# Second-Order Discretization in Space and Time for Radiation-Hydrodynamics

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## 1. Introduction

In this work, we derive, implement, and test a new IMEX scheme for solving the equations of radiation hydrodynamics that is second-order accurate in both space and time. We consider a RH system that combines a 1-D slab model of compressible fluid dynamics with a grey radiation $S_2$ model, given by:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\,, \tag{1a}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial x}(p) = \frac{\sigma_t}{c} F_{r,0}\,, \tag{1b}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[(E+p)\,u\right] = -\sigma_a c\left(aT^4 - E_r\right) + \frac{\sigma_t u}{c} F_{r,0}\,, \tag{1c}$$

$$\frac{1}{c}\frac{\partial \psi^+}{\partial t} + \frac{1}{\sqrt{3}}\frac{\partial \psi^+}{\partial x} + \sigma_t \psi^+ = \frac{\sigma_s}{4\pi} c E_r + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t u}{4\pi c} F_{r,0} + \frac{\sigma_t}{\sqrt{3}\pi} Eu\,, \tag{1d}$$

$$\frac{1}{c}\frac{\partial \psi^-}{\partial t} - \frac{1}{\sqrt{3}}\frac{\partial \psi^-}{\partial x} + \sigma_t \psi^- = \frac{\sigma_s}{4\pi} c E_r + \frac{\sigma_a}{4\pi} acT^4 - \frac{\sigma_t u}{4\pi c} F_{r,0} - \frac{\sigma_t}{\sqrt{3}\pi} Eu\,, \tag{1e}$$

where $\rho$ is the density, $u$ is the velocity, $E = \frac{\rho u^2}{2} + \rho e$ is the total material energy density, $e$ is the specific internal energy density, $T$ is the material temperature, $E_r$ is the radiation energy density,

$$E_r = \frac{2\pi}{c}\left(\psi^+ + \psi^-\right) , \tag{2}$$

$F_r$ is the radiation energy flux,

$$F_r = \frac{2\pi}{\sqrt{3}}\left(\psi^+ - \psi^-\right) \tag{3}$$

and $F_{r,0}$ is an approximation to the comoving-frame flux,

$$F_{r,0} = F_r - \frac{4}{3}E_r u\,. \tag{4}$$

Note that if we multiply Eqs. ((1d)) and ((1e) by $2\pi$ and sum them, we obtain the radiation energy equation:

$$\frac{\partial E_r}{\partial t} + \frac{\partial F_r}{\partial x} = \sigma_a c(aT^4 - E_r) - \frac{\sigma_t u}{c}F_{r,0}\,, \tag{5a}$$

and if we multiply Eq. (1d) by $\frac{2\pi}{c\sqrt{3}}$, multiply Eq. (1e) by $-\frac{2\pi}{c\sqrt{3}}$ and sum them, we get the radiation momentum equation:

$$\frac{1}{c^2}\frac{\partial F_r}{\partial t} + \frac{1}{3}\frac{\partial E_r}{\partial x} = -\frac{\sigma_t}{c}F_{r,0}\,. \tag{5b}$$

Equations (1a) through (1e) are closed in our calculations by assuming an ideal equation of state (EOS):

$$p = \rho e(\gamma - 1)\,, \tag{6a}$$

$$T = \frac{e}{C_v}\,, \tag{6b}$$

where $\gamma$ is the adiabatic index, and $C_v$ is the specific heat. However, our method is compatible with any valid EOS.

