Planck's radiation law

Matthias Taulien, April 2005, August 2017

Lit.: Gerthsen, Physik, 21. Auflage, Springer 2002
Alonso-Finn, Physik III, dt. Ausgabe, Inter. European Editions B.V., 1974

Planck's radiation law (depending on frequency)

The energie density of radiation in the frequency interval $\nu + d\nu$ at the temperature T is:

$$\rho(\nu, T)\cdot d\nu = \frac{8\,\pi\cdot h\cdot \nu^3}{c^3}\cdot\frac{1}{e^{\frac{h\cdot\nu}{k\cdot T}} - 1}\cdot d\nu$$ (Equation 1)

$restart$
$with(plots):$

$$\rho_\nu := (\nu, T) \rightarrow \frac{8\,\pi\cdot h\cdot \nu^3}{c^3}\cdot\frac{1}{e^{\frac{h\cdot\nu}{k\cdot T}} - 1}:$$

$c := 299792453 : \#\ velocity\ of\ light$
$h := 6.626\cdot 10^{-34} : \#\ Planck's\ constant$
$k := 1.380650\cdot 10^{-23} : \#\ Boltzmann's\ constant$

Calculation of the maximum of radiation

$$d\rho_\nu := \frac{d}{d\,\nu}\rho_\nu(\nu, T)$$

$$\frac{1.854173881\ 10^{-57}\,\nu^2}{e^{\frac{4.799188788\ 10^{-11}\,\nu}{T}} - 1} - \frac{2.966176833\ 10^{-68}\,\nu^3\, e^{\frac{4.799188788\ 10^{-11}\,\nu}{T}}}{\left(e^{\frac{4.799188788\ 10^{-11}\,\nu}{T}} - 1\right)^2 T} \quad (1)$$

