# Homework 1

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NUEN 629

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# NUEN 629, Homework 1

Due Date Sept. 17

## 1 Ayzman

(10 points) Henyey and Greenstein (1941) introduced a function which, by the variation of one parameter, $-1 \leq h \leq 1$, ranges from backscattering through isotropic scattering to forward scattering. In our notation we can write this as

$$K(\mu_0, v' \to v) = \frac{1}{2} \frac{1-h^2}{(1+h^2-2h\mu_0)^{3/2}} \delta(v'-v).$$

Verify that this is a properly normalized $f(\mu_0)$ and compute $K_l(v' \to v)$ for $l = 0, 1, 2$ as a function of $h$.

## 2 Bolding

(20 points) In an elastic scatter between a neutron and a nucleus, the scattering angle in the center of mass system is related to the energy change as

$$\frac{E}{E'} = \frac{1}{2}\left((1+\alpha) + (1-\alpha)\cos\theta_c\right),$$

where $E$ is the energy after scattering and $E'$ is the initial energy of the neutron and

$$\alpha = \frac{(A-1)^2}{(A+1)^2}.$$

The scattered angle in the center-of-mass system is related the lab-frame scattered angle as

$$\tan\theta_L = \frac{\sin\theta_c}{A^{-1} + \cos\theta_c}.$$

Also, the distribution of scattered energy is given by

$$P(E' \to E) = \begin{cases} \frac{1}{(1-\alpha)E'}, & \alpha E' \leq E \leq E' \\ 0 & \text{otherwise} \end{cases}.$$

Derive an expression for $K(\mu_0, E' \to E)$, where $\mu_0$ is $\cos\theta_L$. What is the distribution in angle of neutrons of energy in the range $[0.05\,\text{MeV}, 10\,\text{MeV}]$ to energies below 1 eV if the scatter is hydrogen?

## 3

(70 points) Consider an infinite square lattice of infinitely tall cylindrical UO2 fuel pins in water. A quarter of a pin cell looks for a square lattice is shown in Fig. 1 and an infinite hex lattice in Fig. 2. The cross-section data for each is given in Table 1. The neutron transport equation for this problem is given simply by

$$\hat{\Omega} \cdot \nabla \psi(x, y, \hat{\Omega}) + \Sigma_t \psi(x, y, \hat{\Omega}) = \frac{1}{4\pi} \Sigma_s \phi(x, y) + \frac{Q}{4\pi}.$$

You may choose whichever lattice you wish – square or hex. For one or the other, perform the following:

Table 1: Data for Test Problem

| | Fuel | Moderator |
|---|---|---|
| $\Sigma_t$ ($\text{cm}^{-1}$) | 0.1414 | 0.08 |
| $\Sigma_s$ ($\text{cm}^{-1}$) | 0 | 0 |
| $Q$ ($\text{n/cm}^3\cdot\text{s}$) | 1 | 0 |

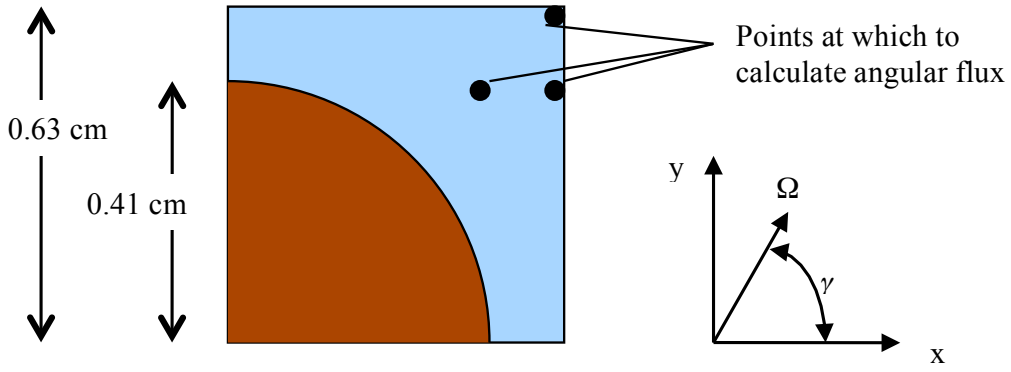



Figure 1: Quarter of a pin cell of infinite square lattice problem. The azimuthal angle $\varphi$ is written as $\gamma$ in the figure.

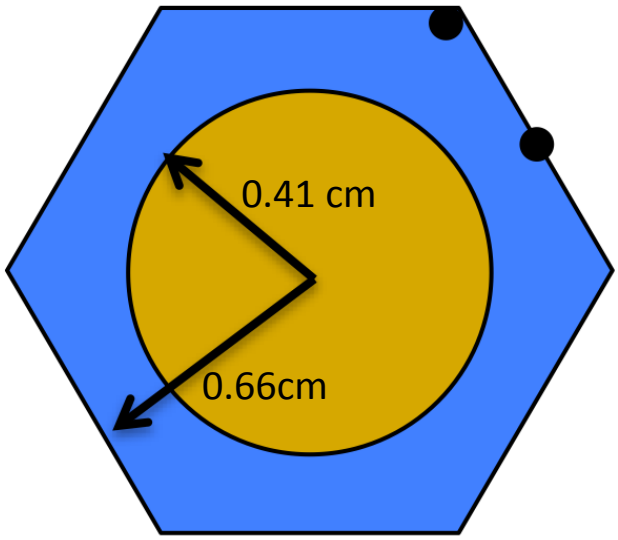



Figure 2: Pin cell of infinite hex lattice problem.

