Assignment 1

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Question 1

a)

Model formulation

$$y = X\beta + e$$

where

$$e \sim \mathcal{N}(0, \sigma^2 I_n)$$

$$\mathbb{E}[y] = \mathbb{E}[X\beta + e] = \mathbb{E}[X\beta] + \mathbb{E}[e] = X\beta + 0 = X\beta$$
$$\text{Var}[y] = \text{Var}[X\beta + e] = \text{Var}[X\beta] + \text{Var}[e] + 2cov(X\beta, e) = 0 + \sigma^2 I_n + 0 = \sigma^2 I_n$$

Therefore

$$y \sim \mathcal{N}(X\beta, \sigma^2 I_n)$$

Density of Y

$$f_Y(y) = (2\pi\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta))$$

$$f_Y(y) = (2\pi\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))$$

Likelihood

$$L(\beta, y, X) \propto (\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))$$

Since $M = M^{-1} = I_{k+1}$
Then the prior distribution is

$$[\beta|\sigma^2] \sim \mathcal{N}_{k+1}(\tilde{\beta}, \sigma^2 M^{-1})$$

Prior density

$$\pi(\beta|\sigma^2) = (2\pi)^{-\frac{k+1}{2}} \det(\sigma^2 M^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta - \tilde{\beta})^T M(\beta - \tilde{\beta}))$$

$$\pi(\beta|\sigma^2) = (2\pi)^{-\frac{k+1}{2}} \det(\sigma^2 M^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta}^T M\tilde{\beta}))$$

$$\pi(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta}^T M\tilde{\beta}))$$

Prior distribution

$$[\sigma^2] \sim \mathcal{IG}(a, b)$$

Prior density

$$\pi(\sigma^2) = \frac{b^a}{\Gamma(a)} \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$

$$\pi(\sigma^2) \propto \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$

Assuming that σ^2 is known

Prior

$$\pi(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta}))$$

Likelihood

$$L(\beta|\sigma^2) \propto exp(-\frac{1}{2\sigma^2}(-2\beta^T X^T y + \beta^T X^T X\beta))$$

Posterior

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T(X^T y + M\tilde{\beta})))$$

Recall that $(X^T X)\hat{\beta} = X^T y$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T((X^T X)\hat{\beta} + M\tilde{\beta})))$$

Let $A = M + X^T X$ and $b = (X^T X)\hat{\beta} + M\tilde{\beta}$

Completing the square using the following identity

$$\beta^T A\beta - 2\beta^T b = (\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta$$

where $\mu_\beta = A^{-1}b$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}((\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta))$$

but $\mu_\beta^T A\mu_\beta$ is independent of β

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$\pi(\beta|y, X) \propto exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T(M + X^T X)(\beta - \mu_\beta))$$

meaning

$$[\beta|\sigma^2, y, X] \sim \mathcal{N}_{k+1}(\mu_\beta, \sigma^2(M + X^T X)^{-1})$$

where

$$\mu_\beta = (M + X^T X)^{-1}((X^T X)\hat{\beta} + M\tilde{\beta})$$

Joint distribution

$$J = [\beta, \sigma^2 | y, X]$$

Consider

$$P = L(\beta, y, X)[\beta|\sigma^2]$$

$$P \propto (\sigma^2)^{-\frac{n}{2}} exp(-\frac{1}{2\sigma^2}(y^T y - 2\beta^T X^T y + \beta^T X^T X\beta))(\sigma^2)^{-\frac{k+1}{2}} exp(-\frac{1}{2\sigma^2}(\beta^T M\beta - 2\beta^T M\tilde{\beta} + \tilde{\beta}^T M\tilde{\beta}))$$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta}^T M\tilde{\beta}))exp(-\frac{1}{2\sigma^2}(\beta^T(M + X^T X)\beta - 2\beta^T((X^T X)\hat{\beta} + M\tilde{\beta}))$$

