Astronomy from 4 Perspectives: the Dark Universe

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Tutorial: the Planet of the Petit Prince

The scope of this tutorial is to create a set of exercises on gravity, and basic mechanics, using as a reference the "Planet of the Petit Prince" by A. de Saint-Exupéry, in order to have students familiarize with the basic scaling of gravitational forces, in a set-environment different from Earth. This should help student understand how to generalize and extrapolate the knowledge they have gotten in school to problems that are unfamiliar. The first set of exercises requires only basic knowledge of classical mechanics (work, and escape velocity) and the inverse square-distance law of gravity. The second set of exercises, more advanced, requires the knowledge of the pendulum law and or Archimedes principle. The third set of exercises focus on the inhomogeneity of the gravitational field, and requires some basic knowledge of tidal forces and basic relativity.

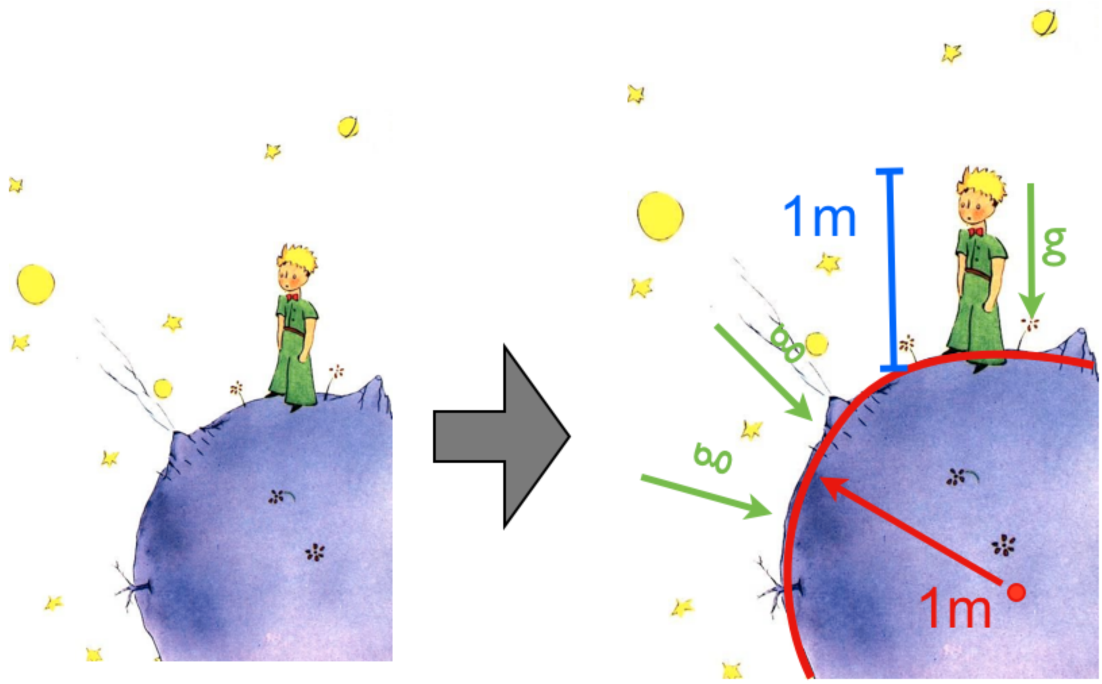

Figure 1: The planet of the Petit Prince and its physical setup

Exercises

1. ***Gravity on the planet of the Petit prince***

   The Petit Prince by A. de Saint-Exupéry lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupéry does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

   (a) What is the density ρ and mass M of the planet, assuming that it is uniform? What astrophysical objects would have similar densities?

(b) What would be the orbital velocity υ of an object at a height of 1 m above the surface? Could the Petit Prince throw an object horizontally and have it orbit his planet?

(c) Can the Petit Prince leave the planet by jumping into space?

(d) Is it possible that the Petit Prince can observe 43 sunsets each day despite the centrifugal force? How many sunsets can one observe at most?

2. ***Devices on the planet of the Petit prince***
Imagine that Saint-Exupéry brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

(a) What's the relation between the oscillation period T of a pendulum clock as a function of height h? Would the oscillation period be independent from the amplitude?

(b) Saint-Exupéry and the Petit Prince have a glass of orange juice with an ice cube. The Petit Prince's ice cube swims higher or not above the surface of the juice compared to Saint-Exupéry's?

3. ***Relativity on the planet of the Petit prince***
Are there relativistic effects of gravity on the planet of the Petit Prince?

(a) What is the tidal gravitational acceleration between the head and the feet of the Petit Prince? Please compute the difference

$$\Delta g = \frac{GM}{R^2} - \frac{GM}{(R+1)^2} \tag{I}$$

(b) What is the gravitational time dilation between the head and the feet of the Petit Prince? Please use the formula

$$\Delta\tau = \sqrt{1 + 2\frac{\Phi}{c^2}}\,\Delta t \tag{II}$$

and approximate the potential as homogeneous, $\Phi = g\Delta r$.

Solutions

1. ***Gravity on the planet of the Petit Prince***

   The Petit Prince by A. de Saint-Exupéry lives on a planet which, according to images, is roughly $R \simeq 1$ m in size and because Saint-Exupéry does not provide any other information, has a value of the surface gravity $g = 9.81$ m/s^2 similar to Earth. But in comparison to Earth where the gradient of the acceleration is almost zero, it is much stronger on the planet of the Petit Prince. Recall that $G = 6.6 \times 10^{-11}$ in SI.

