

Astronomy from 4 Perspectives: the Dark Universe

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Exercise: Dark matter and galaxy rotation curves

1. Harmonic oscillator and energy types

The harmonic oscillator is described by the differential equation $\ddot{x} = -g/l x$, and performs harmonic oscillations $x(t) \propto \exp(\pm i\omega t)$ with $\omega^2 = g/l$.

- (a) Please show that $\langle T \rangle = \langle V \rangle$ with the kinetic energy T and the potential energy V . The brackets $\langle \dots \rangle$ are time averages over one oscillation period τ ,

$$\langle T \rangle = \frac{1}{\tau} \int_0^\tau dt T(t) \quad \text{and} \quad \langle V \rangle = \frac{1}{\tau} \int_0^\tau dt V(t) \quad (\text{I})$$

which is defined as $\tau = 2\pi/\omega$, and the specific energies $T(t) = \dot{x}^2/2$ and $V(t) = gx^2/(2l)$.

- (b) Could you predict the proportionality between $\langle T \rangle$ and $\langle V \rangle$ from the isochrony of the harmonic oscillator?

The probability of finding the oscillator at a certain amplitude x is inversely proportional to the velocity: $dx/dt = v$, such that $\Delta t = \Delta x/v$. If the range of motion is divided into equidistant intervals Δx , the probability p of seeing the oscillator in one of those is proportional to the time it spends there, i.e. proportional to $1/|v|$.

- (c) Please normalise p and draw the function $p(v)$: If you look randomly at a harmonic oscillator, at what stage in its oscillation are you most likely to see it?
- (d) Please define averages

$$\langle T \rangle = \int dv p(v) T(v) \quad \text{and} \quad \langle V \rangle = \int dv p(v) V(v) \quad (\text{II})$$

and compute both integrals. You can use energy conservation for the second integral to express V in terms of the velocity v . Are the results identical to the previous computation? Be careful to take the positive sign of p into account, by using the symmetry of the integrand.

- (e) Why is there no issue with convergence when the probability density $p \rightarrow \infty$ at $v \rightarrow 0$?
- (f) Is the virial relation $\langle T \rangle = \langle V \rangle$ as well valid for a circular orbit in a spherically symmetric harmonic potential?
- (g) Is it valid as well for any other Lissajous-figure?

2. Flat rotation curves

Let's consider the motion of stars inside a galaxy with the density profile of a *singular isothermal sphere*, which is $\rho \propto r^{-2}$. The singular isothermal sphere describes the density of dark matter well on scales of the galactic disc.

- (a) Please show by solving the Poisson equation $\Delta\Phi = 4\pi G\rho$,

$$\Delta\Phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G\rho, \quad (\text{III})$$

for a spherically symmetric density profile $\rho \propto r^{-2}$ that rotation curves are flat.

- (b) Please compute the mean kinetic $\langle T \rangle$ and mean potential energy $\langle V \rangle$ for the circular motion in an isothermal sphere as a function of r .
- (c) Is it possible in this case to decompose the circular orbiting motion into two uncoupled orthogonal harmonic oscillations?
- (d) What would the density profile need to be such that stars would perform harmonic oscillations through the centre of the galaxy, i.e. for the potential to be quadratic, $\Phi \propto r^2$?

3. *MoND, the Solar system and the Milky Way*

Let's assume that we can change the acceleration due to gradients in the gravitational potential $\nabla\Phi$ in an empirical way,

$$\frac{d\Phi}{dr} \rightarrow \frac{d\Phi}{dr} + a_0, \quad (\text{IV})$$

as it would be relevant for a circular motion around the Milky Way centre in a spherically symmetric potential.

- (a) What would be the effect on a rotation curve from the density profile $\rho \propto r^{-\alpha}$?
- (b) The parameter a_0 would need to be chosen small: Please estimate an upper bound on the value of a_0 from the orbital acceleration of the Solar system on its passage around the Milky Way center. You can find all necessary data on Wikipedia.
- (c) Please think of a way to visualise the numerical value of a_0 .
- (d) At what distance from the Earth's surface would the gravitational acceleration be a_0 ?