Second-Order Discretization in Space and Time for Radiation-Hydrodynamics

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1. Introduction

In this work, we derive, implement, and test a new IMEX scheme for solving the equations of radiation hydrodynamics that is second-order accurate in both space and time. We consider a RH system that combines a 1-D slab model of compressible fluid dynamics with a grey radiation S_2 model, given by:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,, \tag{1a}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial x}(p) = \frac{\sigma_t}{c} F_{r,0}\,, \tag{1b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[(E+p)\,u\right] = -\sigma_a c\left(aT^4 - E_r\right) + \frac{\sigma_t u}{c} F_{r,0}\,, \tag{1c}$$

$$\frac{1}{c}\frac{\partial \psi^+}{\partial t} + \frac{1}{\sqrt{3}}\frac{\partial \psi^+}{\partial x} + \sigma_t \psi^+ = \frac{\sigma_s}{4\pi} c E_r + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t u}{4\pi c} F_{r,0} + \frac{\sigma_t}{\sqrt{3}\pi} E u\,, \tag{1d}$$

$$\frac{1}{c}\frac{\partial \psi^-}{\partial t} - \frac{1}{\sqrt{3}}\frac{\partial \psi^-}{\partial x} + \sigma_t \psi^- = \frac{\sigma_s}{4\pi} c E_r + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t u}{4\pi c} F_{r,0} - \frac{\sigma_t}{\sqrt{3}\pi} E u\,, \tag{1e}$$

where ρ is the density, u is the velocity, $E = \frac{\rho u^2}{2} + \rho e$ is the total material energy density, e is the specific internal energy density, T is the material temperature, E_r is the radiation energy density,

$$E_r = \frac{2\pi}{c}\left(\psi^+ + \psi^-\right) \,, \tag{2}$$

F_r is the radiation energy flux,

$$F_r = \frac{2\pi}{\sqrt{3}}\left(\psi^+ - \psi^-\right) \tag{3}$$

and $F_{r,0}$ is an approximation to the comoving-frame flux,

$$F_{r,0} = F_r - \frac{4}{3}E_r u \,. \tag{4}$$

Note that if we multiply Eqs. ((1d)) and ((1e) by 2π and sum them, we obtain the radiation energy equation:

$$\frac{\partial E_r}{\partial t} + \frac{\partial F_r}{\partial x} = \sigma_a c(aT^4 - E_r) - \frac{\sigma_t u}{c}F_{r,0} \,, \tag{5a}$$

and if we multiply Eq. (1d) by $\frac{2\pi}{c\sqrt{3}}$, multiply Eq. (1e) by $-\frac{2\pi}{c\sqrt{3}}$ and sum them, we get the radiation momentum equation:

$$\frac{1}{c^2}\frac{\partial F_r}{\partial t} + \frac{1}{3}\frac{\partial E_r}{\partial x} = -\frac{\sigma_t}{c}F_{r,0} \,. \tag{5b}$$

Equations (1a) through (1e) are closed in our calculations by assuming an ideal equation of state (EOS):

$$p = \rho e(\gamma - 1) \,, \tag{6a}$$

$$T = \frac{e}{C_v} \,, \tag{6b}$$

where γ is the adiabatic index, and C_v is the specific heat. However, our method is compatible with any valid EOS.

2. Linearization of Equations

Consider the case of the non-linear system to be solved for Crank Nicolson over a time step from t_n to t_{n+1}. The changes to the non-linear system for the predictor and corrector time steps will only effect the choice of Δt, the end time state, and the known source terms on the right hand side from previous states in time. The non-linear equations to be solved in this case are

$$\frac{\mathcal{E}^{n+1} - \mathcal{E}}{\Delta t} = \tag{7}$$

$$\begin{aligned}\frac{E^{n+1} - E^*}{\Delta t} &= -\frac{1}{2}\left[\sigma_a c\left(aT^4 - \mathcal{E}\right)\right]^{n+1,k+1} - \frac{1}{2}\left[\sigma_a c\left(aT^4 - \mathcal{E}\right)\right]^{n} \\ &\quad - \frac{1}{2}\left[\sigma_t \frac{u}{c}\left(\frac{4}{3}\mathcal{E}u - \mathcal{F}\right)\right]^{n+1,k} - \frac{1}{2}\left[\sigma_t \frac{u}{c}\left(\frac{4}{3}\mathcal{E}u - \mathcal{F}\right)\right]^{n}\end{aligned} \tag{8}$$

To simplify the algebra, define a source term Q_E for all the known, lagged quantities in the above equation as

$$Q_E^k = \frac{1}{2}\left[\sigma_a c\left(a(T^n)^4 - \mathcal{E}\right)\right]^n - \frac{1}{2}\left[\sigma_t \frac{u}{c}\left(\frac{4}{3}\mathcal{E}u - \mathcal{F}\right)\right]^{n+1,k} - \frac{1}{2}\left[\sigma_t \frac{u}{c}\left(\frac{4}{3}\mathcal{E}u - \mathcal{F}\right)\right]^{n} \tag{9}$$

We then linearize the Planckian function about some temperature near T^{n+1}, denoted T^k. The linearized Planckian is

$$\left(T^{n+1,k+1}\right)^4 = (T^k)^4 + \frac{4(T^k)^3}{\rho^{n+1}c_v^k}\left(e^{n+1,k+1} - e^k\right). \tag{10}$$

For the initial iteration $T^k = T^n$. The above equation is substituted into Eq. (8) and we define $\beta^k = \frac{4a(T^k)^3}{\rho^{n+1}c_v^k}$ for clarity. The resulting equation can be solved for $e^{n+1,k+1} - e^k$ through algebraic manipulation:

$$\begin{aligned}\frac{E^{n+1} - E^*}{\Delta t} &= -\frac{1}{2}\left[\sigma_a^{n+1,k} c\left(a(T^{n+1,k+1})^4 - \mathcal{E}^{n+1,k}\right)\right] + Q_E^k \\ \frac{E^{n+1} - E^*}{\Delta t} &= -\frac{1}{2}\left[\sigma_a^{n+1,k} c\left(a(T^k)^4 + \beta^k(e^{n+1} - e^k) - \mathcal{E}^{n+1,k}\right)\right] + Q_E^k \\ \frac{E^{n+1} - \rho^{n+1}e^k + \rho^{n+1}e^k - E^*}{\Delta t} &= -\frac{1}{2}\left[\sigma_a^{n+1,k} c\left(a(T^k)^4 + \beta^k(e^{n+1} - e^k) - \mathcal{E}^{n+1,k}\right)\right] + Q_E^k\end{aligned}$$

We drop the superscript on ρ because $\rho^{n+1} = \rho^*$. Then, the left hand side can be simplified as

$$\frac{E^{n+1} - \rho e^k + \rho e^k - E^*}{\Delta t} = \frac{\rho}{\Delta t}\left[(e^{n+1} - e^k) + \frac{1}{2}(u^{n+1,2} - u^{*2}) + (e^k - e^*)\right] \tag{11}$$

Solution of the main equation for the desired quantity gives

$$(e^{n+1} - e^k) = \frac{\Delta t\left(\frac{1}{2}\sigma_a c(E^{n+1,k+1} - a(T^k)^4) - \rho(e^k - e^*) - \frac{1}{2}(u^{n+1,2} - u^{*2}) + Q_E^k\right)}{\rho\left[1 + \frac{1}{2}\sigma_a c\Delta t \beta^k\right]} \tag{12}$$