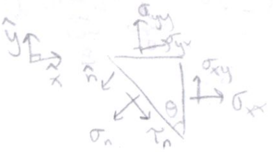
UNIT 2 SUMMARY

σ_{ij}
↑ surface ↖ force component



$\sigma_n = \hat{n}\cdot(\underline{\sigma}\,\hat{n})$
$\tau_n = \hat{n}^{\perp}\cdot(\underline{\sigma}\,\hat{n})$

$$\boxed{\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix}\begin{bmatrix}\hat{n}\end{bmatrix}=\sigma_n\begin{bmatrix}\hat{n}\end{bmatrix}+\tau_n\begin{bmatrix}\hat{n}^{\perp}\end{bmatrix}}$$

(stress as a matrix)

- looking @ force balance, get

$$\underbrace{\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix}}_{\underline{\sigma}}\underbrace{\begin{bmatrix}-\cos\theta\\ -\sin\theta\end{bmatrix}}_{\hat{n}}=\sigma_n\underbrace{\begin{bmatrix}-\cos\theta\\ -\sin\theta\end{bmatrix}}_{\hat{n}}+\tau_n\underbrace{\begin{bmatrix}\sin\theta\\ -\cos\theta\end{bmatrix}}_{\hat{n}^{\perp}}$$

in 3D:

$$\underline{\sigma}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix}$$

tensor

← normal stress on diagonal

shear stress on off-diagonal (if $=0$, 'biaxial stress')

Ignoring moments on each face since:
- internal moments are macroscopic generalizations gotten from a distribution of force by performing an integral
  - ↳ zooming in, just force/area (stress)

- looking @ moment balance, get that $\sigma_{xy}=\sigma_{yx}$
- looking @ $\theta\to\frac{\pi}{2}$ get that dir. of stress comp. flip on parallel box faces

- Stress acting on face cutting 45°?

$$\begin{bmatrix}\sigma_{xx} & 0\\ 0 & \sigma_{yy}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}=\sigma_n\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}+\tau_n\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\underline{\sigma}\hat{n}=\begin{bmatrix}\frac{-\sigma_{xx}}{\sqrt{2}}\\ \frac{-\sigma_{yy}}{\sqrt{2}}\end{bmatrix}=\underbrace{\frac{(\sigma_{xx}+\sigma_{yy})}{2}}_{\sigma_n}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}+\underbrace{\frac{(\sigma_{yy}-\sigma_{xx})}{2}}_{\tau_n}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$$

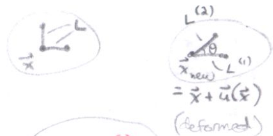
↳ Cases: (some)

1) $\sigma_{xx}=\sigma_{yy}=\sigma,\ \tau_n=0$
   ↳ hydrostatic pressure state
   ~~bi~~axial stress
2) $\sigma_{yy}=0:\ \sigma_n=\frac{\sigma_{xx}}{2},\ \tau_n=\frac{-\sigma_{xx}}{2}$
   ↳ uniaxial tension/compression

- no need to consider gravity in stress matrix derivation since accounted for in balance
  - ↳ cause of pressures/variations in stress
- Stress is a field: @ any $\vec{x}$, draw a small box...
- Strain:
  - displacement field: $\vec{u}(x,y,z)=\begin{bmatrix}u_x(x,y,z)\\ u_y(x,y,z)\\ u_z(x,y,z)\end{bmatrix}$
  - displacement gradient matrix: $\nabla\vec{u}=\begin{bmatrix}\frac{du_x}{dx} & \frac{du_x}{dy} & \frac{du_x}{dz}\\ \frac{du_y}{dx} & \frac{du_y}{dy} & \frac{du_y}{dz}\\ \frac{du_z}{dx} & \frac{du_z}{dy} & \frac{du_z}{dz}\end{bmatrix}$
  - small displacement strain matrix: $\underline{\varepsilon}=\frac{1}{2}(\nabla\vec{u}+(\nabla\vec{u})^T)$

$$=\begin{bmatrix}\frac{du_x}{dx} & \frac{1}{2}\left(\frac{du_y}{dx}+\frac{du_x}{dy}\right) & \frac{1}{2}\left(\frac{du_z}{dx}+\frac{du_x}{dz}\right)\\ \frac{1}{2}\left(\frac{du_y}{dx}+\frac{du_x}{dy}\right) & \frac{du_y}{dy} & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix}$$

