Homework 2

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NUEN 629, Homework 2

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1 Childs

(35 points) Compute three group cross-sections for a homogeneous mixture of graphite and natural uranium where the ratio of graphite to uranium is 150:1. You can assume the Watt-fission spectrum, and that the group bounds are $\{0, 1\,\text{eV}, 100\,\text{keV}, 20\,\text{MeV}\}$.

2 Franklin

(40 points) The enclosed file gives the microscopic cross-sections for ^{1}H in units of barns for 5 groups as calculated by the code `NJOY`. Imagine we have a large, nearly infinite tank of high-pressure hydrogen at 30 atm next to a bare sphere of ^{235}U. Compute the scalar flux ϕ_g and the current $\vec{J}_g$ in the hydrogen using the separable, P1 equivalent, and extended Legendre approximations. Compare your solutions graphically.

3 Geer

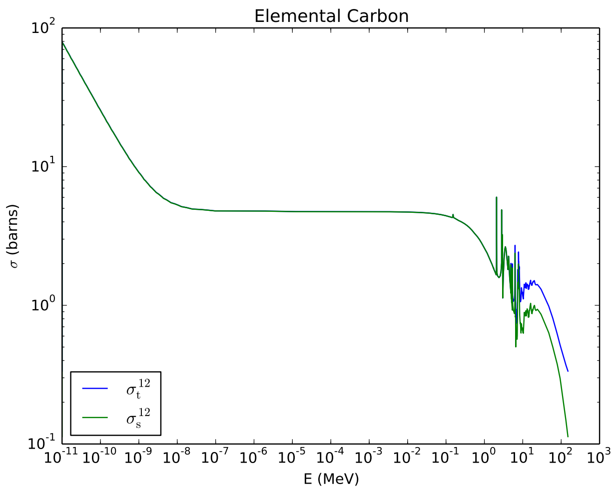
(25 points) Find the solution to the diffusion equation for 1-group, slab geometry with a uniform source, where the material is a pure scatter and the slab width is X under the following conditions

1. Vacuum Marshak conditions
2. Vacuum Mark conditions
3. Vacuum Dirichlet conditions
4. Vacuum Dirichlet condition on the left and albedo on the right at $X/2$, and
5. Vacuum Dirichlet condition on the left and reflecting on the right at $X/2$.

Compare the solutions and comment on the similarities and differences.

Solution 1:

Several approximations were made to simplify the process. First, graphite is approximated as elemental Carbon, with molar mass 12.0107 (g/mol). This was done because there is not a human friendly form of the graphite cross sections available on NNDC and elemental Carbon will have similar scattering properties, except for at low energies where diffraction is possible. A plot of the elastic and total cross sections for elemental Carbon from NNDC are given below.



The ratio of 150:1 for graphite to natural uranium is assumed to be an of atomic ratio. Natural uranium is taken to be 0.72% ^{235}U and the remainder ^{238}U, by atom percentage. The total cross section is assumed to only consist of elastic scattering, fission, and removal events.

We follow a similar procedure to the one in lab. For an infinite medium, with fine-group cross sections, the balance equation becomes

$$N^U \sum_j \gamma_j \sigma_t^j(E)\psi(\mu, E) = N^U \sum_j \gamma_j \frac{1}{2} \int_0^\infty \mathrm{d}E' \sigma_s^j(E' \to E)\phi(E') + N^U \frac{\chi(E)}{2k} \int_0^\infty \mathrm{d}E' \left(\gamma_{238}\overline{\nu}\sigma_f^{238} + \gamma_{235}\overline{\nu}\sigma_f^{235}\right)\phi(E'), \quad (1)$$

where j indicates the j-th isotope, N^U is the atom density of natural uranium in the system, and γ_j is 150, 0.9928, and 0.0072 for Carbon, ^{238}U, and ^{235}U, respectively. It is assumed $\chi(E)$ is the same for ^{238}U and ^{235}U, given by the Watt spectrum from class

$$\chi(E) = 0.4865 \sinh(\sqrt{2E})e^{-E}. \quad (2)$$

We now simplify by normalizing such that the energy integrated fission source has a magnitude of 1. We also assume all scattering events result in the average scattering energy loss, which, assuming isotropic scattering in the center of mass frame, gives an average outgoing energy of

$$\langle E \rangle = \frac{A^2+1}{(A+1)^2} E' \tag{3}$$

in the lab frame. With this simplificiation, only a particular E' governed by the above equation can scatter into E, so the the elastic scattering source for the j-th term in the summation can be simplified as

$$\int_0^\infty \mathrm{d}E' \sigma_s^j(E') P(E' \to E) \phi(E') = \int_0^\infty \mathrm{d}E' \sigma_s^j(E') \delta\left(E' - E\frac{E'}{\langle E \rangle}\right) \phi(E') \tag{4}$$

$$= \sigma_s^j \left(\frac{(A+1)^2}{A^2+1} E\right) \phi \left(\frac{(A+1)^2}{A^2+1} E\right) \tag{5}$$

$$\tag{6}$$

where A is the atomic mass number for the j-th isotope, approximated as 12.0107 for elemental carbon. Substituting back into the original equation and integrating over angle gives the final equation for the scalar flux as

$$\sum_j \gamma_j \sigma_t^j(E) \phi(E) = \sum_j \gamma_j \sigma_s^j \left(\frac{(A+1)^2}{A^2+1} E\right) \phi \left(\frac{(A+1)^2}{A^2+1} E\right) + \chi(E). \tag{7}$$

