Preprint

Uncertainties of Ramp Identification in 100 Equidistant Noisy Data Points

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1 Introduction

1.1 Motivation

The present paper was motivated by a scenario as illustrated in Fig. 1-1. Attached to an industrial robot in constant motion, a linear elastic body collides with a fixed body. Contact forces build up and grow linearly with increasing depth of intrusion while the elastic body is deflecting. An ideal Force-Torque-Sensor would measure a flat signal prior to the collision and a ramping signal afterwards, but noise from electronics, drives, shocks, vibrations obscure the ramp. Best-fit reconstruction of contact point and mechanical stiffness out of a noisy ramp signal comes with some uncertainty. For safer, faster, and more accurate control of collisions in robotics, a ramp parameter estimation method is needed which quantifies this uncertainty at a guaranteed confidence level.

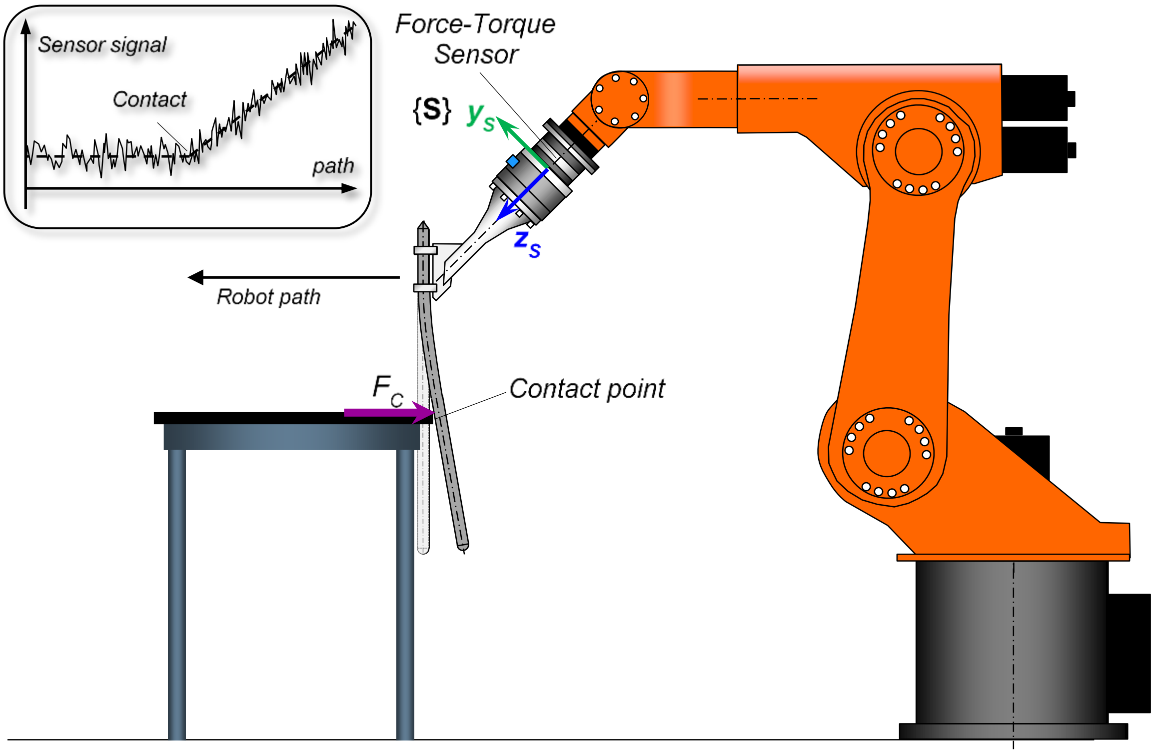


Fig. 1-1: Noisy Force Ramp sensed from an elastic contact.

This paper aims at developing a method, referred to as “Ramp Identification” or short “RI”, that, cyclically and in real-time, observes the signal data coming from a noisy force/torque sensor during linear-elastic collision processes. If a ramp is present, the ideal RI-method would detect it and identify the ramp, i.e.: determine the contact time, ramp gradient and height and estimate uncertainties in all these ramp parameters. Due to randomness in the signal source, the accuracy of parameter estimations will increase with more signal data becoming available, but 100% will never be reached.

The RI-method presented here delivers ramp parameter estimations as-soon-as-possible for quick reaction, and as-late-as-necessary to guarantee the accuracy required by the process designer. First, RI finds out whether-or-not a collision has occurred. In case of a collision, RI estimates with growing accuracy its point in time, ramp height and contact compliance. Later on, RI quantifies the uncertainties in ramp parameter estimations. No prior knowledge about any of the ramp parameters is necessary to run RI on the signal data stream. RI is numerically efficient and stable for use in real-time robotics, thus allows for faster, safer, and more accurate collision control.