- in 2D:



$\vec{x}_{new}=\vec{x}+\vec{u}(\vec{x})$ (deformed)

$\cancel{\underline{\varepsilon}=\begin{bmatrix}\frac{du_x}{dx}=\frac{L^{(1)}-L}{L} & \frac{1}{2}(\end{bmatrix}}$

$$\underline{\varepsilon}=\begin{bmatrix}\frac{du_x}{dx}=\frac{L^{(1)}-L}{L} & \frac{1}{2}\left(\frac{du_y}{dx}+\frac{du_x}{dy}\right)=\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\\ =\frac{1}{2}\left(\frac{\pi}{2}-\theta\right) & \frac{du_y}{dy}=\frac{L^{(2)}-L}{L}\end{bmatrix}$$

angle change   length change

3D elasticity

$$\underline{\sigma}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix}=\begin{bmatrix}\sigma_M & 0 & 0\\ 0 & \sigma_M & 0\\ 0 & 0 & \sigma_M\end{bmatrix}+\underbrace{\begin{bmatrix}\sigma_{xx}-\sigma_M & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy}-\sigma_M & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_M\end{bmatrix}}_{\equiv\underline{\sigma}'}$$

← general case

↖ stress deviator

Mean normal stress/hydrostatic part of the stress: $\sigma_M=\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})=\frac{1}{3}\operatorname{tr}\underline{\sigma}$

↳ $\boxed{\operatorname{tr}\underline{\sigma}'=0}$ by def.

↳ in other words: stress deviator

$\underline{\sigma}=\sigma_M\underline{I}+\underline{\sigma}'$

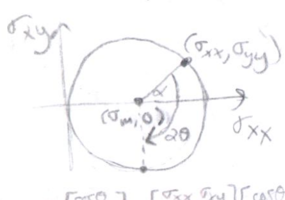
$\underline{\varepsilon}=\varepsilon_M\underline{I}+\underline{\varepsilon}'$, $\varepsilon_M=\frac{1}{3}\operatorname{tr}\underline{\varepsilon}$

- $\underline{\sigma}'$ wants to change shape
- σ_M wants to change vol.
- $\underline{\varepsilon}'$ shape change
- ε_M volume change

for 2D, len. & angle

- engineering shear strain: $\gamma_{xy}=2\varepsilon_{xy},\ \gamma_{xz}=2\varepsilon_{xz},\ldots$
  ↳ just the angle Δ, no $\frac{1}{2}$

Changing basis:
Mohr's Circle:



$\sigma_{x'x'}=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\cdot\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{xy} & \sigma_{yy}\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$

$=\underbrace{\frac{1}{2}(\sigma_{xx}+\sigma_{yy})}_{\sigma_M}+\underbrace{\sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2+\sigma_{xy}^2}}_{R}\cos(\alpha-2\theta)$

principal stresses on horizontal axes

So, $\underline{\sigma}=\underbrace{\sigma_M\underline{I}}_{=3K\varepsilon_M\underline{I}}+\underbrace{\underline{\sigma}'}_{=2G\underline{\varepsilon}'}$

from before   new!

shear modulus: measures resistance to shape change

- (for an isotropic material)

$$\underbrace{\begin{bmatrix}\sigma & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \sigma\end{bmatrix}}_{\underline{\sigma}}=3K\underbrace{\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon\end{bmatrix}}_{\underline{\varepsilon}}$$

