Astronomy from 4 perspectives: the Dark Universe

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exercise: Supernova-cosmology and dark energy
Solutions

1. *light-propagation in FLRW-spacetimes*
Photons travel along null geodesics, $\mathrm{d}s^2 = 0$, in any spacetime.

(a) Let us do the following substitution $dt \to a(t)d\tau$ then the line element can be written $ds^2 = a(t)^2[c^2 d\tau^2 - d\chi^2]$ and the equation ofr the null-geodesic will be $d\chi = \pm c d\tau$.

(b) The cosmic time is the time measured by a cosmic observer synchronized for $t = 0$

$$t = \int_0^t dt' = \int_0^a \frac{da'}{\dot{a}'} \tag{I}$$

The conformal time is tied to the time interval over which an observer at $t = t_0$ sees to happen an event in the past at time t. Now at $t = t_0$ this will coincide with the cosmic time, ence it will be afected by cosmic time dilation.

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} = \frac{1}{a(t)} \int_0^t \frac{a(t)}{a(t')} dt' > \frac{t}{a(t)} \tag{II}$$

(c) Now for the given metric:

$$H = \frac{\dot{a}}{a} = H_o a^{-3(1+w)/2} \Rightarrow \dot{a} = H_o a^{1-3(1+w)/2} \tag{III}$$

$$a(t) = \left(\frac{4}{9}\right)^{3(w+1)} (t + wt)^{2/3(w+1)} \tag{IV}$$

$$\tau_H = \int_0^t \frac{dt'}{a(t')} = \int_0^1 \frac{da'}{\dot{a}a} = \frac{1}{H_o} \int_0^1 a^{3(w+1)/2-2} da = \frac{1}{H_o} \frac{1}{3(w+1)/2 - 1} \tag{V}$$

(d) Isotropy of the universe ensures us that it is not.

2. *light-propagation in perturbed metrics*

$$ds^2 = \left(1 + 2\frac{\Phi}{c^2}\right) c^2 dt^2 - \left(1 - 2\frac{\Phi}{c^2}\right) dx^2 \tag{VI}$$

With $ds^2 = 0$:

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 = \left(1 - \frac{2\Phi}{c^2}\right) dx^2 \tag{VII}$$

$$\frac{dx}{dt} = \pm c \sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \tag{VIII}$$

With $\frac{1}{1-\epsilon} \approx 1 + \epsilon$ for small ϵ:

$$\frac{dx}{dt} \approx \pm c\left(1 + \frac{2\Phi}{c^2}\right) \tag{IX}$$

For a non-zero Φ this is not equal to c!
We assign an effective index of refraction by:

$$n(\Phi) = \frac{dx/dt}{c} \approx \left(1 + \frac{2\Phi}{c^2}\right) \tag{X}$$

3. *classical potentials including a cosmological constant*

The field equation of classical gravity including a cosmological dark energy density λ is given by

$$\Delta\Phi = 4\pi G(\rho + \lambda) \tag{XI}$$

(a) field calculation

Now it is possible to simply integrate the field equation starting with:

$$\begin{aligned}
\Delta\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) && \text{(XII)}\\
&= 4\pi G(\rho(r) + \lambda) && \text{(XIII)}\\
r^2\frac{\partial\Phi}{\partial r} &= \int_0^r \mathrm{d}r' \left(4\pi G[(r')^2\rho(r') + (r')^2\lambda]\right) && \text{(XIV)}\\
&= GM + G\frac{\lambda}{3}r^3 && \text{(XV)}\\
\frac{\partial\Phi}{\partial r} &= \frac{GM}{r^{n-1}} + G\frac{\lambda r}{n} && \text{(XVI)}\\
\Phi &= -\frac{GM}{r} + G\frac{\lambda r^2}{6} && \text{(XVII)}
\end{aligned}$$

(b) power-law solutions
Following the calculation one may see that each source term corresponds to an individual power-law:

$$\begin{aligned}
C(n)G\rho(r) \quad &\Rightarrow \quad -\frac{GM}{r}\\
\lambda \quad &\Rightarrow \quad G\frac{\lambda r^2}{6}
\end{aligned}$$

(c) equilibrium
To find an equilibrium distance one must set $\Phi\left(r_{\mathrm{eq}}\right) = 0$

$$\begin{aligned}
\frac{GM}{r_{\mathrm{eq}}} &= G\frac{\lambda r_{\mathrm{eq}}^2}{6} && \text{(XVIII)}\\
\frac{\lambda r_{\mathrm{eq}}^3}{6} &= M && \text{(XIX)}
\end{aligned}$$

from which follows immediatly:

$$r_{\mathrm{eq}} = \sqrt[3]{6\frac{M}{\lambda}} \tag{XX}$$

(c) if one inputs the number one gets $r_{\mathrm{eq}} = 1.5$ Mpc one hundred times larger than the size of a galaxy.

