Gaussian Processes for Time Series Modelling

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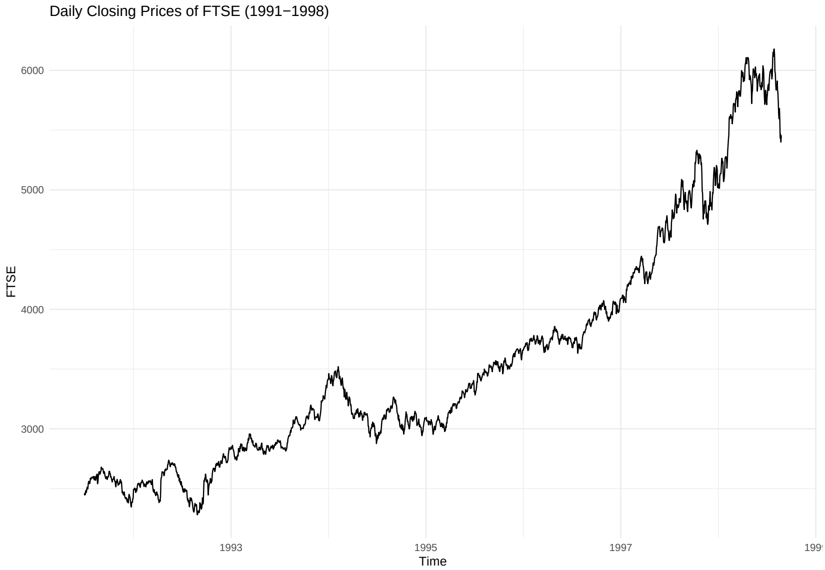
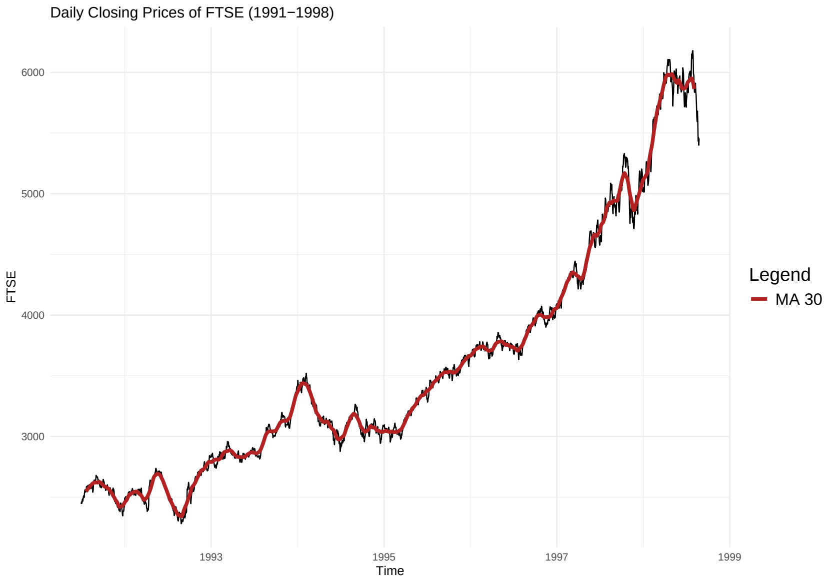
Plot of the dataset