(hydrostatic tension state)

bulk modulus

↳ resistance to volume change

$\Rightarrow\ \underline{\sigma}=K(\operatorname{tr}\underline{\varepsilon})\underline{I}+2G\underline{\varepsilon}'$

↳ constit. rel. for linear isotropic 3D elasticity

$\underline{\varepsilon}=\frac{1}{9K}(\operatorname{tr}\underline{\sigma})\underline{I}+\frac{1}{2G}\underline{\sigma}'$

↳ can get stress & strain so Young's Mod. if pick a dir.
↳ compliance relation

stress deviator: $(\sigma') = \sigma - \sigma_M \cdot I$

↳ $(tr\,\sigma' = 0)$

↳ $\sigma = \sigma' + \sigma_M \cdot I$

$\varepsilon = \varepsilon' + \varepsilon_M I$

$\sigma = K(tr\,\varepsilon)I + 2G\varepsilon'$

↳ relating σ & ε; 3D elasticity

↑ modulus → ↑ harder to change

↳ inverting, get compliance relation:

$\varepsilon = \frac{1}{9K}(tr\,\sigma)I + \frac{1}{2G}\sigma'$

$\varepsilon = \frac{1}{E}[\sigma(1+\nu) - \nu(tr\,\sigma)I]$

· Poisson's ratio:

$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}$ under simple tension

$tr\,\varepsilon = \frac{\Delta volume}{original\ volume}$ → under simple tension $\frac{\Delta V}{V_0} = \varepsilon_{xx} + 2\varepsilon_{yy}$?

↳ simple tension:

$\sigma = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → $E = \frac{9KG}{3K+G}$

→ $\varepsilon = \begin{bmatrix} (\frac{1}{9K}+\frac{1}{3G})\sigma_{xx} & 0 & 0 \\ 0 & (\frac{1}{9K}-\frac{1}{6G})\sigma_{xx} & 0 \\ 0 & 0 & (\frac{1}{9K}-\frac{1}{6G})\sigma_{xx} \end{bmatrix}$

· incompressible → $\Delta V = 0$ → $\nu = \frac{1}{2}, K = \infty$

· $\nu < 0$ → auxetic materals

· $\nu = 0$ → longer w/o coming thinner (cork)

· most $0 \le \nu \le \frac{1}{2}$ → under simple tension

→ $\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{3K-G}{2(G+3K)}$ $\nu \to K$ & G

$K = \frac{E}{3(1-2\nu)}$, $G = \frac{E}{2(1+\nu)}$ K & $G \leftarrow E$ & ν

→ $\sigma = E\left[\frac{1}{3(1-2\nu)}(tr\,\varepsilon)I + \frac{1}{1+\nu}\varepsilon'\right]$

or $\varepsilon = \frac{1}{E}\left[\frac{1-2\nu}{3}(tr\,\sigma)I + (1+\nu)\sigma'\right]$ compliance relation

$= \frac{1}{E}\begin{bmatrix} \sigma_{xx}-\nu(\sigma_{yy}+\sigma_{zz}) & (1+\nu)\sigma_{xy} & (1+\nu)\sigma_{xz} \\ (1+\nu)\sigma_{xy} & \sigma_{yy}-\nu(\sigma_{xx}+\sigma_{zz}) & (1+\nu)\sigma_{yz} \\ (1+\nu)\sigma_{xz} & (1+\nu)\sigma_{yz} & \sigma_{zz}-\nu(\sigma_{xx}+\sigma_{yy}) \end{bmatrix}$

· Convenient to use E,ν over K,G since we can get E,ν from observing a single tension test

Pressure vessels

$\varepsilon_{\theta\theta} = \frac{l_\theta - l_\theta'}{L}$ (Δ circumf. / circumf.) = hoop strain

no $\sigma_{r\theta}, \sigma_{\theta z}, \sigma_{rz}$ due to cond. symmetry

$\sigma_{rr}(R) = -P_{in}$ & $\sigma_{rr}(R+t) = 0$, so $0 \le |\sigma_{rr}| \le P_{in}$ throughout can thickness

