Field induced delocalization in a Koch fractal

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We show that an infinite Koch fractal can support extended electron states in the presence of a suitably chosen magnetic field. The eigenvalues corresponding to these states exhibit interesting transmission characteristics for arbitrarily large versions of the lattice. While one group of eigenvalues makes the lattice completely transparent to an incoming electron, for another energy the transmission coefficient decays with increasing system size following a power law. We obtain an exact expression for the transmission coefficient in every case, and show that, by tuning the value of the magnetic flux appropriately, the transmission coefficient can be made equal to unity in the latter case. [S0163-1829(99)10627-1]

The electronic and statistical properties of fractals are known to be markedly different from either ordered or fully disordered materials.[1–9] Regular fractal lattices are artificial. However, the electronic problem (and also other dynamical models) defined on these lattices, in general, can be solved exactly. The properties on one length scale can be related to the same properties on another length scale by exact recursion relations, and interesting scaling behavior, if any, can be obtained.[10] The energy spectrum of fractals in general is a Cantor set with the single-particle states mostly localized and possessing a high degree of degeneracy.[1,7]

The localization phenomenon in a deterministic fractal, as suggested in the literature,[8,11] may be different (if one says so) from the famous Anderson localization[12] only in the sense that in the latter case the essential randomness causing localization is introduced by varying energy randomly from site to site in an otherwise periodic system (as in Anderson's original model), whereas in a fractal, even with identical site energies, the varying environment around each site mimics a topologically disordered system[13] and is thought to be responsible for localization. Apart from the electronic problems, the effect of this different geometry and topology (compared to a Bravais lattice) on the thermodynamics of certain regular fractals was studied earlier in some detail by Griffiths and Kaufman[13] using different classical spin models on hierarchical lattices.

Our objective however, is to study the effect of an externally applied magnetic field on the localization properties of a fractal. The magnetic field is already known to produce interesting changes in the energy spectrum in two-dimensional lattices.[14] More recently, Vidal, Mosseri, and Ducot[15] have demonstrated an extreme localization effect induced by a magnetic field in two-dimensional periodic, quasiperiodic, and random tilings. Among the popular fractal models, the energy spectrum of a Sierpinski gasket fractal (SGF) has been studied in presence of a magnetic field numerically[16] and in an analytical way as well.[11,17]

In this paper we study a Koch fractal[2] [Fig. 1(a)] in presence of a magnetic field applied perpendicular to the plane of the paper. The spectrum and eigenstates of a Koch fractal without a magnetic field have been studied to a good extent over the past few years using the transfer-matrix formalism and the methods of dynamical systems analysis,[18–20] used earlier by Kohmoto, Kadanoff, and Tang[21] and Kohmoto and Sutherland[22] in investigating the spectral characteristics of quasiperiodic chains. Here, we address two different problems. First, we investigate whether extended eigenstates exist at some special energies on this nonbranching Koch lattice. This particular problem of getting extended electronic states in nontranslationally invariant systems has received some attention over the past few years.[23–26] In addition to this result, power-law localized eigenstates have been known to exist in quasiperiodic chains in general[21,22] and in some branching fractals as well.[1,6,7,10,27,28]

Second, we examine the transmission characteristics of arbitrarily large finite versions of the Koch lattice in the presence of a magnetic field at those particular energies for which the infinite lattice sustains extended eigenstates. In the recent past, we have encountered situations in which we found[28] that in a Vicsek fractal[29] it is possible to find extended electronic states for which the transmission coefficient decays with increasing system size following a power law. In the present work, we obtain analytically exact results

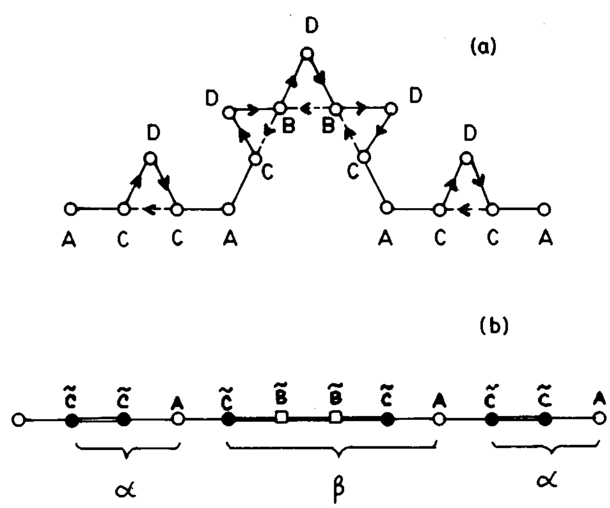

FIG. 1. (a) A Koch fractal in its second generation and (b) its mapping onto an effective one-dimensional chain. Arrows point to the forward direction of hopping. Different sites have been marked A, B, C, and D in (a) and A, $\tilde{\epsilon}_B$, and $\tilde{\epsilon}_C$ in (b), as described in the text.

revealing the behavior of the transmission coefficient (T) as functions of the externally applied flux, as well as the system size.