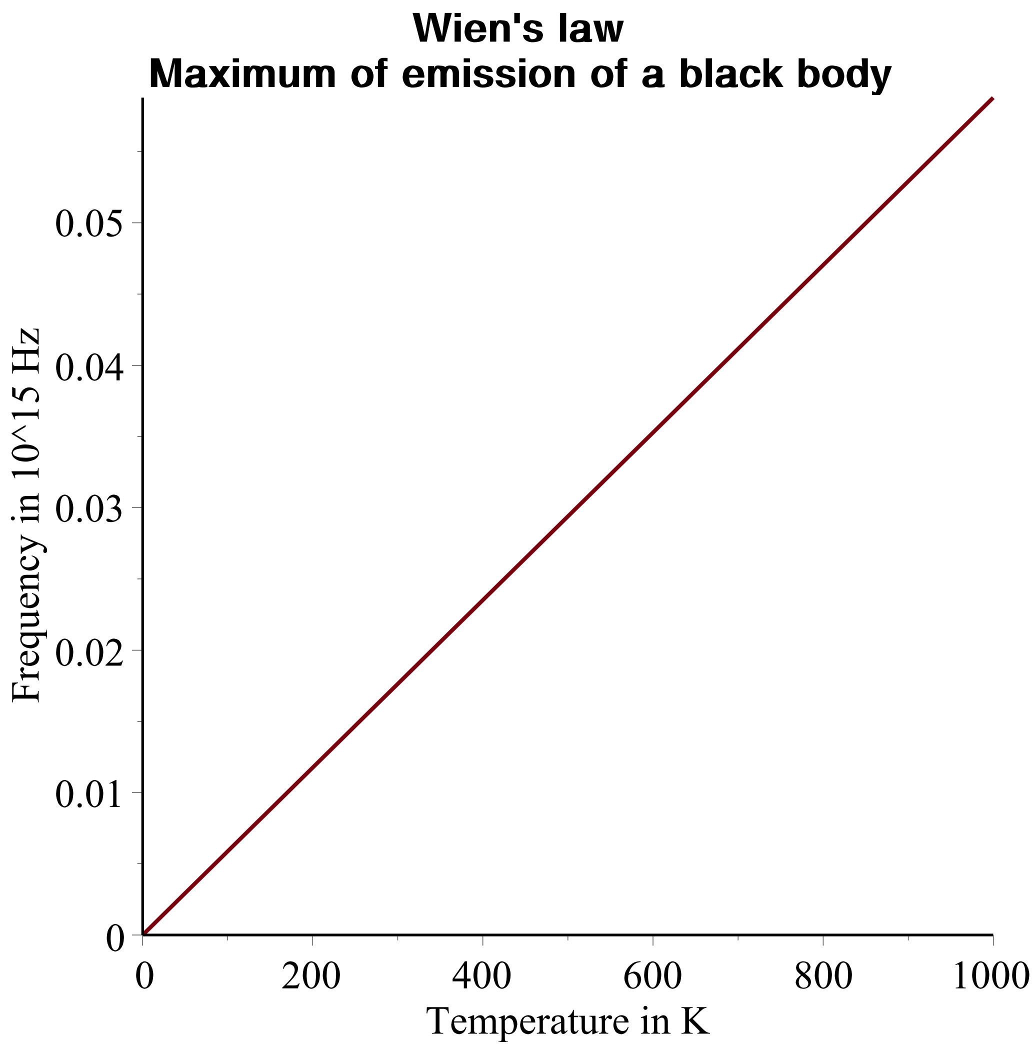
$\max_\nu := solve(d\rho_\nu = 0, \nu)$

$$5.878992258\ 10^{10}\ T, 0. \quad (2)$$

Wien's law (depending on frequency)

$\nu_{Wien} := unapply(op(1, [\max_\nu]), T):$

$plot\left(\nu_{Wien}(T)\cdot 10^{-15}, T = 0\,..10^3, title = \text{"Wien's law\nMaximum of emission of a black body"}, titlefont = [HELVETICA, BOLD, 12], labels = [\text{"Temperature in K"}, \text{"Frequency in 10^15 Hz"}], labeldirections = [HORIZONTAL, VERTICAL]\right)$



Calculation of the maximum of radiation emission

$eq\nu := \nu_{max} = apply\left(\nu_{Wien}, T\right) :$

$const\nu := solve(eq\nu, T) :$

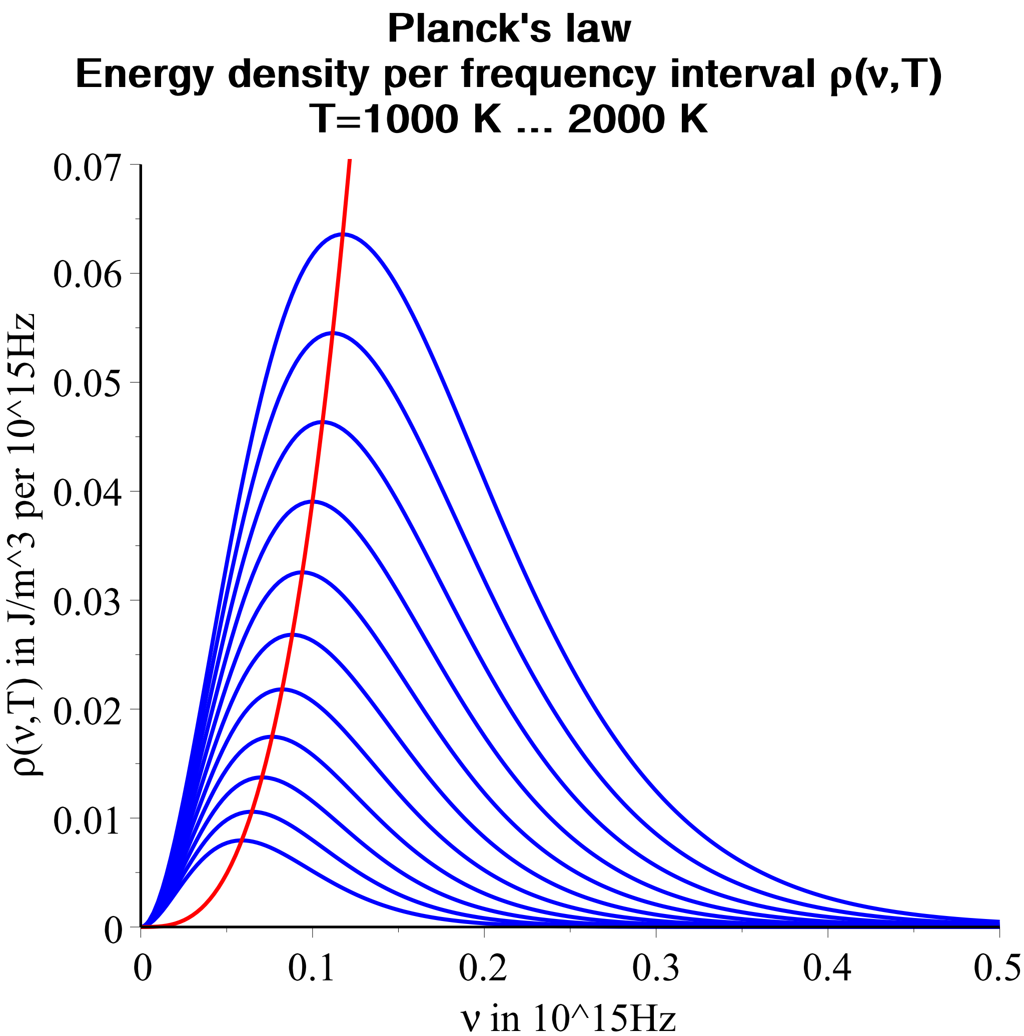
$\rho_{\nu_{max}} := unapply\left(\rho_{\nu}\left(\nu, subs\left(\nu_{max} = \nu, const\nu\right)\right), \nu\right) :$