$P_{in} << \frac{R}{t}P_{in} = \sigma_{\theta\theta}$ since $t << R$, so just say $\sigma_{rr} \approx 0$ compared to $\sigma_{\theta\theta}$

$\varepsilon_{r\theta} = \frac{1}{2}$ angle Δ betwn lines orig. along $\hat{r}$ & $\hat{\theta}$

ε_{rr} = rel. length change of line along $\hat{r}$

☆ don't forget to consider depth

result of example:

$\frac{t-t_0}{t_0} = \varepsilon_{rr} = -\frac{3}{2}\frac{\nu}{E}(P_{in}-P_{out})\frac{R}{t}$

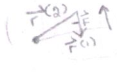
$\frac{\Delta circ}{circ} = \varepsilon_{\theta\theta} = \frac{1-\nu/2}{E}(P_{in}-P_{out})\frac{R}{t}$

$\frac{L_z - L_z}{L_z} = \varepsilon_{zz} = \frac{\frac{1}{2}-\nu}{E}(P_{in}-P_{out})\frac{R}{t}$

1) force balance w/ dif. cross-sections

2) if still too many unks, use compat

'Couple:



(from compliance)

$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$

$G = \frac{E}{2(1+\nu)} \Rightarrow \sigma_{xy} = 2G\varepsilon_{xy} = G\gamma_{xy}$

$= G(2 \cdot \frac{1}{2}(\frac{\pi}{2} - \text{new angle}))$

$\sigma_{xy} = G\gamma_{xy}$ shear angle & stress

Assumptions in torsion analysis:

· nothing moves in axial dirn

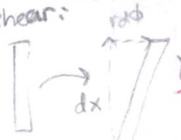
↳ $\varepsilon_{xx} = 0$

· every cross-section of shaft moves only by rotating in its plane some amount ⇒ $\varepsilon_{rr} = 0$, $\varepsilon_{\theta\theta} = 0$, $\varepsilon_{r\theta} = 0$

· $\phi = \phi(x)$ how each cross-section rotates @ a given x

↳ if not constant, twisted

For small deformations column in state of simple shear:

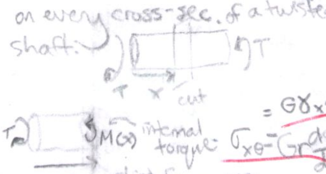


$\gamma_{x\theta} = \tan^{-1}\frac{r\,d\phi}{dx} \approx \frac{r\,d\phi}{dx}$ ($\frac{d\phi}{dx}$ = twist / unit length)

during torsion:

$\varepsilon|_{\{\hat{r},\hat{\theta},\hat{x}\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{r}{2}\frac{d\phi}{dx} \\ 0 & \frac{r}{2}\frac{d\phi}{dx} & 0 \end{bmatrix}$

can use to calc. internal torque on every cross-sec. of a twisted shaft:



$\sigma_{x\theta} = G\gamma_{x\theta} = Gr\frac{d\phi}{dx}$

$M(x) = \int_{cross\text{-}sec.\ thru\ x} r \cdot \sigma_{x\theta}\,dA$ (r: dist. from axis; $\sigma_{x\theta}dA$: force)

$= \int_{area} r \cdot Gr\frac{d\phi(x)}{dx}dA = \boxed{G\frac{d\phi(x)}{dx}\int_{area} r^2 dA}$

$M(x) = GJ\frac{d\phi(x)}{dx}$ ↳ use like you would use $N(x) = EA\frac{du(x)}{dx}$ to do torsion to a rod

$= J$: Polar moment of inertia

J: for solid circular cross-section: $J = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r^2\, r\,dr\,d\theta = \frac{\pi R^4}{2}$

for thin tube of radius R & thickness t: $J = \int_A r^2 dA \cong R^2 2\pi R t = 2\pi R^3 t$