We start by describing a single electron on an infinite Koch fractal by the tight-binding Hamiltonian in presence of an external magnetic field:[30]

$$H=\sum_i \epsilon_i|i\rangle\langle i|+\sum_{\langle ij\rangle}[V_{ij}^f|i\rangle\langle j|+V_{ij}^b|j\rangle\langle i|], \qquad (1)$$

where ϵ_i is the on-site potential at the ith atomic site. A similar Hamiltonian was solved by Andrade and Schellnhuber[18] in the zero field case to analyze the spectrum of a nonbranching Koch curve. In our case we use four different symbols for ϵ_i, viz., ϵ_A, ϵ_B, ϵ_C, and ϵ_D. Corresponding sites are marked in Fig. 1(a). The introduction of a magnetic field breaks the time-reversal symmetry along the edges of a triangle. Accordingly, we define ''forward'' and ''backward'' hopping by V_{ij}^f and V_{ij}^b, respectively. Between the sites C and D [Fig. 1(a)], $V_{ij}^f=t_2e^{i\gamma}$ and between the pairs of sites (C,C) and (C,B), $V_{ij}^f=t_3e^{i\gamma}$. The values of V_{ij}^b are complex conjugates of those of V_{ij}^f. Here, $\gamma=2\pi\phi/\phi_0$, ϕ is the applied flux, and $\phi_0=hc/e$ is the flux-quantum. The forward and backward hopping between the sites A and C is unaffected by the field, and is denoted here by t_1 at the bare length scale. Following Banavar, Kadanoff, and Pruisken[16] we select a flux distribution such that the directions of the arrows in Fig. 1(a) represent ''forward'' hopping. All the triangles enclose the same flux 3ϕ. It is quite straightforward to reduce such a geometry as shown in Fig. 1(a) to a purely one-dimensional chain by expressing the amplitude of the wave function ψ at the vertex D of every triangle in terms of the amplitudes at the two other vertices at the base. The Koch curve now becomes a chain of ''atoms'' with site energies ϵ_A, $\tilde{\epsilon}_C$, and $\tilde{\epsilon}_B$ arranged in a hierarchical pattern. The ''effective'' chain is shown in Fig. 1(b) with the sites properly marked. The renormalized site energies are given by

$$\tilde{\epsilon}_C=\epsilon_C+t_2^2/(E-\epsilon_D), \qquad (2)$$

$$\tilde{\epsilon}_B=\epsilon_B+2t_2^2/(E-\epsilon_D). \qquad (3)$$

Here, E stands for the energy of the electron. In Fig. 1(b), we identify two basic clusters of atoms, one of them being a three-site cluster and the other a five-site cluster, which are arranged in a hierarchical fashion. We designate in Fig. 1(b), the three-site and the five-site clusters by α and β, respectively. We therefore have to analyze an infinite one-dimensional system consisting of α and β blocks arranged in a hierarchical fashion.

We now discuss the occurrence of extended eigenstates from two different viewpoints. In order to avoid unnecessarily complicated mathematical expressions, from now on we specifically discuss our results in terms of a model where all $\epsilon_i=0$ and $t_1=t_2=t_3=1$. This, however, does not affect the generality of the approach.

(a). Resonant tunneling. Once the original fractal structure is reduced to an effective chain of atoms, the transfer-matrix formalism[21] is easily implemented. After constructing the matrix products following the standard rules,[21,22] it can be seen that, if we set $E=\tilde{\epsilon}_B$, then the product transfer matrix for the effective *pair* of $\tilde{B}$-type sites on the equivalent one-dimensional chain [Fig. 1(b)] is identity, apart from a phase factor. In our model, by setting $E=\tilde{\epsilon}_B$, we get $E=\pm\sqrt{2}$. Let us continue the discussion with $E=\sqrt{2}$. The other value of E can be dealt with in a similar fashion. The product matrix M_β across the entire five-site cluster β is related to M_α, the transfer matrix for the three-site cluster, by the relation