Let $A = M + X^T X$ and $b = (X^T X)\hat{\beta} + M\tilde{\beta}$

Completing the square using the following identity

$$\beta^T A\beta - 2\beta^T b = (\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta$$

where $\mu_\beta = A^{-1}b$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta}^T M\tilde{\beta}))exp(-\frac{1}{2\sigma^2}((\beta - \mu_\beta)^T A(\beta - \mu_\beta) - \mu_\beta^T A\mu_\beta))$$

$$P \propto (\sigma^2)^{-\frac{n+k+1}{2}} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta}^T M\tilde{\beta} - \mu_\beta^T A\mu_\beta))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$[\sigma^2] \propto \sigma^{2(-a-1)} exp(-\frac{b}{\sigma^2})$$

$$J \propto L(\beta, y, X)[\beta|\sigma^2][\sigma^2]$$

$$J \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{2\sigma^2}(y^T y + \tilde{\beta}^T M\tilde{\beta} - \mu_\beta^T A\mu_\beta))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))exp(-\frac{b}{\sigma^2})$$

Let $A_2 = y^T y + \tilde{\beta}^T M\tilde{\beta} - \mu_\beta^T A\mu_\beta$

$$J \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))$$

$$[\sigma^2|y, X] = \int_\beta [\beta, \sigma^2|y, X]\, d\beta$$

$$[\sigma^2|y, X] \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2})) \int_\beta exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta)\, d\beta$$

Recall

$$\int_\beta (2\pi)^{-\frac{k+1}{2}} det((\frac{1}{\sigma^2}(M + X^T X))^{-1})^{-\frac{1}{2}} exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta))\, d\beta = 1$$

then

$$I = \int_\beta exp(-\frac{1}{2\sigma^2}(\beta - \mu_\beta)^T A(\beta - \mu_\beta)\, d\beta = (2\pi)^{\frac{k+1}{2}} det((\frac{1}{\sigma^2}(M + X^T X))^{-1})^{\frac{1}{2}})$$

We know that $det(aA) = a^k det(A)$, where A is a k by k matrix.

$$I = (2\pi)^{\frac{k+1}{2}} (\sigma^2)^{\frac{k+1}{2}} det((M + X^T X)^{-1})^{\frac{1}{2}}$$

$$I \propto (\sigma^2)^{\frac{k+1}{2}}$$

$$[\sigma^2|y, X] \propto (\sigma^2)^{-(\frac{n+k+1}{2}+a+1)} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))(\sigma^2)^{\frac{k+1}{2}}$$

$$[\sigma^2|y, X] \propto (\sigma^2)^{-(\frac{n}{2}+a)-1} exp(-\frac{1}{\sigma^2}(b + \frac{A_2}{2}))$$

meaning

$$[\sigma^2|y, X] \sim \mathcal{IG}(a + \frac{n}{2}, b + \frac{A_2}{2})$$

where $a = 1$, $b = 1$ and $A_2 = y^T y + \tilde{\beta}^T M\tilde{\beta} - \mu_\beta^T (M + X^T X)\mu_\beta$

b)