1.2 Basic Terminology on Ramp Identification

Within this paper, terms having specific meanings are written with an initial capital letter. The terms “Controller, Process, Signal, Noise” and “Cycle” are employed as represented in Fig. 1-2. The term “Corner” will be used throughout this paper for the non-steadily differentiable bending point in a ramp function.

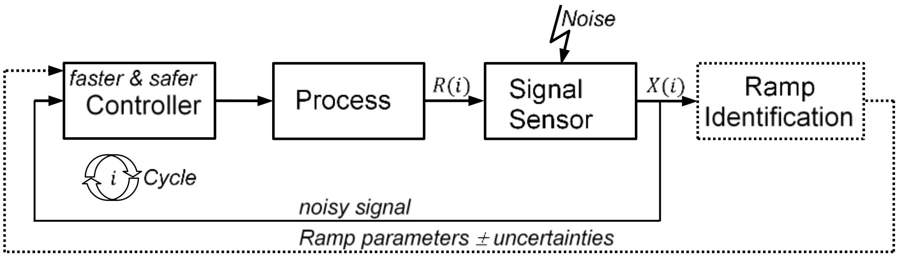


Fig. 1-2: Ramp Identification improves controller design for critical process

At the Corner point, the graph of a ramp function shows a sharp bend between two straight lines: the Floor line on the left and the Ramp line on the right side, see Fig. 1-3. The Ramp begins at the Corner and after a certain Length reaches a certain Height. Gaussian Signal Noise overlaying the Ramp obscures the Corner and misrepresents the actual Height. The physical meaning of “Progress” is given by the process itself, and could be the time, a distance, Cycle number or Increment i.

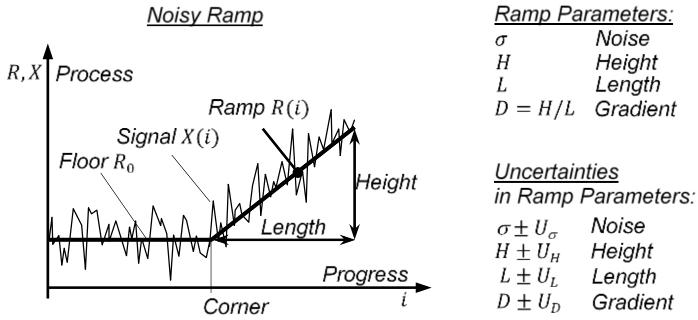


Fig. 1-3: Terms and parameters for Ramp functions

A set of four parameters σ, H, L, D characterizes the “Ground Truth” of the Process Ramp. Throughout this paper, noise will be considered as normal distributed, Gaussian, resp. white noise. Noise constitutes the source of uncertainty in determining the Ramp. Noise is quantified by the standard deviation; thus “Noise” and “standard deviation” will be used as synonyms to denote the randomness in the Signal. The term “Bias” with a capital ‘B’ stands for the mean error over a big number of samples.

The procedure for Ramp parameter estimation was inspired by the Maximum Likelihood method, therefore it is referred to as “ML”. Eventually, for better performance, Ramp parameters H, L, D result from solving a Least Squares problem while the Noise parameter σ is estimated separately. Hence “ML” has lost its original meaning but still refers to the initial Ramp parameter estimation step.

1.3 Objective

The first task of Ramp Identification, (short: “RI”), consist of detecting, with an acceptable confidence, whether or not a Ramp actually exists in a Noisy Signal. Next, it estimates the Ramp parameters Height H, Length L, Gradient D and the Noise level σ in the Noisy Ramp signal. Eventually, RI estimates the levels of uncertainty that the Noise has induced into the estimations of σ, H, L, D, and evaluates the level of confidence associated with these uncertainty estimations.

Final Ramp Identification results are:

σ_{RI}	Standard deviation of signal noise, identified by RI
$H_{RI} = L_{RI} = D_{RI} = 0$	RI responds: no Ramp is present in the Signal data yet
$H_{RI} > 0$	identified Ramp Height
$L_{RI} > 0$	identified Ramp Length
$D_{RI} = \frac{H_{RI}}{L_{RI}} > 0$	identified Ramp Gradient
$U_\sigma = U_H = U_L = U_D = 0$	RI responds: Uncertainties in σ, H, L, D are unknown yet
$U_\sigma > 0$	Uncertainty of σ_{RI} for a desired confidence
U_H, U_L, U_D	Uncertainties of H_{RI}, L_{RI}, D_{RI} and confidence

1.4 RI development outline

The idea behind RI development is, that by adding random numbers to a given Ground Truth Ramp, an infinite number of synthetic Noisy Ramps can be built to study the influence of signal noise. Synthetic Noisy Ramps are built from only two parameters L, D, with Floor level and Noise being normalized to $R_0 = 0$ resp. $\sigma = 1$. The "ML"-solver then estimates Ramp parameters within synthetic signals. Errors in Ramp parameter estimations are compared to Ground Truth knowledge over all synthetic Noisy Ramps and split into "Bias": the mean error $\tilde{\mu}$, and "Noise": the standard deviation $\tilde{\sigma}$, where the tilde accent ˜ stands for "over all synthetic Ramps with same Ground Truth", see Fig. 1-4.