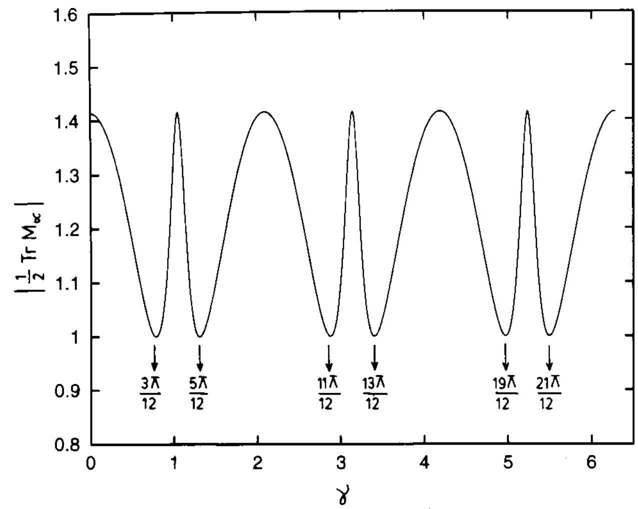

FIG. 2. Variation of $\frac{1}{2}|\mathrm{Tr}\,M_\alpha|$ against γ for $E=\sqrt{2}$.

$$M_\beta=-e^{i\theta}M_\alpha, \qquad (4)$$

where, $\tan\theta=(2\sin 2\gamma-2\sqrt{2}\sin\gamma-\sin 4\gamma)/(2\cos 2\gamma+2\sqrt{2}\cos\gamma+\cos 4\gamma)$. Thus, apart from the phase factor in Eq. (4), the entire fractal now becomes equivalent to a periodic array of $\tilde{C}\tilde{C}A$ atoms that constitute an α cluster. If $E=\sqrt{2}$ now falls within the ''allowed'' band of this effective array of atoms,[22–25] then we have an extended eigenfunction supported by an infinite Koch fractal.

The condition that the calculated energy falls within the *band* of allowed energies for the effective ordered chain is that $\frac{1}{2}|\mathrm{Tr}\,M_\alpha|\leqslant 1$[22] for $E=\sqrt{2}$. As the trace is a function of the magnetic flux now, one can tune the value of γ to see what are the values of γ for which the condition $|\mathrm{Tr}\,M_\alpha|/2\leqslant 1$ is satisfied. To visualize this, we make a plot of $|\mathrm{Tr}\,M_\alpha|/2$ against γ. The variation is shown in Fig. 2. It is seen that $|\mathrm{Tr}\,M_\alpha|/2$ is never less than 1 and that it is precisely equal to 1 only at some special values of γ which occur periodically with a period of $2\pi/3$. Between 0 and 2π, the values of γ are given by $\gamma=(8p+3)\pi/12$ and $\gamma=(8p+5)\pi/12$ with $p=0$, 1, and 2. The above spectrum of γ values is symmetric around $\gamma=0$. The result thus indicates that at $E=\sqrt{2}$ an extended eigenstate is sustained by the *infinite* Koch fractal only at a special set of values of the magnetic field. It is instructive to calculate the product matrix M_α corresponding to the cluster α at $E=\sqrt{2}$ for these special values of γ. We find that for each of these γ values the matrix is given by

$$M_\alpha=\pm e^{\pm i\pi/3}I, \qquad (5)$$

where I is the identity matrix. The case with $E=-\sqrt{2}$ has a similar result. These states are thus expected to make the Koch fractal completely transparent to any incoming wave. The corresponding transmision coefficient for any large but finite fractal will be unity, as will be discussed later. Before we end this section, we may note that in the zero-field case a unique extended state was found by Andrade and Schellnhuber.[18]

(b). The case of commuting matrices. Now, we turn to a second approach to unravel the extended eigenstates at energies other than $E=\pm\sqrt{2}$. To proceed, we inspect whether M_α and M_β commute for any value of energy E. A similar approach was previously adopted by Macia.[31] We find that $[M_\alpha, M_\beta]=0$ if

$$E(E^2-2)(E+2\cos 3\gamma)=0. \tag{6}$$

The required values of the energy are therefore given by $E=0$, $E=\pm\sqrt{2}$, and $E=-2\cos 3\gamma$. The two matrices thus commute, independent of γ for $E=0$ and $E=\pm\sqrt{2}$. Of these, the commutation of M_α and M_β for $E=\pm\sqrt{2}$ is obvious, as both the matrices become equal to identity matrix barring a constant phase factor. The two other values are nontrivial findings. However, it is interesting to find that for $E=0$, $M_\beta=M_\alpha^3$ and for $E=-2\cos 3\gamma$, $M_\beta=-e^{4i\gamma}M_\alpha^2$. The commutation is therefore obvious. Thus, for all these energies, the three-site and the five-site clusters can be arranged alternately resulting in a periodic array of two different blocks with a unit cell comprising an ''$\alpha\beta$'' pair, or even a single array of α blocks. In addition to this, we find that for each energy mentioned above, the absolute value of trace of M_α or $M_\alpha M_\beta$ is precisely equal to 2. Therefore all these energies are allowed in the spectrum of an infinite fractal and correspond to extended eigenstates. The value $E=-2\cos 3\gamma$ implies that for any E lying in the range $-2\leqslant E\leqslant 2$ an extended eigenstate may be obtained by properly tuning the value of the external flux, i.e., by choosing $\gamma=\arccos(-E/2)/3$. Before we end this section, we note that the energy values $E=0$ and $E=\pm\sqrt{2}$ are also present in Ref. 31. However, in our case, $E=\pm\sqrt{2}$ corresponds to extended eigenstates only in the presence of special values of the external flux.