$plot_{\rho_{\nu_{max}}} := plot\left(\rho_{\nu_{max}}\left(\nu \cdot 10^{15}\right) \cdot 10^{15}, \nu = 0 ..0.5, y = 0 ..0.07, color = RED\right) :$

$seq_{\rho_{\nu}} := \left[seq\left(subs\left(T = 1000 + 100 \cdot n, \rho_{\nu}\left(\nu \cdot 10^{15}, T\right) \cdot 10^{15}\right), n = 0 ..10\right)\right]:$

$plot_{\rho_{\nu}} := plot\Big(seq_{\rho_{\nu}}, \nu = 0 ..0.5, color = BLUE, title$
$= \text{"Planck's law\textbackslash nEnergy density per frequency interval ρ(ν,T)\textbackslash nT=1000 K ... 2000 K"}, titlefont$
$= [HELVETICA, BOLD, 12], labels = [\text{"ν in 10\^{}15Hz"}, \text{"ρ(ν,T) in J/m\^{}3 per 10\^{}15Hz"}],$
$labeldirections = [HORIZONTAL, VERTICAL]\Big) :$

$display\left(\left[plot_{\rho_{\nu}}, plot_{\rho_{\nu_{max}}}\right]\right)$



Planck's radiation law (depending on wavelength)

Sometimes you want to calculate the emitted energy depending on the wavelength.

It is $c = \lambda \cdot \nu$ and therefor $\frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$. Furtheron is $E(\nu)\,d\nu = -E(\lambda)\,d\lambda$, because $d\nu$ and $d\lambda$ have different signs.

Setting in Equation 1 you get:

$$E(\lambda)\cdot d\lambda = \frac{8\,\pi\cdot h\cdot c}{\lambda^5}\cdot\frac{1}{e^{\frac{h\cdot c}{\lambda\cdot k\cdot T}}-1}\cdot d\lambda$$ (Equation 2)

$$\rho_\lambda := (\lambda, T) \rightarrow \frac{8\,\pi\cdot h\cdot c}{\lambda^5}\cdot\frac{1}{e^{\frac{h\cdot c}{\lambda\cdot k\cdot T}}-1}:$$

Calculate the maximum of radiation

$$d\rho_\lambda := \frac{d}{d\,\lambda}\,\rho_\lambda(\lambda, T)$$

$$-\frac{2.496215016\ 10^{-23}}{\lambda^6\left(e^{\frac{0.01438760579}{\lambda T}}-1\right)}+\frac{7.182911525\ 10^{-26}\ e^{\frac{0.01438760579}{\lambda T}}}{\lambda^7\left(e^{\frac{0.01438760579}{\lambda T}}-1\right)^2 T} \qquad \textbf{(3)}$$