Table 1: First six rows of σ^2 sample values

σ^2
1.968357
1.695290
1.674221
1.982145
1.534436
1.750102

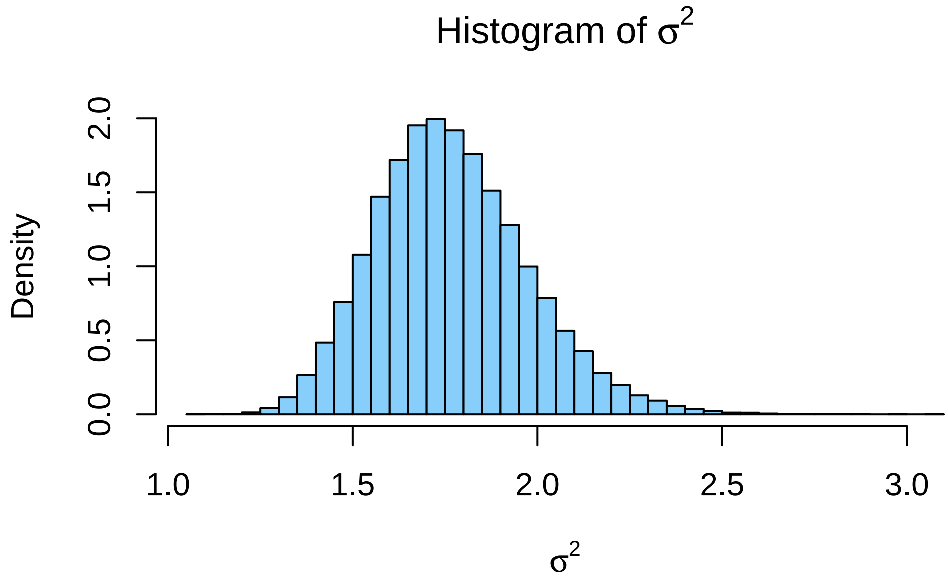
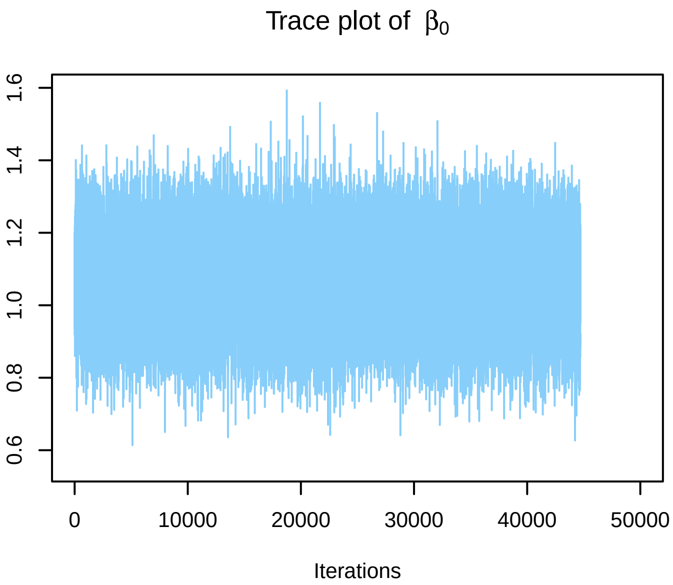


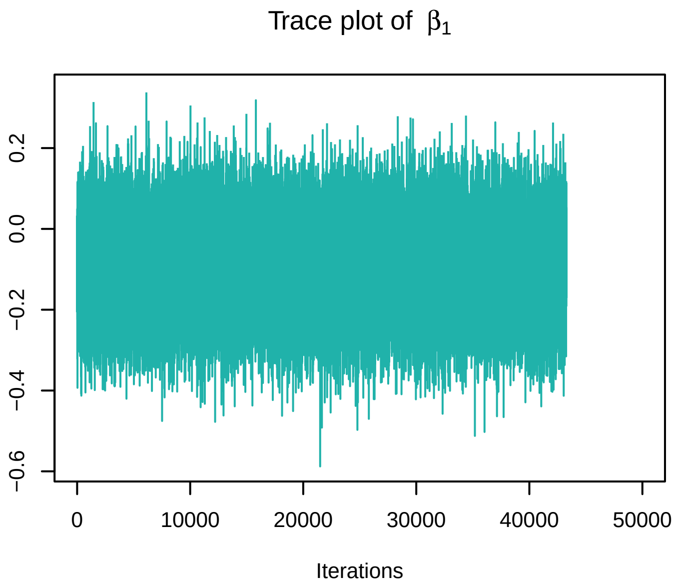
Table 2: First six rows of the β sample values

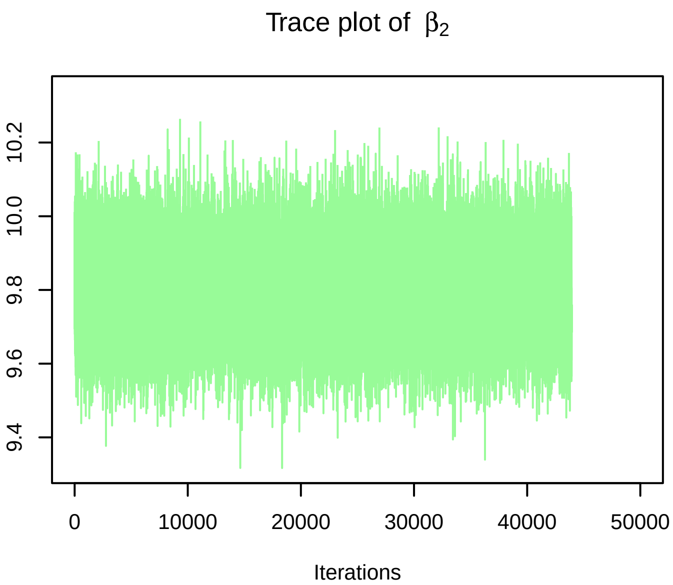
β_0	β_1	β_2	β_3	β_4	β_5
0.9227484	-0.10623460	9.759792	0.10381837	2.051554	-2.839680
1.0567194	-0.20497249	9.788940	-0.09386065	1.772719	-2.809105
1.0122741	-0.04329813	9.801656	0.05378141	2.089081	-2.834065
1.2013402	-0.15871660	9.956266	-0.15968348	1.990569	-2.957680
1.1146270	-0.13473328	9.724995	0.06301181	1.879163	-3.124324
0.9180509	-0.14602992	10.011397	0.08100581	1.804324	-2.854732
1.0698329	0.01746358	9.703786	-0.08617031	1.880744	-2.950838

