Astronomy from 4 perspectives: the Dark Universe

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High-School exercises: Supernova-cosmology and dark energy

1. ***Classical potentials including a cosmological constant***
   The field equation of classical gravity including a cosmological dark energy density λ is given by

$$\Delta\Phi = 4\pi G\,(\rho + \lambda) \tag{I}$$

   (a) Solve the field equation for 3 dimensions outside a spherically symmetric and static matter distribution ρ.
   The expression for the Laplace-operator in spherical coordinates for 3 dimensions is:

$$\Delta\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) \tag{II}$$

   Also, please set as the total baryon mass M

$$M = 4\pi\int_0^r \mathrm{d}r'\,(r')^2\rho(r') \tag{III}$$

   (b) Show, that both source terms individually give rise to power-law solutions for $\Phi(r)$.

   (c) Is there a distance where the baryon part from the ρ-terms is equal to dark energy part the λ-term?

   (d) Assuming a typical galaxy is formed by 10^{11} stars like the Sun each with a mass of 10^{30} kg, and a dark energy density of $\lambda = 10^{-27}$ kg / m^3, find at which distance from a galaxy the dark energy dominates. How does it compare with the typical size of a Galaxy ($\sim 10^4$ pc)?

2. ***Light-propagation in FLRW-spacetimes***
   Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

   (a) Please show that by introducing *conformal time* τ in a suitable definition, one recovers Minkowskian light propagation $c\tau = \pm\chi$ in comoving distance χ and conformal time τ for FLRW-space times,

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2(t)\,\mathrm{d}\chi^2, \tag{IV}$$

   which we have assumed to be spatially flat for simplicity.

   (b) What's the relationship between conformal time τ and cosmic time t? What would the watch of a cosmological observer display?

   (c) Please compute the conformal age of the Universe given a Hubble function $H(a)$,

$$H(a) = H_0\,a^{-\gamma} \qquad \text{with: } \gamma > 0 \tag{V}$$

   (d) In applying $\mathrm{d}s^2 = 0$ to the FLRW-metric we have assumed a radial geodesic - is this a restriction?

   (e) Draw a diagram of a photon propagating from a distant galaxy to us in conformal coordinates for a cosmology of your choice, with markings on the light-cone corresponding to equidistant Δa.

3. Measure cosmic acceleration

The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by

$$d_{\mathrm{lum}}(z) = (1+z)\int_0^z \mathrm{d}z'\,\frac{1}{H(z')} \tag{VI}$$

with the Hubble function $H(z)$. We can prove this relation between the scale factor a and the cosmological red-shift z:

$$a = \frac{1}{1+z} \tag{VII}$$

So we can write the Hubble function $H(z)$:

$$H(z) = H_0\,(1+z)^{\gamma} \tag{VIII}$$

(a) Find the limit value of γ for an accelerating (and non-accelerating) universe.
Use the deceleration parameter given by

$$q = -\frac{\ddot{a}\,a}{\dot{a}^2}$$

from the Hubble-function

$$H = \frac{\dot{a}}{a}$$

to prove it.

(b) Please show that in accelerated universes supernovae appear systematically dimmer, because d_{lum} is always larger than in a non-accelerating universe.