4. ***physics close to the horizon***
Why is it necessary to observe supernovae at the Hubble distance c/H_0 to see the dimming in accelerated cosmologies? Please start at considering the curvature scale of the Universe: A convenient quantisation of curvature might be the Ricci-scalar $R = 6H^2(1-q)$ for flat FLRW-models.

   (a) The Dimension of the Ricci-scalar is $1/s^2$ thus we can define a time and a distance scale by:
   $$\tau = 1/\sqrt{R} \quad \text{and} \quad d = c/\sqrt{R} \approx c/H_0$$
   which gives the curvature scale of the Universe.

   (b) To observe supernovae dimming caused by accelerated cosmic expansion the supernova distance had to be about (or larger than) the curvature scale, because at distances << d the different cosmological distance measures converge.
   For illustration see: *https://en.wikipedia.org/wiki/Distance_measures_(cosmology)*

5. ***measure cosmic acceleration***
The luminosity distance $d_{\mathrm{lum}}(z)$ in a spatially flat FLRW-universe is given by
$$d_{\mathrm{lum}}(z) = (1+z)\int_0^z \mathrm{d}z' \, \frac{1}{H(z')} \tag{XXI}$$
with the Hubble function $H(z)$. Let's assume that the Universe is filled with a cosmological fluid up to the critical density with a fluid with equation of state w, such that the Hubble function is
$$H(z) = H_0(1+z)^{\frac{3(1+w)}{2}}. \tag{XXII}$$

   (a) By definition:
   $$H = \frac{\dot{a}}{a} \text{ and } q = -\frac{\ddot{a}a}{\dot{a}^2}$$
   It follows
   $$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}a}{a^2} - H^2$$
   So we get
   $$\frac{\dot{H}}{H^2} = \frac{\ddot{a}a}{\dot{a}^2} - 1 = -q - 1$$
   $$q = -(\frac{\dot{H}}{H^2} + 1)$$
   We also have
   $$H = H_0 \cdot (1+z)^{\frac{3(1+w)}{2}} = H_0 \cdot a^{\frac{-3(1+w)}{2}}$$
   and
   $$\begin{aligned}\dot{H} &= H_0\left(\frac{-3(1+w)}{2}\right) \cdot a^{\frac{-3(1+w)}{2}} \cdot \dot{a} \\ &= H_0 \cdot a^{\frac{-3(1+w)}{2}} \cdot \frac{\dot{a}}{a} \cdot \left(\frac{-3(1+w)}{2}\right) \\ &= H^2 \cdot \left(\frac{-3(1+w)}{2}\right)\end{aligned}$$

so

$$q = -\left(\frac{-3(1+w)}{2} + 1\right) = \frac{1}{2}(3w+1)$$

and obviously

$$\begin{aligned} q < 0 \text{ for } w < -\frac{1}{3} \\ q > 0 \text{ for } w > -\frac{1}{3} \end{aligned}$$

(b) First, we consider the case $w = -\frac{1}{3}$ (non-accelerating universe):

$$H = H_0(1+z)^{\frac{3(1+w)}{2}} = H_0(1+z)$$

$$\begin{aligned} d_{lum,1} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\ &= (1+z)\int_0^z \frac{1}{H_0(1+z')}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0} ln(1+z) \end{aligned}$$

Now, we consider the case $w < -\frac{1}{3}$ (accelerating universe):

$$\begin{aligned} d_{lum,2} &= (1+z)\int_0^z \frac{1}{H(z')}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0}\int_0^z (1+z')^{\frac{-3(1+w)}{2}}\,\mathrm{d}z' \\ &= \frac{1+z}{H_0}\left[(1+z')^{\frac{-3(1+w)+2}{2}} \cdot \frac{2}{-3(1+w)+2}\right]_0^z \\ &= \frac{1+z}{H_0}\left(\frac{2}{-3(1+w)+2}\right)\left[(1+z')^{\frac{-3(1+w)+2}{2}} - 1\right] \end{aligned}$$

It follows: $d_{lum_2}(z) > d_{lum_1}(z)$, because the exponent $\frac{-3(1+w)+2}{2}$ is positive ($w < -\frac{1}{3}$), so $d_{lum_2}(z)$ is growing faster, than the logarithmic function $d_{lum_1}(z)$.

(c) Yes, as long as the universe is flat. Just try to plot the two functions

(d) The formula still apply,. But in a contracting universe one would see light that is blue-shifted and not red-shifted. Su $z < 0$. But Obviiously $z > -1$ otherwhise one would get negative wavelengths (frequancies) that make no sense. As a consequence, the origin of time in a contracting universe cosrresponds to $z = -1$ for a universe contracting from infinity. Hence the limit of integration must be corrected accordingly.