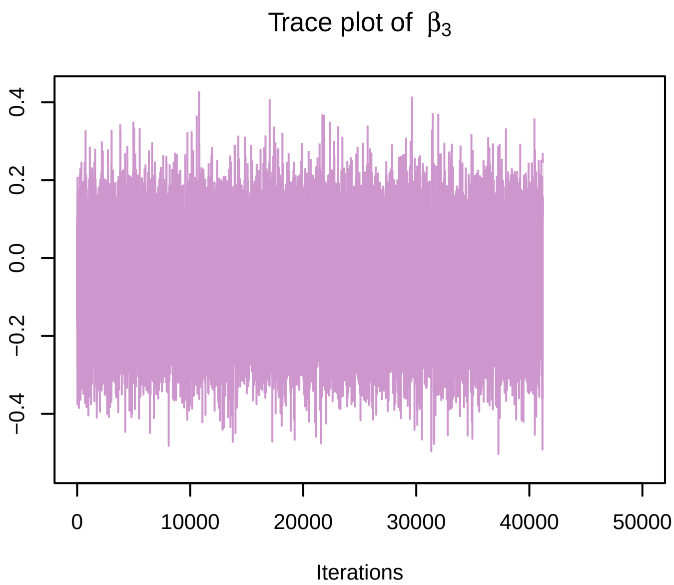
c)

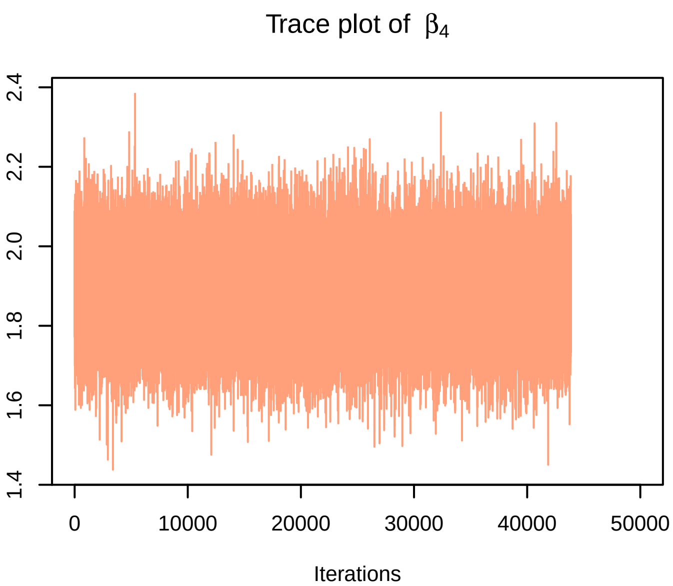
i)