   (a) The relation between the mass $M$and the density ρ of the planet is $M = (4\pi/3)\rho R^3$. The surface gravity is tied to the mass by $g = GM/R^2$. Substituting one in the other, one can solve to find a density $\rho \simeq 3.5 \times 10^{10}$ kg m^{-3}. This is quite close to the density of a White Dwarf.

   (b) The orbital period and velocity can be obtained by equation the gravitational acceleration to the centrifugal acceleration: $GM/(R+1\text{m})^2 = \Omega^2(R+1\text{m})$. This gives $\Omega \simeq 1$ s^{-1} corresponding to a period $P \simeq 6$ sec, and an orbital speed $V = \Omega(R + 1\text{m}) \simeq 4$ m s^{-1}. So yes, the Petit Prince can throw an object fast enough to put it into orbital motion.

   (c) The escape speed is given by equating the specific kinetic energy $V^2/2$ to the potential energy GM/R, and this gives a typical value $V \simeq 4$ m s^{-1}. This is too much for a kid to jump.

   (d) Given that the maximum period is 6 sec (computed above, and our day corresponds to 86400 sec then there will be at most 14000 sunsets/sunrises.

2. ***Devices on the planet of the Petit prince***

   Imagine that Saint-Exupéry brings simple mechanical systems with him, and find out if they behave differently because of the strong gradient $\partial g/\partial r$ in the gravitational acceleration g.

   (a) For small oscillations the formula for the oscillation period of a pendulum is $T = 2\pi\sqrt{L/g}$ where L is the length of the pendulum and g is the gravitational acceleration. On this planet, at an heigh h above the surface it will be: $g = GM/(R + h)^2$. Then $T = 2\pi(R + h)\sqrt{L/GM}$. The period increases proportionally to the height. Given that g is not constant with height, as the pendulum oscillates it will experience different acceleration (stronger at the bottom point, weaker at the edge points of its oscillating trajectory) so the pendulum formula does not apply, and there will be a dependence on the oscillation amplitude. One can try to use a mean values for g. Using some basic trigonometry (see Figure 2) the difference in height between the bottom point and the edge point is $L(1 - \cos\theta)$, where θ is the amplitude of the oscillation. So the average $g \simeq GM/R^2[1+L(1-\cos\theta)]/2R$. substituting this in the equation for the pendulum period T one gets an estimate on how it depends on the amplitude θ.

   (b) Archimedes principle says that a body immersed in a liquid (water in our case) received a lift upward with a force equal to the weight of the volume of the liquid it displaces. With reference to the figure, let us consider an ice cube of an edge of length L, floating in water with a depth equal to h (see Figure 2). Let us call ρ_I the density of ice and ρ_W the density of wager. We know that ice is lighter than water, such that the former floats on latter. For convenience we assume that the ice cube is small enough that the gravity can be taken as uniform over its size. A typical ice cube is $\sim$ 1cm, while on the Petit Price planet the typical scale for the variation of gravity is $\sim$ 1m. The gravitational force acting on the ice cube is equal to $F_g = g\rho_\text{I}L^3$. Archimedes force is instead $F_a = g\rho_\text{W}hL^2$. The cube will float if the two are equal, and this gives $h = L\rho_\text{I}/\rho_\text{W}$, that as one can see does not depend on the gravitational acceleration. So it does not matter if the gravitational field is stronger or weaker. The ice cube of the prince will float as much as the one of Saint Exupéry.

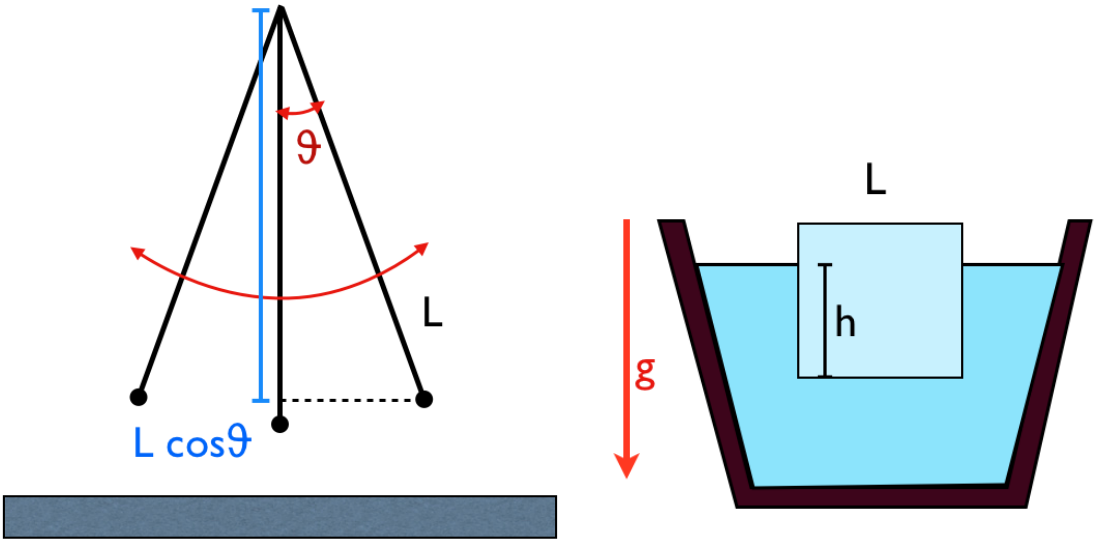

Figure 2: Left: the geometry of a pendulum. Right: a floating ice cube.

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