Astronomy from 4 Perspectives: the Dark Universe

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Exercise: Planck-spectrum and the CMB

1. ***Properties of the Planck-spectrum***
   Let's derive the fundamental properties of the Planck-spectrum,

$$S(\nu) = S_0 \frac{\nu^3}{\exp(h\nu/(k_B T)) - 1} \quad \rightarrow \quad S(\nu) = S_0 \nu^3 \exp(-h\nu/(k_B T)), \tag{I}$$

   by using Wien's approximation (the second expression), which makes the integrals easier. The constant S_0 depends only on numbers, natural and mathematical constants.

   (a) Please compute the total intensity $\int_0^\infty \mathrm{d}\nu\, S(\nu)$ and show that it is $\propto T^4$.

   (b) Show that the position ν_m of the maximum scales $\nu_m \propto T$.

   (c) Please derive the scaling of the mean

$$\langle \nu \rangle = \frac{\int_0^\infty \mathrm{d}\nu\, \nu S(\nu)}{\int_0^\infty \mathrm{d}\nu\, S(\nu)} \tag{II}$$

   and show that it is proportional to T.

   (d) Is there a fixed ratio between $\langle \nu \rangle$ and ν_m?

   (e) In which limit is Wien's approximation applicable?

   (f) Do the scaling behaviours with T derived above depend on the details of the distribution?

2. ***Wien's distribution function***
   Let's stick for a second with Wien's distribution function in n dimensions,

$$S(\nu) = S_0 \nu^n \exp(-h\nu/(k_B T)), \tag{III}$$

   and derive a few general properties, which will hold for the Planck-distribution as well (although the computations are more complicated).

   (a) Please begin by showing that

$$\int_0^\infty \mathrm{d}x\, x^n \exp(-x) = n! \tag{IV}$$

   using n-fold integration by parts.

   (b) Alternatively, please show the recursion relation of the Γ-function,

$$\Gamma(n) = (n-1)\Gamma(n-1), \quad \text{together with} \quad \Gamma(0) = 1. \tag{V}$$

   The Γ-function is defined by

$$\Gamma(n) = \int_0^\infty \mathrm{d}x\, x^{n-1} \exp(-x), \tag{VI}$$

   and is related to the factorial by $\Gamma(n) = (n-1)!$

(c) What scaling of the moments

$$\langle \nu^m \rangle = \frac{\int_0^\infty \mathrm{d}\nu\, \nu^m S(\nu)}{\int_0^\infty \mathrm{d}\nu\, S(\nu)} \tag{VII}$$

with temperature T do you expect?

(d) Please show that the skewness parameter $s = \langle \nu^3 \rangle / \langle \nu^2 \rangle^{3/2}$ and the kurtosis parameter $k = \langle \nu^4 \rangle / \langle \nu^2 \rangle^2$ are independent from the temperature T. What would be the physical interpretation of s and k?

(e) Would an equivalent result be true for the parameter $\langle \nu^{2n} \rangle / \langle \nu^2 \rangle^n$?

3. *Planck-spectra at cosmological distances*
Imagine you observe an object emitting a Planck-spectrum at a cosmological distance, such that all photons arrive with a redshifted frequency $\nu \to a\nu = \nu/(1+z)$ with scale factor a (remember $a < 1$) and redshift z. A couple of students discusses the fact that the temperature scales with $T \propto 1/a$ and that the photons are redshifted: What's your opinion on the different arguments?

(a) Johannes from Heidelberg says: The temperature T of a photon gas is linked to the thermal energy E by $E = k_B T$. Then, the relativistic dispersion relation of the photons assumes $E = cp$ with the momentum p. The momentum p is given by the de Broglie-relationship as $p = h/\lambda$. If now the photon wavelength is changed $\lambda \to a\lambda$, the temperature needs to scale $T \propto 1/a$.

(b) Antonia from Padova says: What about a purely thermodynamical argument? A gas of photons has an adiabatic index of $\kappa = 4/3$, and the Hubble expansion is an adiabatic change of state, because there is no thermal energy created or dissipated. Then, the adiabatic invariant says that $TV^{\kappa-1}$ is conserved, which gives me $T \propto 1/a$ with $V \propto a^3$. And I understand why entropy is conserved but not energy.

(c) Marlene from Jena says: Due to the Hubble-expansion, every point is in recession motion with respect to every other point. If a photon gets scattered into your direction, the scattering particle will necessarily move away from you, leading to a lower perceived energy and a larger wavelength. It's important to view it like that because a photon gas can not change its state without interaction due to the linearity of electrodynamics, and this argument shows that it's a kinematic effect: It's a similarity transform of the Planck-spectrum.

(d) Lorenzo from Florence says: It's important for the Planck-spectrum that the mean particle separation and the thermal wavelength are identical. You can only arrange for that if $T \propto 1/a$ if the particle separation increases $\propto a$. I'm only assuming that the number of photons is conserved, but not their energy, and that everything stays in equilibrium.

4. *CMB as a source of energy*
A Carnot-engine converts thermal energy taken from two reservoirs at different temperatures into mechanical energy at the efficiency $\eta = 1 - T_2/T_1$.

(a) Estimate if one can use the temperature anisotropies in the CMB of $\Delta T/T \simeq 10^{-5}$ to generate mechanical energy. How much power could you realistically generate? Construct a machine that converts radiation power into mechanical energy.

(b) Could you use the time evolution of the CMB-temperature for this purpose? You know already that $T \propto 1/a$, so please construct a machine that produces energy from the CMB.

(c) Why does a solar cell transform the radiation from the Sun into electrical energy? One might argue that the Planck-spectrum is that of thermal equilibrium in which case the mechanical work is zero: Due to the first law, mechanical work can not be performed in thermal equilibrium. (please be careful: trick question)