Moving average smooting



B-splines

Base case

$$B_{i,0}(t) := \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

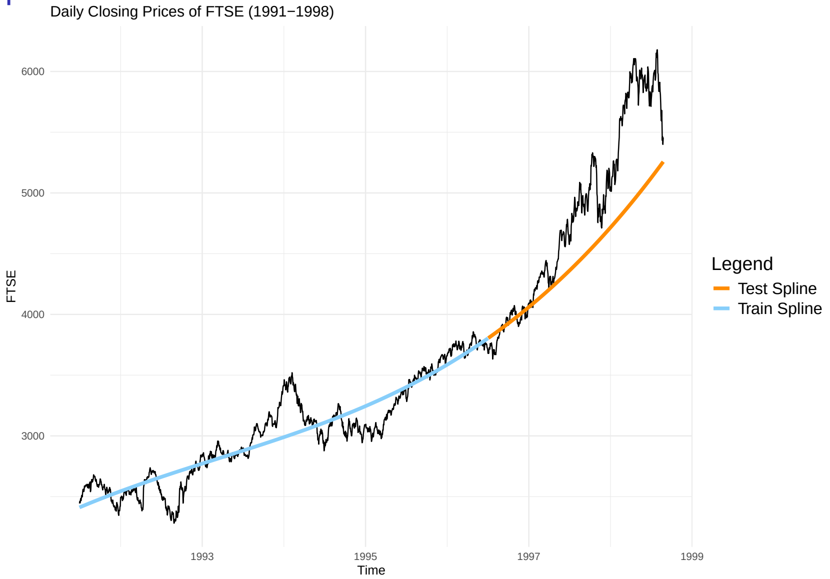
Recursive step

$$B_{i,p}(t) := \frac{t - t_i}{t_{i+p} - t_i} B_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} B_{i+1,p-1}(t)$$

Where

t is the covariate and p is the degree of the polynomial.

B-spline of order 2 fit



Discussion

Problem

Splines do not consider the correlations between data points. They model the immediate shape of the data.

Solution

Use Gaussian processes to model the data generating process. GPs consider all the observations and their correlations.

Gaussian process

Definition

A time continuous stochastic process $\{X_t;\ t \in T\}$ is Gaussian if and only if for every finite set of indices $t_1, ..., t_k$ in the index set T
$\mathbf{X}_{t_1,...,t_k} = (X_{t_1}, ..., X_{t_k})$
is a multivariate Gaussian random variable.

Meaning

$f \sim GP(m, k) \to f_n \sim MVN(\mathbf{m}, \mathbf{K})$
$\pi(y_n; f(x_n), \phi) \to \pi(y_n; f_n, \phi)$

Parameters

$$\mathbf{m} = \begin{bmatrix} \mu(x_1^{\text{obs}}) \\ \vdots \\ \mu(x_{N_{\text{obs}}}^{\text{obs}}) \\ \mu(x_1^{\text{pred}}) \\ \vdots \\ \mu(x_{N_{\text{pred}}}^{\text{pred}}) \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k(x_1^{\text{obs}}, x_1^{\text{obs}}) & \cdots & k(x_1^{\text{obs}}, x_{N_{\text{obs}}}^{\text{obs}}) & & k(x_1^{\text{obs}}, x_1^{\text{pred}}) & \cdots & k(x_1^{\text{obs}}, x_{N_{\text{pred}}}^{\text{pred}}) \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ k(x_{N_{\text{obs}}}^{\text{obs}}, x_1^{\text{obs}}) & \cdots & k(x_{N_{\text{obs}}}^{\text{obs}}, x_{N_{\text{obs}}}^{\text{obs}}) & & k(x_{N_{\text{obs}}}^{\text{obs}}, x_1^{\text{pred}}) & \cdots & k(x_{N_{\text{obs}}}^{\text{obs}}, x_{N_{\text{pred}}}^{\text{pred}}) \\ k(x_1^{\text{pred}}, x_1^{\text{obs}}) & \cdots & k(x_1^{\text{pred}}, x_{N_{\text{obs}}}^{\text{obs}}) & & k(x_1^{\text{pred}}, x_1^{\text{pred}}) & \cdots & k(x_1^{\text{pred}}, x_{N_{\text{pred}}}^{\text{pred}}) \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ k(x_{N_{\text{pred}}}^{\text{pred}}, x_1^{\text{obs}}) & \cdots & k(x_{N_{\text{pred}}}^{\text{pred}}, x_{N_{\text{obs}}}^{\text{obs}}) & & k(x_{N_{\text{pred}}}^{\text{pred}}, x_1^{\text{pred}}) & \cdots & k(x_{N_{\text{pred}}}^{\text{pred}}, x_{N_{\text{pred}}}^{\text{pred}}) \end{bmatrix}$$

Prior realizations

Prior quantiles