Astronomy from 4 perspectives: the Dark Universe

prepared by: Padua participants

exercises: Planck - spectrum and CMB

Solutions

1. Properties of the Planck-spectrum

$$S(\nu) = S_0 \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} \Rightarrow S(\nu) = S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{I}$$

(a)

$$\int_0^\infty S(\nu) = \int_0^\infty S_0 \nu^3 e^{-\frac{h\nu}{kT}} \tag{II}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{III}$$

$$\int_0^\infty S_0 (\frac{kT}{h})^3 e^{-\frac{h\nu}{kT}} (\frac{kT}{h}) \propto T^4 \tag{IV}$$

(b)

$$\frac{dS}{d\nu} = 0 \Rightarrow \frac{dS}{d\nu} = 3S_0 \nu^2 e^{-h\nu kT} + S_0 \nu^3 (-\frac{h}{kT}) e^{-h\nu kT} = \tag{V}$$

$$= S_0 \nu^3 e^{-\frac{h\nu}{kT}} (3 - \frac{h\nu}{kT}) \tag{VI}$$

$$3 - \frac{h\nu}{kT} = 0 \Rightarrow \nu = \frac{3kT}{h} \propto T \tag{VII}$$

(c)

$$< \nu >= \frac{\int_0^\infty \nu S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} = \frac{\int_0^\infty \nu^3 S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{VIII}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{IX}$$

$$\frac{\int_0^\infty S_0 \frac{KT}{h}^4 e^{-x} (\frac{kT}{h}) dx}{\int_0^\infty S_0 \frac{KT}{h}^3 e^{-x} \frac{kT}{h} dx} \tag{X}$$

$$S \propto \frac{T^5}{T^4} \propto T \tag{XI}$$

(d) Yes, as both of them depend on T

(e) Only at high energies

(f) No, as if we consider any power of T in the exponential, any substitution $x = \frac{h\nu}{KT}$ gives a new differential $d\nu\frac{kT^4}{h}$ both in numerator and denominator

(a) ***Wien's distribution function***

$$S(\nu) = S_0\nu^n e^{-\frac{h\nu}{kT}} \quad ; \quad \Gamma(n) = \int_0^\infty dx x^{n-1} e^{-x} \tag{XII}$$

a)

$$\int_0^\infty dx x^n e^{-x} = \cancel{-[e^{-x}x^n]_0^\infty} - n\int_0^\infty dx x^{n-1}e^{-x} = \int_0^\infty dx n x^{n-1} e^{-x} \tag{XIII}$$

$$= \cancel{x[e^{-x}nx^{n-1}]_0^\infty} - \int_0^\infty dx - n(n-1)x^{n-2} = \int_0^\infty dx - n(n-1)x^{n-2}e^{-x} \tag{XIV}$$

$$= [...] = \tag{XV}$$

$$= n!\int_0^\infty dx e^{-x} = n! \tag{XVI}$$

Let's show that $\Gamma(1) = 1$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} \tag{XVII}$$

To demonstrate our relation recursively let's start by showing it works for $n = 2$

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = [-e^{-x}x]_0^\infty - \int_0^\infty -e^{-x} = 1 \tag{XVIII}$$

$$= (2-1)\Gamma(1) = 1 \tag{XIX}$$

We can now show it for $n + 1$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)\int_0^\infty x^{n-2}e^{-x}dx \tag{XX}$$

Let's introduce $m = n - 2 \Rightarrow n = m + 2$

$$\Gamma(n+1) = (m+2)(m+1)\int_0^\infty x^m e^{-x} dx = (m+2)(m+1)m! = (m+2)! = n! \tag{XXI}$$

But

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n! \tag{XXII}$$

(b)

$$< \nu^m >= \frac{\int_0^\infty \nu^m S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \tag{XXIII}$$

$$= \frac{\int_0^\infty \nu^m S_0 \nu^n e^{-\frac{h\nu}{kT}} d\nu}{\int_0^\infty S_0 \nu^n e^{-\frac{h\nu}{kT}}} d\nu \tag{XXIV}$$

$$x = \frac{h\nu}{kT} \quad ; \quad \nu = \frac{kT}{h} x \quad ; \quad d\nu = \frac{kT}{h} dx \tag{XXV}$$

$$< \nu^m >= \frac{\int_0^\infty \frac{kT}{h}^m S(\nu) \frac{kT}{h}^n e^{-x} \frac{kT}{h} d\nu}{\int_0^\infty S_0 \frac{kT}{h}^n e^{-x} \frac{kT}{h} d\nu} \propto \tag{XXVI}$$

$$\propto \frac{T^{m+n+1}}{T^{n+1}} \propto T^m \tag{XXVII}$$

The momenta scale like the power chosen.

(c) From the result on point c, we find:

$$S = \frac{< \nu^3 >}{(< \nu >^2)^{3/2}} \propto \frac{T^3}{T^{2^{3/2}}} \tag{XXVIII}$$

$$k = \frac{< \nu^4 >}{< \nu^2 >^2}^2 \propto \frac{T^4}{(T^2)^2} \tag{XXIX}$$

Neither of which depends on T

(d) Yes,again from the result of point c):

$$\frac{< \nu^{2n} >}{< \nu^2 >^n} \propto \frac{T^{2n}}{(T^2)^n} \tag{XXX}$$

3. ***Planck-spectra at cosmological distances***
   Discussed in the classroom
4. ***CMB as a source of energy***

   (a) The answer a) with the Stephan Boltzmann law, the power per m^2 can be produced if:

$$w = 10^{-5} \sigma T^4 = 3 \cdot 10^{-11} W m^{-2} \tag{XXXI}$$