Astronomy from 4 perspectives: the Dark Universe

Exercises and proceedings from the summer school 2017 in Heidelberg

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Introduction

The Heraeus Summer School series "Astronomy from four perspectives"[1 2] draw together teachers and teacher students, research astronomers, physicists and astronomy students from Germany and Italy. For each summer school, participants gather at one of the four participating nodes: Heidelberg, Padua, Jena, and Florence. One of the main goals is to bring astronomy into schools, which is done by educating and training the teachers and teacher students.
Before and during the fifth edition, which took place in Heidelberg from 26.08.– 02.09.2017, the participants and lecturers worked on exercises and material which could either be used for teachers' education or for pupils directly. They are split into two categories, *tutorials* and *in the classroom*, but experience shows that the transition between those categories is smooth and material from both categories might serve both needs:

Tutorials:	Exercises and computer scripts prepared before the summer school to teach the attendants of the school
In the classroom:	Material developed mainly during the summer school: Lower level material that addresses mainly high school students

Also all the material was split into the four main topics of the summerschool. Those are

- Supernova Cosmology
- Virial Theorem
- Rotation Curves
- Cosmic Microwave Background (CMB) temperature

Tutorials

The tutorials are well prepared exercises which come mostly with solutions. These exercises and (mainly python-)scripts were used to give the attendants of the conference a feeling for the topic. But nevertheless, some exercises are also appropriate for higher class pupils. Below the above mentioned four topics, there are also mostly four subcategories for the different kinds of material:

- *Exercise:* Classical exercise sheets and solutions for the participants

[1]`http://www.physik.uni-jena.de/didaktik_summerschool.html`
[2]`http://www.haus-der-astronomie.de/en/events/heraeus-four-perspectives`

- *Play with data:* Exercises where a computer with a working python-3 programming environment is needed[3]
- *Script:* The python-3 scripts needed in order to solve the *play with data exercises*
- *Question:* More advanced questions one might tackle with the knowledge gained so far

You can find the exercises, play with data and questions as part of this publication.
Additionally there is also a hand written script with notes on how to tackle the Kepler Problem in a fast and elegant way, which might even – in a less formal way – be used for high school students.
And last, but not least, some exercises about the home planet of the *little prince* (french original: Le Petit Prince). These exercises are unrelated to the dark universe, but add up in an easy way to get familiar with nonlinear gradients in the gravitational potential, more about it can be found in Palla et al. (2017).

In the Classroom

As mentioned, these are thoughts and materials developed during the summer school. There is material for *CMB temperature* (a Maple exercise sheet), *supernova cosmology* as well as *the virial theorem.* Additionally some german curricula are added (in german), to support discussions on where to use these ideas.
As hoped for, the present material is quite diverse: There are sheets explaining the physics, typical questions from pupils or sketches of whole teaching units, especially around the topic of *supernova cosmology.*

Accessing the Material

Most of the tutorials material is part of this article. But all material can be found as a github repository[4] and can be used for any non-commercial purposes. In order to give the possibility to change and adapt the content to your needs, almost all PDF-files are also accompanied by the *.tex files which were used to compile the code. Of course, the authors are not responsible for any changes that might then be applied by a third person.
Additionally two 90 minutes talks of the summer school can be found online:

- *The thermal History of the Universe* by Matthias Bartelmann[5]
- *Introducting the expanding universe* by Markus Pössel[6]
  More about this metric-free approach can also be found in Pössel (2017), which was written in the course of the previous Heraeus summer schools

Acknowledgments We thank the *Wilhelm and Else Heraeus foundation* for funding the summer schools and thus making these unique conferences possible.

References

Palla, F., Duvernoy, S., & Tobin, S. 2017, The Little Prince's Universe (NAJS . No art Just Sign)

Pössel, M. 2017, ArXiv e-prints, arXiv:1712.10315

[3]Python is a popular, freely available programming language. One way of getting it is by installing anaconda: https://www.anaconda.com
[4]https://github.com/sbrems/Dark_universe
[5]https://www.youtube.com/watch?v=m35fXJoQLA0
[6]https://www.youtube.com/watch?v=gA-0C-88WbE

Figure 1: Group picture of the summer school 2017 in front of the Haus der Astronomie in Heidelberg.

Exercise sheets and solutions

Astronomy from 4 perspectives: the Dark Universe

prepared by: Florence participants and BMS

exercise: Supernova-cosmology and dark energy

1. ***light-propagation in FLRW-spacetimes***
   Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

   (a) Please show that by introducing *conformal time* τ in a suitable definition, one recovers Minkowskian light propagation $c\tau = \pm\chi$ in comoving distance χ and conformal time τ for FLRW-space times,
   $$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\chi^2, \tag{I}$$
   which we have assumed to be spatially flat for simplicity.

   (b) What's the relationship between conformal time τ and cosmic time t? What would the watch of a cosmological observer display?

   (c) Please compute the conformal age of the Universe given a Hubble function $H(a)$,
   $$H(a) = H_0 a^{-3(1+w)/2} \tag{II}$$
   which is filled up to the critical density with a fluid with a fixed equation of state w.

   (d) In applying $\mathrm{d}s^2 = 0$ to the FLRW-metric we have assumed a radial geodesic - is this a restriction?

   (e) Please draw a diagram of a photon propagating from a distant galaxy to us in conformal coordinates for a cosmology of your choice, with markings on the light-cone corresponding to equidistant Δa.

2. ***light-propagation in perturbed metrics***
   The weakly perturbed ($|\Phi| \ll c^2$) Minkowskian metric is given by
   $$\mathrm{d}s^2 = \left(1 + 2\frac{\Phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - 2\frac{\Phi}{c^2}\right)\mathrm{d}x_i\mathrm{d}x^i \tag{III}$$
   with the Newtonian potential Φ. Please compute the effective speed of propagation $c' = \mathrm{d}\,|x|\,/\mathrm{d}t$ for a photon following a null geodesic $\mathrm{d}s^2 = 0$. Please Taylor-expand the expression in the weak-field limit $|\Phi| \ll c^2$: Can you assign an effective index of refraction to a region of space with a nonzero potential?

3. ***classical potentials including a cosmological constant***
   The field equation of classical gravity including a cosmological dark energy density λ is given by
   $$\Delta\Phi = 4\pi G(\rho + \lambda) \tag{IV}$$

   (a) Solve the field equation for 3 dimensions outside a spherically symmetric and static matter distribution ρ. The expression for the Laplace-operator in spherical coordinates for 3 dimensions is: $\Delta\Phi = r^{-2}\partial_r(r^2\partial_r\Phi)$. Also, please set as the total baryon mass M
   $$M = 4\pi\int_0^r \mathrm{d}r'(r')^2\rho(r') \tag{V}$$

(b) Please show, that both source terms individually give rise to power-law solutions for $\Phi(r)$.

(c) Is there a distance where the baryon part from the ρ-terms is equal to dark energy part the λ-term?

(d) Assuming a typical galaxy is formed by 100 billion stars like the Sun each with a mass of 10^{30} kg, and a dark energy density of 10^{-27} kg / m^3, find at which distance from a galaxy the dark energy dominates. How does it compare with the typical size of a Galaxy ($\sim$ 10000 pc?

4. *physics close to the horizon*

Why is it necessary to observe supernovae at the Hubble distance c/H_0 to see the dimming in accelerated cosmologies? Please start at considering the curvature scale of the Universe: A convenient quantisation of curvature might be the Ricci-scalar $R = 6H^2(1-q)$ for flat FLRW-models.

(a) Can you define a distance scale d or a time scale from R?

(b) What happens on scales $\ll d$, what on scales $\gg d$?

5. *measure cosmic acceleration*

The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by

$$d_{\mathrm{lum}}(z) = (1+z)\int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{VI}$$

with the Hubble function $H(z)$. Let's assume that the Universe is filled with a cosmological fluid up to the critical density with a fluid with equation of state w, such that the Hubble function is

$$H(z) = H_0(1+z)^{\frac{3(1+w)}{2}}. \tag{VII}$$

(a) For this type of cosmology you will obtain acceleration if $w < -1/3$ and deceleration for $w > -1/3$: Please show this by computing the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ from the Hubble-function $H = \dot{a}/a$, with the relation $a = 1/(1+z)$.

(b) Please show that in accelerated universes supernovae appear systematically dimmer, because d_{lum} is always larger than in a non-accelerating universe.

(c) Is it true that d_{lum} is systematically smaller in a decelerating universe?

(d) Would the expression for d_{lum} still be valid if the universe was contracting instead of expanding? What correction would you need to apply?

Astronomy from 4 perspectives: the Dark Universe

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prepared by: Florence participants and BMS

exercise: Supernova-cosmology and dark energy
Solutions

1. ***light-propagation in FLRW-spacetimes***
   Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

   (a) Let us do the following substitution $dt \rightarrow a(t)d\tau$ then the line element can be written $ds^2 = a(t)^2[c^2 d\tau^2 - d\chi^2]$ and the equation ofr the null-geodesic will be $d\chi = \pm c d\tau$.

   (b) The cosmic time is the time measured by a cosmic observer synchronized for $t = 0$

$$t = \int_0^t dt' = \int_0^a \frac{da'}{\dot{a}'} \tag{I}$$

   The conformal time is tied to the time interval over which an observer at $t = t_0$ sees to happen an event in the past at time t. Now at $t = t_0$ this will coincide with the cosmic time, ence it will be afected by cosmic time dilation.

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} = \frac{1}{a(t)} \int_0^t \frac{a(t)}{a(t')} dt' > \frac{t}{a(t)} \tag{II}$$

   (c) Now for the given metric:

$$H = \frac{\dot{a}}{a} = H_o a^{-3(1+w)/2} \Rightarrow \dot{a} = H_o a^{1-3(1+w)/2} \tag{III}$$

$$a(t) = \left(\frac{4}{9}\right)^{3(w+1)} (t + wt)^{2/3(w+1)} \tag{IV}$$

$$\tau_H = \int_0^t \frac{dt'}{a(t')} = \int_0^1 \frac{da'}{\dot{a}a} = \frac{1}{H_o} \int_0^1 a^{3(w+1)/2-2} da = \frac{1}{H_o} \frac{1}{3(w+1)/2 - 1} \tag{V}$$

   (d) Isotropy of the universe ensures us that it is not.

2. ***light-propagation in perturbed metrics***

$$ds^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 dt^2 - \left(1 - 2\frac{\Phi}{c^2}\right) dx^2 \tag{VI}$$

   With $ds^2 = 0$:

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 = \left(1 - \frac{2\Phi}{c^2}\right) dx^2 \tag{VII}$$

$$\frac{dx}{dt} = \pm c \sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \tag{VIII}$$

With $\frac{1}{1-\epsilon} \approx 1+\epsilon$ for small ϵ:

$$\frac{dx}{dt} \approx \pm c\left(1+\frac{2\Phi}{c^2}\right) \tag{IX}$$

For a non-zero Φ this is not equal to c!
We assign an effective index of refraction by:

$$n(\Phi) = \frac{dx/dt}{c} \approx \left(1+\frac{2\Phi}{c^2}\right) \tag{X}$$

3. *classical potentials including a cosmological constant*

The field equation of classical gravity including a cosmological dark energy density λ is given by

$$\Delta\Phi = 4\pi G(\rho+\lambda) \tag{XI}$$

(a) field calculation

Now it is possible to simply integrate the field equation starting with:

$$\begin{aligned}
\Delta\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) && \text{(XII)}\\
&= 4\pi G(\rho(r)+\lambda) && \text{(XIII)}\\
r^2\frac{\partial\Phi}{\partial r} &= \int_0^r \mathrm{d}r' \left(4\pi G[(r')^2\rho(r') + (r')^2\lambda]\right) && \text{(XIV)}\\
&= GM + G\frac{\lambda}{3}r^3 && \text{(XV)}\\
\frac{\partial\Phi}{\partial r} &= \frac{GM}{r^{n-1}} + G\frac{\lambda r}{n} && \text{(XVI)}\\
\Phi &= -\frac{GM}{r} + G\frac{\lambda r^2}{6} && \text{(XVII)}
\end{aligned}$$

(b) power-law solutions
Following the calculation one may see that each source term corresponds to an individual power-law:

$$\begin{aligned}
C(n)G\rho(r) \quad &\Rightarrow \quad -\frac{GM}{r}\\
\lambda \quad &\Rightarrow \quad G\frac{\lambda r^2}{6}
\end{aligned}$$

(c) equilibrium
To find an equilibrium distance one must set $\Phi\left(r_{\mathrm{eq}}\right) = 0$

$$\begin{aligned}
\frac{GM}{r_{\mathrm{eq}}} &= G\frac{\lambda r_{\mathrm{eq}}^2}{6} && \text{(XVIII)}\\
\frac{\lambda r_{\mathrm{eq}}^3}{6} &= M && \text{(XIX)}
\end{aligned}$$

from which follows immediatly:

$$r_{\mathrm{eq}} = \sqrt[3]{6\frac{M}{\lambda}} \tag{XX}$$

(c) if one inputs the number one gets $r_{\mathrm{eq}} = 1.5$ Mpc one hundred times larger than the size of a galaxy.