$\max_\lambda := solve\left(d\rho_\lambda = 0, \lambda\right)$

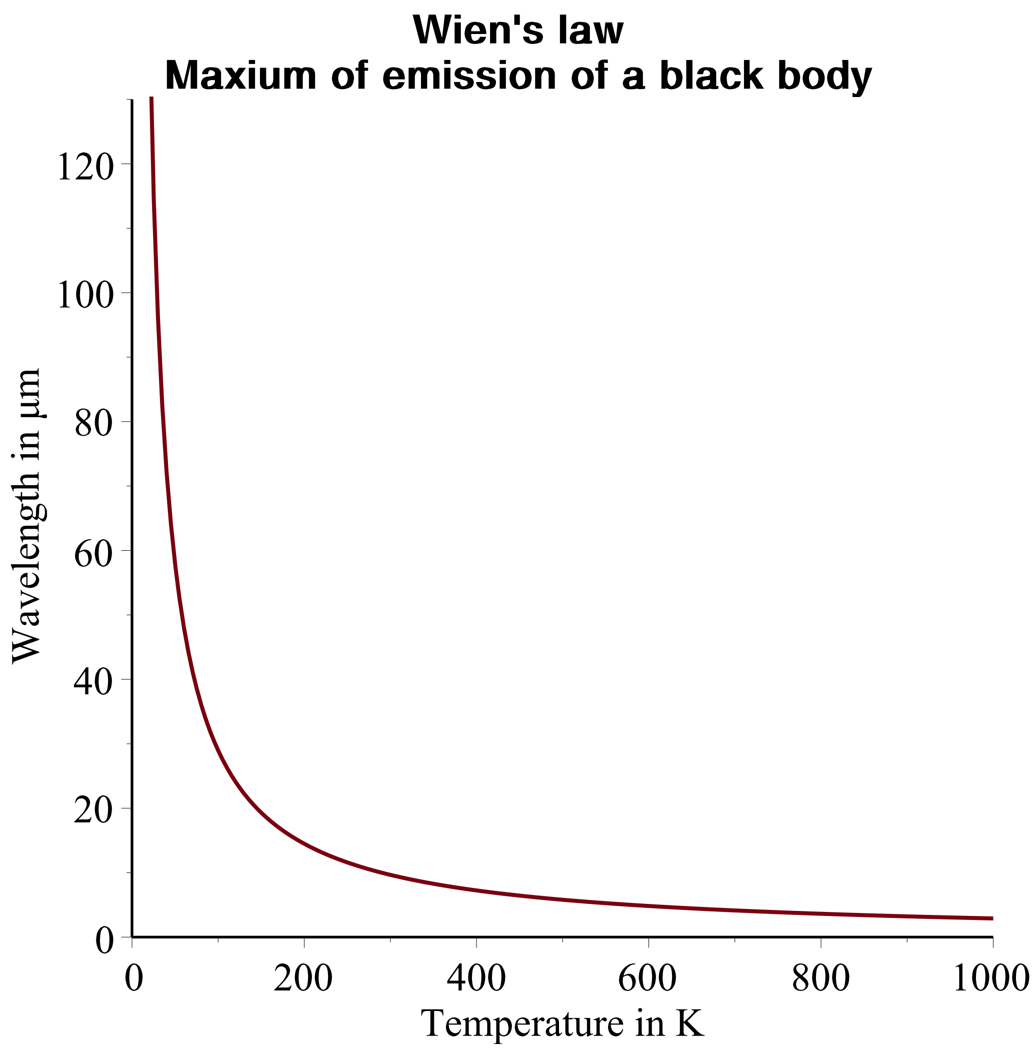
Warning, solutions may have been lost

$$\frac{0.002897739130}{T} \qquad \textbf{(4)}$$

Wien's law (depending on wavelength)

$\lambda_{Wien} := unapply\left(\max_\lambda, T\right)$:

$plot\left(\lambda_{Wien}(T)\cdot 10^6, T = 0..10^3, title = \text{"Wien's law\nMaxium of emission of a black body"}, titlefont = [HELVETICA, BOLD, 12], labels = [\text{"Temperature in K"}, \text{"Wavelength in μm"}], labeldirections = [HORIZONTAL, VERTICAL]\right)$



$eq\lambda := \lambda_{\max} = apply\left(\lambda_{Wien}, T\right)$:

$const\lambda := solve(eq\lambda, T)$:

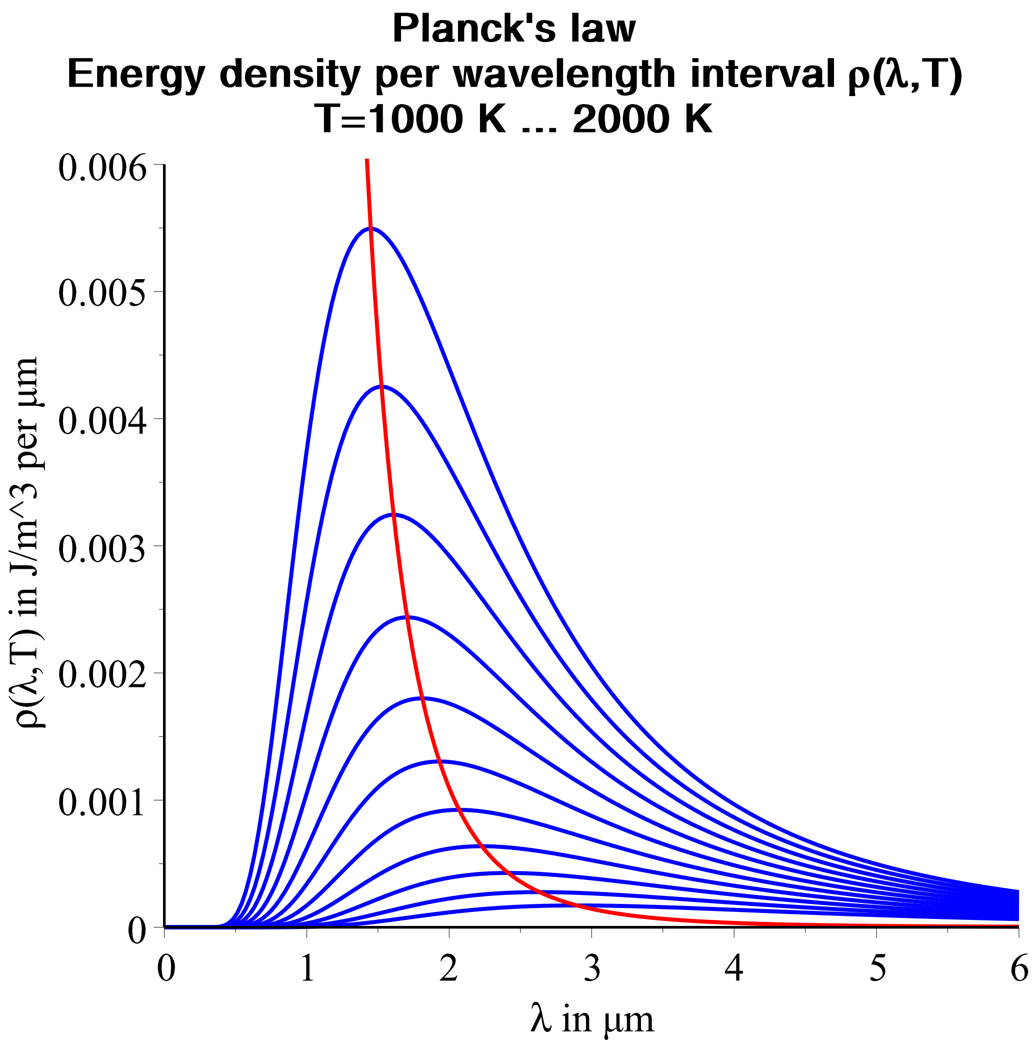
$$\rho_{\lambda_{max}} := unapply\left(\rho_{\lambda}\left(\lambda, subs\left(\lambda_{max} = \lambda, const\lambda\right)\right), \lambda\right) :$$

$$plot_{\rho_{\lambda_{max}}} := plot\left(\rho_{\lambda_{max}}\left(\lambda \cdot 10^{-6}\right) \cdot 10^{-6}, \lambda = 0 ..6, y = 0 ..6 \cdot 10^{-3}, color = RED\right) :$$

$$seq_{\rho_{\lambda}} := \left[seq\left(subs\left(T = 1000 + 100 \cdot n, \rho_{\lambda}\left(\lambda \cdot 10^{-6}, T\right) \cdot 10^{-6}\right), n = 0 ..10\right)\right] :$$

$plot_{\rho_{\lambda}} := plot\Big(seq_{\rho_{\lambda}}, \lambda = 0 ..6, color = BLUE, title$
= "Planck's law\nEnergy density per wavelength interval ρ(λ,T)\nT=1000 K ... 2000 K", *titlefont*
= [*HELVETICA*, *BOLD*, 12], *labels* = ["λ in μm", "ρ(λ,T) in J/m^3 per μm"], *labeldirections*
= [*HORIZONTAL*, *VERTICAL*]$\Big)$:

$$display\left(\left[plot_{\rho_{\lambda}}, plot_{\rho_{\lambda_{max}}}\right]\right)$$



Stefan Boltzmann's law

The total energy density

ρ_{tot} in a cavity totally filled with black radiation of temperature T integrated over all frequencies corresponds to the area beneath the Planck-curve

$\rho_{tot} = \int_0^\infty \rho(\nu, T)\, d\nu$ (ρ_{tot} in $\frac{J}{m^3}$).

For to calculate the integral in an easy way you set $x = \frac{h \cdot \nu}{k \cdot T}$ and therefor is $\nu = \frac{k \cdot T}{h} \cdot x$ and $d\nu = \frac{k \cdot T}{h} \cdot dx$.

$\rho_x := unapply\left(subs\left(\nu = \frac{k \cdot T}{h} \cdot x, \rho_\nu(\nu, T)\right), (x, T)\right) :$

$\rho_{tot} := unapply\left(\int_0^\infty \rho_x(x, T) \cdot \frac{k \cdot T}{h}\, dx, T\right)$

$$T \rightarrow 7.565995849\ 10^{-16}\ T^4 \tag{5}$$

$a := \rho_{tot}(1)$

$$7.565995849\ 10^{-16} \tag{6}$$

Therefor is $\rho_{tot} = a \cdot T^4$ (a in $\frac{J}{m^3 \cdot K^4}$).