One of the most common methods for solving the radiation diffusion equation in time is the Crank-Nicholson method, also known as the Trapezoid Rule. This is a well-known, implicit method that is second-order accurate; however, its principal drawback is that it can become highly oscillatory for stiff systems. An alternative to this is a linear-discontinuous Galerkin method in time. Despite the fact that this scheme is more accurate than the Crank-Nicholson method and damps oscillations quickly, it has a much higher computational cost that is roughly equivalent to that of solving two Crank-Nicholson systems simultaneously over each time step [1]. In this work, we use the TR/BDF2 scheme for discretizing the radiation $S_2$ and energy exchange terms in time. The TR/BDF2 scheme is a one-step, two-stage [1] implicit method that was first derived in [2]. There is actually a family of such schemes, but one member of the family can be shown to be optimal in a certain sense. A simple version of this method that is near-optimal was applied to the equations of radiative transfer in [3], where it is shown to be both L-stable, accurate, and efficient. In [3], the near-optimal TR/BDF2 scheme is used to solve the equations of radiative transfer. It consists of a Crank-Nicholson step over half the time step and, using that solution, a BDF2 step over the remainder of the time step. The TR/BDF2 method has a computational cost that is roughly equivalent to that of solving two Crank-Nicholson systems sequentially over each time step.

A critical issue for radiation transport spatial discretizations is the preservation of the diffusion limit. Radiation-hydrodynamics problems often contain highly diffusive regions. In any type of calculation it is generally expected that accurate solutions will be obtained whenever the spatial variation of the solution is well-resolved by the mesh. However, use of a consistent transport discretization scheme in a highly diffusive problem will not guarantee such behavior. To ensure this behavior, a consistent discretization scheme must "preserve" the diffusion limit or "be asymptotic preserving" [4]. A consistent discretization that does not preserve the diffusion limit will only yield accurate results in highly diffusive problems if the spatial cells are small with respect to a mean-free-path. Since the diffusion length can a arbitarily large with respect to a mean-free-path, discretization schemes that are not asymptotic preserving can be prohibitively expensive to use in problems with highly diffusive regions. Thus preservation of the radiation diffusion limit is an essential property of any radiation-hydrodynamics scheme. Although we only use an $S_2$ radiation treatment, our overall coupling and solution scheme is applicable with an $S_n$ treatment of arbitrary order. The only caveat is that a higher order $S_n$ model will require a standard iterative solution technique for the $S_n$ equations themselves. Thus we are able to investigate preserva-

[1]Here we use the term "stage" to refer to an implicit equation that must be solved within each time step in a discretization scheme.

tion of the diffusion limit assuming an LDFEM spatial discretization for a $S_2$ treatment that will be valid for an $S_n$ treatment of arbitrary order.

The MHM includes spatial differencing for the advection equations and incorporates a linear interpolation from cell-averaged values to compute the slopes. However, Lowrie and Morel show in [5] that interpolation schemes which only depend on the mesh geometry and do not incorporate additional physical data, e.g. cross-section values, fail to have the diffusion limit. Furthermore, the differences in spatial discretization between the advection and $S_2$ equations present considerable complications due to the fact that, in the MHM, the slopes are determined from interpolations of the cell-centered unknowns; whereas, in the LDFEM, the slopes are computed as part of the solution to the discretized spatial moment equations. To add to these complications, the internal energy of the material represents an unknown in both the material advection and radiation diffusion equations. The easy solution to this problem is to recompute the internal energy and radiation slopes at the beginning of each time step using the MHM limiter. Doing this, we were able to show that our method maintained the diffusion limit in 1D and reproduced shock solutions accurately. However, standard 2D and 3D hydrodynamics limiters use a spatial representation that will not maintain the radiation diffusion limit [6]. In particular, a simple linear dependence for the solution is assumed but a bilinear (2D) or a trilinear dependence (3D) is required [7]. Thus, to overcome this limitation, the method we present here preserves the slopes computed by the LDFEM from one time step to the next. We use reconstructed slopes as determined in the MUSCL-Hancock method only to compute the advection fluxes, and we use the preserved LDFEM slopes to initialize the implicit calculations for the radiation energy density and flux and for the material temperature update. This allows our method to reduce to its standard constituent methods when the contributions from coupled physics are negligible, and we believe it will also allow us to preserve the diffusion limit in the future extension of our method to 2D and 3D. Of course, this remains to be demonstrated.

