

Homework 1

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NUEN 629

September 13, 2015

Problem 1:

Henyey and Greenstein (1941) introduced a function which, by the variation of one parameter, $1 \leq h \leq 1$, ranges from backscattering through isotropic scattering to forward scattering. In our notation we can write this as

$$K(\mu_0, v' \rightarrow v) = \frac{1}{2} \frac{1 - h^2}{(1 + h^2 - 2h\mu_0)^{3/2}} \delta(v' - v). \quad (1)$$

Verify that this is a properly normalized $f(\mu_0)$ and compute $K_l(v' \rightarrow v)$ for $l = 0, 1, 2$ as a function of h .

Solution:

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- First, expand scattering Kernel in Legendre Polynomials.
- Begin w/ Legendre generating fn:

$$\frac{1}{\sqrt{1-2xh+h^2}} = \sum_{n=0}^{\infty} P_n(x) h^n$$

(1) [wiki]

- Take a derivative w.r.t. x

$$\frac{h}{(1-2xh+h^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(x) h^n$$

(2)

- Multiply both sides by $\frac{(1-h^2)}{2h}$:

$$K(x) = \frac{1}{2} \frac{(1-h^2)}{(1-2xh+h^2)^{3/2}} = \frac{1}{2} \sum_{n=0}^{\infty} P'_n(x) h^{n-1} (1-h^2)$$

(3)

- (3) is our scattering kernel. Now we need to eliminate the derivative in terms of $P_n(x)$. Rewrite sum by letting $n \rightarrow n+1$

$$\frac{1}{2} \sum_{n=0}^{\infty} P'_n(x) h^{n-1} (1-h^2) = \frac{1}{2} \sum_{n=0}^{\infty} P'_{n+1}(x) h^n (1-h^2)$$

(4)

- Note that the sum still starts at zero because $P'_0(x) = 0$, so we can trivially add this term. From Abramowitz:

$$P'_{n+1}(x) = \sum_{m \text{ even}}^n (2m+1) P_m(x)$$

(5)

- (5) \rightarrow (4) and write as two sums:

$$K(x) = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{m \\ \text{even}}}^n (2m+1) P_m(x) h^n - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{m \\ \text{even}}}^n (2m+1) P_m(x) h^{n+2}$$

(6)

* Note: Here orthogonality is defined as $\int P_m P_n = \frac{\delta_{mn} 2}{2n+1}$ *

• Shift the second infinite sum by $n' = n+2$

$$K = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{m \\ \text{even}}}^n (2m+1) P_m(x) h^n - \frac{1}{2} \sum_{n=2}^{\infty} \sum_{\substack{m \\ \text{even}}}^{n-2} (2m+1) P_m(x) h^{n'}$$

• Combine the two series, writing out $n=0,1$ & letting $n' \rightarrow n$

$$K = \frac{1}{2} \sum_{n=0}^1 (2n+1) P_n(x) h^n + \frac{1}{2} \sum_{n=2}^{\infty} \left(\sum_{\substack{m \\ \text{even}}}^n (2m+1) P_m(x) - \sum_{\substack{m \\ \text{even}}}^{n-2} (2m+1) P_m(x) \right) h^n$$

• For the inner sums all terms but n cancel, adding back energy:

$$K(\mu_0, v' \rightarrow v) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) P_n(\mu_0) h^n \delta(v'-v) \quad (7)$$

• Taking Legendre moments:

$$K_e(v' \rightarrow v) = \frac{\delta(v'-v)}{2} \sum_{n=0}^{\infty} (2n+1) h^n \int_{-1}^1 d\mu_0 P_n(\mu_0) P_e(\mu_0)$$

$$K_e(v' \rightarrow v) = \delta(v'-v) \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \right) h^n \frac{\delta_{ne} 2}{(2n+1)}$$

$$K_e(v' \rightarrow v) = h^e$$

• Noting $\int_0^{\infty} dv \int_{-1}^1 d\mu_0 = \int_0^{\infty} dv \int_{-1}^1 P_0(\mu_0) d\mu_0$, $\int_0^{\infty} dv \int_{-1}^1 d\mu_0 K(\mu_0, v' \rightarrow v) = h^0 = 1$,
so normalization is correct.

$$\begin{aligned} K_0 &= 1 \\ K_1 &= h \\ K_2 &= h^2 \end{aligned}$$

Problem 2:

In an elastic scatter between a neutron and a nucleus, the scattering angle in the center of mass system is related to the energy change as

$$\frac{E}{E'} = \frac{1}{2} ((1 + \alpha) + (1 - \alpha) \cos \theta_c) \quad (2)$$

where E is the energy after scattering and E' is the initial energy of the neutron and

$$\alpha = \frac{(A - 1)^2}{(A + 1)^2}. \quad (3)$$

The scattered angle in the center-of-mass system is related the lab-frame scattered angle as

$$\tan \theta_L = \frac{A \sin \theta_c}{1 + A \cos \theta_c} \quad (4)$$

Also, the distribution of scattered energy is given by

$$P(E' \rightarrow E) = \begin{cases} \frac{1}{(1-\alpha)E'} & E'\alpha \leq E \leq E' \\ 0 & \text{otherwise} \end{cases}. \quad (5)$$

Derive an expression for $K(\mu_0, E' \rightarrow E)$, where μ_0 is $\cos \theta_L$. What is the distribution in angle of neutrons of energy in the range [0.05 MeV, 10 MeV] to energies below 1 eV if the scatter is with hydrogen?

Solution :

Due to Eq. (2), for a fixed A , a given value of E and E' fully define μ_c . As a result, the shape of the doubly differential scattering cross section in the center of mass (COM) system is fully defined by the probability density function (PDF) $P(E' \rightarrow E)$. Thus, it is possible to write the scattering cross section in the COM frame as [1]

$$\Sigma_s(E, \mu_0) = \sigma_s(E') (P(E' \rightarrow E) \delta(\mu_c - f(E, E'))) \quad (6)$$

where $f(E, E')$ is the value of μ_c that satisfies Eq. (2) for a given E , i.e.,

$$f(E, E') = \quad (7)$$

Since we are interested in s , we only need to transform the PDF $P(E' \rightarrow E)$ into a density function $P(\mu_c)$. From Eq. (??), there is a one-to-one relationship between E and $\mu_c = \cos(\Theta_c)$, thus

$$P(E' \rightarrow E) dE = P(\mu_c) d\mu_c. \quad (8)$$

or

$$P(\mu_c) = P(E' \rightarrow E) \frac{dE}{d\mu_c}. \quad (9)$$

Differentiation of Eq. (2) and multiplication by E' yields

$$\frac{dE}{d\mu_c} = \frac{1}{2} (1 - \alpha) E' \quad (10)$$

Evaluating μ_c for E at the limits $\alpha E'$ and E' gives the support for $P(\mu_c)$, defined for $\mu_c \in [-1, 1]$. Substitution of the above equation and Eq. (5) into Eq. (9) gives the desired PDF

$$P(\mu_c) = \frac{1}{(1 - \alpha) E'} \left(\frac{1}{2} (1 - \alpha) E' \right) = \frac{1}{2}, \quad \mu \in [-1, 1] \quad (11)$$

References

- [1] W.L. Dunn and J.K. Shultis, *Exploring Monte Carlo Methods*, 2012.