1. Calculate the angular flux as a function of the azimuthal angle, $\varphi$, at the spatial points indicated in the figure (two points for the hex lattice; three for the square). Use two different polar angles: $\pi/2$, which means the neutrons are traveling in the x-y plane, and $\pi/8$. Use the dimensions and cross sections from Table 1. Note that to simplify the problem we have abolished scattering. In the highest energy group of a fine-group set, there is very little within-group scattering, so it does not change the problem very much to ignore scattering. We have also assumed that the neutrons are born uniformly in the fuel – a flat radial profile. This isn't precisely true, but again, the simplification does not change the character of the solution that we wish to study. You will need to write a simple computer program for this in whatever language you'd like. You will need to trace rays and compute points of intersection. When you reach the boundary of a pin cell you use a periodic boundary condition to translate the ray across the cell, and then you continue. You will need a strategy to know how far a ray must be traced before you say "enough." Your code should accept as input:

   (a) the number of values of the azimuthal angle, $\varphi$ at which to calculate the angular flux;

   (b) the precision to which to calculate the angular flux at a given spatial point and given $\varphi$. (This tells you when you can say "enough." You can say "enough" when you've traced through $\tau$ mean free paths, where $\exp(-\tau) =$ the requested precision.)

2. Convince me that your code calculates the angular flux correctly.

3. Plot the angular flux as a function of $\varphi$ for each of the two polar angles, for each of the three spatial points (two if hex lattice). Use enough $\varphi$ values to convince yourself that you have resolved all the significant bumps and wiggles in the angular flux. Discuss your plots, and in particular compare them against what was shown in the notes for square pins. Do the circles make things smoother? Be prepared to present your solutions to the class, and (see part above) be prepared to argue that they are correct.

**Problem 1:**

Henyey and Greenstein (1941) introduced a function which, by the variation of one parameter, $1 \leq h \leq 1$, ranges from backscattering through isotropic scattering to forward scattering. In our notation we can write this as

$$K(\mu_0, v' \to v) = \frac{1}{2} \frac{1 - h^2}{(1 + h^2 - 2h\mu_0)^{3/2}} \delta(v' - v). \tag{1}$$

Verify that this is a properly normalized $f(\mu_0)$ and compute $K_l(v' \to v)$ for $l = 0, 1, 2$ as a function of $h$.

**Solution:**

Solution:

- First, expand scattering kernel in Legendre Polynomials.
- Begin w/ Legendre generating fn:

$$\frac{1}{\sqrt{1-2xh+h^2}^1} = \sum_{n=0}^{\infty} P_n(x) h^n \qquad (1)\ [\text{wiki}]$$

- Take a derivative w.r.t. $x$

$$\frac{h}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x) h^n \qquad (2)$$

- Multiply both sides by $\frac{(1-h^2)}{2h}$:

$$K(x) = \frac{1}{2}\frac{(1-h^2)}{(1-2xh+h^2)^{3/2}} = \frac{1}{2}\sum_{n=0}^{\infty} P_n'(x) h^{n-1}(1-h^2) \qquad (3)$$

- (3) is our scattering kernel. Now we need to eliminate the derivative in terms of $P_n(x)$. Rewrite sum by letting $n \to n+1$

$$\frac{1}{2}\sum_{n=0}^{\infty} P_n'(x) h^{n-1}(1-h^2) = \frac{1}{2}\sum_{n=0}^{\infty} P_{n+1}'(x) h^n (1-h^2) \qquad (4)$$

- Note that the sum still starts at zero because $P_0'(x) = 0$, so we can trivially add this term. From Abromowitz:

$$P_{n+1}'(x) = \sum_{m\ \text{even}}^{n} (2m+1) P_m(x) \qquad (5)$$

- (5)→(4) and write as two sums:

$$K(x) = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m\ \text{even}}^{n} (2m+1) P_m(x) h^n - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m\ \text{even}}^{n} (2m+1) P_m(x) h^{n+2} \qquad (6)$$

* Note: Here orthogonality is defined as $\int P_m P_n = \frac{\delta_{mn}\, 2}{2n+1}$ *

- Shift the second infinite sum by $n' = n+2$

$$K = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{m \\ \text{even}}}^{n}(2m+1)P_m(x)h^n - \frac{1}{2}\sum_{n'=2}^{\infty}\sum_{\substack{m \\ \text{even}}}^{n'-2}(2m+1)P_m(x)h^{n'}$$

- Combine the two series, writing out $n=0,1$ & letting $n' \to n$

$$K = \frac{1}{2}\sum_{n=0}^{1}(2n+1)P_n(x)h^n + \frac{1}{2}\sum_{n=2}^{\infty}\left(\sum_{\substack{m \\ \text{even}}}^{n}(2m+1)P_m(x) - \sum_{\substack{m \\ \text{even}}}^{n-2}(2m+1)P_m(x)\right)h^n$$