The behavior of $\tilde{\mu}$ and $\tilde{\sigma}$ is analyzed and cast into polynomial and exponential functions $f(\sigma_{ML}, H_{ML}, L_{ML}, D_{ML})$ of the ML-estimated Ramp parameters. Standard deviations of errors $\tilde{\sigma}$ are further cast into an interval of Uncertainty $\pm U$ and an associated Confidence level fd e.g., 99,73%. "fd" denotes the probability that ML-estimated Ramp parameters fall within the interval: $Ground\ Truth - \tilde{\mu} \pm U$. While Ground Truth is available for synthetic Noisy Ramps, it generally not for runtime controlled real-world processes. Thus, RI results, i.e. formulae for $\tilde{\mu}$, $\tilde{\sigma}$ and U, must not depend on Ground Truth knowledge either.

For RI development, a window of 100 Data points incrementally shifts over the synthetic Ramp, whereas during Runtime Ramp analysis, signal data would be pushed incrementally into the RI algorithm as it occurs, see Fig. 1-4. RI is applicable to Runtime data that exhibits white noise over 100 evenly spaced increments, begins with a Floor part and may-or-may-not contain a Ramp part. In addition, the uncertainties of Ramp parameters are estimated according to the level of confidence which is requested by process specifications.

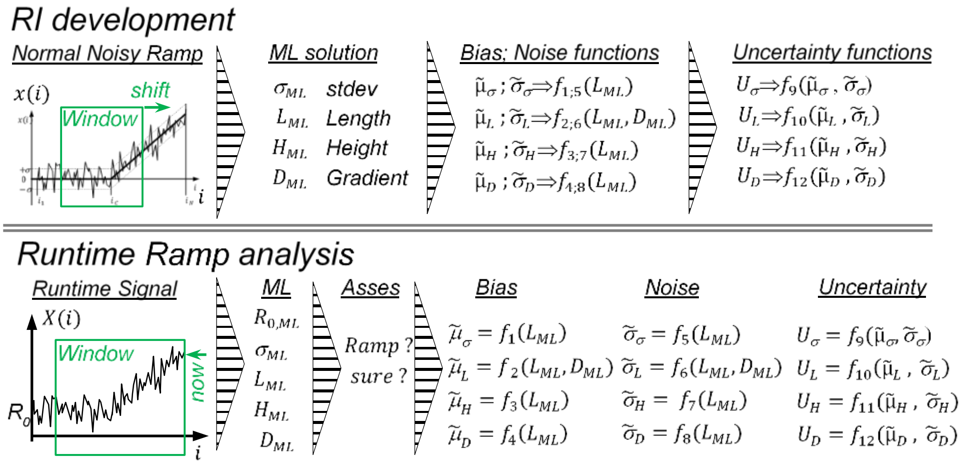


Fig. 1-4: Ramp Identification: development and application

Chapter 2 will coin more terms used in this paper and lays out some theoretical background. Objectives and expectations for Ramp Identification are presented in chapter 3. Chapter 4 develops an experimentation plan to produce data sets for further analysis. Chapter 5 presents a runtime optimized solver for the underlying numerical problem. The results of 4+5 are then analyzed in chapter 6 to find statistically stable patterns that deliver Ramp Parameter magnitudes and uncertainties without using Ground Truth. The main findings of this paper can be found in chapter 6.3 and 6.4, Application instructions for Ramp Identification method found, are given in chapter 7.

2 Models and Methods

2.1 Runtime Ramp Signal model

Ramp Identification works for Noisy Ramp Signals, denoted as $X(i)$, that typically come from a signal source in an experiment, a process or a simulation. The true physical process behind the signal is suspected of showing a Corner at an unknown point in time, followed by a positively growing, linear Ramp with unknown gradient.