The power of the emitted radiation of a black body at temperature T through a surface A is $P = \sigma \cdot A \cdot T^4$ with $\sigma = \frac{c \cdot a}{4}$ (Lambert's law).

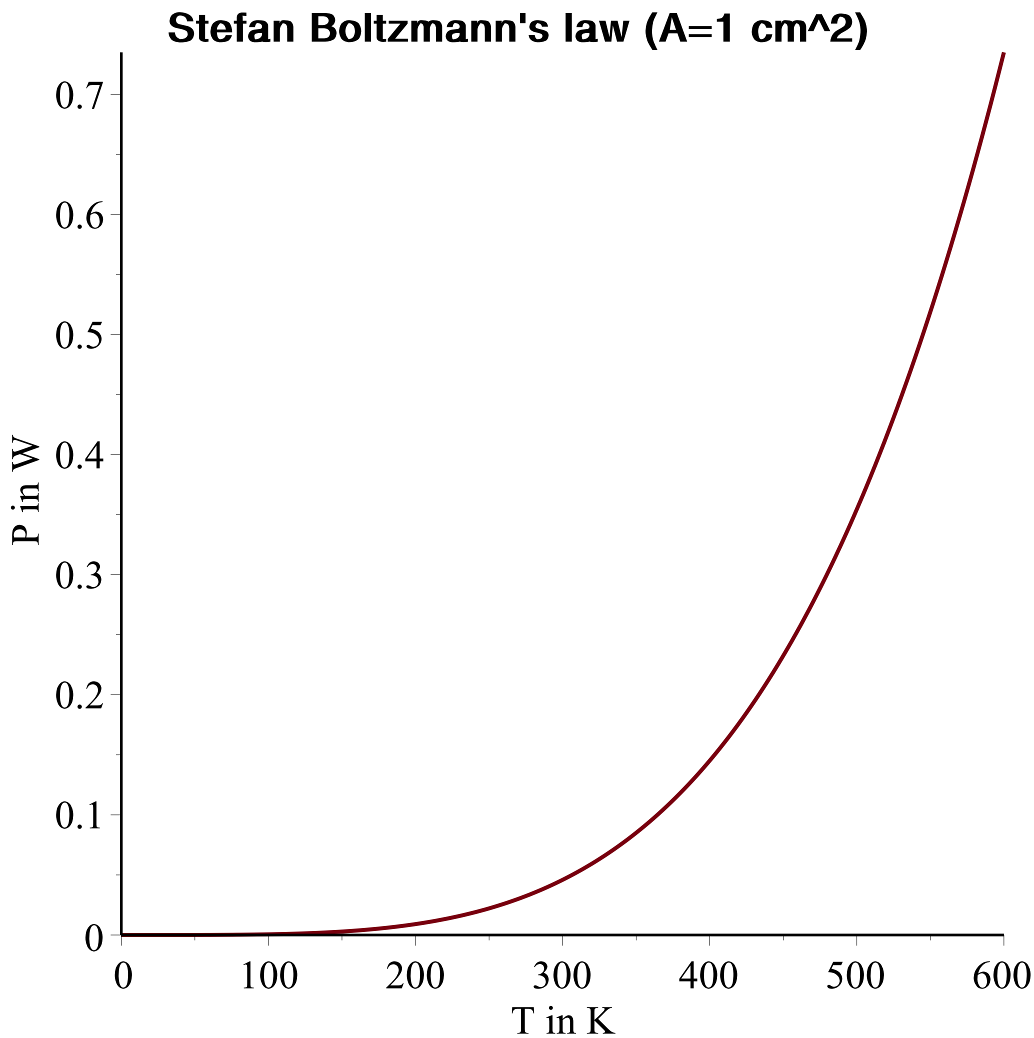
It is possible to calculate the temperature of a black body by messure the radiation intensity per square unit.

$\sigma := \frac{1}{4} \cdot c \cdot a$

$$5.670571138\ 10^{-8} \tag{7}$$

$P := unapply(\sigma \cdot A \cdot T^4, (A, T)) :$

$plot(P(10^{-4}, T), T = 0..600, title = \text{"Stefan Boltzmann's law (A=1 cm^2)"}, titlefont = [HELVETICA, BOLD, 12], labels = [\text{"T in K"}, \text{"P in W"}], labeldirections = [HORIZONTAL, VERTICAL])$