Torsional stiffness: (GJ) ↑ → harder to twist

$F = EA\frac{du(x)}{dx}$ $A = \int_{cross\text{-}sec} dA$ $\delta = N \cdot \int_0^L \frac{dx'}{E(x)A(x)}$

$M = GJ\frac{d\phi(x)}{dx}$ $J = \int_{cross\text{-}sec} r^2 dA$

$\delta = \frac{NL}{EA}$ $k = \frac{EA}{L}$ $N = k\delta$

· Dist. loads: $\vec{r}^* \times \vec{F}_{sum} = \vec{M}_{sum}$

$x_c = \frac{\int_0^L x f(x)dx}{\int_0^L f(x)dx}$

$\Rightarrow N = (\frac{EA}{L})\delta \Rightarrow \underbrace{\frac{N}{A}}_{\sigma} = E\underbrace{\frac{\delta}{L_0}}_{\varepsilon} = E\frac{du(x)}{dx}$

· 2-force members: objects held static by 2 pin joints only (forces @ ends & no moment)

↳ reason: M-bal. abt. one end ↳ truss made of these

1) FBD: isolate structure & draw ext. F's

2) eq. eqns 3) relate deformations/displ. 4) constit.

· composites: $GJ = G_1J_1 + G_2J_2$, $N = N_1 + N_2$

· $\nabla(\varepsilon - \Omega) = \vec{0}$ - draw how it would move

system potential $\varepsilon = \sum \frac{1}{2}k^{(i)}\delta^{(i)2}$ $\Omega = (\sum F \cdot (u_x^{(i)}, u_y^{(i)}))$

· forces: make cut @ every joint & balance

$\frac{|\vec{v}+\vec{\varepsilon}| - |\vec{v}|}{|\vec{v}|}$ = strain, $\frac{\vec{v}\cdot\vec{\varepsilon}}{|\vec{v}|} = \delta$

Principles	Axial loading	Torsion	Closed Pressure Vessel
Equilibrium	$\Sigma F_x = 0$, $N = F = \int_{A(x)} \sigma_{xx} dA$	$\Sigma M_x = 0$, $M_x = \int_{A(x)} r\sigma_{x\theta} dA$	$\sigma_{\theta\theta} = \frac{P_{in}R}{t}$, $\sigma_{zz} = \frac{P_{in}R}{2t}$, $\sigma_{rr} \approx 0$
Constit. rel.	$\sigma_{xx} = E\varepsilon_{xx}$ OR $N = EA\frac{du}{dx}$	$\sigma_{x\theta} = G\gamma_{x\theta}$ OR $M = GJ\frac{d\phi}{dx}$	(applying compliance) $\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu(\sigma_{zz}+\sigma_{\theta\theta}))$; $\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu(\sigma_{rr}+\sigma_{zz}))$; $\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{rr}+\sigma_{\theta\theta}))$
compat	$u(x) = \int\frac{du}{dx}dx$ ⇒ $\delta = \int_{x_0}^{x_f}\frac{du}{dx}dx$	$\phi(x) = \int\frac{d\phi}{dx}dx$, $\Delta\phi(x) = \int_{x_0}^{x}\frac{d\phi}{dx}dx$	$\Delta t = \varepsilon_{rr}t_0$, $\Delta R = \varepsilon_{\theta\theta}R_0$, $\Delta L_z = \varepsilon_{zz}L_{z0}$

$J = \int_A r^2 dA$, $J_{solid\ cylinder} = \frac{\pi}{2}R^4$, $J_{cylindrical\ shell} = 2\pi R^3 t$

1) force bal., compliance, compat., consider dif. cuts

Torsion: $M(x) = \int_{cross\text{-}sec} r \cdot G(r) r\frac{d\phi}{dx}dx$ (in e.g.)

· compat. in shafts: angle

· stiffnesses: AE, GJ, EI

· pure bending: normal stress

· plane stress: thin-walled, ⊥ comp. negligible

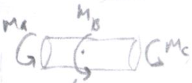
· plane strain: thick members

☆ beam eqns

☆ buckling w/ def. BCs

shaft: torsion

· sum of entries on main diag. doesn't change if rotate bases of stress eq.