We now present our results on the transmission coefficient (T) separately. We adopt here the method used earlier by Douglas Stone, Joannopoulos, and Chadi.[32]

(i). T for $E=\sqrt{2}$. As we have already pointed out, the condition under which $E=\sqrt{2}$ corresponds to an extended eigenstate of an infinite fractal is satisfied only for a special set of values of γ. For each of these γ, $M_\alpha=\pm e^{\pm i\pi/3}I$, I being the identity matrix. It is then easy to show after a simple algebra following Ref. 32 that, for a finite sized fractal of any generation, $T=1$ for the special values of the flux that correspond to extended states in the infinite system. However, in a finite lattice, it is likely that there should be some contribution to T for other values of the flux as well, but these contributions will die out as we approach the thermodynamic limit. The value of T at $E=\sqrt{2}$ and for any arbitrary flux γ can be calculated exactly for any size of the lattice, and is given by

$$T=\frac{4(3+2\sqrt{2}\cos 3\gamma)^{N_l}}{[(3+2\sqrt{2}\cos 3\gamma)^{N_l}+1]^2}. \tag{7}$$

Here, N_l, the number of α blocks, is given by $N_l=4N_{l-1}-2$ for $l\geqslant 2$ with $N_1=1$. T for any value of N_l is found to be unity at the special γ values $\gamma=(8p+3)\pi/12$ and $(8p+3)\pi/12$ within 0 and 2π. The periodicity in T is clearly $2\pi/3$. The contribution from other γ values turns out to be zero for large values of N_l. We have checked that even for a fourth-generation fractal with 172 sites, the features of an infinite chain are clearly visible.

(ii). T for $E=0$. The matrices M_α for $E=0$ is given by

$$M_\alpha=\begin{pmatrix} e^{-2i\gamma} & 0 \\ -e^{-5i\gamma}+e^{i\gamma} & e^{-2i\gamma} \end{pmatrix}. \tag{8}$$

The transmission coefficient for any lth generation fractal can now be shown to be given by

$$T=\frac{1}{1+(4N_l-3)^2\cos^2 3\gamma}. \tag{9}$$

T, in this case, has a periodicity of $\pi/3$, which is twice its period in the case with $E=\sqrt{2}$. Further, it is interesting to observe that for $3\gamma=(2p+1)\pi/2$, $p=0,1,2\ldots$, i.e., $\phi=(2p+1)\phi_0/4$, we get $T=1$. For any other value of the flux, the transmission coefficient exhibits a power-law decay: $T\sim N_l^{-2}$ as $l\rightarrow\infty$. Thus the magnetic field is shown to induce a *transition* in T from a power-law decay to completely unattenuated transmission through a Koch fractal. It may be remarked that a similar behavior of T has recently been reported in the case of a Vicsek fractal.[28]

(iii). T for $E=-2\cos 3\gamma$. For $E=-2\cos 3\gamma$, it can be shown that, in any lth generation fractal, the product matrix (P) for the equivalent one-dimensional chain is given by

$$P=(-1)^{N_l-1}e^{4i\gamma(N_l-1)}M_\alpha^{3N_l-2}M_0, \tag{10}$$

where M_0 is the matrix corresponding to the atom at the extreme left. Once again following the method illustrated in Ref. 32, it is quite straightforward to show that for any value of the electron energy satisfying $E=-2\cos 3\gamma$, $T=1$. This implies that for any value of E lying within ± 2, we can have complete transmission irrespective of the system size, as well as the value of the field strength.

Before we end, it should be appreciated that the presence of randomness in the on-site potentials or in the values of the hopping integrals results in an effectively randomly disordered one-dimensional chain. Naturally, the transfer matrices do not commute and we do not get any extended eigenstate, as is well known. The geometry of the Koch fractal actually results in a positional correlation between different atoms which finally leads to the extended eigenfunctions in such a structure.

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