Astronomy from 4 Perspectives: the Dark Universe

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Exercise: Planck spectrum and CMB
Solutions

1. Properties of the Planck Spectrum

$$S(\nu) = S_0 \frac{\nu^3}{e^{-\frac{h\nu}{kT}} - 1} \Rightarrow S(\nu) = S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{I}$$

(a)

$$\int_0^\infty S(\nu) = \int_0^\infty S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{II}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{III}$$

$$\int_0^\infty S_0 \left(\frac{kT}{h}\right)^3 e^{-\frac{h\nu}{kT}} \left(\frac{kT}{h}\right) \propto T^4 \tag{IV}$$

(b)

$$\frac{dS}{d\nu} = 0 \Rightarrow \frac{dS}{d\nu} = 3S_0 \nu^2 e^{-\frac{h\nu}{kT}} + S_0 \nu^3 \left(-\frac{h}{kT}\right) e^{-\frac{h\nu}{kT}} = \tag{V}$$

$$= S_0 \nu^3 e^{-\frac{h\nu}{kT}} \left(3 - \frac{h\nu}{kT}\right) \tag{VI}$$

$$3 - \frac{h\nu}{kT} = 0 \Rightarrow \nu = \frac{3kT}{h} \propto T \tag{VII}$$

(c)

$$< \nu > = \frac{\int_0^\infty \nu S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} = \frac{\int_0^\infty \nu^3 S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{VIII}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad d\nu = \frac{kT}{h} dx \tag{IX}$$

$$\frac{\int_0^\infty S_0 \frac{kT^4}{h} e^{-x} \frac{kT}{h} dx}{\int_0^\infty S_0 \frac{kT^3}{h} e^{-x} \frac{kT}{h} dx} \tag{X}$$

$$S \propto \frac{T^5}{T^4} \propto T \tag{XI}$$

(d) Yes, as both of them depend on T

(e) Only at high energies

(f) No, as if we consider any power of T in the exponential, any substitution $x = \frac{h\nu}{kT}$ gives a new differential $d\nu \frac{kT^4}{h}$, both in numerator and denominator

2. *Wien's distribution function*

$$S(\nu) = S_0 \nu^n \, \mathrm{e}^{-\frac{h\nu}{kT}} \quad ; \quad \Gamma(n) = \int_0^\infty dx \, x^{n-1} \, \mathrm{e}^{-x} \tag{XII}$$

(a)

$$\int_0^\infty dx \, x^n \, \mathrm{e}^{-x} = \cancel{-[e^{-x} x^n]_0^\infty} - n \int_0^\infty dx \, x^{n-1} e^{-x} = \int_0^\infty dx \, n x^{n-1} \, \mathrm{e}^{-x} \tag{XIII}$$

$$= \cancel{x[\mathrm{e}^{-x} n x^{n-1}]_0^\infty} - \int_0^\infty dx \, -n(n-1)x^{n-2} = \int_0^\infty dx \, -n(n-1)x^{n-2} \, \mathrm{e}^{-x} \tag{XIV}$$

$$= [\ldots] = \tag{XV}$$

$$= n! \int_0^\infty dx \, \mathrm{e}^{-x} = n! \tag{XVI}$$

Let's show that $\Gamma(1) = 1$:

$$\Gamma(1) = \int_0^\infty x^0 \, \mathrm{e}^{-x} \tag{XVII}$$

To demontstrate our relation recursively, let's start by showing it works for n=2:

$$\Gamma(2) = \int_0^\infty x \, \mathrm{e}^{-x} \, dx = [-\mathrm{e}^{-x} \, x]_0^\infty - \int_0^\infty dx \, (-\mathrm{e}^{-x}) = 1 \tag{XVIII}$$

$$= (2-1)\Gamma(1) = 1 \tag{XIX}$$

We can now show it for $n+1$:

$$\Gamma(n+1) = n \, \Gamma(n) = n \, (n-1) \, \Gamma(n-1) = n(n-1) \int_0^\infty x^{n-2} \, \mathrm{e}^{-x} \, dx \tag{XX}$$

Let's introduce $m = n - 2 \quad \Rightarrow \quad n = m + 2$

$$\Gamma(n+1) = (m+2)(m+1) \int_0^\infty x^m \, \mathrm{e}^{-x} \, dx = (m+2)(m+1)m! = (m+2)! = n! \tag{XXI}$$

But

$$\Gamma(n+1) = \int_0^\infty x^n \, \mathrm{e}^{-x} \, dx = n! \tag{XXII}$$

(b)

$$< \nu^m >= \frac{\int_0^\infty \nu^m S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{XXIII}$$

$$= \frac{\int_0^\infty \nu^m S_0 \nu^n \, \mathrm{e}^{-\frac{h\nu}{kT}} \, d\nu}{\int_0^\infty S_0 \nu^n \, \mathrm{e}^{-\frac{h\nu}{kT}} \, d\nu} \tag{XXIV}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{XXV}$$

$$< \nu^m >= \frac{\int_0^\infty \left(\frac{kT}{h}\right)^m S(\nu) \left(\frac{kT}{h}\right)^n \mathrm{e}^{-x} \, \frac{kT}{h} d\nu}{\int_0^\infty S_0 \left(\frac{kT}{h}\right)^n \mathrm{e}^{-x} \, \frac{kT}{h} d\nu} \tag{XXVI}$$

$$\propto \frac{T^{m+n+1}}{T^{n+1}} = T^m \tag{XXVII}$$

$\Rightarrow$ The momenta scale like the power chosen.

(c) From the result on point 1(c), we find:

$$S = \frac{< \nu^3 >}{(< \nu >^2)^{3/2}} \propto \frac{T^3}{(T^2)^{3/2}} \tag{XXVIII}$$

$$k = \frac{< \nu^4 >^2}{< \nu^2 >^2} \propto \frac{T^4}{(T^2)^2} \tag{XXIX}$$

Neither of which depends on T.

(d) Yes, again from the result of point 1(c):

$$\frac{< \nu^{2n} >}{< \nu^2 >^n} \propto \frac{T^{2n}}{(T^2)^n} \tag{XXX}$$

3. ***Planck Spectra at cosmological distances***
The solution to this question was discussed live in the classroom

4. ***CMB as a source of energy***

(a) The answer (a) with the Stephan Boltzmann law, the power per m^2 can be produced if:

$$w = 10^{-5} \sigma T^4 = 3 \cdot 10^{-11} \ \mathrm{Wm^{-2}} \tag{XXXI}$$