Calculating the Solar constant

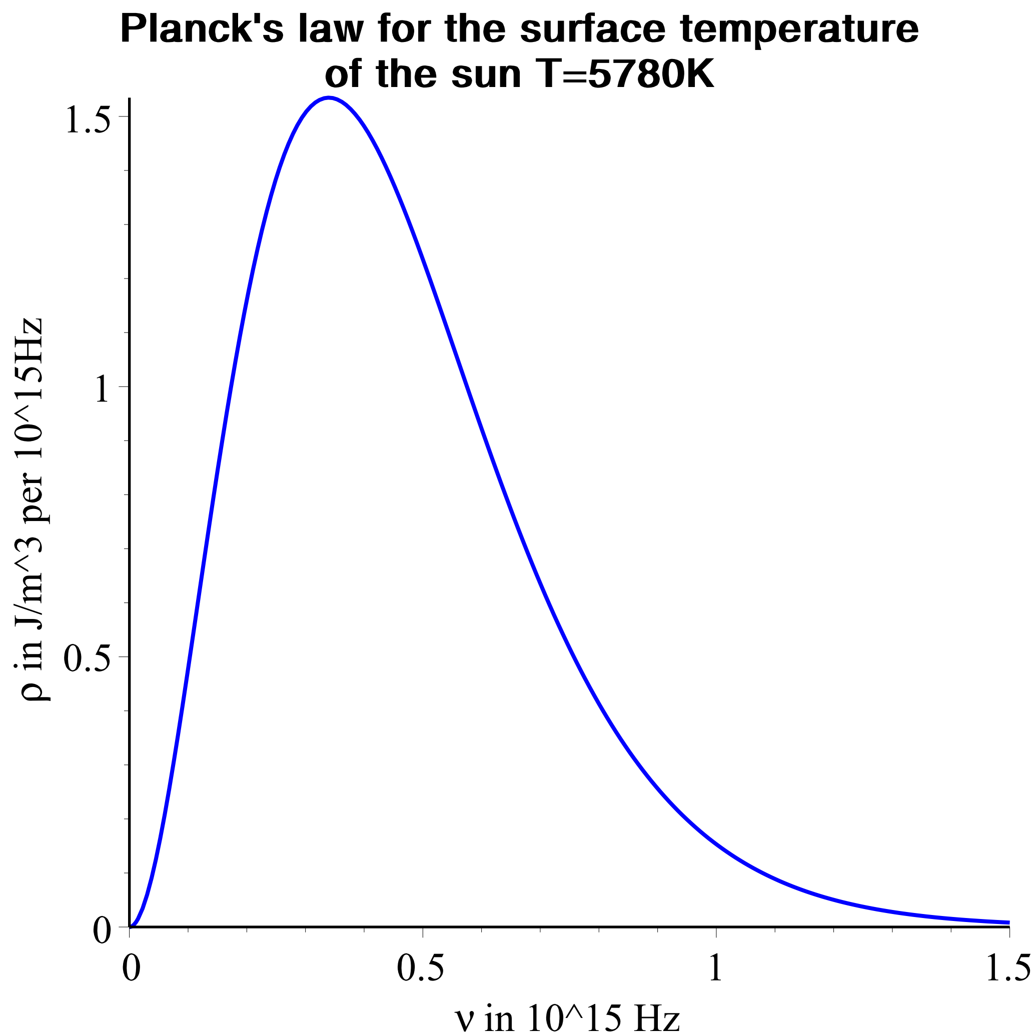
$T_{sun} := 5780$: *# Surface temperature of the sun in Kelvin*

$r_{sun} := 695 \cdot 10^6$: *# Radius of the sun in meter*

$r_{earth} := 6.378 \cdot 10^6$: *# Radius of the earth in meter*

$r_{orbit} := 149.6 \cdot 10^9$: *# Radius of the earth's orbit in meter*

$plot\left(\rho_\nu\left(\nu \cdot 10^{15}, T_{sun}\right) \cdot 10^{15}, \nu = 0 ..1.5, color = BLUE, title\right.$
$= cat\left(\text{"Planck's law for the surface temperature\textbackslash nof the sun T="}, T_{sun}, \text{"K"}\right), titlefont$
$= [HELVETICA, BOLD, 12], labels = [\text{"}\nu\text{ in 10^15 Hz"}, \text{"}\rho\text{ in J/m^3 per 10^15Hz"}], labeldirections$
$\left.= [HORIZONTAL, VERTICAL]\right)$