4. ***physics close to the horizon***
   Why is it necessary to observe supernovae at the Hubble distance c/H_0 to see the dimming in accelerated cosmologies? Please start at considering the curvature scale of the Universe: A convenient quantisation of curvature might be the Ricci-scalar $R = 6H^2(1-q)$ for flat FLRW-models.

   (a) The Dimension of the Ricci-scalar is $1/s^2$ thus we can define a time and a distance scale by:
   $$\tau = 1/\sqrt{R} \quad \text{and} \quad d = c/\sqrt{R} \approx c/H_0$$
   which gives the curvature scale of the Universe.

   (b) To observe supernovae dimming caused by accelerated cosmic expansion the supernova distance had to be about (or larger than) the curvature scale, because at distances << d the different cosmological distance measures converge.
   For illustration see: *https://en.wikipedia.org/wiki/Distance_measures_(cosmology)*

5. ***measure cosmic acceleration***
   The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by
   $$d_{\mathrm{lum}}(z) = (1+z)\int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{XXI}$$
   with the Hubble function $H(z)$. Let's assume that the Universe is filled with a cosmological fluid up to the critical density with a fluid with equation of state w, such that the Hubble function is
   $$H(z) = H_0(1+z)^{\frac{3(1+w)}{2}}. \tag{XXII}$$

   (a) By definition:
   $$H = \frac{\dot{a}}{a} \text{ and } q = -\frac{\ddot{a}a}{\dot{a}^2}$$
   It follows
   $$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}a}{a^2} - H^2$$
   So we get
   $$\frac{\dot{H}}{H^2} = \frac{\ddot{a}a}{\dot{a}^2} - 1 = -q - 1$$
   $$q = -(\frac{\dot{H}}{H^2} + 1)$$
   We also have
   $$H = H_0 \cdot (1+z)^{\frac{3(1+w)}{2}} = H_0 \cdot a^{\frac{-3(1+w)}{2}}$$
   and
   $$\begin{aligned}\dot{H} &= H_0\left(\frac{-3(1+w)}{2}\right) \cdot a^{\frac{-3(1+w)}{2}} \cdot \dot{a} \\ &= H_0 \cdot a^{\frac{-3(1+w)}{2}} \cdot \frac{\dot{a}}{a} \cdot \left(\frac{-3(1+w)}{2}\right) \\ &= H^2 \cdot \left(\frac{-3(1+w)}{2}\right)\end{aligned}$$

so

$$q = -\left(\frac{-3(1+w)}{2} + 1\right) = \frac{1}{2}(3w+1)$$

and obviously

$$q < 0 \text{ for } w < -\frac{1}{3}$$
$$q > 0 \text{ for } w > -\frac{1}{3}$$

(b) First, we consider the case $w = -\frac{1}{3}$ (non-accelerating universe):

$$H = H_0(1+z)^{\frac{3(1+w)}{2}} = H_0(1+z)$$

$$\begin{aligned} d_{lum,1} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\ &= (1+z)\int_0^z \frac{1}{H_0(1+z')}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0} ln(1+z) \end{aligned}$$

Now, we consider the case $w < -\frac{1}{3}$ (accelerating universe):

$$\begin{aligned} d_{lum,2} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0}\int_0^z (1+z')^{\frac{-3(1+w)}{2}}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0}\left[(1+z')^{\frac{-3(1+w)+2}{2}} \cdot \frac{2}{-3(1+w)+2}\right]_0^z \\ &= \frac{1+z}{H_0}\left(\frac{2}{-3(1+w)+2}\right)\left[(1+z')^{\frac{-3(1+w)+2}{2}} - 1\right] \end{aligned}$$

It follows: $d_{lum_2}(z) > d_{lum_1}(z)$, because the exponent $\frac{-3(1+w)+2}{2}$ is positive ($w < -\frac{1}{3}$), so $d_{lum_2}(z)$ is growing faster, than the logarithmic function $d_{lum_1}(z)$.

(c) Yes, as long as the universe is flat. Just try to plot the two functions

(d) The formula still apply,. But in a contracting universe one would see light that is blue-shifted and not red-shifted. Su $z < 0$. But Obviiously $z > -1$ otherwhise one would get negative wavelengths (frequancies) that make no sense. As a consequence, the origin of time in a contracting universe cosrresponds to $z = -1$ for a universe contracting from infinity. Hence the limit of integration must be corrected accordingly.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Florence participants and BMS

play with data: Supernova-cosmology and dark energy

The Supernova Cosmology Project observed supernovae of the type Ia in distant galaxies, determined the distance modulus by measuring the apparent brightness as well as the redshift of the host galaxy. From the relation between distance and redshift one can measure the density parameters Ω_X and the dark energy equation of state w. The exercise sheet uses data from A. Goobar et al., PhST, 85, 47 (2000).

1. ***distance redshift-relationships in FLRW-universes***
   In this exercise you can play with SCP-data and explore the sensitivity of the supernova brightness on the cosmological parameters. Please have a look at the python-script `supernova_plot.py`, which reads the data file from SCP and plots distance modulus μ as a function of redshift z. The cosmological model is a spatially flat FLRW-cosmology with the Hubble function

$$H(z) = H_0 \sqrt{\Omega_m(1+z) + (1-\Omega_m)(1+z)^{3(1+w)}} \tag{I}$$

   where the density of the dark energy component is automatically set to $\Omega_X = 1 - \Omega_m$ to enforce flatness. Setting $w = -1$ recovers the case of the cosmological constant Λ, in which case $\Omega_X = \Omega_\Lambda$.

   (a) Start by guessing different values for Ω_m to find the best value: What's Ω_Λ?

   (b) What's your explanation why Λ makes the supernovae dimmer?

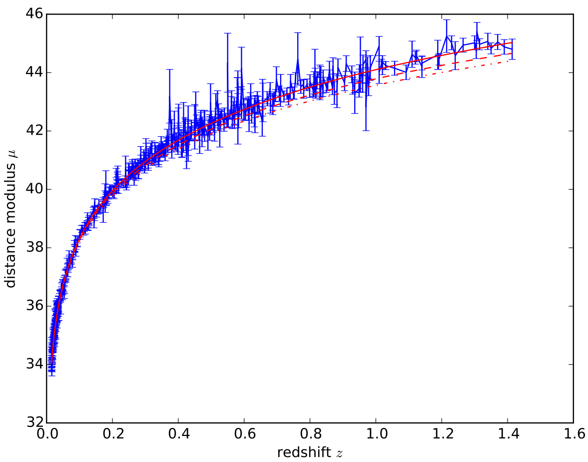

Figure 1: data from the SCP and distance redshift relationships with varying Ω_m

2. ***fitting a FLRW-cosmology***
   The script `supernova_fit.py` does a proper regression of a model $\mu(z)$ to the data, by minimising the squared difference between data and model, in units of the measurement error.

   (a) What is the best fitting value for Ω_m?

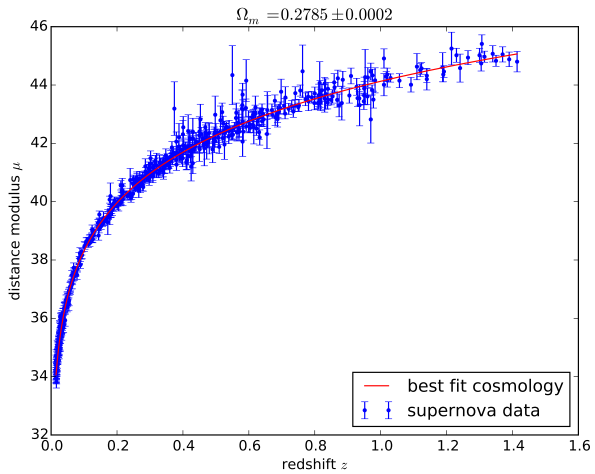

Figure 2: fit to the SCP-data in a FLRW-cosmology: Λ is not zero

(b) What's the certainty that $\Omega_\Lambda \neq 0$?

3. ***precision of the measurement***
In running the script `supernova_likelihood.py` you can simultaneously fit Ω_m and w to the data. It evaluates the likelihood $\mathcal{L}(\Omega_m, w) \propto \exp(-\chi^2(\Omega_m, w)/2)$, with

$$\chi^2(\Omega_m, w) = \sum_{i=1}^{n_\text{data}} \left(\frac{\mu_i - \mu(z_i, \Omega_m, w)}{\sigma_i}\right)^2 \tag{II}$$

for the n_data data points μ_i at the redshifts z_i. The probability that a parameter choice is true is reflected by the density of points.

(a) What are the statistical errors on Ω_m and on w?

(b) Why do you require more negative w if Ω_m is larger?

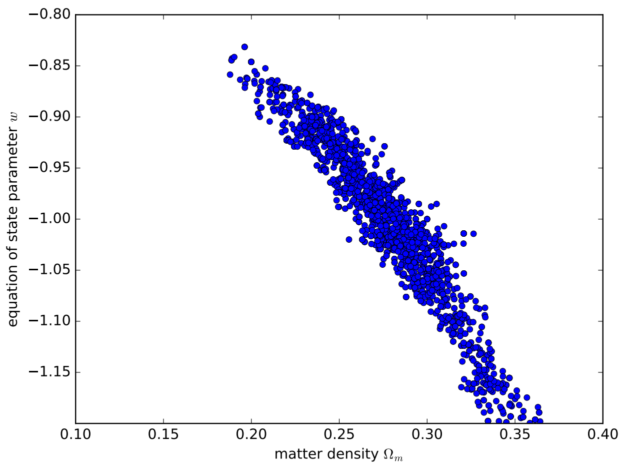

Figure 3: simultaneous measurement of Ω_Λ and w

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prepared by: Florence participants

questions: Supernova-cosmology and dark energy

1. ***FLRW-models and the equation of state***

   (a) What are the ingredients (fields?) the enter into a FLRW metric?
   (b) How does each field behave as the universe expands?

2. ***light propagation in relativity***

   (a) What is the equation that describes light propagation in a curved space-time?
   (b) How does the presence of a body exerting gravity affect light paths?
   (c) How curvature affect light propagation?
   (d) How expansion affects light propagation?

3. ***distance measures***

   (a) How do we measure distance in a curved space-time?
   (b) What is the angular distance?
   (c) How we disentangle red-shift from curvature?

4. ***supernova-cosmology***

   (a) Why SN IA are standard candles?
   (b) Why is it important to observe them at high redshift?
   (c) Why are they not enough to fully constrain the cosmology?