- For the inner sums all terms but $n$ cancel, adding back energy:

$$\boxed{K(\mu_0, \nu' \to \nu) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\mu_0)h^n\delta(\nu'-\nu)} \qquad (7)$$

- Taking legendre moments:

$$K_\ell(\nu' \to \nu) = \frac{\delta(\nu'-\nu)}{2}\sum_{n=0}^{\infty}(2n+1)h^n\int_{-1}^{1}d\mu_0\,P_n(\mu_0)P_\ell(\mu_0)$$

$$K_\ell(\nu' \to \nu) = \delta(\nu'-\nu)\sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)h^n\frac{\delta_{n\ell}2}{(2\ell+1)}$$

$$\boxed{K_\ell(\nu' \to \nu) = h^\ell}$$

- Noting $\int_0^\infty d\nu\int_{-1}^{1}d\mu_0(\cdot) = \int_0^\infty d\nu\int_{-1}^{1}P_0(\mu_0)(\cdot)$; $\int_0^\infty d\nu\int_{-1}^{1}d(\mu_0)\,K(\mu_0,\nu'\to\nu) = h^0 = 1$,

so normalization is correct.

$$\boxed{\begin{aligned} K_0 &= 1\,\delta(\nu-\nu') \\ K_1 &= h\,\delta(\nu-\nu') \\ K_2 &= h^2\delta(\nu-\nu') \end{aligned}}$$

**Problem 2:**

In an elastic scatter between a neutron and a nucleus, the scattering angle in the center of mass system is related to the energy change as

$$\frac{E}{E'} = \frac{1}{2}\left((1+\alpha) + (1-\alpha)\cos\theta_c\right) \tag{2}$$

where $E$ is the energy after scattering and $E'$ is the initial energy of the neutron and

$$\alpha = \frac{(A-1)^2}{(A+1)^2}. \tag{3}$$

The scattered angle in the center-of-mass system is related the lab-frame scattered angle as

$$\tan\theta_L = \frac{A\sin\theta_c}{1+A\cos\theta_c} \tag{4}$$

Also, the distribution of scattered energy is given by

$$P(E'\to E) = \begin{cases} \frac{1}{(1-\alpha)E'} & E'\alpha \le E \le E' \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Derive an expression for $K(\mu_0, E'\to E)$, where $\mu_0$ is $\cos\theta_L$. What is the distribution in angle of neutrons of energy in the range [0.05 MeV, 10 MeV] to energies below 1 eV if the scatter is with hydrogen?

**Solution:**

**Scattering Kernel Derivation**

Due to Eq. (2), for a fixed $A$, a given value of $E$ and $E'$ fully define $\mu_c$; the lab frame cosine of the scattering angle $\mu_0$ is also fully defined through Eq. (4). As a result, the shape of the doubly differential scattering cross section is fully defined by the probability density function (PDF) $P(E'\to E)$. Thus, it is possible to write the scattering cross section in the COM frame as [1]

$$\Sigma_s(\mu_0, E'\to E) = \Sigma_s(E')P(E'\to E)\delta(\mu_c - f_\mu(E,E')) \tag{6}$$

where $f_\mu(E,E')$ is the value of $\mu_c$ that satisfies Eq. (2) for a given $E$, i.e.,

$$f_\mu(E,E') = \frac{2(\frac{E}{E'}) - (1+\alpha)}{(1-\alpha)} \tag{7}$$

Because we are interested in the scattering kernel as a function of the lab frame cosine $\mu_0$, we define the scattering cross section in an equivalent form

$$\Sigma_s(\mu_0, E'\to E) = \Sigma_s(E')P(\mu_0)\delta(E - f_E(\mu_c(\mu_0), E')) \tag{8}$$

where $P(\mu_0)$ is a PDF for $\mu_0$ given a certain value of $E'$, $f_E$ is defined as

$$f_E(\mu_C, E') = \frac{E'}{2}\left((1+\alpha) + (1-\alpha)\mu_c\right), \tag{9}$$

and $\mu_c$ as a function of $\mu_0$ will be derived later in Eq. (21). The scattering kernel is defined as

$$K(\mu_0, E' \to E) = \frac{\Sigma_s(E' \to E, \mu_0)}{\int_0^\infty \mathrm{d}E \int_{-1}^{1} \mathrm{d}\mu_0 \, \Sigma_s(E' \to E, \mu_0)} \tag{10}$$

