Assignment 1

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Model formulation

$$y = X\beta + e$$

where

$$e \sim \mathcal{N}(0, \sigma^2 I_n)$$

$$\mathbb{E}[y] = \mathbb{E}[X\beta + e] = \mathbb{E}[X\beta] + \mathbb{E}[e] = X\beta + 0 = X\beta$$
$$\text{Var}[y] = \text{Var}[X\beta + e] = \text{Var}[X\beta] + \text{Var}[e] + 2cov(X\beta, e) = 0 + \sigma^2 I_n + 0 = \sigma^2 I_n$$

Therefore

$$y \sim \mathcal{N}(X\beta, \sigma^2 I_n)$$

Density of Y

$$f_Y(y) = (2\pi\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta))$$
$$f_Y(y) = (2\pi\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))$$

Likelihood

$$L(\beta, y, X) \propto (\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))$$

Since $M = M^{-1} = I_{k+1}$

Then the prior distribution is

$$[\beta|\sigma^2] \sim \mathcal{N}_{k+1}(\tilde{\beta}, \sigma^2 M^{-1})$$

Prior density

$$\pi(\beta|\sigma^2) = (2\pi)^{-\frac{k+1}{2}} \det(\sigma^2 M^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta - \tilde{\beta})^T M(\beta - \tilde{\beta}))$$
$$\pi(\beta|\sigma^2) = (2\pi)^{-\frac{k+1}{2}} \det(\sigma^2 M^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta}^T M\tilde{\beta}))$$
$$\pi(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta}^T M\tilde{\beta}))$$

Prior distribution

$$[\sigma^2] \sim \mathcal{IG}(a, b)$$

Prior density

$$\pi(\sigma^2) = \frac{b^a}{\Gamma(a)} \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$
$$\pi(\sigma^2) \propto \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$

Assuming that σ^2 is known

Prior

$$\pi(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta}))$$

Likelihood

$$L(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(-2\beta^T X^T y + \beta^T X^T X\beta))$$

Posterior

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T(X^T y + M\tilde{\beta})))$$

Recall that $(X^T X)\hat{\beta} = X^T y$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T((X^T X)\hat{\beta} + M\tilde{\beta})))$$

Let $A = M + X^T X$ and $b = (X^T X)\hat{\beta} + M\tilde{\beta}$

Completing the square using the following identity

$$\beta^T A\beta - 2\beta^T b = (\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta$$

where $\mu_\beta = A^{-1}b$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}((\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta))$$

but $\mu_\beta^T A\mu_\beta$ is independent of β

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T (M + X^T X)(\beta - \mu_\beta))$$

meaning

$$[\beta|\sigma^2, y, X] \sim \mathcal{N}_{k+1}(\mu_\beta, \sigma^2(M + X^T X)^{-1})$$

where

$$\mu_\beta = (M + X^T X)^{-1}((X^T X)\hat{\beta} + M\tilde{\beta})$$

Joint distribution

$$J = [\beta, \sigma^2 | y, X]$$

Consider

$$P = L(\beta, y, X)[\beta|\sigma^2]$$

$$P \propto (\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))(\sigma^2)^{-\frac{k+1}{2}} exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta^T} M\tilde{\beta}))$$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta^T} M\tilde{\beta}))exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T((X^T X)\hat{\beta} + M\tilde{\beta}))$$

Let $A = M + X^T X$ and $b = (X^T X)\hat{\beta} + M\tilde{\beta}$

Completing the square using the following identity

$$\beta^T A\beta - 2\beta^T b = (\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta$$

where $\mu_\beta = A^{-1}b$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta^T} M\tilde{\beta}))exp(-\frac{1}{2\sigma^2}((\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta))$$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta^T} M\tilde{\beta} - \mu_\beta^T A\mu_\beta))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$[\sigma^2] \propto \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$

$$J \propto L(\beta, y, X)[\beta|\sigma^2][\sigma^2]$$

$$J \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta^T} M\tilde{\beta} - \mu_\beta^T A\mu_\beta))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))exp(-\frac{b}{\sigma^2})$$

Let $A_2 = y^T y + \tilde{\beta^T} M\tilde{\beta} - \mu_\beta^T A\mu_\beta$

$$J \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$[\sigma^2|y, X] = \int_\beta [\beta, \sigma^2|y, X]\, d\beta$$

$$[\sigma^2|y, X] \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2})) \int_\beta exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta)\, d\beta$$

Recall

$$\int_\beta (2\pi)^{-\frac{k+1}{2}} det((\frac{1}{\sigma^2}(M + X^T X))^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))\, d\beta = 1$$

then

$$I = \int_{\beta} exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta)\, d\beta = (2\pi)^{\frac{k+1}{2}} det((\frac{1}{\sigma^2}(M + X^T X))^{-1})^{\frac{1}{2}})$$

We know that $det(aA) = a^k det(A)$, where A is a k by k matrix.

$$I = (2\pi)^{\frac{k+1}{2}} (\sigma^2)^{\frac{k+1}{2}} det((M + X^T X)^{-1})^{\frac{1}{2}}$$

$$I \propto (\sigma^2)^{\frac{k+1}{2}}$$

$$[\sigma^2 | y, X] \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))(\sigma^2)^{\frac{k+1}{2}}$$

$$[\sigma^2 | y, X] \propto (\sigma^2)^{-(\frac{n}{2}+a)-1} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))$$

meaning

$$[\sigma^2 | y, X] \sim \mathcal{IG}(a + \frac{n}{2}, b + \frac{A_2}{2})$$

where $a = 1$, $b = 1$ and $A_2 = y^T y + \tilde{\beta}^T M \tilde{\beta} - \mu_\beta^T (M + X^T X)\mu_\beta$