The remainder of this paper is structured as follows. In Section 2, we describe our second-order accurate radiation-hydrodynamics method in detail. In Section 3, we use the method of manufactured solutions to show that our method is second-order accurate in both space and time in the equilibrium diffusion limit as well as in the streaming limit. Then, in Section 4, we demonstrate the capability of our method to accurately compute radiation-hydrodynamic shocks by reproducing semi-analytic shock solutions. Finally, in Section 5, we summarize our results and present our conclusions and recommendations for future work.

## 2. Radiation-Hydrodynamics Method

This scheme consists of two predictor-corrector cycles, each of which includes a full MHM step to compute the material advection components. The predictor step of each cycle generates radiation-updated hydrodynamic unknowns, which are used to compute the advection fluxes in the corrector step, and in the first cycle, the predicted radiation quantities are used to update the corrector step material momentum. In addition to the MHM Godunov step, each corrector includes an explicit update for radiation momentum deposition to the fluid, and an implicit solve to compute the radiation diffusion and material energy exchange. In the first cycle, the radiation solve is computed using the Crank-Nicholson scheme, and in the second cycle, the radiation solve is computed using the BDF2 scheme.

One advantage to applying the full MHM over each half time step is that, if the time step size is being determined by the Courant limit, we can take twice the usual time step. In this case, the cost of two diffusion solves per time step is mitigated. Furthermore, the scheme is designed in such a way that, if the radiation contributions to the hydrodynamics are negligible, the standard MHM solution is obtained over each half time step, and if the hydrodynamics contributions to the radiation diffusion are negligible, the standard TR/BDF2 solution for radiative transfer is obtained over the full time step. An outline of our two-cycle system is given as follows:

### *2.1. Cycle 1*

In this section, we define the first cycle of our hybrid MUSCL-Hancock TR/BDF2 scheme in further detail. We begin our IMEX scheme by linearly reconstructing the hydro unknowns, $U_i^n$:

$$U_{L,i}^n = U_i^n - \frac{\Delta_i^n}{2}; \quad U_{R,i}^n = U_i^n + \frac{\Delta_i^n}{2}\,, \tag{7}$$

where

$$U_i = \begin{bmatrix} \rho_i \\ (\rho u)_i \\ E_i \end{bmatrix}\,, \tag{8}$$

and $\Delta_i$ is some slope constructed from the cell-centered data. Next, we evolve the hydro unknowns over a quarter time-step:

$$U_i^* = U_i^n + \frac{\Delta t}{4\Delta x}\left(F_{L,i}^n - F_{R,i}^n\right) , \tag{9}$$

where $F_{L/R,i}$ is the hydro flux computed as $F(U_{L/R,i})$. Note that a Riemann solver is not used to define these fluxes because they are computed using only unknowns within cell $i$. Further note that $\rho_i^{n+1/4} = \rho_i^*$. We continue by updating the fluid momentum in the predictor with the cell-averaged, explicit radiation momentum deposition:

$$\begin{aligned}\frac{4\rho_i^{n+1/4}\left(u_i^{n+1/4} - u_i^*\right)}{\Delta t} =& \frac{1}{2}\frac{\sigma_{t,L,i}^n}{c}\left(F_{r,L,i}^n - \frac{4}{3}E_{r,L,i}^n u_{L,i}^n\right) \\ &+ \frac{1}{2}\frac{\sigma_{t,R,i}^n}{c}\left(F_{r,R,i}^n - \frac{4}{3}E_{r,R,i}^n u_{R,i}^n\right) .\end{aligned} \tag{10}$$