Astronomy from 4 perspectives: the Dark Universe

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prepared by: Bucciantini N.

Quantum Mechanics, Relativity and Supernovae

This tutorial is aimed at advanced students, to whom the teacher has already introduced the basic concepts of quantum mechanics and relativity. In particular the only notion of QM required is the **uncertainty principle of position and momentum** and the only notion from SR the **relativistic relation between energy and momentum**. Some familiarity with thermodynamics and hydrostatic is also required.

We are going to show how with simples arguments using basic quantum mechanics and special relativity one can find that there is a limit to the mass of a quantistic star.

The starting points are:

1. From quantum mechanics the uncertainty principle of momentum and position $\Delta x \Delta p = \hbar$. The teacher should have introduced it before and explained its meaning and implications.

2. From special relativity the relativistic relation between energy and momentum of a relativistic particle $e = cp$. One can use photons to show that they carry energy (obvious - think of solar panels that produce electricity) and also momentum (the solar mill is a good example).

3. From thermodynamics the relation between pressure of a gas and the energy density. This can be introduced classically with a discussion of the relation $P = nkT = U$, recalling that kT is the thermal energy of a particle of a gas.

4. From hydrostatic the equilibrium relation $F = -\nabla P$, stating that any external force (force density to be more precise) must be balanced by a pressure gradient. As an example one can discuss how pressure changes going under water, by simply showing that at any depth the pressure must be equal to the force exerted by the overlying column of water. And use this argument to derive the hydrostatic relation.

We begin by showing how a relativistic gas behaves.

First consider a volume V containing N particles. Then the average volume per particle is $V/N = 1/n$ where we have introduced the particle density n. Then one can define the average distance between particles is $(V/N)^{1/3} = 1/n^{1/3}$. Using the uncertainty principle, and taking as a typical uncertainty over the distance the average distance between particles, we get a typical momentum $p = \hbar n^{1/3}$. Now use the relativistic energy momentum relation to get a typical energy $e = c\hbar n^{1/3}$. Finally the energy density of this system of particles will be $U = ne = c\hbar n^{4/3}$. Recall from classical thermodynamics that the pressure is of the order of the energy density then the pressure of our quantistic and relativistic system is $P = c\hbar n^{4/3}$. And setting equal to m the typical mass of a particle $P = c\hbar(\rho/m)^{4/3}$, where we have introduced the mas density ρ.

We now turn to hydrostatic equilibrium. For a star the equation at any depth where the density is ρ will look like:

$$G\frac{M\rho}{r^2} = -\nabla P = -\frac{dP}{dr} \qquad \text{(I)}$$

given that stars are spherically symmetric. At this point we are going to simplify the treatments looking only at the scaling of the equation. This is done replacing some of the quantities with their simplest approximations: the local radius $r \to R$ the stellar radius; the density $\rho \to MR^{-3}$ where M is the mass of the star; the pressure derivative $dP/dr \to P/R$. Then the differential equation turns into an algebraic one that the students should be more familiar with.

$$G\frac{M^2R^{-3}}{R^2} = -\frac{P}{R} \tag{II}$$

$$G\frac{M^2}{R^5} = \frac{\hbar c}{R}\left(\frac{M}{mR^3}\right)^{4/3} \tag{III}$$

The student should see immediately that the radius simplifies out of the equation. This means that the equilibrium is independent of the radius. Or stated in other words that if the equilibrium (the above equation) is not satisfied, the star will start to expand (explode) or collapse (implode), and you cannot avoid this by adjusting the radius; it is a catastrophic process. The student should also see that the equilibrium equation gives the following solution of the mass:

$$M_{\text{eq}} = \frac{1}{m^2}\left(\frac{c\hbar}{G}\right)^{3/2} \tag{IV}$$

Now let us put the numbers: the speed of light $c = 3 \times 10^9$ m s^{-1}; m is the mass of a proton 1.6×10^{-27} kg; $G = 6.6 \times 10^{-11}$ m^3 s^{-2} kg^{-1}; $\hbar = 10^{-34}$ J s. Substituting these values one gets:

$$M_{\text{eq}} = 3.8 \times 10^{30}\text{kg} \tag{V}$$

This is about 2 times that mass of the Sun $M_{\text{Sun}} = 2 \times 10^{30}$ kg (doing the correct model the astro-physicists get 1.4 times the mass of the Sun - be happy with such simple equation you got only a 25% difference). This equilibrium mass is known as **Chandrasekhar mass**. If the mass is smaller, the star will expand until the physical processes of the gas change so much that somehow the star reaches a new equilibrium. If the stellar mass gets bigger than the gravity wins and the star will start to collapse, driving a catastrophic evolution, that can either end with a black hole or with a stellar detonation, known as Supernova.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants

exercise: Dark matter and the virial theorem

1. ***empirical approach to the virial theorem***
   Please complete this table and compute the specific kinetic energy T, the specific potential energy V and the ratio between the two. Does the virial law hold as well for specific kinetic and potential energies? You find the necessary data on all planets on Wikipedia, and please assume that the planets follow circular orbits.

planet	distance r	orbital period t	kinetic energy T	potential energy V	ratio T/V
Mercury					
Venus					
Earth					
Mars					
Jupiter					
Saturn					
Uranus					
Neptune					

2. ***Kepler orbits and the virial theorem***
   Why do the planets follow orbits with a fixed ratio between kinetic and potential energy?

   (a) Please start by deriving a relationship between orbital velocity υ and distance from a Newtonian calculation for a circular orbit.

   (b) For that orbit, predict the kinetic energy from the potential energy. Is there a fixed ratio between the two?

   (c) The virial theorem is only valid for time-averaged quantities: Why are the energies constant for a circular orbit, implying that you don't have to average?

3. ***relationship to flat rotation curves***
   An important model for the distribution of (dark) matter inside a halo is the density profile $\rho \propto r^{-2}$ (called isothermal sphere), which has a number of important consequences:

   (a) Please compute the potential Φ of a density profile $\rho \propto r^{-2}$. This type of density profile is typical for dark matter haloes at intermediate distances. The solution for Φ follows from the Poisson equation $\Delta\Phi = 4\pi G\rho$ assuming spherical symmetry,

$$\Delta\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho. \tag{I}$$

   (b) Now, derive a relationship between the velocity $\upsilon(r)$ of a star following a circular orbit at the distance r from the halo centre: Do you find that $\rho \propto r^{-2}$ enforces $\upsilon(r) = \text{const}$?

   (c) Next, solve the equation of motion $\ddot{r} = -\nabla\Phi$ for a star oscillating in that halo through the centre. Do you find a consequence of the specific profile $\rho \propto r^{-2}$?

(d) What's the escape velocity from a singular isothermal sphere?

4. ***virial theorem for the harmonic oscillator***
Please show for a harmonic pendulum $\ddot{x} = -g/l\, x$ (with the gravitational acceleration g and the pendulum length l) that the

(a) total energy is conserved at every instant t.

(b) average kinetic and potential energies are equal. Please use

$$\langle x^2 \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, x^2(t) \quad \text{and} \quad \langle \dot{x}^2 \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, \dot{x}^2(t) \tag{II}$$

as definitions of the average, with the oscillation period $\tau = 2\pi\sqrt{l/g}$.

5. ***mechanical similarity and the virial theorem***
Mechanical similarity implies the relationship $r^{2-n} \propto t^2$ between the length scale r and the time scale t in mechanical systems with a potential $\Phi \propto r^n$. Collecting results for the 4 most common potentials leads to:

system	potential	similarity	remark
Kepler-problem	$\Phi \propto r^{-1}$	$r^3 \propto t^2$	Kepler's law
flat potential	$\Phi = \text{const}$	$r \propto t$	inertial motion
inclined plane	$\Phi \propto r$	$r \propto t^2$	constant acceleration
pendulum	$\Phi \propto r^2$	$t = \text{const}$	isochrony

(a) Why can the virial theorem only be applied to the first and the last case?

(b) Can you guess with you knowledge of the Kepler law that kinetic and potential energy need to be proportional to each other?

(c) Boosting into another frame by doing a Galilei-transform changes the kinetic energy: Would this affect the virial theorem?

6. ***application to clusters***
The galaxies inside a cluster have kinetic energies that are a factor of ~ 100 too large, if only the visible matter gravitates: Could you reconcile this by changing the gravitational potential from $\Phi \propto 1/r$ to $\Phi \propto 1/r^n$? Can you predict a number for n from the virial theorem?

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants

exercise: Dark matter and the virial theorem
Solutions

1. ***empirical approach to the virial theorem***
   Please complete this table and compute the specific kinetic energy T, the specific potential energy V and the ratio between the two. Does the virial law hold as well for specific kinetic and potential energies? You find the necessary data on all planets on Wikipedia, and please assume that the planets follow circular orbits.

   The specific kinetic energy i. e. kinetic energy divided by mass can be obtained by

$$T = \frac{1}{2}\left(\frac{2\pi r}{t}\right)^2 \tag{I}$$

   and the specific potential energy i. e. potential energy divided by mass by

$$V = -\frac{GM}{r} \tag{II}$$

   where G is Newton's gravitational constant and M the mass of the sun.

planet	distance r 10^9 m	orbital period t days	kinetic energy T J/kg	potential energy V J/kg	ratio T/V
Mercury	58	88	$1,2 \cdot 10^9$	$-2,3 \cdot 10^9$	-0,50
Venus	108	225	$6,1 \cdot 10^8$	$-1,2 \cdot 10^9$	-0,49
Earth	150	365	$4,5 \cdot 10^8$	$-8,9 \cdot 10^8$	-0,50
Mars	228	687	$2,9 \cdot 10^8$	$-5,9 \cdot 10^8$	-0,50
Jupiter	778	4330	$8,5 \cdot 10^7$	$-1,7 \cdot 10^8$	-0,50
Saturn	1434	10585	$4,8 \cdot 10^7$	$-9,3 \cdot 10^7$	-0,52
Uranus	2872	30660	$2,3 \cdot 10^7$	$-4,6 \cdot 10^7$	-0,50
Neptune	4498	60225	$1,5 \cdot 10^7$	$-3,0 \cdot 10^7$	-0,50

2. ***Kepler orbits and the virial theorem***

   (a) The gravitational force F_G acts as centripetal force F_c:

$$F_c = F_G$$
$$\frac{mv^2}{r} = G\frac{mM}{r^2}$$

   with r as radius of the circle, v as velocity, m as mass of the planet and M as mass of the sun. It follows

$$v^2 = \frac{GM}{r}$$

(b) The kinetic energy is $T = \frac{1}{2}mv^2$. So if we use the formula of the potential energy

$$V = -G\frac{mM}{r}$$

and the result a), it follows

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{GM}{r} = -\frac{1}{2}V$$

(c) For a circular orbit, the radius r is constant, thus also the potential energy V.
The total energy $E = T + V$ is also a constant.
It follows that T is constant.

3. *relationship to flat rotation curves*

(a) The Poisson equation reads:

$$\nabla\Phi = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\Phi}{dr}) = 4\pi G\rho$$

For the density of a SIS profile, we have $\rho \propto r^{-2}$ and hence $\rho = \rho_0 \cdot \frac{r_0}{r}^2$.
This yields:

$$\begin{aligned}
\frac{d}{dr}(r^2\frac{d\Phi}{dr}) &= 4\pi G\rho_0 r_0^2 \\
r^2\frac{d\Phi}{dr} &= \int 4\pi G\rho_0 dr = 4\pi G\rho_0 r_0^2 r \\
\frac{d\Phi}{dr} &= 4\pi G\rho_0 r_0^2 r^{-1} \\
\Phi(r) &= \int 4\pi G\rho_0 r_0^2 r^{-1} dr
\end{aligned}$$

This leads to our final result:

$$\Phi(r) = 4\pi G\rho_0 r_0^2 \ln\frac{r}{r_0}$$

(b) The orbital velocity is obtained by equating gravitational and centripetal force:

$$\begin{aligned}
F_c &= -F_g \\
m\frac{v(r)^2}{r} &= m\nabla\Phi = m\frac{d}{dr}4\pi G\rho_0 r_0^2 \ln\frac{r}{r_0}
\end{aligned}$$

This yields:

$$\begin{aligned}
m\frac{v(r)^2}{r} &= m4\pi G\rho_0 r_0^2\frac{1}{r} \\
v(r) = \sqrt{4\pi G\rho_0 r_0^2}
\end{aligned}$$

This result is independent of r; hence, the orbital velocity for objects in a SIS halo is constant even at large radii.