$M(x) = -M_A$ if $x < L_{AB}$

$M(x) = -M_A - M_B$ if $x > L_{AB}$

ϕ zero @ walls $0 = \int_0^{L_{AB}+L_{BC}}\frac{d\phi}{dx}dx$

$\Rightarrow M_A = -\frac{L_{BC}}{L_{AB}+L_{BC}}M_B$ & $M_C = -M_A - M_B$

for $\phi(x)$, $\int_0^x \frac{M(x')}{GJ}dx'$

for composites: $(GJ)_{equiv} = G_1J_1 + G_2J_2$

$r = \sqrt{x^2+y^2}$

J for rect. cross-sec: $J = \frac{bh^3}{12}$

UNIT 1

- to see orig. doesn't matter $\vec{r}^{(i)} = \vec{u} + \vec{r}_*^{(i)}$ & calc. $\sum \vec{r}^{(i)} \times \vec{F}^{(i)}$
- couple: $\vec{M} = \vec{d} \times \vec{F}_1$ · dist. load: $x_c = \frac{\int_0^L x f(x)dx}{\int_0^L f(x)dx}$
- (small deformations) $N = k\delta$ ← elong. axial stiffness

$k = \frac{EA}{L_0} \Rightarrow \frac{N}{A} = E\frac{\delta}{L_0} \Rightarrow \sigma = E\varepsilon$ ← axial strain; Young's Modulus; normal stress (constitutive rel. for 1D elastic solids)

- 2-force members: objects held static by 2 pin joints only
  ↳ pin forces = & opposite & along line connecting pins

$\varepsilon(x) = \frac{du(x)}{dx}$ $\quad u(x) = \int_0^x \frac{du}{dx'}dx' = \int_0^x \frac{N}{E(x')A(x')}dx' \Rightarrow \delta = N\int_0^{L_0}\frac{dx'}{E(x')A(x')}$

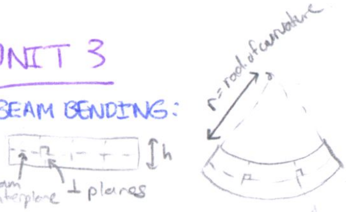
$k_{eff} = \frac{E_1A_1}{L_0} + \frac{E_2A_2}{L_0}$ $\quad$ 1) F & M bal. (cuts) 2) compat $\delta = \frac{\vec{v}\cdot\vec{\omega}}{|\vec{v}|}$ 3) constit. $F = \frac{AE}{L}\delta$ 4) plug into orig.

$= |\vec{v}+\vec{\omega}| - |\vec{v}|$

Energy method: in terms of u's
1. compat (δ)
2. const. equ. $N = k\delta$ ← like spring, $E = \frac{1}{2}k\delta^2$ everything in terms of u's
3. energy eq. $\nabla(E - W) = 0$ $\quad W = \Omega = \sum \vec{F}^{(i)}\cdot\vec{u}^{(i)}$ system potential
   ↳ grad. wrt. each u

UNIT 3

BEAM BENDING:



Kinematic assumptions:
- ⊥s stay ⊥
- center plane does not stretch
  ↳ if 'pure' bending state

- $\tilde{y}$ (from arbitrary origin)

$\frac{\ell}{L} = \frac{\Delta\theta\cdot(r-\tilde{y})}{\Delta\theta\cdot r} = \frac{r-\tilde{y}}{r} = 1 - \frac{\tilde{y}}{r}$

$\Rightarrow \varepsilon_{xx}(\tilde{y}) = \frac{\ell - L}{L} = -\frac{\tilde{y}}{r} = -k\tilde{y}$

- top & bottom edges not touching anything
  $\Rightarrow \sigma_{\tilde{y}\tilde{y}}$ & $\sigma_{z\tilde{y}} = 0$, & since thin, not much room to grow from zero $\Rightarrow \sigma_{xy}, \sigma_{yy} \ll \sigma_{xx}$
  ↳ like simple tension on each lateral fiber of beam

