Probability

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1 Basics

Sample Space: The sample space S is the set of all possible outcomes

Event: A subset of the sample space

Example: A coin is flipped two times. We can define the sample space S = {HH, TT, HT, TH}. A possible event would be where there was at least one heads flipped, A = {HH, HT, TH}. This is a subset of the sample space.

Probability: The basic definition of probability is, given a sample space S and an event A, the probability of A occurring or P(A) = $\frac{\text{number of outcomes in A}}{\text{total number of outcomes in S}}$

Complement: The complement of an event A is the outcomes where A does not occur, or 1 - P(A).

2 Calculating Probability

Multiplication Rule: For an experiment with two sub-experiments A and B, which have a and b possible outcomes respectively, the main experiment has a*b total outcomes.

Examples:

Suppose an experiment consists of flipping a coin and rolling a die. The coin flip has two possible outcomes, heads or tails, while the dice roll has 6. Thus, the experiment has 12 total outcomes.

Assume that there is a race with 10 people competing and three winners. There are 10 possibilities for who can take first place. Since someone has taken first place, there are 9 possibilities for who can come in second place. Thus, there are 8 possibilities for who can come in third. By the multiplication rule, there are 10*9*8 = 720 total outcomes.

No Replacement: As shown in the previous example, when there are n objects and choose k from them without replacing each time, there are n * (n - 1) * (n - 2) * * * (n - k + 1) total possible outcomes.

Permutation: A permutation of objects is an arrangement of them in some order. For n objects, there are n! permutations, as shown in the previous example.

Binomial Coefficient Formula: For nonnegative numbers k and n, the binomial coefficient $\binom{n}{k}$ is the number of subsets of size k in a set of size n. For k $\leq$ n, we have $\binom{n}{k} = \frac{\text{n * (n - 1) * (n - 2)* * * (n - k + 1)}}{k!} = \frac{n!}{(n-k)!k!}$. When k > n, $\binom{n}{k} = 0$.

Example: How many different words of length 8 can be created using 5 A's and 3 L's? To determine the number of permutations, we need to choose where the L's will go in this word or where the A's will go. Thus, the number of words is equal to $\binom{8}{5} = \binom{8}{3} = \frac{8!}{5!3!}$ = 56 permutations.

3 Properties of Probability

General Definitions of Probability: A probability space consists of a sample space S and a probability function P, which takes an event A $\subseteq$ S as an input and returns P(A), a number between 0 and 1, as the output. The function P must satisfy the following:

1. P($\emptyset$) = 0, P(S) = 1
2. If the events A_1, A_2,... are disjoint, P$(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Properties of Probability: Probability has the following properties, for any events A and B:

1. P(A^c) = 1 - P(A)
2. If A $\subseteq$ B, then P(A) $\leq$ P(B)
3. P(A$\cup$B) = P(A) + P(B) - P(A$\cap$B)

Inclusion-Exclusion: Following from the third property of probability, for any events A_1, A_2, ..., A_n,
$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_i) + \sum_{i<j<k}^{n} P(A_i \cap A_i \cap A_k) - \ldots + (-1)^{n+1} P(A_i \cap \ldots \cap A_n)$.
This formula ensures that each intersection of the events is only included once in the final probability.

Example: Suppose that a coin is flipped twice and a die is rolled once. The sample space of this event has 24 possible outcomes. Consider the events A_1 where a head is rolled exactly once, A_2 where the dice roll results in a 4, and A_3 where there is at least one tails flipped. $P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{6}$, and $P(A_3) = \frac{3}{4}$. By considering the intersection of these events, $P(A_1 \cap A_2) = \frac{1}{12}$, $P(A_2 \cap A_3) = \frac{1}{8}$, $P(A_1 \cap A_3) = \frac{1}{2}$, and $P(A_1 \cap A_2 \cap A_3) = \frac{1}{12}$. By the inclusion-exclusion formula, $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) = \frac{1}{2} + \frac{1}{6} + \frac{3}{4} - \frac{1}{12} - \frac{1}{8} - \frac{1}{2} + \frac{1}{12} = \frac{19}{24}$.

4 Conditional Probability

Conditional Probability: If A and b are events and $P(B) > 0$, then the probability of A occurring given that B occurs is $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Example: A man has two children, and at least one of them is a girl. What is the probability that both children are girls? The event A is that both children are girls and B is that at least one child is a girl. We are asked to calculate $P(A \mid B)$. $P(B) =$ the probability that one child is a girl + the probability that both children are girls $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. The probability of both children being girls is $\frac{1}{4}$. Thus $P(A|B) = \frac{0.25}{0.75} = \frac{1}{3}$.

Probability of the Intersection of n Events: For any 2 events A and B with positive probabilities, $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$. For n events A_1, A_2, ..., A_n with positive probabilities, $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$.

Baye's Rule: For any two events A and B where $P(B) > 0$, $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. This follows from the probability of the intersection of 2 events in the previous definitions.