Differentiation on R^n

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1 Single variable derivative

Definition 1.1. *Let A be a subset of R; Let*

$$\phi'(a) = \lim_{t \to 0} \frac{\phi(a+t) - \phi(a)}{t}$$

provided the limit exists, we say that ϕ is differentiable at a.

The following facts are inmediat consequence:

- Differentiable functions are continuous.
- Composites of differentiable functions are differentiable.

We seek to define the derivativo of a function f mapping a subset of R^m into R^n which preserves the properties that we have previously mentioned.

2 Multivariable differentiation

Definition 2.1. *Let $A \in R^m$; lef $f : A \to R^n$. Suppose A contains a neighbourhood of a. Given $u \in R^m$ with $u \neq 0$, define*

$$f'(a; u) = \lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

provided the limit exists. This limit is called ***directional derivative*** *of f at a with respecto to the vector u.*

Definition 2.2. *Let $A \in R^m$, let $f : A \to R^n$. Suppose A contains a nighborhood of a. We say that f is* ***differentiable at a*** *if there is a n by m matrix B such that*

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{||h||} = 0$$

The matrix B, whisch is unique, is called the derivative of f at a; it is denoted $Df(a)$.

Next, we will prove the following facts about differentiable functions:

- Differentiable functions are continuous.
- Composites of differentiable functions are differentiable.
- Differentiablilty of f at a implies the existence of all the directional derivatives of f at a.

Theorem 2.1. *Let $A \in R^m$; let $f : A \to R^n$. If f is differentiable at a, then all directional derivatives of f at a exist, and*

$$f'(a; u) = Df(a) \cdot u$$

.

Proof. Let $B = Df(a)$. Set $h = tu$ in the definition of differentiability, where $t \neq 0$. Then by hypothesis,

$$^* \lim_{t \to 0} \frac{f(a+tu) - f(a) - Btu}{||tu||} = \lim_{t \to 0} \frac{f(a+tu) - f(a) - tBu}{|t| \cdot ||u||} = 0$$

If $t \to 0$ through positive values, we multiply * by $||u||$ to conclude that

$$\lim_{t \to 0} \frac{f(a+tu) - f(a)}{t} - Bu = 0$$

as $t \to 0$ as desired. If t approaches 0 through negative values, we multiply (*) by $-||u||$ to reach the same conclusion.

$\square$

Theorem 2.2. *Let $A \in R^m$; let $f : A \to R^n$. If f is differentiable at a, then f is continuous at a.*

Proof. Let $B = Df(a)$. For h near 0, write

$$f(a+h) - f(a) = ||h|| \cdot \left[\frac{f(a+h) - f(a) - B \cdot ||h||}{||h||}\right] - B||h||$$

By hypothesis, the expression in brackets approaches 0 as h approaches 0. Then

$$\lim_{h \to 0} [f(a+h) - f(a)] = 0$$

□

We now center our attention to the concept of partial derivatives. Which will be of great use calculating derivatives of aribitrary order.

Definition 2.3. *Let $A \in R^m$; let $f : A \to R$. We define the j^{th}* ***partial derivative*** *of f at a to be the directional derivative of f at a with respect to the vector e_j. Provided this derivative exists we denote it by $D_j f(a)$.*

Theorem 2.3. *Let $A \in R^m$; let $f : A \to R$. If f is differentiable at a, then*

$$Df(a) = [D_1 f(a), D_2 f(a), ..., D_m f(a)]$$

Proof. By hypothesis, $Df(a)$ exists and is a matrix of size 1 by m. Let

$$Df(a) = [\lambda_1, \lambda_2, ..., \lambda_m]$$

It follows (Theorem 2.1) that

$$D_j f(a) = f'(a; e_j) = Df(a) \cdot e_j = \lambda_j$$

□