(c) The equation of motion is given by:

$$\ddot{r} = -\nabla\Phi$$

Setting $k = 4\pi G\rho_0 r_0^2$, we obtain:

$$\begin{aligned}
\ddot{r} &= -\nabla\Phi \\
\ddot{r} &= -\frac{\partial}{\partial r} k \ln\frac{r}{r_0} \\
\ddot{r} &= -\frac{k}{r}
\end{aligned}$$

This differential equation is solved by the following expression, where $erf(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} e^{-t^2} dt$ denotes the error function:

$$r(t) = \exp\left[\frac{c_0 - 2\, erf^{-1}\left(-\sqrt{\frac{2}{\pi}}\sqrt{ke^{-\frac{c_0}{k}}(c_1+t)^2}\right)^2}{2k}\right]$$

(d) The escape velocity can be obtained by equating the kinetic energy of the escaping body with the work required to move the body from r_0 to infinity against the gravitational force:

$$E_{kin} = \frac{m}{2}v_{esc}^2 = W = -\int_{r_0}^{\infty} F_g\, dr = m\int_{r_0}^{\infty} \nabla\Phi r\, dr = m\int_{r_0}^{\infty} \nabla\Phi dr$$

However, this turns out to be infinite:

$$W = 4\pi G\rho_0 r_0^2 m \int_{r_0}^{\infty} \frac{1}{r} dr$$

Hence, the escape velocity from a SIS halo is infinite. This is due to the unphysical density singularity at the halo centre, which means that the total mass of the halo is also divergent.

4. *virial theorem for the harmonic oscillator*

The general solution for the harmonic oscillator

$$\ddot{x} = -\omega^2 x \tag{I}$$

with $\omega^2 = g/l$ and the initial conditions ($x(0) = x_0, \dot{x}(0) = v_0$) is:

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t \tag{II}$$

(a) energy conservation

The kinetic energy is:

$$\begin{aligned}
T &= \frac{1}{2}\dot{x}^2 && \text{(III)}\\
&= \frac{1}{2}\left(v_0\cos\omega t - x_0\omega\sin\omega t\right)^2 && \text{(IV)}\\
&= \frac{1}{2}\left(v_0^2\cos^2\omega t - 2x_0v_0\omega\sin\omega t\cos\omega t + x_0^2\omega^2\sin^2\omega t\right) && \text{(V)}\\
&= \frac{v_0^2 + x_0^2\omega^2}{4} + \frac{v_0^2 - x_0^2\omega^2}{4}\cos 2\omega t - \frac{x_0v_0\omega}{2}\sin 2\omega t && \text{(VI)}
\end{aligned}$$

Whereas the potential energy satisfies:

$$\begin{aligned}
V &= \frac{1}{2}\omega^2x^2 && \text{(VII)}\\
&= \frac{1}{2}\omega^2\left(x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t\right)^2 && \text{(VIII)}\\
&= \frac{1}{2}\left(\omega^2x_0^2\cos^2\omega t + 2x_0v_0\omega\sin\omega t\cos\omega t + v_0^2\sin^2\omega t\right) && \text{(IX)}\\
&= \frac{x_0^2\omega^2 + v_0^2}{4} + \frac{x_0^2\omega^2 - v_0^2}{4}\cos 2\omega t + \frac{x_0v_0\omega}{2}\sin 2\omega t && \text{(X)}
\end{aligned}$$

from which follows, that:

$$\begin{aligned}
E &= T + V && \text{(XI)}\\
&= \frac{v_0^2 + x_0^2\omega^2}{2} && \text{(XII)}
\end{aligned}$$

which is constant for all times t because it is time independent.

(b) energy equalities

To calculate the average, one may use the identities obtained above using $\tau = 2\pi/\omega$

$$\begin{aligned}
\left\langle x^2\right\rangle &= \frac{1}{\tau}\int_0^\tau \mathrm{d}t\; x^2(t) && \text{(XIII)}\\
&= \frac{1}{\tau}\int_0^\tau \mathrm{d}t\left(\frac{x_0^2 + v_0^2/\omega^2}{2} + \frac{x_0^2 - v_0^2/\omega^2}{2}\cos 2\omega t + \frac{x_0v_0}{\omega}\sin 2\omega t\right) && \text{(XIV)}\\
&= \frac{\omega}{2\pi}\left[\frac{x_0^2 + v_0^2/\omega^2}{2}t + \frac{x_0^2 - v_0^2/\omega^2}{4\omega}\sin 2\omega t - \frac{x_0v_0}{2\omega^2}\cos 2\omega t\right]_0^{2\pi/\omega} && \text{(XV)}\\
&= \frac{\omega}{2\pi}\left[\frac{x_0^2 + v_0^2/\omega^2}{2}\frac{2\pi}{\omega}\right] && \text{(XVI)}\\
&= \frac{x_0^2 + v_0^2/\omega^2}{2} && \text{(XVII)}
\end{aligned}$$

And for the kinetic terms:

$$\left\langle \dot{x}^2 \right\rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\ \dot{x}^2(t) \tag{XVIII}$$

$$= \frac{1}{\tau}\int_0^\tau \mathrm{d}t \left(\frac{v_0^2 + x_0^2\omega^2}{2} + \frac{v_0^2 - x_0^2\omega^2}{2}\cos 2\omega t - x_0 v_0 \omega \sin 2\omega t\right) \tag{XIX}$$

$$= \frac{\omega}{2\pi}\left[\frac{v_0^2 + x_0^2\omega^2}{2}t + \frac{v_0^2 - x_0^2\omega^2}{4\omega}\sin 2\omega t - \frac{x_0 v_0}{2}\cos 2\omega t\right]_0^{2\pi/\omega} \tag{XX}$$

$$= \frac{\omega}{2\pi}\left[\frac{v_0^2 + x_0^2\omega^2}{2}\frac{2\pi}{\omega}\right] \tag{XXI}$$

$$= \frac{v_0^2 + x_0^2\omega^2}{2} \tag{XXII}$$

So one may see that with:

$$\langle T \rangle = \frac{1}{2}\left\langle \dot{x}^2 \right\rangle \tag{XXIII}$$

$$= \frac{v_0^2 + x_0^2\omega^2}{4} \tag{XXIV}$$

$$\langle V \rangle = \frac{1}{2}\omega^2 \left\langle x^2 \right\rangle \tag{XXV}$$

$$= \frac{x_0^2\omega^2 + v_0^2}{4} \tag{XXVI}$$

The virial theorem $\langle T \rangle = \langle V \rangle$ holds stand.

5. *mechanical similarity and the virial theorem*

(a) Why can the virial theorem only be applied to the first and last case?

For the virial theorem to be applied the motion must be constraint in space and momentum, because of the averaging. (see below for more information)

Kepler	ellipse $\Rightarrow \exists\ r_{max}, p_{max} < \infty$
flat potential	$r \xrightarrow{t\to\infty} \infty$, if $p \neq 0$
inclined plane	$r, p \xrightarrow{t\to\infty} \infty$
pendulum	$\exists\ \theta_{max}, p_{max} < \infty$

Derivation of the virial theorem:

$$2T = m\dot{x}^2 = p\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}(px) - \dot{p}x = \frac{\mathrm{d}}{\mathrm{d}t}(px) + x\frac{\partial V}{\partial x} \tag{I}$$

$$\Rightarrow 2T - x\frac{\partial V}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}(px) \tag{II}$$

$$\Rightarrow \langle 2T - x\frac{\partial V}{\partial x}\rangle = \langle \frac{\mathrm{d}}{\mathrm{d}t}(px)\rangle = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t' \frac{\mathrm{d}}{\mathrm{d}t'}(px) \tag{III}$$

$$= \lim_{t\to\infty}\frac{1}{t}\left[p(t)x(t) - p(0)x(0)\right] = 0 \text{ (If p and x are constrained)} \tag{IV}$$

$$\Rightarrow 2\langle T\rangle = \langle x\frac{\partial V}{\partial x}\rangle \tag{V}$$

$$\Rightarrow 2\langle T\rangle = k\langle V\rangle \text{ (If V is homogeneous of grade k)} \tag{VI}$$

(b) Can you guess with you knowledge of the Kepler law that kinetic and potential energy need to be proportional to each other?
From mechanical similarity follows $r^3 \propto t^2 \Rightarrow (t/r)^2 \propto r^{-1}$

(c) Boosting into another frame by doing a Galilei-transform changes the kinetic energy: Would this affect the virial theorem?
Yes. For an observer moving relative to the system, the motion is in general not constraint.

$$T' = T + T_{boost} \Rightarrow \langle T'\rangle = \langle T\rangle + T_{boost}$$

6. ***application to clusters***
For $\Phi \propto r^{-1}$ the virial theorem is

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle \tag{I}$$

$$\propto -\frac{1}{2}r^{-1} \tag{II}$$

$$\tag{III}$$

If we measure kinetic energies too large by a foctor ≈ 100 we get $\langle T'\rangle = 100\langle T\rangle$ and can explain this by changing the potential to $\Phi \propto r^{-n}$:

$$\langle T'\rangle = -\frac{n}{2}\langle V'\rangle \tag{IV}$$

$$100\langle T\rangle \propto -\frac{n}{2}r^{-n} \tag{V}$$

Because this must hold for all r we predict $n = 100$ to explain the measured kinetic energies.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants and BMS

play with data: virial theorem and periodic motion

In these exercises we will explore the virial theorem by solving equations of motions numerically and measuring averaged energies from the solutions.

1. ***virial theorem and the harmonic oscillator***
The script `harmonic.py` generates a solution to the harmonic oscillator equation $\ddot{x} = -x$ by transforming it with the definition $y = \dot{x}$ into a coupled first order equation,
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{I}$$
where the angular frequency is set to $\omega = 1$ for the numerics.
   (a) Is the total energy conserved?
   (b) Please run the script and measure the average kinetic and potential energies: Do you find $\langle T\rangle = \langle V\rangle$ for the harmonic oscillator?
   (c) Is $\langle T\rangle = \langle V\rangle$ true for any initial condition?

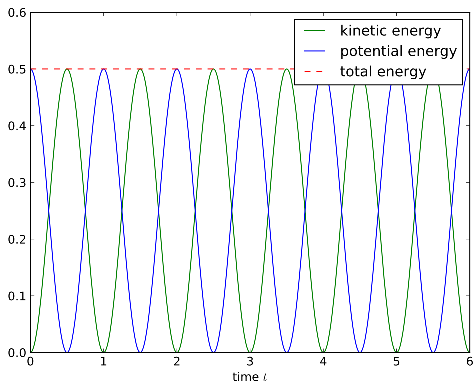

Figure 1: numerical solutions to the harmonic oscillator

2. ***virial relationship in the anharmonic oscillator***
The virial theorem makes a prediction for the average kinetic and potential energies in any system with a scale-free potential, for instance the anharmonic oscillator with the potential $\Phi \propto x^{2n}$. The script `anharmonic.py` solves the equation of motion. The constant of proportionality is set to 1.
   (a) What's the equation of motion for the potential $\Phi = x^{2n}/(2n)$, and what's the corresponding coupled first order system?
   (b) Why are there with increasing n phases of constant velocity and a sawtooth pattern in position?
   (c) Please measure the average kinetic and potential energies over many oscillations: What's their ratio r as a function of n? Why does $r = \langle T\rangle/\langle V\rangle$ increase with n?