We solve this equation with the Jacobi iteration

$$\sum_j \gamma_j \sigma_t^j(E) \phi^{(k)}(E) = \sum_j \gamma_j \sigma_s^j \left(\frac{(A+1)^2}{A^2+1} E\right) \phi^{(k-1)} \left(\frac{(A+1)^2}{A^2+1} E\right) + \chi(E). \tag{8}$$

with an initial guess of $\phi^{(0)}(E) = 0$. To approximate the continuous energy cross sections and $\phi(E)$ we simply evaluate the above iteration at each of the energy points of the fine group cross sections. The points are defined using the union of the total cross section energy grids of all isotopes. A linear interpolation (python interp1D default interpolation) is used between energy points when one cross section is coarser than others. For evaluation above the maximum energy for a given cross section, the value of the cross section at the maximum energy is used.

3 Solution

- Since a pure scatterer, D.E. reduces to (for 1 speed, slabs)

$$-D\frac{d^2\phi}{dx^2} = Q \qquad x \in [-\tfrac{X}{2}, \tfrac{X}{2}] \qquad (1)$$

- w/ the general Brunner B.C.

$$A\phi + BD(\hat{n}\cdot\hat{i})\frac{\partial\phi}{\partial x} = C, \qquad x \in \partial V$$

- Solving for $\phi(x)$ on interior from (1):

$$\int -D\frac{d^2\phi}{dx^2}dx = \int Q\,dx$$

$$\frac{d\phi}{dx} = \frac{Qx}{D} + K_1$$

$$\phi(x) = \frac{Qx^2}{2D} + K_1x + K_2 \qquad (2)$$

- For all B.C. of interest, in Brunner form, $C=0$:

$$A_L\phi\left(-\tfrac{X}{2}\right) - B_L D\frac{d\phi}{dx}\Big|_{x=-\frac{X}{2}} = 0 \qquad (3)$$

$$A_R\phi\left(\tfrac{X}{2}\right) + B_R D\frac{d\phi}{dx}\Big|_{x=\frac{X}{2}} = 0$$

- (2)→(3)

$$A_L\left(\frac{QX^2}{8D} - \frac{K_1X}{2} + K_2\right) + B_L\left(\frac{QX}{2} - K_1D\right) = 0$$

$$A_R\left(\frac{QX^2}{8D} + \frac{K_1X}{2} + K_2\right) + B_R\left(\frac{QX}{2} + K_1D\right) = 0$$

Or $$\left(+\frac{A_L X}{2}+B_L D\right)K_1 - (A_L)K_2 = \frac{A_L Q X^2}{8D} + \frac{B_L Q X}{2}$$

$$\left(\frac{A_R X}{2}+B_R D\right)K_1 + A_R K_2 = -\left(\frac{A_R Q X^2}{8D} + \frac{B_R Q X}{2}\right)$$

- solving Algebraically gives:

$$K_1 = \frac{\left(\frac{A_L Q X^2}{8D} + \frac{B_L Q X}{2}\right) - \left(\frac{A_L}{A_R}\right)\left(\frac{A_R Q X^2}{8D} + \frac{B_R Q X}{2}\right)}{\left(\frac{A_L X}{2} + B_L D\right) + \left(\frac{A_L}{A_R}\right)\left(\frac{A_R X}{2} + B_R D\right)} \tag{4}$$

$$K_2 = \frac{-\left(\frac{A_L Q X^2}{8D} + \frac{B_L Q X}{2} + \left(\frac{\frac{A_L X}{2} + B_L D}{\frac{A_R X}{2} + B_R D}\right)\left(\frac{A_R Q X^2}{8D} + \frac{B_R Q X}{2}\right)\right)}{A_L + \left(\frac{\frac{A_L X}{2} + B_L D}{\frac{A_R X}{2} + B_R D}\right) A_R} \tag{5}$$

- The above expressions are evaluated and simplified for each case, The below table summarizes results on the next page.

A summary of the solutions obtained for each of the boundary conditions is given in the table below. Each solution was checked to ensure they satisfy the boundary conditions.

Table 1: Solutions with different boundary conditions for a pure scatter for slab of width X centered at $x = 0$.

Left BC	Right BC	$\phi(x)$
Vacuum Marshak	Vacuum Marshak	$\phi(x) = Q\left(\frac{X^2}{8D} + X - \frac{x^2}{2D}\right)$
Vacuum Mark	Vacuum Marshak	$\phi(x) = Q\left(\frac{X^2}{8D} + \frac{X\sqrt{3}}{2} - \frac{x^2}{2D}\right)$
Vacuum Dirichlet	Vacuum Dirichlet	$\phi(x) = \frac{Q}{2D}\left(\frac{X^2}{4} - x^2\right)$
Vacuum Dirichlet	Albedo	$\phi(x) = -\frac{Qx^2}{2D} + QxX\left(\frac{1+\frac{(1-\alpha)}{2(1+\alpha)}\frac{X}{2D}}{\frac{(1-\alpha)}{2(1+\alpha)}X+D} - \frac{1}{2D}\right) + Q\frac{X^2}{2}\left(\frac{1+\frac{(1-\alpha)}{2(1+\alpha)}\frac{X}{2D}}{\frac{(1-\alpha)}{2(1+\alpha)}X+D} - \frac{1}{4D}$
Vacuum Dirichlet	Reflecting	$\phi(x) = \frac{Q}{2D}\left(\frac{3X^2}{4} + xX - x^2\right)$