The exact Corner position along the $i-$axis is denoted interchangeably as either C or i_C, see Fig. 2-1. The $i-$axis is discretized into equidistant increments, numbered, for convenience by integers. The term “increment”, literally meaning “step up by one”, brings ambiguity if sub-incremental resolution of the $i-$axis matters. In the present paper the graduation marks along the $i-$axis are referred to as “increment points”, the associated integer as “increment number”. The interval between two subsequent increment points is referred to as “increment interval”. Increment intervals also carry a number: the increment point number at their left border. Expression “at increment” means the $i-$point, whereas “in increment” means the $i-$interval. The term “increment” refers to the increment number i alone.

If integer resolution is sufficient, the Corner position C is rounded down to the closest lower increment point, expressed by floor brackets $\lfloor C \rfloor \equiv \lfloor i_C \rfloor$. As a result, the number $\lfloor i_C \rfloor$ also denotes the increment interval which contains the actual Corner position C.

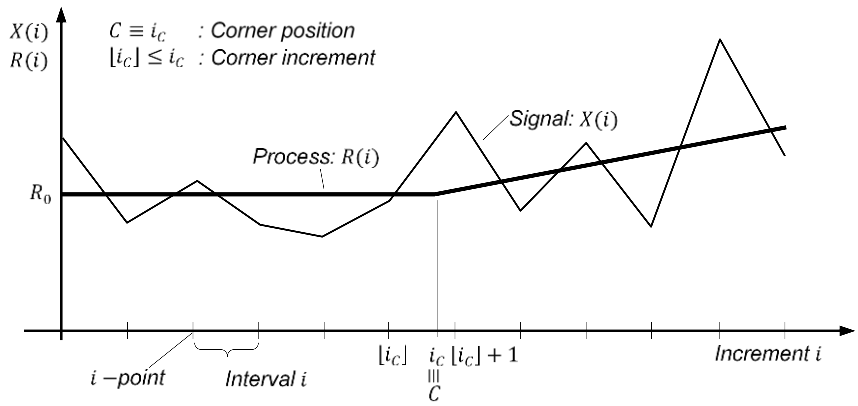


Fig. 2-1: Corner position C vs. Corner increment i_C

Noisy Ramp Signals are represented as the sum of an unknown pure Ramp function $R(i)$ and Gaussian Noise, see Fig. 2-2:

- A Ramp function is defined as an ordered data set of N_S values of a variable $R(i)$ with $i = i_{1S} \dots i_S$, where $R(i)$ describes a Ramp. i is referred to as ‘increment’. The data set contains one value $R(i)$ for each increment i. Increments are equidistant.
- Before reaching the Corner increment i_C, the Ramp function equals R_0. At Corner increment i_C, the Ramp function $R(i)$ begins to increase at constant rate $D = \frac{\partial R(i)}{\partial i}$ Since i is a dimensionless integer, the gradient D has the dimension of R.
- The Ramp function $R(i)$ is:

$$R(i) = \begin{cases} i < i_C: & R_0 \\ i \geq i_C: & R_0 + D \cdot (i - i_C) \end{cases} \tag{2-1}$$

- Signal data exists for interval $[i_{1S} \dots i_S]$ which contains $N_S = i_S - i_{1S} + 1$ data points.
- Gaussian noise with a constant standard deviation σ adds to the entire Ramp function to produce the Noisy Ramp Signal data points $X(i)$.

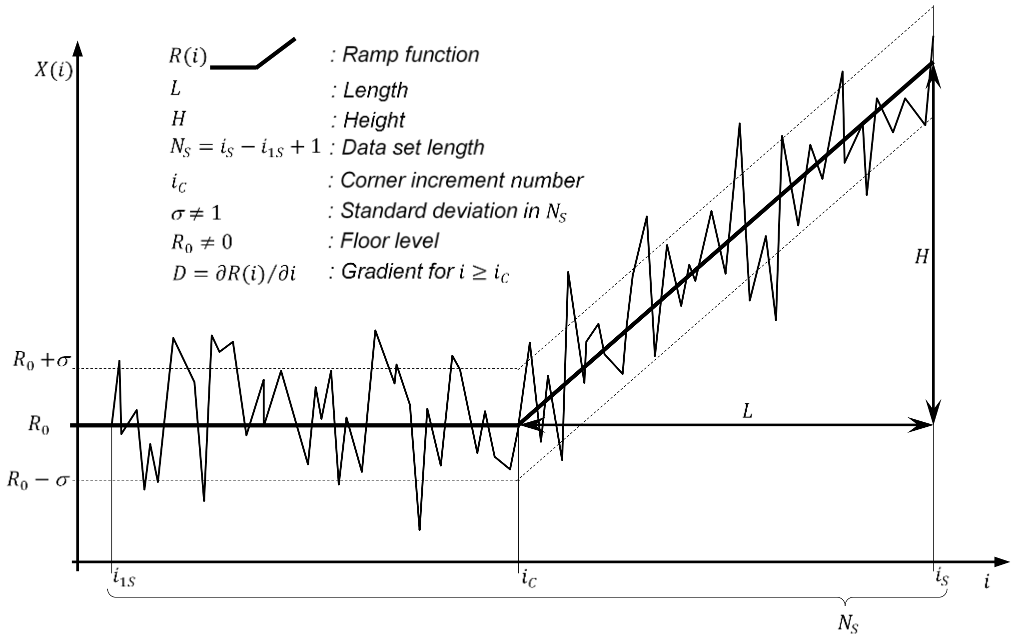


Fig. 2-2: Parameters of a Noisy Runtime Signal and a Ground Truth Ramp to be found.