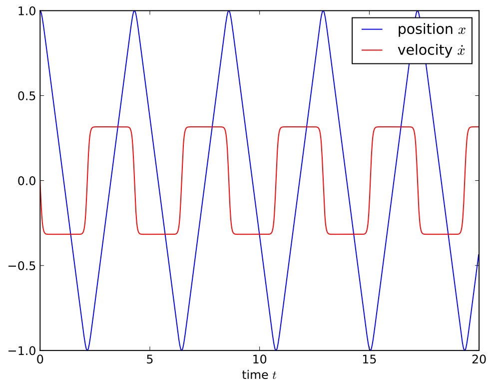

Figure 2: numerical solutions to the anharmonic oscillator for $n = 10$

3. virial theorem in the (generalised) Kepler-problem

Please simulate Kepler orbits with the script `kepler.py`: The equation of motion of a particle in the potential $\Phi \propto 1/r^\alpha$ can be derived from the Lagrange density

$$\mathcal{L} = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GM}{r^\alpha} \tag{II}$$

Please derive the equation of motion with the Euler-Lagrange-equations and convert the resulting second-order equations in a set of first coupled order equations.

(a) Change the total energy of the planet by setting δ to a value unequal to 0 and observe the change in the orbit. The product constants GM is set to $GM = 1$ for the numerics.

(b) Change the value of α to a number different from 1: Are the orbits still closed? NB: The problem becomes unstable if α is too large, try to experiment in the range $\alpha = 0.8 \ldots 1.2$.

(c) Can you generate precession motion and orbits lagging behind by choosing a suitable α?

(d) Is it possible to have bound systems for $\alpha \geq 2$? What's the physical reason?

(e) What's the virial relationship and how does it depend on α?

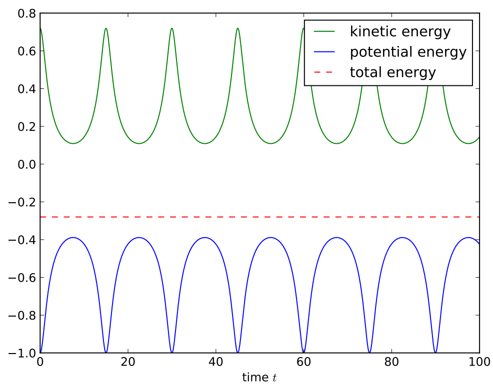

Figure 3: kinetic and potential energy in Kepler-orbits

Astronomy from 4 perspectives: the Dark Universe

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prepared by: Heidelberg participants

questions: Dark matter and the virial theorem

1. ***theory behind the virial theorem***
   (a) What's the idea of the virial theorem?
   (b) What's the difference to energy conservation?
   (c) For what kind of system can you use the virial theorem?
2. ***mechanical similarity***
   (a) Why is mechanical similarity restricted to potentials of the shape $\Phi \propto r^\alpha$?
   (b) Does α have to be integer?
   (c) What's the generalisation of the Kepler-law for a potential of the form $\Phi \propto r^\alpha$?
3. ***harmonic oscillator***
   (a) What can you say about different energy types in the harmonic oscillator on average?
4. ***Kepler-problem and planetary motion***
   (a) Can you show that Kepler's third law implies that potential and kinetic energy are proportional to each other?
   (b) Is this true in general?
5. ***virial theorem in galaxies***
   (a) Could one explain the rotation curves by assuming a different gravitational law?

Astronomy from 4 perspectives: the Dark Universe

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prepared by: Jena participants

exercise: Dark matter and galaxy rotation curves

1. ***harmonic oscillator and energy types***

   The harmonic oscillator is described the the differential equation $\ddot{x} = -g/l\, x$, and performs harmonic oscillations $x(t) \propto \exp(\pm i\omega t)$ with $\omega^2 = g/l$.

   (a) Please show that $\langle T \rangle = \langle V \rangle$ with the kinetic energy T and the potential energy V. The brackets $\langle \ldots \rangle$ are time averages over one oscillation period τ,

   $$\langle T \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, T(t) \quad \text{and} \quad \langle V \rangle = \frac{1}{\tau}\int_0^\tau \mathrm{d}t\, V(t) \tag{I}$$

   which is defined as $\tau = 2\pi/\omega$, and the specific energies $T(t) = \dot{x}^2/2$ and $V(t) = gx^2/(2l)$.

   (b) Could you predict the proportionality between $\langle T \rangle$ and $\langle V \rangle$ from the isochrony of the harmonic oscillator?

   The probability of finding the oscillator at a certain amplitude x is inversely proportional to the velocity: $\mathrm{d}x/\mathrm{d}t = \upsilon$, such that $\Delta t = \Delta x/\upsilon$. If the range of motion is divided into equidistant intervals Δx, the probability p of seeing the oscillator in one of those is proportional to the time it spends there, i.e. proportional to $1/|\upsilon|$.

   (c) Please normalise p and draw the function $p(\upsilon)$: If you look randomly at a harmonic oscillator, at what stage in its oscillation are you most likely to see it?

   (d) Please define averages

   $$\langle T \rangle = \int \mathrm{d}\upsilon\, p(\upsilon) T(\upsilon) \quad \text{and} \quad \langle V \rangle = \int \mathrm{d}\upsilon\, p(\upsilon) V(\upsilon) \tag{II}$$

   and compute both integrals. You can use energy conservation for the second integral to express V in terms of the velocity υ. Are the results identical to the previous computation? Be careful to take the positive sign of p into account, by using the symmetry of the integrand.

   (e) Why is there no issue with convergence when the probability density $p \to \infty$ at $\upsilon \to 0$?

   (f) Is the virial relation $\langle T \rangle = \langle V \rangle$ as well valid for a circular orbit in a spherically symmetric harmonic potential?

   (g) Is it valid as well for any other Lissajous-figure?

2. ***flat rotation curves***

   Let's consider the motion of stars inside a galaxy with the density profile of a *singular isothermal sphere*, which is $\rho \propto r^{-2}$. The singular isothermal sphere describes the density of dark matter well on scales of the galactic disc.

   (a) Please show by solving the Poisson equation $\Delta\Phi = 4\pi G\rho$,

   $$\Delta\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho, \tag{III}$$

   for a spherically symmetric density profile $\rho \propto r^{-2}$ that rotation curves are flat.

(b) Please compute the mean kinetic $\langle T \rangle$ and mean potential energy $\langle V \rangle$ for the circular motion in an isothermal sphere as a function of r.

(c) Is it possible in this case to decompose the circular orbiting motion into two uncoupled orthogonal harmonic oscillations?

(d) What would the density profile need to be such that stars would perform harmonic oscillations through the centre of the galaxy, i.e. for the potential to be quadratic, $\Phi \propto r^2$?

3. ***MoND, the Solar system and the Milky Way***

Let's assume that we can change the acceleration due to gradients in the gravitational potential $\nabla\Phi$ in an empirical way,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} \to \frac{\mathrm{d}\Phi}{\mathrm{d}r} + a_0, \tag{IV}$$

as it would be relevant for a circular motion around the Milky Way centre in a spherically symmetric potential.

(a) What would be the effect on a rotation curve from the density profile $\rho \propto r^{-\alpha}$?

(b) The parameter a_0 would need to be chosen small: Please estimate an upper bound on the value of a_0 from the orbital acceleration of the Solar system on its passage around the Milky Way center. You can find all necessary data on Wikipedia.

(c) Please think of a way to visualise the numerical value of a_0.

(d) At what distance from the Earth's surface would the gravitational acceleration be a_0?

Astronomy from 4 perspectives: the Dark Universe

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prepared by: Jena participants and BMS

play with data: rotation curves of galaxies

Observations of the rotation of disc galaxies is nowadays done in the Hα-line of hydrogen, because it reaches to much larger distances from the galaxy centre compare to the stellar light. In this exercise we have a look at a data set on low surface-brightness galaxies by W. de Blok, S. McGaugh and V. Rubin, Astronomical Journal 122, 2381 (2001).

1. flat rotation curves

Let's start by exploring Hα-data for low surface-brightness galaxies.

(a) Why is it a clever idea to focus on galaxies with a low (optical) surface brightness?

(b) What is the general relationship between rotation curve $\upsilon(r)$ and mass profile $\rho(r)$?

(c) A isothermal sphere with a core has the density profile $\rho(r)$,

$$\rho(r) = \rho_0\left(1 + \left(\frac{r}{r_c}\right)^2\right)^{-1}, \tag{I}$$

with the central density ρ and the core radius r_c. The corresponding velocity profile $\upsilon(r)$,

$$\upsilon(r)^2 = 4\pi G\rho_0 r_c^2\left(1 - \frac{r_c}{r}\arctan\left(\frac{r}{r_c}\right)\right), \tag{II}$$

with the gravitational constant $G \simeq 10^{-11}$ m^3/kg/s^2.

(d) Please show that the asymptotic value for υ for $r \to \infty$ is $\upsilon_\infty = \sqrt{4\pi G\rho_0 r_c^2}$. Please check the units of the relation between υ_∞ and r_c and ρ_0.

(e) Please use the script `rotplot.py` and plot a couple of rotation curves: Do they show the expected behaviour?

2. luminous and dark matter

With the script `rotfit.py` you can fit a model rotation curve to data. Take care to read off the distance d to the galaxy from the table, in order to convert r from arcseconds to kpc.

(a) Are the curves from the isothermal-sphere model providing a good fit to data?

(b) What are typical velocities υ_∞, central densities ρ_0 and core radii r_c?

(c) What is the role of ϵ in the script? Why are the results not affected if ϵ is small enough?

Please continue by completing the table.

(d) Please try to find out if the mass to light-ratio M/L is large: For that purpose, estimate the total mass M in units of the solar mass $M_\odot = 10^{30}$ kg,

$$M = 4\pi\int_0^\infty r^2 \mathrm{d}r\,\rho(r), \tag{III}$$

and compare it to the total luminosity. For the integral, you can use the result

$$\int \mathrm{d}x\,\frac{x^2}{1+x^2} = \arctan(x) + \text{const.} \tag{IV}$$

Please truncate the integration at the tidal radius $10r_c$. With the expression for the mass, please verify the relationship between orbital velocity υ and distance r.

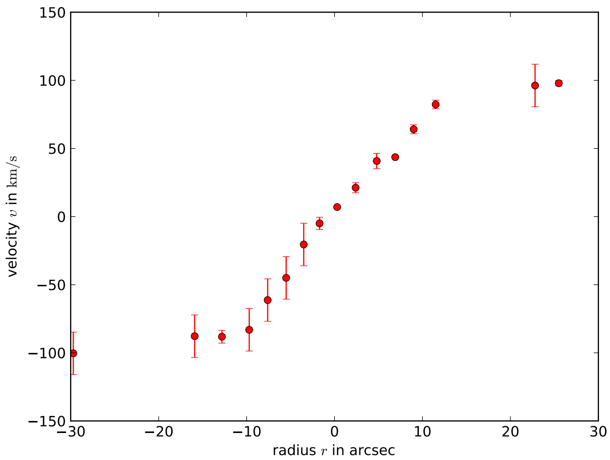


Figure 1: rotation curve $\upsilon(r)$ of the galaxy F568

(e) Then, please express the mass to light-ratio M/L in units of solar masses per solar luminosities $M_\odot/L_\odot$: The luminosity L in units of the solar luminosity $L_\odot$ follows from the difference of the absolute magnitudes,

$$\frac{L}{L_\odot} = 10^{0.4(\mathrm{Mag}_\odot - \mathrm{Mag})}, \tag{V}$$

you can find the values for Mag of the galaxies in the table, and use the literature value for $\mathrm{Mag}_\odot = 5.45$ in the same band (R-band) from the literature.