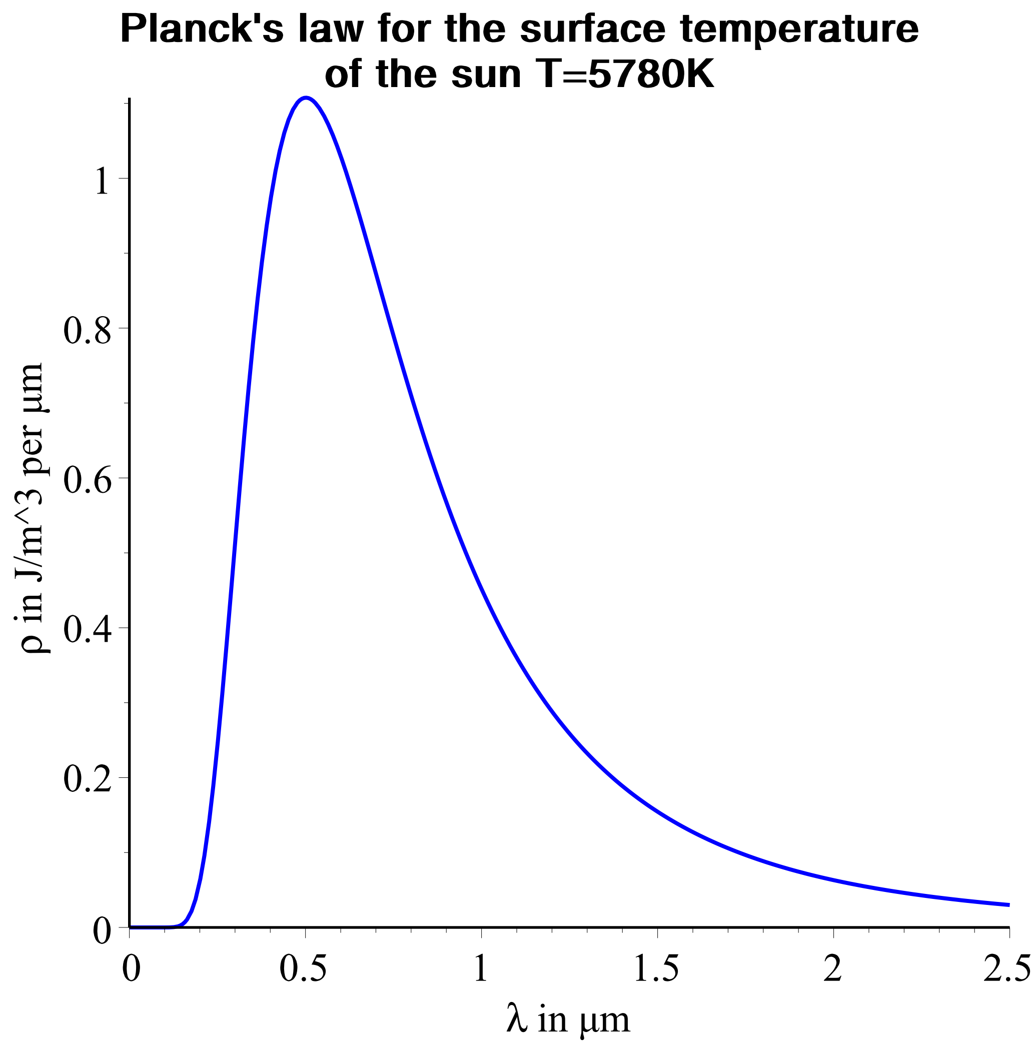
$\nu_{\max_{sun}} := \nu_{Wien}(T_{sun})$

$$3.398057525\ 10^{14} \tag{8}$$

$\lambda_{\max_{sun}} := \dfrac{c}{\nu_{\max_{sun}}}$

$$8.822465505\ 10^{-7} \tag{9}$$

$plot\left(\rho_{\lambda}\left(\lambda \cdot 10^{-6}, T_{sun}\right) \cdot 10^{-6}, \lambda = 0 \,..2.5, color = BLUE, title\right.$
$= cat\left(\text{"Planck's law for the surface temperature\nof the sun T=", } T_{sun}, \text{"K"}\right), titlefont$
$= [HELVETICA, BOLD, 12], labels = [\text{"λ in μm", "ρ in J/m^3 per μm"}], labeldirections$
$\left.= [HORIZONTAL, VERTICAL]\right)$



The power of radiation of the sun through its surface in W:

$$P_{sun} := P\left(4 \cdot \pi \cdot r_{sun}^{2}, T_{sun}\right)$$

$$3.841648539\ 10^{26} \quad \textbf{(10)}$$

The Solar constant is the power of radiation of the sun on 1 m^2 of the earth's surface in $\frac{W}{m^2}$:

$$S := \frac{P_{sun}}{4 \cdot \pi \cdot r_{orbit}^{2}}$$

$$1365.980749 \quad \textbf{(11)}$$

Power of radiation of the sun on the total earth's surface perpendicular to the direction of the sun in W:

$$P_{earth} := S \cdot \pi \cdot r_{earth}^{2}$$

$$1.745675358\ 10^{17} \quad \textbf{(12)}$$