The above definition relies on 4 free parameters R_0, i_C, D, σ and 2 derived parameters L, H to represent a Noisy Ramp Signal. It is referred to as “RTRamp” or Runtime Ramp, i.e. a model for a data stream received from a real-time Process. Runtime Ramp parameters use capital letters, except for $'i'$, the dimensionless increment number and $'\sigma'$ the standard deviation.

2.2 Normal Noisy Ramp

To facilitate the development of a Ramp Identification method, the number of free Ramp parameters is further reduced by normalizing:

$\sigma = 1$

$R_0 = 0$

The two remaining free parameters i_C, D describe the actual Ramp behind the noise. A Noisy Ramp which complies with the above conditions is referred to as “Normal Noisy Ramp”, short: “NNRamp”, where “Normal” means “normalized”. To develop a Ramp Identification method, a big number of synthetic NNRamps are synthesized by choosing values for i_C, D and adding Gaussian noise. Thus, for any NNRamp, the 4 Ramp parameters used for synthesizing constitute the ‘Ground Truth’ knowledge behind the noisy signal. To mark the difference with Runtime Ramps that have no known ‘Ground Truth’, NNRamp parameters are written in lower case letters. Index ‘GT’ denotes the actual Ground Truth values of these parameters:

Ground Truth Corner increment	$i_{C,GT}$
Ground Truth Ramp Gradient	d_{GT}
Ground Truth Normal parameters	$\sigma_{GT} = 1$, $R_{0,GT} = 0$
Ground Truth Normal Ramp	$r_{GT}(i) \begin{cases} i < i_{C,GT}: & 0 \\ i \geq i_{C,GT}: & d_{GT} \cdot (i - i_{C,GT}) \end{cases}$

Length and Height of a Ramp are derived for the latest observed increment i_W

Observation Increment 'noW'	i_W
Ground Truth Ramp Length	$L_{GT} = i_W - i_{C,GT}$
Ground Truth Ramp Height	$H_{GT} = r_{GT}(i_W)$

Let $randn(\sigma_{GT}, i)$ be a function that delivers a normal distributed random numbers with standard deviation $\sigma_{GT} = 1$. A signal data series $x(i)$ of a NNRamp is synthesized by executing the following statement once for every increment i:

Ground Truth Normal Noise	$n_{GT}(i) = randn(\sigma_{GT} = 1, i)$
Normal Noisy Ramp Signal	$x(i) = r_{GT}(i) + n_{GT}(i)$

Note that $x(i)$ and $r_{GT}(i)$ are dimensionless Ramp data points.

2.3 Correlations NNRamp - RTRamp

Based on a big set of synthetic NNRamps, different methods for Ramp Identification can be tried out and their results compared to the actual Ground Truth.

For comparability between Runtime Ramp data $X(i)$ and synthetic NNRamp data $x(i)$, RTRamps must be made similar, e.g. by alignment to Floor level R_0 and division by the standard deviation σ. In a Runtime case however σ is unknown and $R_0 = 0$ cannot always be guaranteed. Instead, both need to be estimated, referred to as σ_{ML} and $R_{0,ML}$, from the available, finite set of signal data:

Normalized RTRamp Signal $\frac{X(i)-R_{0,ML}}{\sigma_{ML}}$

However, the estimations σ_{ML} and $R_{0,ML}$ come with stochastic errors and bias. By consequence, applying 'Ramp Identification' to an Normalized RTRamp Signal will not show the same error patterns as 'Ramp Identification' applied directly to NNRamp Signals. Instead, NNRamps must be treated in the same way as RTRamps, in particular using σ_{ML} and $R_{0,ML}$ determined on the NNRamp signal $x(i)$. To put it formally, one can state that:

$RI\left(\frac{X(i)-R_{0,ML}}{\sigma_{ML}}\right)$ is similar to $RI\left(\frac{x(i)-R_{0,ML}}{\sigma_{ML}}\right)$ but is not similar to $RI\big(x(i)\big)$

Read $RI()$ as "apply 'Ramp Identification' method" to something. Understand 'Similar' in the sense: "shows same average error patterns". Note that the ratio between the Ramp Height and the Ramp Noise H/σ expresses the height-to-noise ratio of a Noisy Ramp Signal over the observed period.

