Astronomy from 4 perspectives: the Dark Universe

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High-School exercises: Supernova-cosmology and dark energy Solutions

1. ***Classical potentials including a cosmological constant***
The field equation of classical gravity including a cosmological dark energy density λ is given by

$$\Delta\Phi = 4\pi G(\rho + \lambda) \tag{I}$$

(a) Field calculation

Now it is possible to simply integrate the field equation starting with:

$$\begin{aligned}
\Delta\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) && \text{(II)}\\
&= 4\pi G(\rho(r) + \lambda) && \text{(III)}\\
r^2\frac{\partial\Phi}{\partial r} &= \int_0^r \mathrm{d}r' \left(4\pi G[(r')^2\rho(r') + (r')^2\lambda]\right) && \text{(IV)}\\
&= 4\pi G\left(M + \frac{r^3}{3}\lambda\right) && \text{(V)}\\
\frac{\partial\Phi}{\partial r} &= 4\pi G\left(\frac{M}{r^2} + \frac{\lambda r}{3}\right) && \text{(VI)}\\
\Phi &= 4\pi G\left(-\frac{GM}{r} + \frac{\lambda r^2}{6}\right) && \text{(VII)}
\end{aligned}$$

(b) ”Equilibrium”

To find an equilibrium distance one must set $\Phi\left(r_{\mathrm{eq}}\right) = 0$

$$\begin{aligned}
\frac{GM}{r_{\mathrm{eq}}} &= G\,\frac{\lambda\, r_{\mathrm{eq}}^2}{6} && \text{(VIII)}\\
\frac{\lambda\, r_{\mathrm{eq}}^3}{6} &= M && \text{(IX)}
\end{aligned}$$

from which follows immediatly:

$$r_{\mathrm{eq}} = \sqrt[3]{6\,\frac{M}{\lambda}} \tag{X}$$

If one inputs the number one gets r_{eq} few Mpc one hundred times larger than the size of a galaxy. We can really observe the Dark-Energy effect at this distance?

2. ***Light-propagation in FLRW-spacetimes***
Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

(a) Let us do the following substitution

$$dt \to a(t)d\tau$$

then the line element can be written

$$ds^2 = a(t)^2 \left[c^2 d\tau^2 - d\chi^2\right]$$

and the equation of the null-geodesic will be

$$d\chi = \pm c d\tau$$

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(b) The cosmic time is the time measured by a cosmic observer synchronized for $t = 0$

$$t = \int_0^t dt' = \int_0^a \frac{da'}{\dot{a}'} \tag{XI}$$

The conformal time is tied to the time interval over which an observer at $t = t_0$ sees to happen an event in the past at time t. Now at $t = t_0$ this will coincide with the cosmic time, ence it will be affected by cosmic time dilation.

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} = \frac{1}{a(t)} \int_0^t \frac{a(t)}{a(t')} dt' > \frac{t}{a(t)} \tag{XII}$$

(c) Now for the given metric:

$$H = \frac{\dot{a}}{a} = H_0\, a^{-\gamma} \Rightarrow \dot{a} = H_0\, a^{1-\gamma} \tag{XIII}$$

We can solve this equation:

$$\frac{da}{dt} = H_0\, a^{1-\gamma} \tag{XIV}$$

$$a^{\gamma-1}\, da = H_0\, dt \tag{XV}$$

$$\int_0^a (a')^{\gamma-1} da' = H_0 \int_0^t dt = H_0\, t \tag{XVI}$$

$$\frac{a^\gamma}{\gamma} = H_0\, t \tag{XVII}$$

$$a(t) = \sqrt[\gamma]{\gamma\, H_0\, t} \tag{XVIII}$$

So we can obtain for the Age of the Universe

$$\tau_H = \int_0^t \frac{dt'}{a(t')} = \int_0^1 \frac{da}{\dot{a}\, a} = \frac{1}{H_0} \int_0^1 a^{\gamma-2}\, da = \frac{1}{H_0} \frac{1}{\gamma - 1} \tag{XIX}$$

(d) Isotropy of the universe ensures us that it is not.

3. Measure cosmic acceleration

The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by

$$d_{\mathrm{lum}}(z) = (1+z) \int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{XX}$$

with the Hubble function $H(z)$:

(a) By definition of the Hubble Function H and deceleration parameter q:

$$H = \frac{\dot{a}}{a}$$
$$q = -\frac{\ddot{a}a}{\dot{a}^2}$$

It follows

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}a}{a^2} - H^2$$

So we get:

$$\frac{\dot{H}}{H^2} = \frac{\ddot{a}a}{\dot{a}^2} - 1 = -q - 1$$
$$q = -\left(\frac{\dot{H}}{H^2} + 1\right)$$

If we use $H(a) = H_0\, a^{-\gamma}$:

$$\begin{aligned}\dot{H} &= -\gamma H_0 \cdot a^{-\gamma-1} \cdot \dot{a} \\ &= -\gamma H_0 \cdot a^{-\gamma} \cdot \frac{\dot{a}}{a} \\ &= -\gamma H^2\end{aligned}$$

And then we get:

$$\frac{\dot{H}}{H^2} = -\gamma$$

Now we can substitute in the deceleration parameter expression:

$$q = -(-\gamma + 1) = \gamma - 1$$

and obviously

$$\begin{aligned} q < 0 \quad &\text{for } \gamma < 1 \\ q > 0 \quad &\text{for } \gamma > 1 \end{aligned}$$

(b) First, we consider the case $\gamma = 1$:

$$H = H_0(1+z)^\gamma = H_0(1+z)$$

Then we find this expression for the luminosity distance

$$\begin{aligned} d_{lum,1} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\ &= (1+z)\int_0^z \frac{1}{H_0(1+z')}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0}\, ln(1+z) \end{aligned}$$

Now, we consider the case $\gamma < 1$ (accelerating universe):

$$\begin{aligned}
d_{lum,2} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\
&= \frac{1+z}{H_0}\int_0^z (1+z')^{-\gamma}\,\mathrm{d}z' \\
&= \frac{1+z}{H_0}\left[(1+z')^{-\gamma+1}\cdot\frac{1}{-\gamma+1}\right]_0^z \\
&= \frac{1+z}{H_0}\left(\frac{1}{-\gamma+1}\right)\left[(1+z)^{-\gamma+1}-1\right]
\end{aligned}$$

It follows: $d_{lum_2}(z) > d_{lum_1}(z)$, because the exponent $(-\gamma+1)$ is positive ($\gamma < 1$), so $d_{lum_2}(z)$ is growing faster, than the logarithmic function $d_{lum_1}(z)$.