(f) Is there evidence for dark matter?

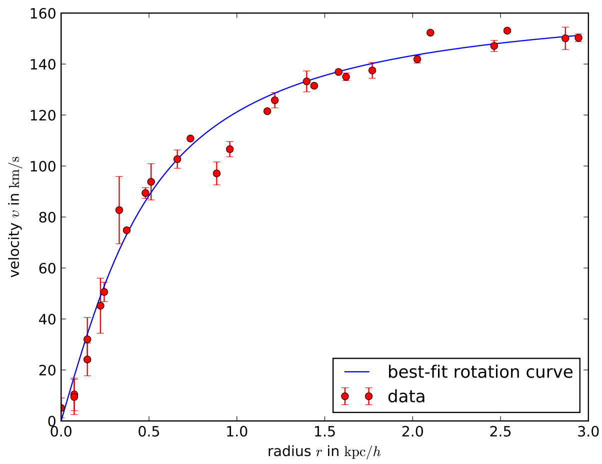


Figure 2: fit of a isothermal sphere rotation curve $\upsilon(r)$ to the galaxy U11557 at 22 Mpc distance, with the values $r_c = 0.412$ kpc and $\upsilon_\infty = 169$ km/s.

galaxy	d in Mpc	v_∞ in km/s	r_c in kpc	M in $M_\odot$	Mag	M/L in $M_\odot/L_\odot$
E0140040	212				-21.6	
E0840411	80				-18.1	
E1200211	15				-15.6	
E1870510	18				-16.5	
E2060140	60				-19.2	
E3020120	69				-19.1	
E3050090	11				-17.3	
E4250180	86				-20.5	
E4880049	22				-16.8	
F563-1	45				-17.3	
F568-3	77				-18.3	
F571-8	48				-17.6	
F579-V1	85				-18.8	
F583-1	32				-16.5	
F583-4	49				-16.9	
U4115	3.2				-12.4	
U5750	56				-18.7	
U6614	85				-20.3	
U11454	91				-18.6	
U11557	22	169	0.412		-20.0	
U11583	5				-14.0	
U11616	73				-20.3	
U11648	48				-21.0	
U11748	73				-22.9	
U11819	60				-20.3	

Task sheet to rotation curves

Friedrich Schiller University Jena participants

09/01/2017

Task 1

Think about how the galaxy should be orientated to be observed?
Here are some pictures as example:

Inclination angle $i = 0°$
Galaxy: NGC 6814
Credit: Abb. 1

Inclination angle $i \approx 60°$
Galaxy: NGC 7606
Credit: Abb. 2

Inclination angle $i = 90°$
NGC 4762
Credit: Abb. 3

Task 2

Calculate the radial velocities from the measured wavelengths and plot them over the distance from the galaxy center! Use $\lambda_0 = 21.106$Å and 1pc$= 3.1 \cdot 10^{16}$m!

λ in Å	Radius R in Mpc	$v_{rotation}$ in $\frac{\text{km}}{\text{s}}$
21.1195	1	
21.1130	2	
21.1173	5	
21.1194	7	
21.1201	10	
21.1208	15	
21.1211	20	
21.1215	22	
21.1213	25	

Task 3

Derive for circular orbits the formula for the velocity v in dependence of the distance r. Assume a radially symmetric mass distribution.

Task 4

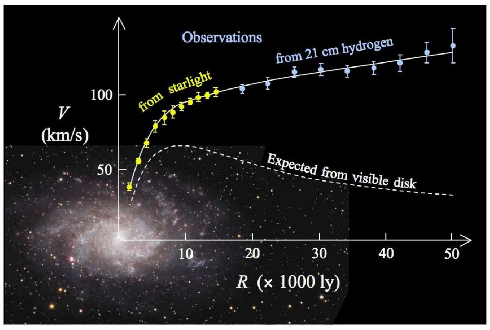
Assuming circular orbits, compute the velocities of the planets in our solar system. Plot the resulting rotation curve v over r.

Task 5

Formulate an expectation for the rotation curve of the Milky Way, assuming that the mass in the bulge to be $1.6 \cdot 10^{10} M_\odot$ and in the disk to be $4 \cdot 10^{10} M_\odot$

Task 6

The observed rotation curves of spiral galaxies are of the following form:



This cannot be explained by visible mass alone. Assuming that dark matter is the source of the difference between the observed and the predicted rotation curves, please calculate the mass of the dark matter depending on the velocities $v(r)$ and $v_{axis}(r)$

Task 7

To find out how dark matter is distributed through out a spiral galaxy, please consider a simple rotation curve consisting of a linear and a constant branch. Assume a spherically symmetric mass distribution of the form

$$\mu(r) \sim r^k$$

For the mass use the formula

$$M(r) = 4\pi \int_{r_0}^{r} \mu(\rho)\rho^2 \mathrm{d}\rho$$

a) Please calculate the mass of the bulge in dependence of k.

b) Using the formula for v from Task 3 and the result from a please determine the exponent k for the bulge $(v\ r)$. Calculate the mass of the bulge in dependence of r and the complete mass M_B of the bulge.

c) To determine k for the halo ($v = \text{const}$), consider the total mass to be composed of the mass of the bulge M_B and the mass of the halo M_H.

$$M(r) = M_B + M_H(r)$$

Calculate the mass outside of the bulge with the integral for the mass. Determine the exponent k using the results of b) and Task 3. Find a formula for the mass of the halo in dependence of r.

d) Compare the rotation curve of the bulge to the rotation curve of a rigid body.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Padova participants and BMS

exercise: Planck-spectrum and the CMB

1. ***properties of the Planck-spectrum***
   Let's derive the fundamental properties of the Planck-spectrum,

   $$S(\nu) = S_0 \frac{\nu^3}{\exp(h\nu/(k_B T)) - 1} \quad \rightarrow \quad S(\nu) = S_0 \nu^3 \exp(-h\nu/(k_B T)), \tag{I}$$

   by using Wien's approximation (the second expression), which makes the integrals easier. The constant S_0 depends only on numbers, natural and mathematical constants.

   (a) Please compute the total intensity $\int_0^\infty \mathrm{d}\nu\, S(\nu)$ and show that it is $\propto T^4$.

   (b) Show that the position ν_m of the maximum scales $\nu_m \propto T$.

   (c) Please derive the scaling of the mean

   $$\langle \nu \rangle = \frac{\int_0^\infty \mathrm{d}\nu\, \nu S(\nu)}{\int_0^\infty \mathrm{d}\nu\, S(\nu)} \tag{II}$$

   and show that it is proportional to T.

   (d) Is there a fixed ratio between $\langle \nu \rangle$ and ν_m?

   (e) In which limit is Wien's approximation applicable?

   (f) Do the scaling behaviours with T derived above depend on the details of the distribution?

2. ***Wien's distribution function***
   Let's stick for a second with Wien's distribution function in n dimensions,

   $$S(\nu) = S_0 \nu^n \exp(-h\nu/(k_B T)), \tag{III}$$

   and derive a few general properties, which will hold for the Planck-distribution as well (although the computations are more complicated).

   (a) Please begin by showing that

   $$\int_0^\infty \mathrm{d}x\, x^n \exp(-x) = n! \tag{IV}$$

   using n-fold integration by parts.

   (b) Alternatively, please show the recursion relation of the Γ-function,

   $$\Gamma(n) = (n-1)\Gamma(n-1), \quad \text{together with} \quad \Gamma(0) = 1. \tag{V}$$

   The Γ-function is defined by

   $$\Gamma(n) = \int_0^\infty \mathrm{d}x\, x^{n-1} \exp(-x), \tag{VI}$$

   and is related to the factorial by $\Gamma(n) = (n-1)!$

(c) What scaling of the moments

$$\langle \nu^m \rangle = \frac{\int_0^\infty \mathrm{d}\nu\, \nu^m S(\nu)}{\int_0^\infty \mathrm{d}\nu\, S(\nu)} \tag{VII}$$

with temperature T do you expect?

(d) Please show that the skewness parameter $s = \langle \nu^3 \rangle / \langle \nu^2 \rangle^{3/2}$ and the kurtosis parameter $k = \langle \nu^4 \rangle / \langle \nu^2 \rangle^2$ are independent from the temperature T. What would be the physical interpretation of s and k?

(e) Would an equivalent result be true for the parameter $\langle \nu^{2n} \rangle / \langle \nu^2 \rangle^n$?

3. *Planck-spectra at cosmological distances*

Imagine you observe an object emitting a Planck-spectrum at a cosmological distance, such that all photons arrive with a redshifted frequency $\nu \to a\nu = \nu/(1+z)$ with scale factor a (remember $a < 1$) and redshift z. A couple of students discusses the fact that the temperature scales with $T \propto 1/a$ and that the photons are redshifted: What's your opinion on the different arguments?

(a) Johannes from Heidelberg says: The temperature T of a photon gas is linked to the thermal energy E by $E = k_B T$. Then, the relativistic dispersion relation of the photons assumes $E = cp$ with the momentum p. The momentum p is given by the de Broglie-relationship as $p = h/\lambda$. If now the photon wavelength is changed $\lambda \to a\lambda$, the temperature needs to scale $T \propto 1/a$.

(b) Antonia from Padova says: What about a purely thermodynamical argument? A gas of photons has an adiabatic index of $\kappa = 4/3$, and the Hubble expansion is an adiabatic change of state, because there is no thermal energy created or dissipated. Then, the adiabatic invariant says that $TV^{\kappa-1}$ is conserved, which gives me $T \propto 1/a$ with $V \propto a^3$. And I understand why entropy is conserved but not energy.

(c) Marlene from Jena says: Due to the Hubble-expansion, every point is in recession motion with respect to every other point. If a photon gets scattered into your direction, the scattering particle will necessarily move away from you, leading to a lower perceived energy and a larger wavelength. It's important to view it like that because a photon gas can not change its state without interaction due to the linearity of electrodynamics, and this argument shows that it's a kinematical effect: It's a similarity transform of the Planck-spectrum.

(d) Lorenzo from Florence says: It's important for the Planck-spectrum that the mean particle separation and the thermal wavelength are identical. You can only arrange for that if $T \propto 1/a$ if the particle separation increases $\propto a$. I'm only assuming that the number of photons is conserved, but not their energy, and that everything stays in equilibrium.

4. *CMB as a source of energy*

A Carnot-engine converts thermal energy taken from two reservoirs at different temperatures into mechanical energy at the efficiency $\eta = 1 - T_2/T_1$.

(a) Estimate if one can use the temperature anisotropies in the CMB of $\Delta T/T \simeq 10^{-5}$ to generate mechanical energy. How much power could you realistically generate? Construct a machine that converts radiation power into mechanical energy.

(b) Could you use the time evolution of the CMB-temperature for this purpose? You know already that $T \propto 1/a$, so please construct a machine that produces energy from the CMB.