2.4 Probability Density Function of a Noisy Ramp

The probability for a particular data pair (x, i) to appear in the x-i-plane is expressed as probability density function $P(x, i)$. The probability density function $P(x, i)$ of a Normal Noisy Ramp defined by parameters i_C, σ, d is:

$$P(x,i) = \begin{cases} i < i_C: & \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(x(i)\right)^2}{2\sigma^2}} \\ i \geq i_C: & \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(x(i)+d(i-i_C)\right)^2}{2\sigma^2}} \end{cases} \tag{2-2}$$

A Normal Noisy Ramp's data set can be synthesized from given Ground Truth values of its three parameters. Inversely, it is not possible to derive the exact Ground Truth parameter values from a non-infinite Signal data set. Instead, i_C, σ, d must be estimated using methods such as the Maximum Likelihood or Least Squares.

2.5 Maximum Likelihood method

For a given set of Signal data $x(i)$, Maximum Likelihood method aims at finding values for $i_{C,ML}, \sigma_{ML}, d_{ML}$ that best fit the data set as a Ramp.

The Likelihood function measures the resemblance between a probability density function and a data set. The probability density function must be chosen carefully, to be a valid model of the process that generates the data set. Its free parameters are the levers to optimize the resemblance. The Likelihood function has its maximum if the model parameters fit best to the data set.

As stated before, the probability density function of a Noisy Ramp has three model parameters: i_C, σ, d. The Likelihood function L is defined as the product of the probabilities of N data points in a set. As a measure for its probability, the data point is processed by the probability density function. The Likelihood function in the present case is $L(i_C, \sigma, d)$:

$$L(i_C, \sigma, d) = \prod_{i=i_1}^{i_N} P(x(i), i)$$

$$= \prod_{i=i_1}^{i_N} \begin{cases} i < i_C: & \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x(i))^2}{2\sigma^2}} \\ i \geq i_C: & \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(x(i)-d\cdot(i-i_C)\right)^2}{2\sigma^2}} \end{cases}$$

With the synthesized Ramp function $r(i)$ derived from (2-1),

$$\mathrm{r}(i) = \begin{cases} i < i_C: & 0 \\ i \geq i_C: & d \cdot (i - i_C) \end{cases} \tag{2-3}$$

L becomes:

$$L(i_C, \sigma, d) = \prod_{i=i_1}^{i_N} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x(i)-\mathrm{r(i)})^2}{2\sigma^2}}$$

The Likelihood function reaches its maximum if the data set best-possibly fits the model function. Finding the optimal model parameters means finding the most likely model function and its randomness w.r.t the data set $\{x(i) | i = i_1 \dots i_N\}$ with N, the number of data points available.

$$\max_{i_C, \sigma, d} L(i_C, \sigma, d)$$

Resulting values of the Likelihood function are typically very small and hardly representable as double precision number on a computer. By applying the logarithm to the Likelihood function,

- the location of the maximum remains unchanged.
- the product becomes a sum.
- the terms of the sum are of manageable magnitude

Thus, instead of L, its logarithm $\ln(L)$ is analyzed:

$$\ln(L) = \sum_{i=i_1}^{i_N} ln\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x(i)-\mathrm{r(i)})^2}{2\sigma^2}}\right)$$

$$= \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2} \sum_{i=i_1}^{i_N} (x(i) - \mathrm{r(i)})^2$$

Solving the maximum likelihood problem (ML) consists of finding three parameters: $i_{C.ML}, \sigma_{ML}, d_{ML}$ that maximize the likelihood function L respectively its logarithm $\ln(L)$:

$$\max_{i_C,\sigma,d}\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)-\frac{1}{2\sigma^2}\sum_{i=i_1}^{i_N}(x(i)-\mathrm{r(i)})^2\right) \tag{2-4}$$

Note that parameters $i_{C.ML}, d_{ML}$ have quadratic influence on the ML problem, while σ_{ML} has inverse quadratic influence. The Likelihood function is not steadily differentiable if i_C is integer. It potentially has local maxima.

Numerical differentiation of $\ln(L)$ w.r.t. σ requires multiple executions of $\ln()$-function which, in case of real-time applications, might become a significant computing time issue. The problem would disappear if σ could be determined separately.

2.6 Standard deviation for non-normal Noisy Ramps

2.6.1 Ground Truth standard deviation

In some cases, Ground Truth knowledge about signal noise is available, for example if the noise being present in the process is well understood and/or its magnitude is predictable. Another case comes with processes in a steady state that have been observed over a very long period of time before the actual ramp shows up.

