Astronomy from 4 Perspectives: the Dark Universe

---

prepared by: Niccolò Bucciantini

Quantum Mechanics, Relativity and Supernovae

This tutorial is aimed at advanced students, to whom the teacher has already introduced the basic concepts of quantum mechanics and relativity. In particular the only notion of QM required is the **uncertainty principle of position and momentum** and the only notion from SR the **relativistic relation between energy and momentum**. Some familiarity with thermodynamics and hydrostatic is also required.

We are going to show how with simple arguments using basic quantum mechanics and special relativity one can find that there is a limit to the mass of a quantum star.

The starting points are:

1. From quantum mechanics the uncertainty principle of momentum and position $\Delta x \Delta p = \hbar$. The teacher should have introduced it before and explained its meaning and implications.

2. From special relativity the relativistic relation between energy and momentum of a relativistic particle $e = cp$. One can use photons to show that they carry energy (obvious - think of solar panels that produce electricity) and also momentum (the solar mill is a good example).

3. From thermodynamics the relation between pressure of a gas and the energy density. This can be introduced classically with a discussion of the relation $P = nkT = U$, recalling that kT is the thermal energy of a particle of a gas.

4. From hydrostatic the equilibrium relation $F = -\nabla P$, stating that any external force (force density to be more precise) must be balanced by a pressure gradient. As an example one can discuss how pressure changes going under water, by simply showing that at any depth the pressure must be equal to the force exerted by the overlying column of water. And use this argument to derive the hydrostatic relation.

We begin by showing how a relativistic gas behaves.

First consider a volume V containing N particles. Then the average volume per particle is $V/N = 1/n$ where we have introduced the particle density n. Then one can define the average distance between particles is $(V/N)^{1/3} = 1/n^{1/3}$. Using the uncertainty principle, and taking as a typical uncertainty over the distance the average distance between particles, we get a typical momentum $p = \hbar n^{1/3}$. Now use the relativistic energy momentum relation to get a typical energy $e = c\hbar n^{1/3}$. Finally the energy density of this system of particles will be $U = ne = c\hbar n^{4/3}$. Recall from classical thermodynamics that the pressure is of the order of the energy density then the pressure of our relativistic quantum system system is $P = c\hbar n^{4/3}$. And setting equal to m the typical mass of a particle $P = c\hbar(\rho/m)^{4/3}$, where we have introduced the mas density ρ.

We now turn to hydrostatic equilibrium. For a star the equation at any depth where the density is ρ will look like:

$$G\frac{M\rho}{r^2} = -\nabla P = -\frac{dP}{dr} \tag{I}$$

given that stars are spherically symmetric. At this point we are going to simplify the treatments looking only at the scaling of the equation. This is done replacing some of the quantities with their simplest approximations: the local radius $r \to R$ the stellar radius; the density $\rho \to MR^{-3}$ where M is the mass of the star; the pressure derivative $dP/dr \to P/R$. Then the differential equation turns into an algebraic one that the students should be more familiar with.

$$G\frac{M^2R^{-3}}{R^2} = -\frac{P}{R} \tag{II}$$

$$G\frac{M^2}{R^5} = \frac{\hbar c}{R}\left(\frac{M}{mR^3}\right)^{4/3} \tag{III}$$

The student should see immediately that the radius simplifies out of the equation. This means that the equilibrium is independent of the radius. Or stated in other words that if the equilibrium (the above equation) is not satisfied, the star will start to expand (explode) or collapse (implode), and you cannot avoid this by adjusting the radius; it is a catastrophic process. The student should also see that the equilibrium equation gives the following solution of the mass:

$$M_{\mathrm{eq}} = \frac{1}{m^2}\left(\frac{c\hbar}{G}\right)^{3/2} \tag{IV}$$

Now let us put the numbers: the speed of light $c = 3 \times 10^9$ m s^{-1}; m is the mass of a proton 1.6×10^{-27} kg; $G = 6.6 \times 10^{-11}$ m^3 s^{-2} kg^{-1}; $\hbar = 10^{-34}$ J s. Substituting these values one gets:

$$M_{\mathrm{eq}} = 3.8 \times 10^{30}\mathrm{kg} \tag{V}$$

This is about 2 times that mass of the Sun $M_{\mathrm{Sun}} = 2 \times 10^{30}$ kg (doing the correct model the astrophysicists get 1.4 times the mass of the Sun - be happy with such simple equation you got only a 25% difference). This equilibrium mass is known as **Chandrasekhar mass**. If the mass is smaller, the star will expand until the physical processes of the gas change so much that somehow the star reaches a new equilibrium. If the stellar mass gets bigger than the gravity wins and the star will start to collapse, driving a catastrophic evolution, that can either end with a black hole or with a stellar detonation, known as Supernova.