_{-1}^{1}d\mu_0(\cdot)=\int_0^\infty d\nu\int_{-1}^{1}P_0(\mu_0)(\cdot)$; $\int_0^\infty d\nu\int_{-1}^{1}d(\mu_0)K(\mu_0,\nu'\to\nu)=h^0=1$,

So normalization is correct.

$$\boxed{\begin{aligned}K_0&=1\\K_1&=h\\K_2&=h^2\end{aligned}}$$

Problem 2:

In an elastic scatter between a neutron and a nucleus, the scattering angle in the center of mass system is related to the energy change as

$$\frac{E}{E'} = \frac{1}{2}\left((1+\alpha) + (1-\alpha)\cos\theta_c\right) \tag{2}$$

where E is the energy after scattering and E' is the initial energy of the neutron and

$$\alpha = \frac{(A-1)^2}{(A+1)^2}. \tag{3}$$

The scattered angle in the center-of-mass system is related the lab-frame scattered angle as

$$\tan\theta_L = \frac{A\sin\theta_c}{1 + A\cos\theta_c} \tag{4}$$

Also, the distribution of scattered energy is given by

$$P(E' \to E) = \begin{cases} \frac{1}{(1-\alpha)E'} & E'\alpha \leq E \leq E' \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Derive an expression for $K(\mu_0, E' \to E)$, where μ_0 is $\cos\theta_L$. What is the distribution in angle of neutrons of energy in the range [0.05 MeV, 10 MeV] to energies below 1 eV if the scatter is with hydrogen?

Solution:

Due to Eq. (2), for a fixed A, a given value of E and E' fully define μ_c. As a result, the shape of the doubly differential scattering cross section in the center of mass (COM) system is fully defined by the probability density function (PDF) $P(E' \to E)$. Thus, it is possible to write the scattering cross section in the COM frame as [1]

$$\Sigma_s(E, \mu_0) = \sigma_S(E')(P(E' \to E)\delta(\mu_c - f(E, E')) \tag{6}$$

where $f(E, E')$ is the value of μ_c that satisfies Eq. (2) for a given E, i.e.,

$$f(E, E') = \mu_c = \frac{1}{2}\left[(A+1)\sqrt{\frac{E'}{E}} - (A-1)\sqrt{\frac{E}{E'}}\right]. \tag{7}$$

Since we are interested in s, we only need to transform the PDF $P(E' \to E)$ into a density function $P(\mu_0)$. From Eq. (**??**), there is a one-to-one relationship between E and $\mu_c = \cos(\Theta_c)$, thus

$$P(E' \to E)\mathrm{d}E = P(\mu_c)\mathrm{d}\mu_C. \tag{8}$$

or

$$P(\mu_c) = P(E' \to E)\frac{\mathrm{d}E}{\mathrm{d}\mu_c}. \tag{9}$$

Differentation of Eq. (2) and multiplication by E' yields

$$\frac{\mathrm{d}E}{\mathrm{d}\mu_c} = \frac{1}{2}(1-\alpha)E' \tag{10}$$

Evaluating μ_c for E at the limits $\alpha E'$ and E' gives the support for $P(\mu_c)$, defined for $\mu_c \in [-1, 1]$. Substitution of the above equation and Eq. (5) into Eq. (9) gives the desired PDF

$$P(\mu_c) = \frac{1}{(1-\alpha)E'}\left(\frac{1}{2}(1-\alpha)E'\right) = \frac{1}{2}, \quad \mu \in [-1, 1] \tag{11}$$

References

[1] W.L. Dunn and J.K. Shultis, *Exploring Monte Carlo Methods*, 2012.