Then, we perform our nonlinear iterations for the predictor, in which we implicitly solve for the radiation energy density and flux and update the material energy using the Crank-Nicholson method. We neglect the spatial indexing in these equations for simplicity since we have separate equations for the left and right unknowns in each spatial cell together with coupling between cells due to spatial derivatives:

$$\begin{aligned}\frac{4\left(E_r^{n+1/4,k+1} - E_r^n\right)}{\Delta t} =& -\frac{1}{2}\left(\frac{\partial F^{n+1/4,k+1}}{\partial x} + \frac{\partial F^n}{\partial x}\right) \\ &+ \frac{\sigma_a^{n+1/4,k}c}{2}\left(a(T^{n+1/4,k+1})^4 - E_r^{n+1/4,k+1}\right) \\ &+ \frac{\sigma_a^n c}{2}\left(a(T^n)^4 - E_r^n\right) + \sigma_t^n \frac{u^n}{c}\left(\frac{4}{3}E_r^n u^n - F_r^n\right) ,\end{aligned} \tag{11a}$$

$$\begin{aligned}\frac{1}{3}\frac{\partial E_r^{n+1/4,k+1}}{\partial x} + \frac{1}{3}\frac{\partial E_r^n}{\partial x} + \frac{\sigma_t^{n+1/4,k}}{c}F^{n+1/4,k+1} + \frac{\sigma_t^n}{c}F^n = \\ \sigma_t^{n+1/4,k}\frac{4}{3}E^{n+1/4,k+1}\frac{u^n}{c} + \sigma_t^n\frac{4}{3}E^n\frac{u^n}{c} ,\end{aligned} \tag{11b}$$

$$\frac{4\left(E^{n+1/4,k+1}-E^{*}\right)}{\Delta t} = -\frac{\sigma_a^{n+1/4,k}c}{2}\left(a(T^{n+1/4,k+1})^4 - E_r^{n+1/4,k+1}\right) - \frac{\sigma_a^n c}{2}\left(a(T^n)^4 - E_r^n\right) - \sigma_t^n \frac{u^n}{c}\left(\frac{4}{3}E_r^n u^n - F_r^n\right) . \tag{11c}$$

In order to solve these equations, we first linearize the Planck function in Eq. (11a) and Eq. (11c):

$$(T^{n+1/4,k+1})^4 = (T^k)^4 + \frac{4(T^k)^3}{C_v^k}\left(e^{n+1/4,k+1} - e^{n+1,k}\right) , \tag{12}$$

where we have allowed for a non-constant specific heat for generality. Then we solve Eq. (11c) for $T^{n+1/4,k+1}$, and substitute that expression into Eq. (11a). This eliminates $(T^{n+1/4,k+1})^4$ from from Eq. (11a), leaving a 7-diagonal system for the radiation energy density and flux corresponding to Eqs. ((11a)) and ((11b)). This system can be directly inverted during each Newton iteration. Once the radiation energy density has been obtained, Eq. (11c) can be locally solved within eack cell for the new material internal energy. Since the material density and the material velocity are computed before the total energy and the radiation field, the only unknowns in the total energy equation are the internal energies. Thus solution of this equation within each Newton iteration yields new internal energies, from which new total energies and temperatures can be calculated. This process is repeated with each Newton iteration until $E^{n+1/4}$ and $E_r^{n+1/4}$ are converged.

To begin the corrector, we reconstruct the hydro variables, again following the implicit update and use these, in conjunction with a Riemann solver, to compute the quarter-step cell-edge fluxes for the hydro variables, $F_{i+1/2}^{n+1/4}$. These fluxes allow us to compute a second-order approximation of the advection component of the rad-hydro system at $t^{n+1/2}$ using a Godunov update:

$$U_i^{**} = U_{M,i}^n + \frac{\Delta t}{2\Delta x}\left(F_{i-1/2}^{n+1/4} - F_{i+1/2}^{n+1/4}\right) . \tag{13}$$

Once this is computed, we update the fluid momentum in the corrector explicitly using the cell-averaged radiation momentum deposition at $t^{n+1/4}$.