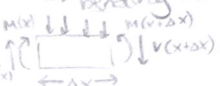
$F_{\tilde{x}} = \int_{cross\text{-}sec} \sigma_{\tilde{x}\tilde{x}}dA = \int_0^b\int_{-h/2}^{h/2} -E\tilde{y}/r\, d\tilde{y}dz = 0 \Rightarrow$ no net tension in pure bending

$M = \int_{cross\text{-}sec}(0,\tilde{y})\times(0,1)\sigma_{\tilde{x}\tilde{x}}dA = \frac{E}{r}\left(\int_{cross\text{-}sec}\tilde{y}^2dA\right)\hat{z}$ $\quad I_{rect} = \int_0^b\int_{-h/2}^{h/2}\tilde{y}^2d\tilde{y}dz = \frac{bh^3}{12}$

Let $\frac{1}{r} = K$ = curvature $\quad M = EIK = EI\frac{d^2w}{dx^2}$ bending stiffness; 'Moment-Curvature relation'

- small deflection approx:
  - centerline same length before & after
  - $u_x \ll u_y \Rightarrow \vec{u} \approx (0, u_y(x))$ ↳ $w(x) = u_y(x)$
  $\Rightarrow K \approx \frac{d^2w}{dx^2}$

- pure bending → just internal moment (bending load causes normal stress)
- general bending → can also induce a shear force $V(x)$ ↳ aka not noticeable $V(x)$
  ↳ but no tensile force $N(x) = 0$
- $|\sigma_{xy}| \ll |\sigma_{xx}|$; since $E \sim G$, avg. $\gamma_{xy} \ll \varepsilon_{xx}$ (otw ⊥s wouldn't stay ⊥)
  $\Rightarrow$ bending motion dominates over shear motion



y-force bal & Taylor expand $\Rightarrow \frac{dV}{dx} = -P(x)$ $\quad$ balance moment $\Rightarrow \frac{dM}{dx} = V(x) = EI\frac{d^3w}{dx^3}$ $\quad$ plug in moment-curvature rel. for M: $P(x) = -\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)$ load; Beam Equation

$\Rightarrow M(x) = EI\frac{d^2w}{dx^2}$

- Beam eqns:

$V(x) = EI\frac{d^3w}{dx^3}$ (shear force)
$M(x) = EI\frac{d^2w}{dx^2}$ (moment)
$\theta(x) = \frac{dw}{dx}$ (slope)
$w(x) = w$ (displacement)

- Free end: $M(x) = 0$, $V(x) = 0$
- clamped: $\theta(x) = 0$, $w(x) = 0$
- M_0 @ end: $M(x) = M_0$, $V(x) = 0$
  ↳ (remember to right like $M(x=L)$)

Fundamental Beam sol'ns:

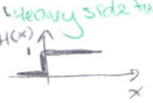
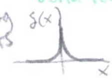
a)  $\Rightarrow w(x) = \frac{-Px^2}{6EI}(3L - x)$

b)  $\Rightarrow w(x) = \frac{-fx^2}{24EI}(x^2 - 4Lx + 6L^2)$

c) $\Rightarrow w(x) = \frac{M_0x^2}{2EI}$

- can use superposition since beam theory based on linear elasticity

- Graphs:

'Heavy side function'  $\quad$ 'Delta function'  $\quad \delta(x) = 0$ if $x \neq 0$; $\int_{-a}^{a}\delta(x)dx = 1$; $\int_{-b}^{a} f(x)\delta(x)dx = f(0)\int_{-b}^{a}\delta(x)dx$

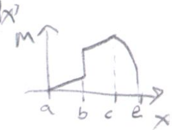
not including moments

Shear force diagram: Since $\frac{dV}{dx} = -P(x)$, $V(x) = \int_0^x -P(x')dx'$

also, $\frac{dM}{dx} = V(x) = \int_{0^-}^{x}\left(a\delta(x') - c\delta(x'-x_c) - dH(x'-x_c) + e\delta(x'-x_e)\right)dx'$