The denominator is evaluated as

$$\int_{-1}^{1} \mathrm{d}\mu_0 \int_0^\infty \mathrm{d}E \; \Sigma_s(E')P(\mu_0)\delta(E - f_E(\mu_c(\mu_0), E)) = \Sigma_s(E') \int_{-1}^{1} \mathrm{d}\mu_0 P(\mu_0) = \Sigma_s(E') \tag{11}$$

where the first equality is true because the argument of the delta function is zero for the value of $\mu_0$ and $E'$ that satisfy $f$, which in this case gives the $\mu_0$ that is the integration variable of the outer integral. The scattering Kernel is then just

$$K(\mu_0, E' \to E) = P(\mu_0)\delta(E - f_E(\mu_C(\mu_0), E)). \tag{12}$$

We now need to transform the PDF $P(E' \to E)$ into a density function $P(\mu_0)$. From Eq. (2), there is a one-to-one relationship between $E$ and $\mu_c = \cos(\Theta_c)$ in the range of $E \in [\alpha E', E']$, thus

$$P(E' \to E)\mathrm{d}E = P(\mu_c)\mathrm{d}\mu_c \tag{13}$$

or

$$P(\mu_c) = P(E' \to E)\frac{\mathrm{d}E}{\mathrm{d}\mu_c}. \tag{14}$$

Multiplication of Eq. (2) by $E'$, followed by differentiation, yields

$$\frac{\mathrm{d}E}{\mathrm{d}\mu_c} = \frac{1}{2}(1-\alpha)E' \tag{15}$$

Evaluating $\mu_c$ for $E$ at the limits $\alpha E'$ and $E'$ gives the support for $P(\mu_c)$, defined for $\mu_c \in [-1, 1]$. Substitution of the above equation and Eq. (5) into Eq. (14) gives the PDF in the COM frame

$$P(\mu_c) = \frac{1}{(1-\alpha)E'}\left(\frac{1}{2}(1-\alpha)E'\right) = \frac{1}{2}, \quad \mu_c \in [-1, 1] \tag{16}$$

We must now transform to the lab frame scattering cosine $\mu_0$. First, we solve Eq. (4) for $\mu_0$ in terms of $\mu_c$ as follows:

$$\tan^2\theta_L = \left(\frac{A\sin\theta_c}{1+A\cos\theta_c}\right)^2 \tag{17}$$

$$\sec^2\theta_L - 1 = \left(\frac{A\sin\theta_c}{1+A\cos\theta_c}\right)^2 \tag{18}$$

$$\mu_0^{-2} = \frac{A^2(\sin^2\theta_c + \cos^2\theta_c) + 1 + 2A\mu_c}{(1+A\mu_c)^2} \tag{19}$$

$$\mu_0 = \frac{1+A\mu_c}{\sqrt{1+2\mu_c A + A^2}}. \tag{20}$$

Solution of the above equation for $\mu_c$ in terms of $\mu_L$ gives

$$\mu_c = -\frac{1}{A}(1-\mu_0^2) + \mu_0\sqrt{1-\frac{1}{A^2}(1-\mu_0^2)}\ . \tag{21}$$

Eq. (20) demonstrates a one-to-one relationship between $\mu_0$ and $\mu_C$. As before,

$$P(\mu_0) = P\left(\mu_C(\mu_0)\right)\frac{\mathrm{d}\mu_c}{\mathrm{d}\mu_0}. \tag{22}$$

Differentiation of Eq. (21) with respect to $\mu_0$ and algebraic manipulation ultimately yields

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}\mu_0} = \frac{2\mu_0}{A} + \frac{1-\frac{1}{A^2}(1-2\mu_0^2)}{\sqrt{1-\frac{1}{A^2}(1-\mu_0^2)}}. \tag{23}$$

Substitution of the above equation and Eq. (16) into Eq. (22) gives an expression for $P(\mu_0)$. The final expression for the scattering kernal is, for $A > 1$

$$\boxed{K(\mu_0, E'\to E) = \begin{cases} \left[\dfrac{\mu_0}{A} + \dfrac{1-\frac{1}{A^2}(1-2\mu_0^2)}{2\sqrt{1-\frac{1}{A^2}(1-\mu_0^2)}}\right]\delta(E - f_E\left(\mu_c(\mu_0), E'\right)), & \mu_0\in[-1,1] \\ 0, & \text{otherwise} \end{cases}}. \tag{24}$$

where the support is from evaluation of Eq. (20) at $\mu_c = -1, 1$. The case of $A = 1$ must be treated separately. This can be seen, for instance, because evaluation of Eq. (20) at $\mu_c = -1$ results in an indeterminant $0/0$. Evaluation of Eq. (21) for $A = 1$ gives a non-indeterminant expression for $\mu_0$ as

$$\mu_0 = \sqrt{\frac{1+\mu_c}{2}} \tag{25}$$

Thus, the support becomes $\mu_0 \in [0, 1]$. The kernel also simplifies significantly at $A = 1$. The final scattering kernel, for the case of $A = 1$, is

$$\boxed{K(\mu_0, E'\to E) = \begin{cases} 2\mu_0\,\delta(E - f_E\left(\mu_c(\mu_0), E'\right)), & \mu_0\in[0,1] \\ 0, & \text{otherwise} \end{cases}}. \tag{26}$$

which is a PDF normalized over $\mu_0$ and $E$. The support of $E$ is implicitly defined by the value of $E'$ and the support of $\mu_0$.