$$\frac{2\rho_i^{n+1/2}\left(u_i^{n+1/2}-u_i^*\right)}{\Delta t} = \frac{1}{2}\frac{\sigma_{t,L,i}^{n+1/4}}{c}\left(F_{r,L,i}^{n+1/4}-\frac{4}{3}E_{r,L,i}^{n+1/4}u_{L,i}^{n+1/4}\right) + \frac{1}{2}\frac{\sigma_{t,R,i}^{n+1/4}}{c}\left(F_{r,R,i}^{n+1/4}-\frac{4}{3}E_{r,R,i}^{n+1/4}u_{R,i}^{n+1/4}\right). \qquad (14)$$

Then, we solve the radiative transfer equations for the corrector step, computing the radiation energy density and radiation current and updating the material internal energy, material total energy and material temperature using the Crank-Nicholson method:

$$\frac{2\left(E_r^{n+1/2,k+1}-E_r^n\right)}{\Delta t} = -\frac{1}{2}\left(\frac{\partial F^{n+1/2,k+1}}{\partial x}+\frac{\partial F^n}{\partial x}\right)+\frac{\sigma_a^n c}{2}\left(a(T^n)^4-E_r^n\right) + \frac{\sigma_a^{n+1/2,k}c}{2}\left(a(T^{n+1/2,k+1})^4-E_r^{n+1/2,k+1}\right) + \sigma_t^{n+1/4}\frac{u^{n+1/4}}{c}\left(\frac{4}{3}E_r^{n+1/4}u^{n+1/4}-F_r^{n+1/4}\right), \qquad (15a)$$

$$\frac{1}{3}\frac{\partial E_r^{n+1/2,k+1}}{\partial x}+\frac{1}{3}\frac{\partial E_r^n}{\partial x}+\frac{\sigma_t^{n+1/2,k}}{c}F^{n+1/2,k+1}+\frac{\sigma_t^n}{c}F^n = \sigma_t^{n+1/2,k}\frac{4}{3}E^{n+1/2,k+1}\frac{u^{n+1/4}}{c}+\sigma_t^n\frac{4}{3}E^n\frac{u^n}{c}, \qquad (15b)$$

$$\frac{2\left(E^{n+1/2,k+1}-E^{**}\right)}{\Delta t} = -\frac{\sigma_a^{n+1/2,k}c}{2}\left(a(T^{n+1/2,k+1})^4-E_r^{n+1/2,k+1}\right) - \frac{\sigma_a^n c}{2}\left(a(T^n)^4-E_r^n\right) - \sigma_t^{n+1/4}\frac{u^{n+1/4}}{c}\left(\frac{4}{3}E_r^{n+1/4}u^{n+1/4}-F_r^{n+1/4}\right). \qquad (15c)$$

Once $E^{n+1/2}$ and $E_r^{n+1/2}$ are converged, Cycle 1 is complete.

*2.2. Cycle 2*

In this section, we detail the second cycle of our hybrid MUSCL-Hancock TR/BDF2 scheme. This cycle is very similar to the first cycle with the exception that we use a BDF2 step to solve for the radiation energy density and to update the material energy in the corrector. Like the first cycle, we begin by linearly reconstructing the hydro unknowns, $U_i^{n+1/2}$:

$$U_{L,i}^{n+1/2} = U_i^{n+1/2} - \frac{\Delta_i^{n+1/2}}{2}; \quad U_{R,i}^{n+1/2} = U_i^{n+1/2} + \frac{\Delta_i^{n+1/2}}{2}. \tag{16}$$

Next, we evolve the hydro unknowns over another quarter time-step:

$$U_{M,i}^{*} = U_{M,i}^{n+1/2} + \frac{\Delta t}{4\Delta x}\left(F_{L,i}^{n+1/2} - F_{R,i}^{n+1/2}\right). \tag{17}$$

