Astronomy from 4 Perspectives: the Dark Universe

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Exercise: Supernova cosmology and dark energy

1. ***Light-propagation in FLRW-spacetimes***
   Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

   (a) Please show that by introducing *conformal time* τ in a suitable definition, one recovers Minkowski light propagation $c\tau = \pm\chi$ in comoving distance χ and conformal time τ for FLRW-space times,
   $$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\chi^2, \tag{I}$$
   which we have assumed to be spatially flat for simplicity.

   (b) What's the relationship between conformal time τ and cosmic time t? What would the watch of a cosmological observer display?

   (c) Please compute the conformal age of the Universe given a Hubble function $H(a)$,
   $$H(a) = H_0 a^{-3(1+w)/2} \tag{II}$$
   which is filled up to the critical density with a fluid with a fixed equation of state w.

   (d) In applying $\mathrm{d}s^2 = 0$ to the FLRW-metric we have assumed a radial geodesic - is this a restriction?

   (e) Please draw a diagram of a photon propagating from a distant galaxy to us in conformal coordinates for a cosmology of your choice, with markings on the light-cone corresponding to equidistant Δa.

2. ***Light-propagation in perturbed metrics***
   The weakly perturbed ($|\Phi| \ll c^2$) Minkowski metric is given by
   $$\mathrm{d}s^2 = \left(1 + 2\frac{\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}x_i\mathrm{d}x^i \tag{III}$$
   with the Newtonian potential Φ. Please compute the effective speed of propagation $c' = \mathrm{d}\,|x|\,/\mathrm{d}t$ for a photon following a null geodesic $\mathrm{d}s^2 = 0$. Please Taylor-expand the expression in the weak-field limit $|\Phi| \ll c^2$: Can you assign an effective index of refraction to a region of space with a nonzero potential?

3. ***Classical potentials including a cosmological constant***
   The field equation of classical gravity including a cosmological dark energy density λ is given by
   $$\Delta\Phi = 4\pi G(\rho + \lambda) \tag{IV}$$

   (a) Solve the field equation for 3 dimensions outside a spherically symmetric and static matter distribution ρ. The expression for the Laplace-operator in spherical coordinates for 3 dimensions is: $\Delta\Phi = r^{-2}\partial_r(r^2\partial_r\Phi)$. Also, please set as the total baryon mass M
   $$M = 4\pi\int_0^r \mathrm{d}r'(r')^2\rho(r') \tag{V}$$

(b) Please show that both source terms individually give rise to power-law solutions for $\Phi(r)$.

(c) Is there a distance where the baryon part from the ρ-terms is equal to dark energy part the λ-term?

(d) Assuming a typical galaxy is formed by 100 billion stars like the Sun each with a mass of 10^{30} kg, and a dark energy density of 10^{-27} kg / m^3, find at which distance from a galaxy the dark energy dominates. How does it compare with the typical size of a Galaxy ($\sim$ 10000 pc?

4. ***Physics close to the horizon***
Why is it necessary to observe supernovae at the Hubble distance c/H_0 to see the dimming in accelerated cosmologies? Please start at considering the curvature scale of the Universe: A convenient measure for the curvature might be the Ricci-scalar $R = 6H^2(1 - q)$ for flat FLRW-models.

   (a) Can you define a distance scale d or a time scale from R?

   (b) What happens on scales $\ll d$, what on scales $\gg d$?

5. ***Measuring cosmic acceleration***
The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by

$$d_{\mathrm{lum}}(z) = (1 + z) \int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{VI}$$

with the Hubble function $H(z)$. Let's assume that the Universe is filled with a cosmological fluid up to the critical density with a fluid with equation of state w, such that the Hubble function is

$$H(z) = H_0(1 + z)^{\frac{3(1+w)}{2}}. \tag{VII}$$

   (a) For this type of cosmology you will obtain acceleration if $w < -1/3$ and deceleration for $w > -1/3$: Please show this by computing the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ from the Hubble-function $H = \dot{a}/a$, with the relation $a = 1/(1 + z)$.

   (b) Please show that in accelerated universes supernovae appear systematically dimmer, because d_{lum} is always larger than in a non-accelerating universe.

   (c) Is it true that d_{lum} is systematically smaller in a decelerating universe?

   (d) Would the expression for d_{lum} still be valid if the universe was contracting instead of expanding? What correction would you need to apply?