Astronomy from 4 Perspectives: the Dark Universe

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Play with data: virial theorem and periodic motion

In these exercises, we will explore the virial theorem by solving equations of motions numerically and measuring averaged energies from the solutions.

1. ***Virial theorem and the harmonic oscillator***
The script `harmonic.py` generates a solution to the harmonic oscillator equation $\ddot{x} = -x$ by transforming it with the definition $y = \dot{x}$ into a coupled first order equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{I}$$

where the angular frequency is set to $\omega = 1$ for the numerics.

(a) Is the total energy conserved?

(b) Please run the script and measure the average kinetic and potential energies: Do you find $\langle T\rangle = \langle V\rangle$ for the harmonic oscillator?

(c) Is $\langle T\rangle = \langle V\rangle$ true for any initial condition?

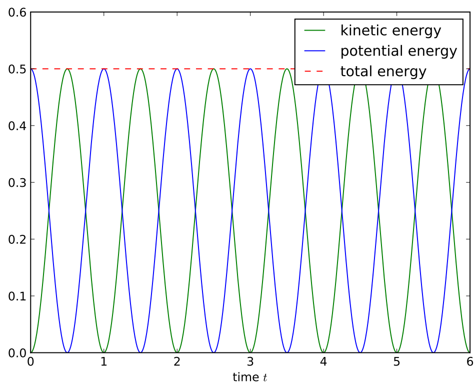

Figure 1: numerical solutions to the harmonic oscillator

2. ***Virial relationship in the anharmonic oscillator***
The virial theorem makes a prediction for the average kinetic and potential energies in any system with a scale-free potential, for instance the anharmonic oscillator with the potential $\Phi \propto x^{2n}$. The script `anharmonic.py` solves the equation of motion. The constant of proportionality is set to 1.

(a) What's the equation of motion for the potential $\Phi = x^{2n}/(2n)$, and what's the corresponding coupled first order system?

(b) Why are there with increasing n phases of constant velocity and a sawtooth pattern in position?

(c) Please measure the average kinetic and potential energies over many oscillations: What's their ratio r as a function of n? Why does $r = \langle T\rangle/\langle V\rangle$ increase with n?

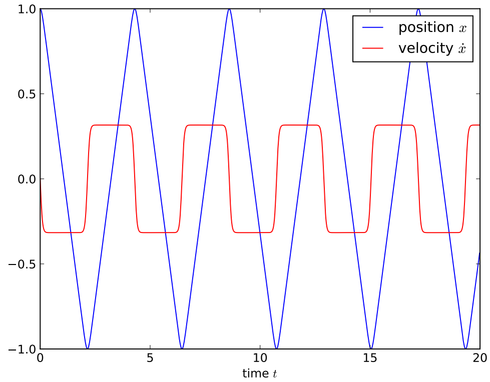

Figure 2: numerical solutions to the anharmonic oscillator for $n = 10$

3. ***Virial theorem in the (generalised) Kepler-problem***

Please simulate Kepler orbits with the script `kepler.py`: The equation of motion of a particle in the potential $\Phi \propto 1/r^\alpha$ can be derived from the Lagrange density

$$\mathcal{L} = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{GM}{r^\alpha} \tag{II}$$

Please derive the equation of motion with the Euler-Lagrange-equations and convert the resulting second-order equations in a set of first coupled order equations.

(a) Change the total energy of the planet by setting δ to a value unequal to 0 and observe the change in the orbit. The product constants GM is set to $GM = 1$ for the numerics.

(b) Change the value of α to a number different from 1: Are the orbits still closed? NB: The problem becomes unstable if α is too large, try to experiment in the range $\alpha = 0.8 \ldots 1.2$.

(c) Can you generate precession motion and orbits lagging behind by choosing a suitable α?

(d) Is it possible to have bound systems for $\alpha \geq 2$? What's the physical reason?

(e) What's the virial relationship and how does it depend on α?

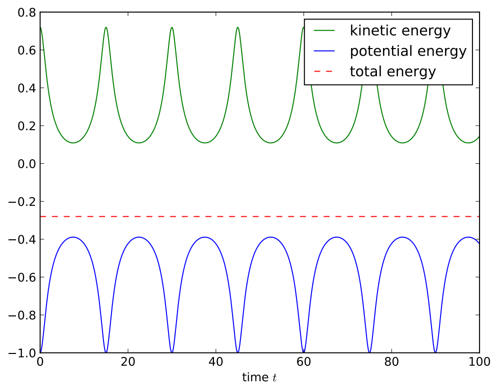

Figure 3: kinetic and potential energy in Kepler-orbits