Again, note that $\rho_i^{n+3/4} = \rho_i^*$. We update the fluid momentum in the predictor of the second cycle:

$$\begin{aligned}\frac{4\rho_i^{n+3/4}\left(u_i^{n+3/4} - u_i^*\right)}{\Delta t} =& \frac{1}{2}\frac{\sigma_{t,L,i}^{n+1/2}}{c}\left(F_{r,L,i}^{n+1/2} - \frac{4}{3}E_{r,L,i}^{n+1/2}u_{L,i}^{n+1/2}\right)\\ &+ \frac{1}{2}\frac{\sigma_{t,R,i}^{n+1/2}}{c}\left(F_{r,R,i}^{n+1/2} - \frac{4}{3}E_{r,R,i}^{n+1/2}u_{R,i}^{n+1/2}\right).\end{aligned} \tag{18}$$

Then, we enter our nonlinear iterations for the second-cycle predictor. As in the first-cycle predictor, in this loop we implicitly solve for the radiation energy density and flux and update the material energy using the Crank-Nicholson method:

$$\begin{aligned}\frac{4\left(E_r^{n+3/4,k+1} - E_r^n\right)}{\Delta t} =& -\frac{1}{2}\left(\frac{\partial F^{n+3/4,k+1}}{\partial x} + \frac{\partial F^n}{\partial x}\right)\\ &+ \frac{\sigma_a^{n+3/4,k}c}{2}\left(a(T^{n+3/4,k+1})^4 - E_r^{n+3/4,k+1}\right)\\ &+ \frac{\sigma_a^{n+1/2}c}{2}\left(a(T^{n+1/2})^4 - E_r^{n+1/2}\right)\\ &+ \sigma_t^{n+1/2}\frac{u^{n+1/2}}{c}\left(\frac{4}{3}E_r^{n+1/2}u^{n+1/2} - F_r^{n+1/2}\right),\end{aligned} \tag{19a}$$

$$\frac{1}{3}\frac{\partial E_r^{n+3/4,k+1}}{\partial x} + \frac{1}{3}\frac{\partial E_r^{n+1/2}}{\partial x} + \frac{\sigma_t^{n+3/4,k}}{c} F^{n+3/4,k+1} + \frac{\sigma_t^{n+1/2}}{c} F^{n+1/2} = \sigma_t^{n+3/4,k}\frac{4}{3}E^{n+3/4,k+1}\frac{u^{n+1/2}}{c} + \sigma_t^{n+1/2}\frac{4}{3}E^{n+1/2}\frac{u^{n+1/2}}{c}\,, \tag{19b}$$

$$\begin{aligned}\frac{4\left(E^{n+3/4,k+1} - E^{*}\right)}{\Delta t} = &- \frac{\sigma_a^{n+3/4,k} c}{2}\left(a(T^{n+3/4,k+1})^4 - E_r^{n+3/4,k+1}\right) \\ &- \frac{\sigma_a^{n+1/2} c}{2}\left(a(T^{n+1/2})^4 - E_r^{n+1/2}\right) \\ &- \sigma_t^{n+1/2}\frac{u^{n+1/2}}{c}\left(\frac{4}{3}E_r^{n+1/2}u^{n+1/2} - F_r^{n+1/2}\right)\,.\end{aligned} \tag{20}$$

Then, Eq. (19) and Eq. (20) are repeatedly solved until $E^{n+3/4}$ and $E_r^{n+3/4}$ are converged. To begin the cycle 2 corrector, we reconstruct the hydro variables, again, following the implicit update and use these, in conjunction with a Riemann solver, to compute the three quarter-step cell-edge fluxes for the hydro variables, $F_{i+1/2}^{n+3/4}$. Using these fluxes, we compute the advection component of the rad-hydro system at $t^{n+1}$ using a Godunov update:

$$U_i^{**} = U_{M,i}^{n+1/2} + \frac{\Delta t}{2\Delta x}\left(F_{i-1/2}^{n+3/4} - F_{i+1/2}^{n+3/4}\right)\,. \tag{21}$$