**Plots for A=1**

To plot the desired distributions, we need to know the equivalent $\mu_0$ corresponding to a given $E$ and $E'$. Evaluation of Eq. (2) at $A = 1$ gives

$$\frac{E}{E'} = \frac{1 + \mu_c}{2}. \tag{27}$$

Then, using Eq. (25), $\mu_0$ in terms of $E$ and $E'$ is

$$\mu_0 = \sqrt{\frac{E}{E'}}. \tag{28}$$

For a given $E' = E_i$, we can get the distribution in angle $P(\mu_0)$ by integrating the kernel over the range of desired outgoing energies $E \in [0, E_{\max}]$. The kernel is a joint PDF in $E$ and $\mu_0$ (for scattering into d$E$ about $E$ and d$\mu_0$ about $\mu_0$), whereas $E'$, as we have written it, is just a parameter of the distribution. Thus, performing the integration, for a particular $E'$, gives

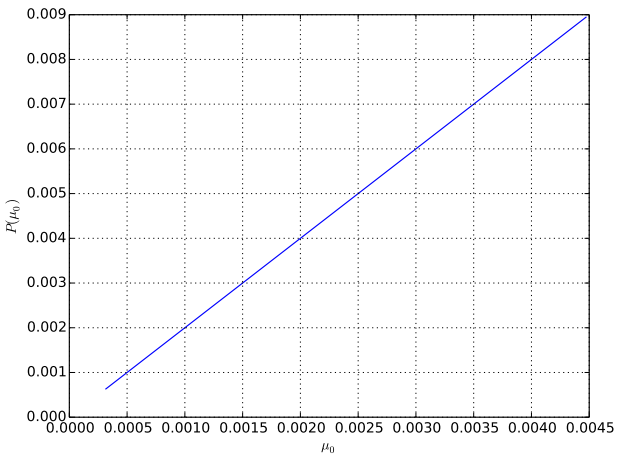
$$P(\mu_0, 0 \leq E \leq E_{\min}; E' = E_i) = \int_0^{E_{\min}} \mathrm{d}E 2\mu_0 \delta(E - f_E(E, E')) \tag{29}$$

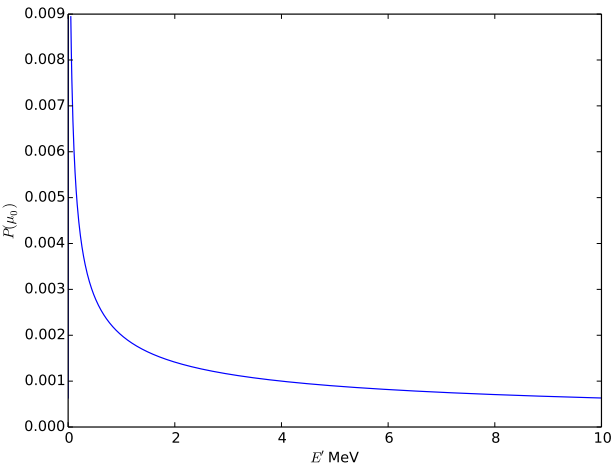
The delta function argument is now only zero for values of $\mu_0$ and $E'$ that satisfy $0 \leq E <\leq E_{\min}$, thus it limits the support of $\mu_0$. So the PDF becomes, using Eq. (28),

$$P(\mu_0, 0 \leq E \leq E_{\min}; E' = E_i) = 2\mu_0, \quad 0 \leq \mu_0 \leq \sqrt{\frac{E_{\min}}{E'}}. \tag{30}$$

Physically, if $E'$ is too much larger than $E$, it cannot undergo a scatter at low angles and lose energy to $E$.

A plot of $P(\mu_0, 0 \leq E \leq 1 \text{ eV})$ vs $\mu_0$ and $P(\mu(E', E))$ vs $E'$ are given below. The values of $E'$ range between 0.05 MeV to 10 MeV, with $E$ fixed at 1 eV. As expected, lower energy neutrons are more likely to scatter to 1 eV.




# References


[1] W.L. Dunn and J.K. Shultis, *Exploring Monte Carlo Methods*, 2012.

**Problem 3:**

The problem details are given on the second page.

**Solution:**

**Description of code**

The angular flux $\psi$ is computed by tracing characteristics as discussed in class. To compute points of intersection, the ray and surfaces of intersection are written in parametric form. The position of a particle in the projected $x-y$ plane is denoted $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. Since we want to trace upstream, the parametric equation for the particle position is given by

$$\mathbf{r} = (x_{i-1} - \Omega_x s)\hat{\mathbf{i}} + (y_{i-1} - \Omega_y s)\hat{\mathbf{j}} \tag{31}$$

where $\mathbf{r}_{i-1}$ is the previous location, $s$ is a parameter that corresponds to the signed distance the ray has traversed, and

$$\Omega_x = \sin(\theta)\cos(\phi) \tag{32}$$
$$\Omega_y = \sin(\theta)\sin(\phi). \tag{33}$$

The parametric equation for each of the surfaces in the problem as a function of $x$ and $y$ are given in Table 1. These equations are then evaluated with $x = x_{i-1} - \Omega_x s$ and $y = y_{i-1} - \Omega_y s$ and solved for $s$ algebraically. This produces a collection of values $\{s_m\}$, where $s_m$ represent the signed distance to the $m$-th surface (excluding surfaces where a solution does not exist). The smallest positive value of $s_m$ corresponds to the next point of intersection. If we were already at a surface, then that equation will give $s_m = 0$. Care is taken to exclude this solution, accounting for potential roundoff.