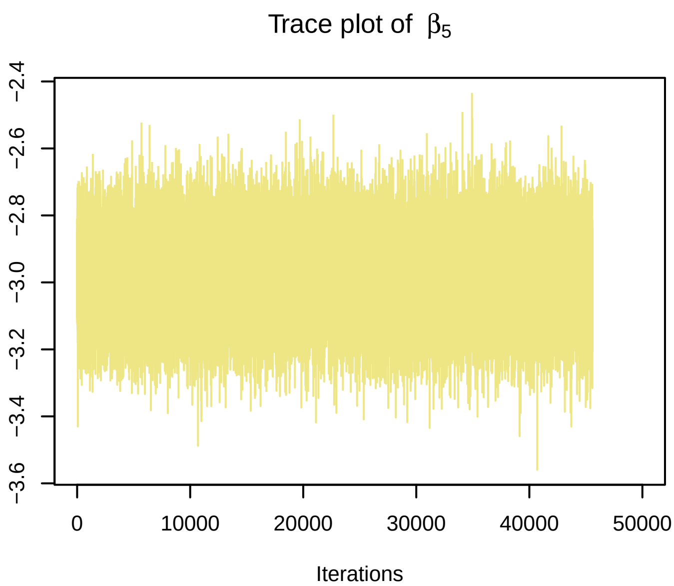






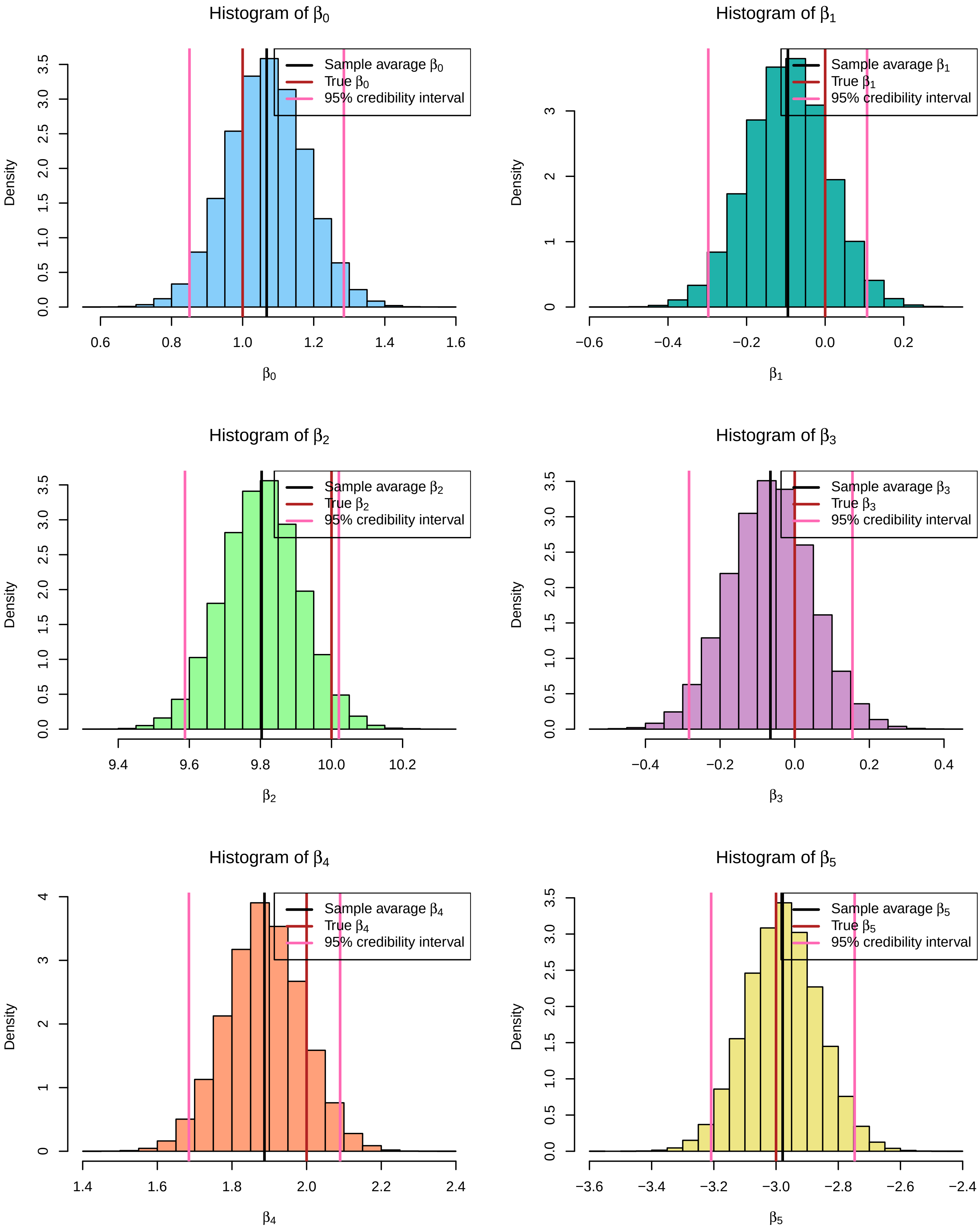






All of the trace plots appear as random scatter, indicating stationarity. This provides evidence for the convergence of the Markov Chains.

ii)



The beta coefficients are approximately normally distributed.