Computing this, we update the fluid momentum in the corrector explicitly using radiation values at $t^{n+3/4}$.

$$\frac{\rho_i^{n+1}\left(u_i^{n+1} - u_i^{**}\right)}{\Delta t} = \frac{\sigma_{t,i}^{n+1/2}}{c}\left(F_{r,i}^{n+1/2} - \frac{4}{3}E_{r,i}^{n+1/2}u_i^{n+1/2}\right)\,. \tag{22}$$

Finally, we enter the nonlinear iterations for the corrector step of Cycle 2. Here, we implicitly solve for the radiation energy density and current using the BDF2 method:

$$\frac{\left(E_r^{n+1,k+1} - E_r^n\right)}{\Delta t} = -\frac{1}{3}\left(\frac{\partial F^{n+1,k+1}}{\partial x} + \frac{\partial F^{n+1/2}}{\partial x} + \frac{\partial F^n}{\partial x}\right)$$
$$+ \frac{\sigma_a^{n+1,k} c}{3}\left(a(T^{n+1,k+1})^4 - E_r^{n+1,k+1}\right)$$
$$+ \frac{\sigma_a^{n+1/2} c}{3}\left(a(T^{n+1/2})^4 - E_r^{n+1/2}\right) + \frac{\sigma_a^n c}{3}\left(a(T^n)^4 - E_r^n\right)$$
$$+ \sigma_t^{n+1/2}\frac{u^{n+1/2}}{c}\left(\frac{4}{3}E_r^{n+1/2}u^{n+1/2} - F_r^{n+1/2}\right), \tag{23a}$$

$$\frac{1}{3}\frac{\partial E_r^{n+1,k+1}}{\partial x} + \frac{1}{3}\frac{\partial E_r^{n+1/2}}{\partial x} + \frac{1}{3}\frac{\partial E_r^n}{\partial x} + \frac{\sigma_t^{n+1,k}}{c}F^{n+1,k+1} + \frac{\sigma_t^{n+1/2}}{c}F^{n+1/2} + \frac{\sigma_t^n}{c}F^n =$$
$$\sigma_t^{n+1,k}\frac{4}{3}E^{n+1,k+1}\frac{u^{n+1/2}}{c} + \sigma_t^{n+1/2}\frac{4}{3}E^{n+1/2}\frac{u^{n+1/2}}{c} + \sigma_t^n\frac{4}{3}E^n\frac{u^{n+1/2}}{c}, \tag{23b}$$

Using these values, we compute the full-step material energy. Because this is a full-step calculation, instead of updating the values computed in Eq. (21), we treat the hydrodynamic fluxes, $F^{n+1/4}$ and $F^{n+3/4}$, from the first and second cycles as sources for the BDF2 equation.

$$\frac{\left(E^{n+1,k+1} - E^n\right)}{\Delta t} = -\frac{\sigma_a^{n+1,k} c}{3}\left(a(T^{n+1,k+1})^4 - E_r^{n+1,k+1}\right)$$
$$- \frac{\sigma_a^{n+1/2} c}{3}\left(a(T^{n+1/2})^4 - E_r^{n+1/2}\right) - \frac{\sigma_a^n c}{3}\left(a(T^n)^4 - E_r^n\right)$$
$$- \sigma_t^{n+1/2}\frac{u^{n+1/2}}{c}\left(\frac{4}{3}E_r^{n+1/2}u^{n+1/2} - F_r^{n+1/2}\right)$$
$$- \frac{1}{2}\left(\frac{\partial F^{n+1/4}}{\partial x} + \frac{\partial F^{n+3/4}}{\partial x}\right). \tag{24}$$

We iterate Eq. (23) and Eq. (24) until $E^{n+1}$ and $E_r^{n+1}$ are converged, and the solution over the full time step is complete.