Table 1: Parametric equations for surfaces in problem.

| Surface | $f(x,y) = 0$ |
|---|---|
| Fuel | $x^2 + y^2 - R^2_{\text{fuel}} = 0$ |
| Left Boundary | $x - x_{\min} = 0$ |
| Right Boundary | $x - x_{\max} = 0$ |
| Bottom Boundary | $y - y_{\min} = 0$ |
| Top Boundary | $y - y_{\max} = 0$ |

The current position of the ray is updated, and the number of mean free paths traveled $\tau_i = s_i \Sigma_t(x,y)$ along the $i$-th path is computed. The total number of MFP traveled up to the latest point $s_i$ is accumulated as $\tau_{\text{tot},i} = \sum_{k=1}^{i} \tau_k$.

Because the transport equation is linear, we can consider the contribution from each fuel element to the angular flux separately. If the $i$-th path traced to point $\mathbf{r}_i$ was across a fuel

element, then a contribution is made to the flux. If the path of length $s_i$ crossed the $j$-th fuel element, the contribution to the flux from that fuel element is computed as

$$\psi_j = \frac{Qe^{-\tau_{\text{tot},i-1}}}{4\pi\Sigma_{t,F}}\left(1 - e^{-\Sigma_{t,F}s_i}\right) \tag{34}$$

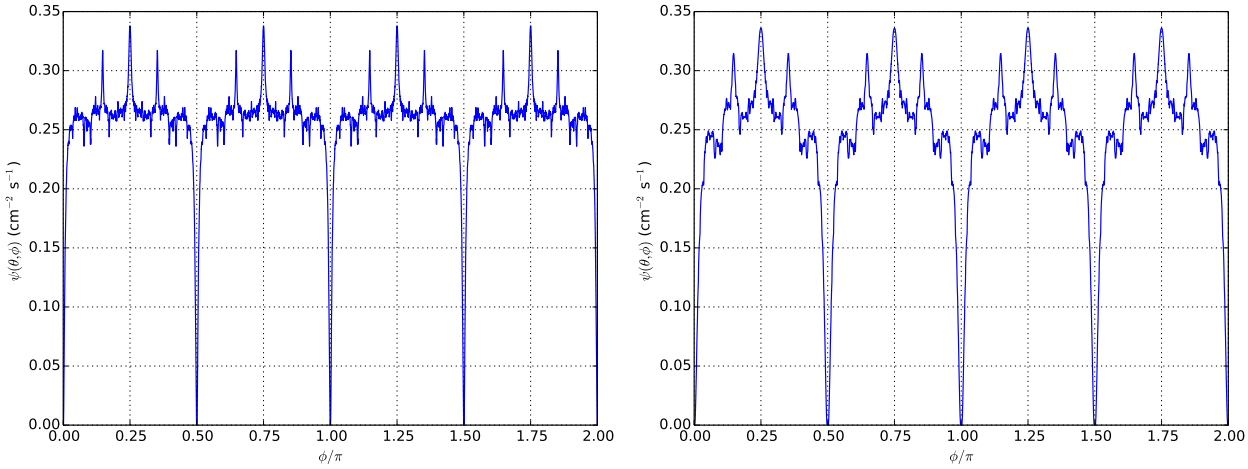
where $\tau_{\text{tot},i-1}$ does not include $s_i\Sigma_{t,F}$ because that attenuation was accounted for in derivation of the term in parenthesis. The ray tracing is then continued from this point until $\tau_{\text{tot,i}} > \tau_{\max}$.

Finally, after computing potential contributions to $\psi$, if the ray hits a boundary, the corresponding coordinate is translated to the opposing boundary. For example, if the right boundary is hit at point $\mathbf{r} = x_{\max}\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$, the new position is $\mathbf{r} = x_{\min}\hat{\mathbf{i}} + y_1\hat{\mathbf{j}}$. Tracing then continues as before, by computing new distances to intersections, with $\hat{\Omega}$ unchanged. Care is taken to handle roundoff issues and corners.

The final solution for $\psi$, at the location and direction of interest, will be

$$\psi(\mathbf{r}, \hat{\Omega}) = \sum_{j}^{N_{\text{fuel}}} \psi_j \tag{35}$$

where $N_{\text{fuel}}$ is the total number of fuel elements crossed during the ray tracing. The process outlined above is repeated for all equally spaced $\phi \in [0, 2\pi)$, for the polar angle and position of interest.