Confidence interval are the relative frequencies of stating valid bounds if you were to re-sample the data. Credibility intervals are the probability that the true parameter is within the stated bounds. The true beta parameter values should be in the closed interval of these credibility intervals indicated on the plot.

Question 2

a)

Given:

$Z_i|Y_i \sim Ber(Y_ip_i) = (Y_ip_i)^z(1-y_ip_i)^{1-z}$

$Y_i \sim Ber(\theta_i) = (\theta_i)^y(1-\theta_i)^{1-y}$

Derivation of equation 2:

$P(Z_i = 0) = \sum_{y=0}^{1} P(Z_i = 0|Y_i = y)P(Y_i = y)$

$= (p_i)^0(1-pi)^1(\theta_i)^1(1-\theta_i)^0 + (0p_i)^0(1-0p_i)^1(\theta_i)^0(1-\theta_i)^1$

$= (1-p_i)(\theta_i) + (1-\theta_i) = 1 - p_i\theta_i$

$P(Y_i = 1) = (\theta_i)^1(1-\theta_i)^0$

$= \theta_i$

$P(Z_i = 0|Y_i = 1) = (p_i)^0(1-p_i)^1$

Therefore :

$P(Y_i = 1|Z_i = 0) = \frac{P(Z_i=0|Y_i=1)P(Y_i)}{P(Z_i=0)}$

$= \frac{(1-p_i)\theta_i}{1-p_i\theta_i}$

Derivation of equation 4 :

If the fisherman is in cell j then searching cell i gives no information about j ,so we are guaranteed to not find the fisherman.

Therefore:

$P(Z_i = 0|Y_j = 1) = 1$

$P(Y_j = 1) = (\theta_j)^1(1-\theta_j)^0$

$= \theta_j$

As proved above :

$P(Z_i = 0) = 1 - p_i\theta_i$

Therefore:

$\theta_{j,new} = P(Y_j = 1|Z_i = 0) = \frac{P(Z_i=0|Y_j=1)P(Y_j)}{P(Z_i=0)}$

$\theta j, new = \frac{\theta_{j,old}}{1-p_i\theta_{i,old}}$

b)

Equation 2 is the posterior probability the fisherman is in the cell given that we fail to detect him.Although we don't detect the fisherman he may still be in the cell.So we reduce the probability of occurring in the cell rather than ruling it out.To show this decrease in probability over time we update the new prior (probability of occurrence $\theta_{i,new}$) using the old prior(probability of occurence$\theta_{i,old}$).Therefore Equation 3 shows how the occurrence probability is adjusted over time as we gain more evidence through the bayesian search.

c)

Approach:

1. **Initialize Prior and True Location using Jakes provided functions:**
   - Generate the initial prior probability distribution using `generate_lost()`.
   - Generate the true location of the fisherman using `generate_fisherman()`.
   - Store the fisherman's coordinates as `rowf` and `colf`.
2. **Variable initialization:**
   - Create a posterior tracker vector of size 48 to record the posterior probability of the fisherman's true location at each time step.
   - Set posterior equal to the initial prior.
   - Initialise a boolean called fishermanfound to false initially and use this to track whether the fisherman is found or not.
3. **Create a for loop which loops through the amount of hours(48):**
   - Merge the "prior" and detection probability to create a search grid
   - Select the cell with the highest probability of successful detection
   - If the chosen cell is the fisherman's true location, simulate detection using the detection probability associated with the cell of interest(use rbinom).
   - Otherwise, detection is automatically set to 0.
   - Record the current posterior probability of the true location in `post_tracker[i]`.
4. **Update Posterior if Fisherman Not Found:**

   If the fisherman is not detected use Bayes theorem to update the probability theorem:

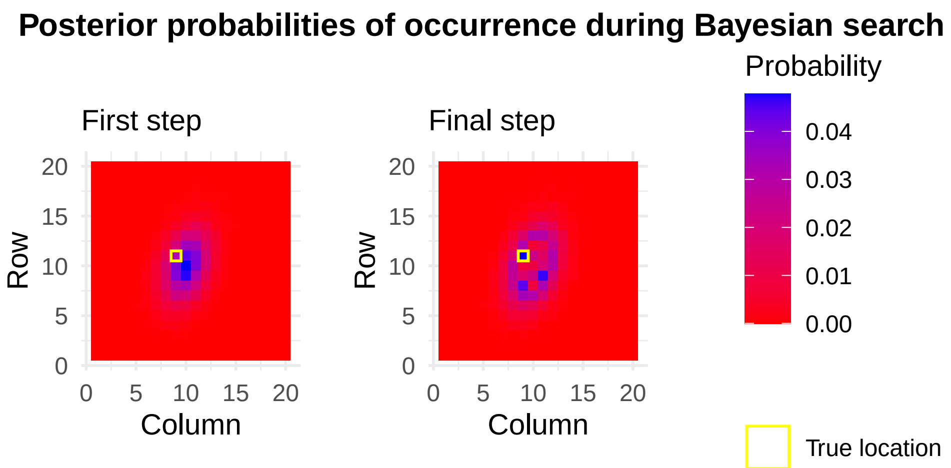
   - Update the probability in the searched cell using:

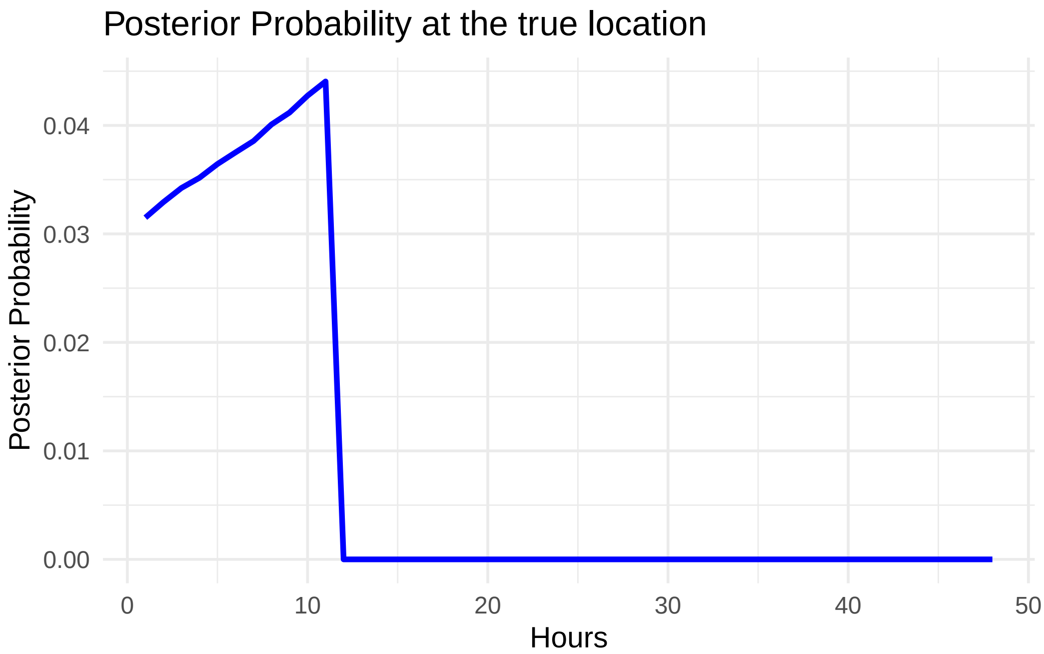
   $$\theta_{i,\text{new}} = \frac{(1-p_i)\theta_{i,\text{old}}}{1-p_i\theta_{i,\text{old}}}$$

   - Update all other cells:

   $$\theta_{j,\text{new}} = \frac{\theta_{j,\text{old}}}{1-p_i\theta_{i,\text{old}}} \quad \text{for } j \neq i$$

5. **Successful Detection:**
   - If detection is successful, break the loop and set the boolean fisherman found to true.





d)

If p_i is constant across cells, detection probability no longer varies by location. In this case, the search strategy would focus on cells with the highest prior probability, rather than the highest probability of successful detection. If detection fails we still update the posterior probability using Bayes Theorem and the occurrence probabilities would also still need to be updated.Therefore the only notable change would be which cell we search.