The exact standard deviation of a signal is determined from the entire signal data set. Hence, Ramp Identification is restricted to a set of $N_W = 100$ samples, referred to as Observation Window. The standard deviation of Ground Truth noise in an Observation Window denoted $\sigma_{GT,W}$. In σ_{GT} index 'W' is omitted to express that it was determined from the entire data set:

$$\sigma_{GT,W} = \sqrt{\frac{1}{N_W-1}\sum_{i=i_1}^{N_W} n_{GT}^2(i)} \approx \sigma_{GT}$$

2.6.2 Standard deviation using ML as a filter

The Noisy Ramp is defined as the sum of a ramp function plus white noise. With given Ground Truth data sets for a ramp $r_{GT}(i)$ and noise $n_{GT}(i)$, the Noisy Ramp data set becomes $x(i)$:

$$x(i) = r_{GT}(i) + n_{GT}(i) \qquad \text{with } n_{GT}(i) = randn(\sigma_{GT}), i = i_1 \dots i_W$$

If the Ramp Identification solver ML finds valid Ramp parameters $i_{C,ML}, d_{ML}$, the Ramp function $r_{ML}(i)$ is represented by (2-3) using $i_{C,ML}, d_{ML}$. This Ramp can be used as a filter to extract a noise function $n_{ML}(i)$ which is an estimation of the original noise $n_{GT}(i)$:

$$n_{ML}(i) = x(i) - r_{ML}(i) \approx n_{GT}(i) \tag{2-5}$$

The standard deviation σ_{ML} of noise $n_{ML}(i)$ over the Observation Window should be close to the standard deviation of the Ground Truth noise over the same Observation Window, $\sigma_{GT,W}$.

$$\sigma_{ML} = \sqrt{\frac{1}{N_W-1}\sum_{i=i_1}^{N_W} n_{ML}^2(i)} \approx \sigma_{GT,W} \tag{2-6}$$

2.6.3 A sample

To give an impression of whether the assumption (2-6) and more importantly $\sigma_{ML} \approx \sigma_{GT}$ is sufficiently accurate, σ_{ML} and $\sigma_{GT,W}$ were calculated for a great number of Noisy Ramps and compared to known Ground Truth σ_{GT} in the following way:

One thousand Noisy Ramps were synthesized for a sample Ground Truth Ramp which contains 50 data points on Floor level and $L = 50$ on the actual Ramp. Each data set has a different random number series (RS) added to the Ground Truth Ramp. All random number series $RS = \{1 \dots 1000\}$ were generated from white noise with $\sigma_{GT} = 1$. For each 1000 RS,

$\sigma_{ML}(RS)$ and $\sigma_{GT,W}(RS)$ were determined. The mean value of $\sigma_{GT,W}(RS)$ over 1000 RS is referred to as $\tilde{\mu}_{GT,W}$, see Fig. 2-3. The standard deviation of $\sigma_{GT,W}(RS)$ over 1000 RS is referred to as $\tilde{\sigma}_{GT,W}(RS)$. Vertical lines in Fig. 2-3 represent the total range of values found: $\min_{RS}(\sigma(RS))$ to $\max_{RS}(\sigma(RS))$. A box represents the mean value $\pm$ the standard deviation, i.e., $\tilde{\mu}(RS) \pm \tilde{\sigma}(RS)$.

The mean values of $\sigma_{ML}(RS)$ and $\sigma_{GT,W}(RS)$ are close to Ground Truth $\sigma_{GT} = 1$, also marked in Fig. 2-3 The uncertainties of σ_{ML} and $\sigma_{GT,W}$, expressed by mean value $\tilde{\mu}$ and standard deviation $\tilde{\sigma}$ over 1000 *RS* are similar. Thus, the ML-filtered standard deviation σ_{ML} appears to be a satisfactory estimation for Ground Truth standard deviation inside the observation window $\sigma_{GT,W}$.

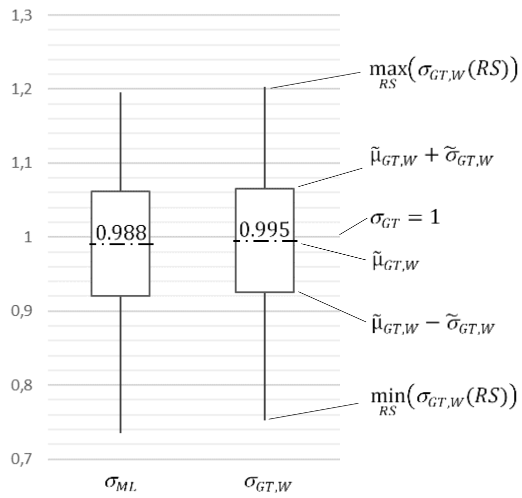


Fig. 2-3: Uncertainties in sigma estimates

2.7 Least Squares Solution to ML

Let σ_W be the standard deviation of a Noisy Ramp inside the Observation Window. σ_W is either determined from data points inside the Window or from Ground Truth, if available.