(c) Why does a solar cell transform the radiation from the Sun into electrical energy? One might argue that the Planck-spectrum is that of thermal equilibrium in which case the mechanical work is zero: Due to the first law, mechanical work can not be performed in thermal equilibrium. (please be careful: trick question)

Astronomy from 4 perspectives: the Dark Universe

prepared by: Padua participants

exercises: Planck - spectrum and CMB

Solutions

1. Properties of the Planck-spectrum

$$S(\nu) = S_0 \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} \Rightarrow S(\nu) = S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{I}$$

(a)

$$\int_0^\infty S(\nu) = \int_0^\infty S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{II}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{III}$$

$$\int_0^\infty S_0 (\frac{kT}{h})^3 e^{-\frac{h\nu}{kT}} (\frac{kT}{h}) \propto T^4 \tag{IV}$$

(b)

$$\frac{dS}{d\nu} = 0 \Rightarrow \frac{dS}{d\nu} = 3 S_0 \nu^2 e^{-h\nu kT} + S_0 \nu^3 (-\frac{h}{kT}) e^{-h\nu kT} = \tag{V}$$

$$= S_0 \nu^3 e^{-\frac{h\nu}{kT}} (3 - \frac{h\nu}{kT}) \tag{VI}$$

$$3 - \frac{h\nu}{kT} = 0 \Rightarrow \nu = \frac{3kT}{h} \propto T \tag{VII}$$

(c)

$$< \nu >= \frac{\int_0^\infty \nu S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} = \frac{\int_0^\infty \nu^3 S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{VIII}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{IX}$$

$$\frac{\int_0^\infty S_0 \frac{KT}{h}^4 e^{-x} (\frac{kT}{h}) dx}{\int_0^\infty S_0 \frac{KT}{h}^3 e^{-x} \frac{kT}{h} dx} \tag{X}$$

$$S \propto \frac{T^5}{T^4} \propto T \tag{XI}$$

(d) Yes, as both of them depend on T

(e) Only at high energies

(f) No, as if we consider any power of T in the exponential, any substitution $x = \frac{h\nu}{KT}$ gives a new differential $d\nu\frac{kT^4}{h}$ both in numerator and denominator

(a) ***Wien's distribution function***

$$S(\nu) = S_0\nu^n e^{-\frac{h\nu}{kT}} \quad ; \quad \Gamma(n) = \int_0^\infty dx x^{n-1}e^{-x} \tag{XII}$$

a)

$$\int_0^\infty dx x^n e^{-x} = \cancel{-[e^{-x}x^n]_0^\infty} - n\int_0^\infty dx x^{n-1}e^{-x} = \int_0^\infty dx n x^{n-1}e^{-x} \tag{XIII}$$

$$= \cancel{x[e^{-x}nx^{n-1}]_0^\infty} - \int_0^\infty dx - n(n-1)x^{n-2} = \int_0^\infty dx - n(n-1)x^{n-2}e^{-x} \tag{XIV}$$

$$= [...] = \tag{XV}$$

$$= n!\int_0^\infty dx e^{-x} = n! \tag{XVI}$$

Let's show that $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} \tag{XVII}$$

To demonstrate our relation recursively let's start by showing it works for $n = 2$

$$\Gamma(2) = \int_0^\infty x e^{-x}dx = [-e^{-x}x]_0^\infty - \int_0^\infty -e^{-x} = 1 \tag{XVIII}$$

$$= (2-1)\Gamma(1) = 1 \tag{XIX}$$

We can now show it for $n + 1$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)\int_0^\infty x^{n-2}e^{-x}dx \tag{XX}$$

Let's introduce $m = n - 2 \Rightarrow n = m + 2$

$$\Gamma(n+1) = (m+2)(m+1)\int_0^\infty x^m e^{-x}dx = (m+2)(m+1)m! = (m+2)! = n! \tag{XXI}$$

But

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = n! \tag{XXII}$$

(b)

$$< \nu^m >= \frac{\int_0^\infty \nu^m S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{XXIII}$$

$$= \frac{\int_0^\infty \nu^m S_0 \nu^n e^{-\frac{h\nu}{kT}} d\nu}{\int_0^\infty S_0 \nu^n e^{-\frac{h\nu}{kT}}} d\nu \tag{XXIV}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{XXV}$$

$$< \nu^m >= \frac{\int_0^\infty \frac{kT}{h}^m S(\nu) \frac{kT}{h}^n e^{-x} \frac{kT}{h} d\nu}{\int_0^\infty S_0 \frac{kT}{h}^n e^{-x} \frac{kT}{h} d\nu} \propto \tag{XXVI}$$

$$\propto \frac{T^{m+n+1}}{T^{n+1}} \propto T^m \tag{XXVII}$$

The momenta scale like the power chosen.

(c) From the result on point c, we find:

$$S = \frac{< \nu^3 >}{(< \nu >^2)^{3/2}} \propto \frac{T^3}{T^{2^{3/2}}} \tag{XXVIII}$$

$$k = \frac{< \nu^4 >}{< \nu^2 >^2}^2 \propto \frac{T^4}{(T^2)^2} \tag{XXIX}$$

Neither of which depends on T

(d) Yes,again from the result of point c):

$$\frac{< \nu^{2n} >}{< \nu^2 >^n} \propto \frac{T^{2n}}{(T^2)^n} \tag{XXX}$$

3. ***Planck-spectra at cosmological distances***
   Discussed in the classroom
4. ***CMB as a source of energy***

(a) The answer a) with the Stephan Boltzmann law, the power per m^2 can be produced if:

$$w = 10^{-5} \sigma T^4 = 3 \cdot 10^{-11} W m^{-2} \tag{XXXI}$$

Astronomy from 4 perspectives: the Dark Universe

prepared by: Padova participants and BMS

play with data: Planck-spectrum and the CMB

The satellite COBE observed the cosmic microwave background from 1989 to 1993. One experiment, FIRAS (Far Infrared Absolute Spectrophotometer), took a very precise measurement of the Planck-shape of the CMB, see D.J. Fixsen et al., Astrophysical Journal 420, 445 (1994).

1. ***CMB-temperature***
   In this exercise you can play with COBE-data and explore the properties of the Planck-spectrum. Please have a look at the python-script `planck_plot.py`, which reads the data file from COBE and plots flux $S(\nu)$ as a function of frequency ν. In addition, it plots Planck-spectra $S(\nu, T)$ for a given temperature T.

   (a) What's your measurement for the CMB-temperature T_{CMB}?

   (b) In what range can you vary T such that the data is well described by the Planck-curve?

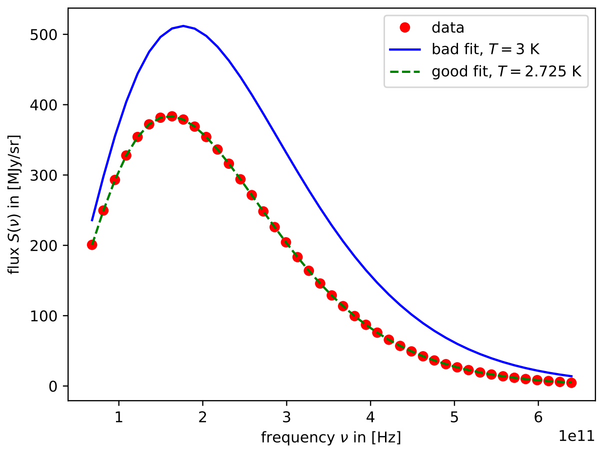


Figure 1: Planck-spectra for different temperatures T superimposed on the COBE-data

2. ***different radiation laws***
   The script `planck_fit.py` does a proper regression of a model $S(\nu, T)$ to the data, by minimising the squared difference between data and model, in units of the noise. There are two models for $S(\nu, T)$, the Planck-spectrum and the simplified Wien-spectrum.

   (a) Carry out a fit to the data with both models: What are the temperatures T?

   (b) Which model is better at explaining the data?

3. ***precision of the measurement***
   In running the script `planck_likelihood.py` you can estimate which range of values for T would be a good fit. It plots the likelihood $\mathcal{L}(T) \propto \exp(-\chi^2(T)/2)$, with

$$\chi^2(T) = \sum_{i=1}^{n_{\mathrm{data}}} \left(\frac{S_i - S(\nu_i, T)}{\sigma_i} \right)^2 \tag{I}$$

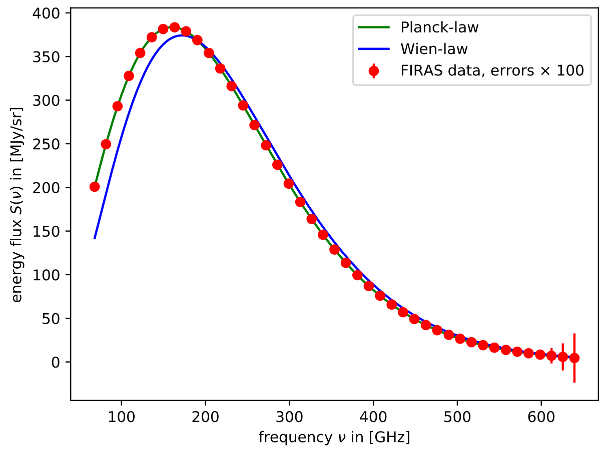


Figure 2: fits of the Planck- and Wien-radiation laws $S(\nu, T)$ to COBE-data

for the n_{data} data points S_i at the frequencies ν_i. The statistical error is given by the width of the resulting Gauss-curve. Would it be possible to measure the Planck-constant $\hbar$ parallel to the temperature?

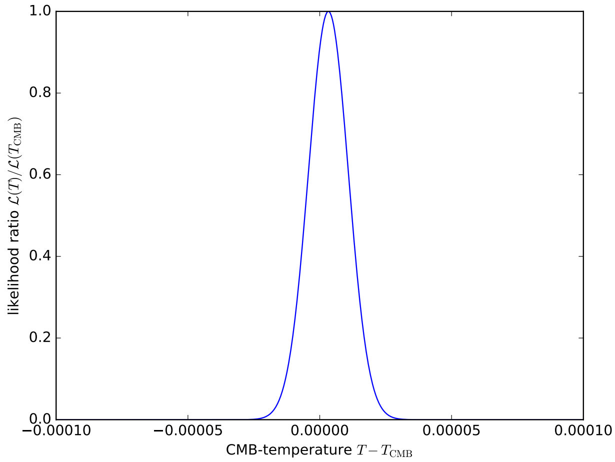


Figure 3: likelihood of the CMB-temperature T for the COBE-data

4. Solar spectrum

The script `solar_plot.py` plots the spectrum of the Sun: Determine the surface temperature $T_\odot$ of the Sun by using Wien's displacement law and the factor that you have determined in the exercises, and estimate the error in your measurement of $T_\odot$.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants and BMS

exercise: the planet of the Petit Prince

1. ***gravity on the planet of the Petit prince***
   The Petit Prince by A. de Saint-Exupery lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupery does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

   (a) What is the density ρ and mass M of the planet, assuming that it is uniform? What astrophysical objects would have similar densities?

   (b) What would be the orbital velocity υ of an object at a height of 1 m above the surface? Could the Petit Prince throw an object horizontally and have it orbit his planet?

   (c) Can the Petit Prince leave the planet by jumping into space?

   (d) Is it possible that the Petit Prince can observe 43 sunsets each day despite the centrifugal force? How many sunsets can one observe at most?

2. ***devices on the planet of the Petit prince***
   Imagine that Saint-Exupery brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

   (a) What's the relation between the oscillation period T of a pendulum clock as a function of height h? Would the oscillation period be independent from the amplitude?

   (b) Saint-Exupery and the Petit Prince have a glass of orange juice with an ice cube. The Petit Prince's ice cube swims higher or not above the surface of the juice compared to Saint-Exupery's?

3. ***relativity on the planet of the Petit prince***
   Are there relativistic effects of gravity on the planet of the Petit Prince?

   (a) What is the tidal gravitational acceleration between the head and the feet of the Petit Prince? Please compute the difference
   $$\Delta g = \frac{GM}{R^2} - \frac{GM}{(R+1)^2} \tag{I}$$

   (b) What is the gravitational time dilation between the head and the feet of the Petit Prince? Please use the formula
   $$\Delta\tau = \sqrt{1 + 2\frac{\Phi}{c^2}}\,\Delta t \tag{II}$$
   and approximate the potential as homogeneous, $\Phi = g\Delta r$.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants and BMS

Solutions: the planet of the Petit Prince

1. ***Gravity on the planet of the Petit prince***
   The Petit Prince by A. de Saint-Exupery lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupery does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

   (a) The relation between the mass and the density of the planet is $M = (4\pi/3)R^3$. The surface gravity is tied to the mass by $g = GM/R^2$. Substituting one in the other one can solve to find a density $\rho \simeq 3.5 \times 10^{10}$ kg m^{-3}. This is quite close to the density of a White Dwarf.