(a) $\theta = \pi/2$ (b) $\theta = \pi/8$

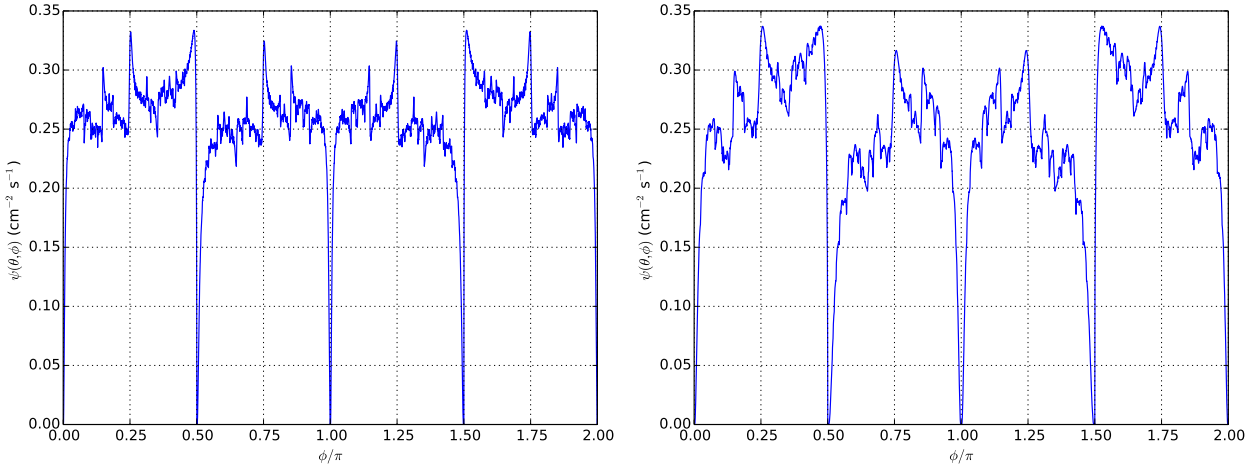
Figure 1: Angular flux results at $x = 0.63$ cm $y = 0.63$ cm for 8000 azimuthal angles.

## Results

To verify the code works, the algorithm was modified to handle a source in the moderator, and the cross sections were set to uniform values.

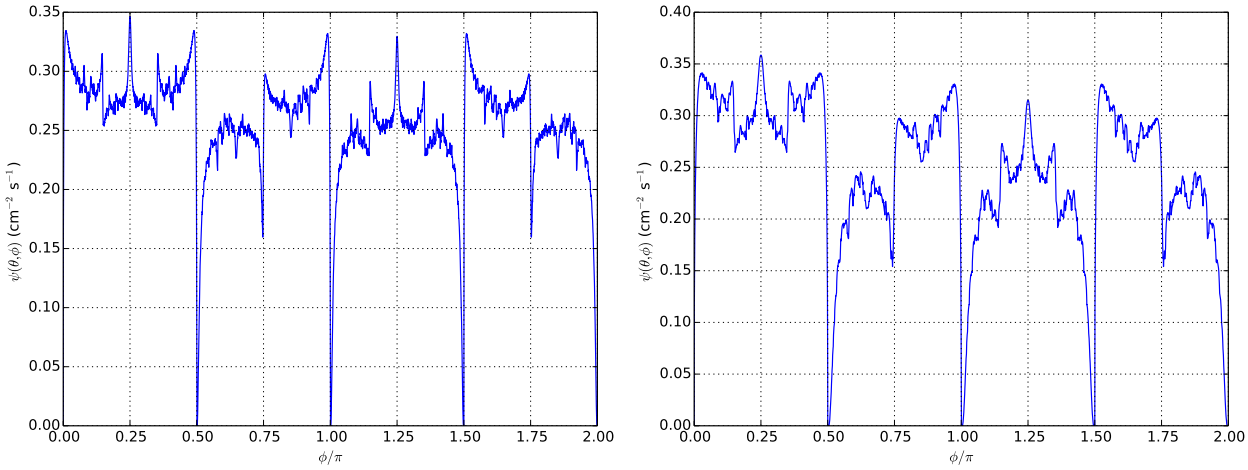
In addition, the code was checked against other students codes who developed algorithms independently and were found to agree. Although this is not proof the solution method is correct, it indicates that both students likely implemented the algorithm correctly.

```
import numpy as np
import re
from math import *
from scipy.optimize import fsolve
import matplotlib.pyplot as plt

def main(n_azimuth, polar_ang, tol=1.e-12):

    # Define geometry parameters based on origin is center of fuel pin
    x_left  = -0.63
    x_right =  0.63
    y_top   =  0.63
    y_bot   = -0.63
    radius  =  0.41

    x_start = 0.62999999
    y_start = 0.41

    #Must be 'definitely' inside or algorithm will fail
    if abs(abs(x_start)-x_right) < 1.E-10 or abs(abs(y_start)-y_top) < 1.E-10:
        raise IOError("Must start at a point inside boundary or problems will occur")

    #cross sections
    sigma_f = 0.1414
    sigma_m = 0.08
    Q_f_tot = 1.
    q_f     = Q_f_tot/(4.*pi)
    q_mod   = 0.0

    debug_mode = False
    if debug_mode:
        sigma_m = sigma_f
```




(a) $\theta = \pi/2$ (b) $\theta = \pi/8$

Figure 2: Angular flux results at $x = 0.41$ cm $y = 0.63$ cm for 8000 azimuthal angles.