- jumps @ pt. forces

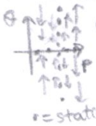
Moment diagram: $M(x) = \int_0^x \left(V(x') + M_{ext}(x')\right)dx$



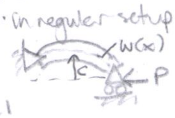
$\Rightarrow$ can set $w(x)$ & $w'(x)$ equal

- w'' can jump & kink
- w' (beam slope) kink but not jump
- w (deflection) can't kink/jump
- w' continuous $\Rightarrow$ ⊥s stay ⊥
- w continuous $\Rightarrow$ beam doesn't split

- Buckling: when axial compression → instability & causes bending
- Stable: a small perturbation induces motion that brings it back towards the same static sol'n
- from bal. x-forces in spring + pin setup, for small θ, $(2P - kL)\cdot\theta = 0$
  ↳ x-force is $(2P - kL)\cdot\varepsilon = ma$; if $a > 0 \Rightarrow$ rightward $\Rightarrow 2P - kL > 0$; $P_{crit} = \frac{kL}{2}$


(bifurcation diagram)
= static sol'ns

- in regular setup:



(small def.) x-force: $P = -N$; y-force: $V = 0$; moment: $M - w(x)N + ... = 0$, $M = -Pw(x)$

$\Rightarrow$ moment curvature: $M = EI\frac{d^2w}{dx^2} = -Pw$

ODE for w → $EIw'' + Pw = 0$

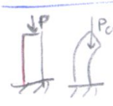
$\downarrow L \Rightarrow \uparrow P_{crit}$
$\uparrow P_{crit} \Rightarrow \uparrow EI$

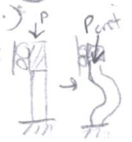
$L = n\pi$; $n = 1$; $c_1 = 0$; unlikely in bending config.

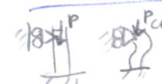
$w(x) = c_1\sin\left(\sqrt{\frac{P}{EI}}x\right) + c_2\cos\left(\sqrt{\frac{P}{EI}}x\right)$ $\quad$ Euler load: $P_{cr} = \frac{\pi^2EI}{L^2}$

↳ pin-pin BC's: $w(0) = 0 \Rightarrow c_2 = 0$
$\sqrt{\frac{P}{EI}}L = n\pi \rightarrow w(L) = 0$

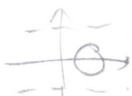
- @ node ⇒ don't influence forces
- form of P_{crit}: $\frac{\pi^2EI}{(KL)^2}$ effective length factor (det. by end cond.)

- fixed-free bc's  $P_{crit} = \frac{\pi^2EI}{(2L)^2}$

- clamp-clamp bc's: $P_{crit} = \frac{\pi^2EI}{(\frac{1}{2}L)^2}$ 

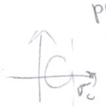
- clamp-pin bc's  $P_{crit} = \frac{\pi^2EI}{(0.7L)^2}$

- visualizing failure criteria:
  plastic def. occurs if any internal plane exists on which the value of shear stress reaches a critical value τ_y
  $\Rightarrow$ no plastic yielding if $|\sigma_{x'y'}| < \tau_y$ for all $\hat{x}'$ (Tresca's Criterion)



- slip lines
  - can't have cont. def. process where shear stress $> \tau_y$ since it just defs. accordingly when τ_y reached

principal directions: shear stresses → 0



- to model fracture, ask if loading state causes there to be a plane/line in 2D on which tensile stress $= \sigma_c$
- Mohr-Coulomb: good for predicting plastic deformation of granular material (gravel, sand) $|\sigma_{x'y'}| < \mu(-\sigma_{x'x'})$
  $\Rightarrow$ elastic flow when: $|\sigma_{x'y'}| = \mu(-\sigma_{x'x'})$