   (b) The orbital period and velocity can be obtained by equation the grvitational accelelration to the centrifugal acceleration: $GM/(R+1\text{m})^2 = \Omega^2(R+1\text{m})$. This gives $\Omega \simeq 1$ s^{-1} corresponding to a period $P \simeq 6$ sec, and an orbital speed $V = \Omega(R + 1\text{m}) \simeq 4$ m s^{-1}. Yes the Petit Prince can throw an object fast enough to put it into orbital motion.

   (c) The escape speed is given by equating the specific kinetic energy $V^2/2$ to the potential energy GM/R, and this gives a typical value $V \simeq 4$ m s^{-1}. This is too much for a kid to jump.

   (d) Given that the maximum period is 6 sec (computed above, and our day corresponds to 86400 sec then there will be at most 14000 sunsets/sunrises.

2. ***Devices on the planet of the Petit prince***
   Imagine that Saint-Exupery brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

   (a) What's the relation between the oscillation period T of a pendulum clock as a function of height h? Would the oscillation period be independent from the amplitude?

   (b) Saint-Exupery and the Petit Prince have a glass of orange juice with an ice cube. The Petit Prince's ice cube swims higher or not above the surface of the juice compared to Saint-Exupery's?

3. ***relativity on the planet of the Petit prince***
   Are there relativistic effects of gravity on the planet of the Petit Prince?

   (a) What is the tidal gravitational acceleration between the head and the feet of the Petit Prince? Please compute the difference
   $$\Delta g = \frac{GM}{R^2} - \frac{GM}{(R+1)^2} \tag{I}$$

   (b) What is the gravitational time dilation between the head and the feet of the Petit Prince? Please use the formula
   $$\Delta\tau = \sqrt{1 + 2\frac{\Phi}{c^2}}\,\Delta t \tag{II}$$
   and approximate the potential as homogeneous, $\Phi = g\Delta r$.

Astronomy from 4 perspectives: the Dark Universe

prepared by: Heidelberg participants and BMS

Tutorial: the Planet of the Petit Prince

Scope of the tutorial The scope of this tutorial is to create a set of exercises on gravity, and basic mechanics, using as a reference the "Planet of the Petit Prince" by A. de Saint-Exupery, in order to have students familiarize with the basic scaling of gravitational forces, in a set-environment different from Earth. This should help student understand how to generalize and extrapolate the knowledge they have gotten in school to problems that are unfamiliar. The first set of exercises requires only basic knowledge of claassical mechanics (work, and escape velocity) and the inverse square-distance law of gravity. The second set of exercizes, more advanced, requires the knowledge of the pendulum law and or Archimesdes principle. The third set of exercises focus on the inhomogeneity of the graviational field, and requires some basic knowledge of tidal forces and basic relativity.

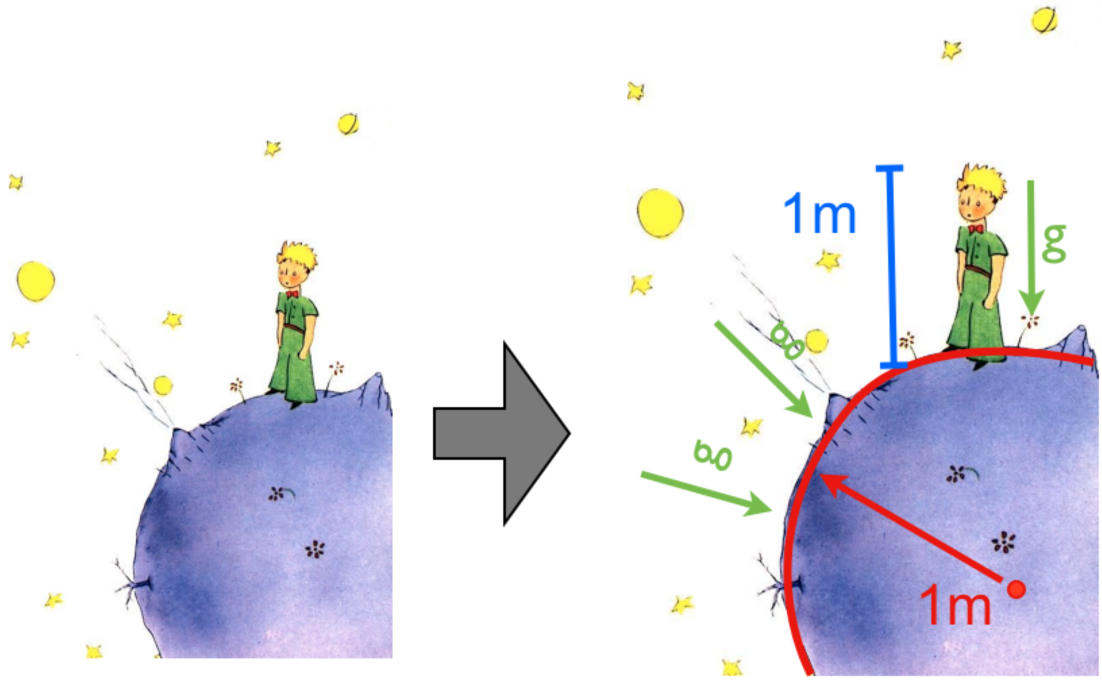


Figure 1: The planet of the petit price and its physical setup

Exercises

1. ***Gravity on the planet of the Petit prince***
   The Petit Prince by A. de Saint-Exupery lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupery does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

   (a) What is the density ρ and mass M of the planet, assuming that it is uniform? What astrophysical objects would have similar densities?

(b) What would be the orbital velocity υ of an object at a height of 1 m above the surface? Could the Petit Prince throw an object horizontally and have it orbit his planet?

(c) Can the Petit Prince leave the planet by jumping into space?

(d) Is it possible that the Petit Prince can observe 43 sunsets each day despite the centrifugal force? How many sunsets can one observe at most?

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Imagine that Saint-Exupery brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

(a) What's the relation between the oscillation period T of a pendulum clock as a function of height h? Would the oscillation period be independent from the amplitude?

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Solutions

1. ***Gravity on the planet of the Petit prince***
The Petit Prince by A. de Saint-Exupery lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupery does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

    (a) The relation between the mass $M$and the density ρ of the planet is $M = (4\pi/3)\rho R^3$. The surface gravity is tied to the mass by $g = GM/R^2$. Substituting one in the other, one can solve to find a density $\rho \simeq 3.5 \times 10^{10}$ kg m^{-3}. This is quite close to the density of a White Dwarf.

    (b) The orbital period and velocity can be obtained by equation the gravitational acceleration to the centrifugal acceleration: $GM/(R+1\text{m})^2 = \Omega^2(R+1\text{m})$. This gives $\Omega \simeq 1$ s^{-1} corresponding to a period $P \simeq 6$ sec, and an orbital speed $V = \Omega(R + 1\text{m}) \simeq 4$ m s^{-1}. Yes the Petit Prince can throw an object fast enough to put it into orbital motion.

    (c) The escape speed is given by equating the specific kinetic energy $V^2/2$ to the potential energy GM/R, and this gives a typical value $V \simeq 4$ m s^{-1}. This is too much for a kid to jump.

    (d) Given that the maximum period is 6 sec (computed above, and our day corresponds to 86400 sec then there will be at most 14000 sunsets/sunrises.

2. ***Devices on the planet of the Petit prince***
Imagine that Saint-Exupery brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

    (a) For small oscillations the formula for the oscillation period of a pendulum is $T = 2\pi\sqrt{L/g}$ where L is the length of the pendulum and g is the gravitational acceleration. On this planet, at an heigh h above the surface it will be: $g = GM/(R + h)^2$. Then $T = 2\pi(R + h)\sqrt{L/GM}$. The period increases proportionally to the height. Given that g is not constan with height, as the pendulum oscillates it will experience different acceleration (stronger at the bottom point, weaker at the edge points of its oscillating trajectory) so the pendulum formula does not apply, and there will be a dependence on the oscillation amplitude. One can try to use a mean values for g. Using some basic trigonomentry (see Figure 2) the difference in heigth between the bottom point and the edge point is $L(1 - \cos\theta)$, where θ is the amplitude of the oscillation. So the average $g \simeq GM/R^2[1+L(1-\cos\theta)]/2R$. substituting this in the equation for the pendulum period T one gets an estimate on how it depends on the apmplitude θ.

    (b) Archimedes principle says that a body immersed in a liquid (water in our case) received a lift upward with a force equal to the weight of the volume of the liquid it displaces. With reference to the figure, let us consider an icecube of an edge of length L, floating in water with a depth equal to h (see Figure 2). Let us call ρ_{I} the density of ice and ρ_{W} the density of wager. We know that ice is lighter than water, such that the former flots on latter. For conveneince we assume that the icecube is small enough that the gravity can be taken as uniform over its size. A typical icecube is $\sim$ 1cm, while on the Petit Price planet the typical scale for the variation of gravity is $\sim$ 1m. The gravitatinal force acting on the icecube is equal to $F_g = g\rho_{\mathrm{I}}L^3$. Archimedes force is instead $F_a = g\rho_{\mathrm{W}}hL^2$. The cube will float if the two are equal, and this gives $h = L\rho_{\mathrm{I}}/\rho_{\mathrm{W}}$, that as one can see does not depend on the graviational acceleration. So it does not matter if the graviational field is stronger or weaker. The icecube of the prince will float as much as the one of Saint Exupery.

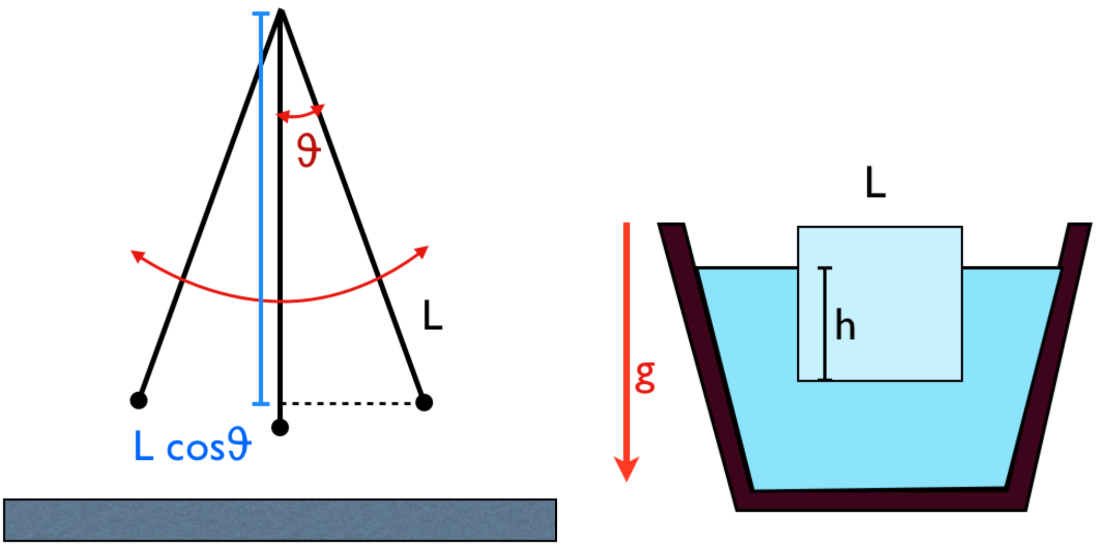

Figure 2: Left: the geometry of a pendulum. Right: a floating icecube.

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