(a) $\theta = \pi/2$ (b) $\theta = \pi/8$

Figure 3: Angular flux results at $x = 0.63$ cm $y = 0.63$ cm for 8000 azimuthal angles.

```
33          q_mod = q_f
34
35
36      phi_list = np.linspace(0.,2.*pi,num=n_azimuth+1) #add one to get endpoints
37      psi_list = []
38
39      #loop over azimuthal angles
40      for phi in phi_list:
41
42          #Pick the point of interest and trace upstream from it
43          x_prev = x_start
44          y_prev = y_start
45
46          #Pick direction
47          theta = polar_ang
48          xcos =  sin(theta)*cos(phi)
49          ycos =  sin(theta)*sin(phi)
50
51          print "Tracing phi (Omega) (max mfp)", phi, -1.*log(tol)
52
53          #Num of mfp we've traveled, and angular flux contribution to this point
54          psi = 0.0
55          n_mfp = 0.0
56          max_mfp = -1.*log(tol)
57
58          #We are ray tracing upstream, so flip cosines
59          xcos *= -1.
60          ycos *= -1.
61
62          #s is parametric length of vector
63          #f_circ can be used to check if circle hit or not
64          f_circ = lambda s: (x_prev + xcos*s)**2 + (y_prev + ycos*s)**2 - radius**2
65
66          while (n_mfp < max_mfp):
67
68              #Calculate all boundary intersections
69              if xcos == 0.:
70                  s_left = -99
71                  s_right = -99
72              else:
73                  s_left  = (x_left - x_prev)/xcos
74                  s_right = (x_right - x_prev)/xcos
75
76              if ycos == 0.:
77                  s_top = -99
78                  s_bot = -99
79              else:
80                  s_top   = (y_top  - y_prev)/ycos
81                  s_bot   = (y_bot - y_prev)/ycos
82
83              #Roots of parametric equation for circle
84              A = xcos**2 + ycos**2
85              B = 2.*xcos*x_prev + 2.*ycos*y_prev
86              C = x_prev**2 + y_prev**2 - radius**2
87              det = B*B - 4.*A*C
88
89              #Determine if we hit the circle, if so this overrides the boundary
90              if (det > 0): #We hit the circle
91
92                  #Roots of quadratic eq
93                  s_circ1 = (-1.*B + sqrt(det))/(2.*A)
94                  s_circ2 = (-1.*B - sqrt(det))/(2.*A)
95
96              else: #We didnt hit the circle, so we must have left
97
98                  s_circ1 = -99
99                  s_circ2 = -99
100
```

```
            #Find the min s that is positive, this is the face we hit
            intersects = [s_left, s_right, s_top, s_bot, s_circ1,s_circ2]
            s_min = min(i for i in intersects if i > 1.E-15 ) #ignore very small roots
            face_id = intersects.index(s_min)
            face_map = ["left","right","top","bot","circ1","circ2"]
            face = face_map[face_id]

            #Check if center of path is in fuel, in this case we were in the fuel
            r_cent = sqrt((x_prev + xcos*s_min*0.5)**2 + (y_prev + ycos*s_min*0.5)**2)
            if (r_cent < radius): #in the fuel:

                #Contribution to psi from this source is based on flux leaving fuel and how
                #many mfp it traveled to get to this point
                mfp_fuel = s_min*sigma_f
                psi     += q_f/(sigma_f)*(1.-exp(-1.*mfp_fuel))*exp(-1.*n_mfp)
                n_mfp   += mfp_fuel

            else: #just traveling in moderator

                n_mfp += s_min*sigma_m #We had to have been in moderator

                #For debugging we have a moderator source that we add in. It will contribute
                #however much is at previous point and how far it has had to attenuate
                if debug_mode:
                    psi    += q_mod/(sigma_f)*(1.-exp(-1.*s_min*sigma_m))*exp(-1.*(n_mfp - s_min*sigma_m))

            #Move to the new coordinates
            x_prev = x_prev + xcos*s_min
            y_prev = y_prev + ycos*s_min

            #Determine if we hit a boundary. Either we hit a circle or boundary
            if not re.search("circ",face):

                #Move to opposite face
                if face == "left":
                    x_prev = x_right
                elif face == "right":
                    x_prev = x_left
                elif face == "bot":
                    y_prev = y_top
                elif face == "top":
                    y_prev = y_bot
                else:
                    raise ValueError("Something wrong in faces")

                #Check if we are in a corner, based on symmetry, requires attention
                if x_left == y_bot: #Check for symmetry
                    if abs(abs(x_prev) - abs(y_prev)) < 1.E-13*abs(x_left):

                        #flip the face we haven't flipped yet
                        if face == "left" or face == "right":
                            y_prev *= -1.
                        else:
                            x_prev *= -1.


        #Done tracing for this phi
        if debug_mode:
            print "Desired solution: ", q_mod/sigma_f, q_mod, q_f
            print "error: ", (q_mod/sigma_f - psi)
            print "tol:   ", psi*tol

        psi_list.append(psi)

    #plot angular flux as a function of azimuthal angle
    plot_phi = [i/pi for i in phi_list]
    plt.plot(plot_phi,psi_list)
    plt.xlabel("$\phi/\pi$")
```

```
169         ax = plt.gca()
170         ax.set_xticks([0,0.25,0.5,0.75,1.0,1.25,1.5,1.75,2.0])
171         plt.ylabel(r"$\psi(\theta,\phi)$ (cm$^{-2}$ s$^{-1}$) ")
172         plt.grid()
173         plt.savefig("plot.pdf",bbox_inches='tight')
174 
175 
176 if __name__ == "__main__":
177         main(8000,pi/2.,tol=2.06115362e-13)
```