$$\sigma_W = \begin{cases} if\ Ground\ Truth\ is\ known: & \sigma_{GT} \\ if\ Ground\ Truth\ is\ unknown: & \sigma_{ML} \end{cases}$$

In particular, σ_W has now become a constant to the Maximum Likelihood problem. The probability density function with σ_W:

$$P(x,i) = \frac{1}{\sqrt{2\pi \cdot \sigma_W{}^2}} \cdot e^{-\frac{(x(i)-\mathrm{r(i)})^2}{2\cdot \sigma_W{}^2}}$$

With σ_W already being determined, the Likelihood function L has two remaining model parameters i_C, d. Its logarithm is

$$ln(L) = ln\left(\frac{1}{\sqrt{2\pi \cdot \sigma_W{}^2}}\right) - \frac{1}{2\cdot \sigma_W{}^2} \sum_{i=\mathrm{i_{1W}}}^{\mathrm{i_W}} (x(i) - \mathrm{r(i)})^2$$

By eliminating constant terms, the Maximum Likelihood problem develops into the least-squares formulation of the same problem and contains no logarithmic terms any more:

$$\max_{i_C,d} ln(L) = \min_{i_C,d} \sum_{i=\mathrm{i_{1W}}}^{\mathrm{i_W}} \begin{cases} i < i_C: & \left(x(i)\right)^2 \\ i \geq i_C: & (x(i) - \mathrm{r(i)})^2 \end{cases} \tag{2-7}$$

The Goal function (2-7) is not steadily differentiable over the Corner position i_C, see Fig. 2-4. Minima can be found exactly at an increment or in-between, and several local minima might occur. A standard solver sometimes picks the minimum closest to its starting increment, disregarding whether it is a local or global minimum to the Goal function.

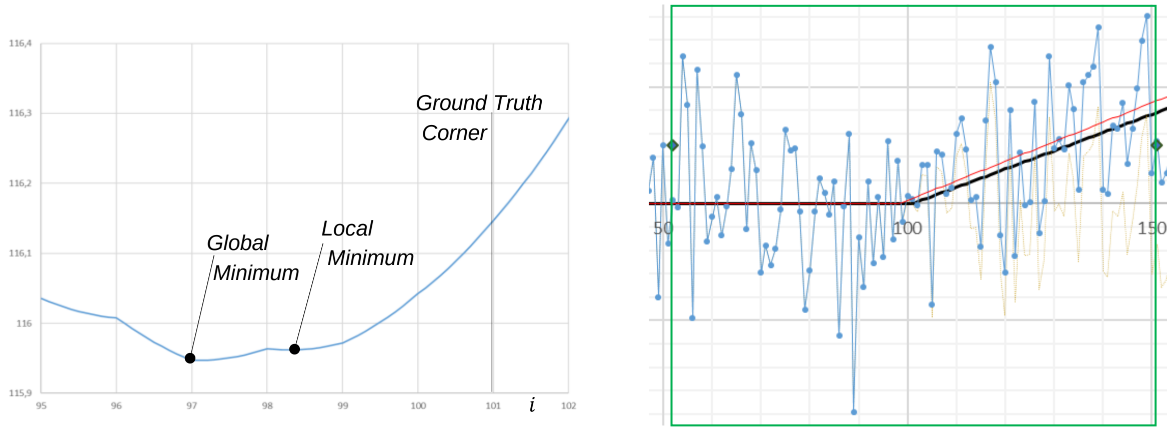


Fig. 2-4: Sensitivity of the Goal function to the Corner position i_C

2.8 Uncertainty and Confidence

Ramp Identification not only aims at estimating Ramp Parameters in Noisy Ramp data sets, but also at determining the maximum error that these estimates might come with. The maximum error is crucial to implementing robot processes at maximum speed without jeopardizing safety.

The term “uncertainty” expresses the maximum possible error between a measurement result and its true value. For any measurement process with Gaussian noise, no absolute error maximum exists that could be trusted blindly forever. Sooner or later any maximum will be broken. The relative number of samples that stay closer than the Uncertainty distance from the truth, is referred to as Confidence level. Eventually, Uncertainty is defined as a symmetric interval centered by the expected result, such that the true value is contained by this interval at a certain confidence level, e.g.:

$$(\text{result} - \text{uncertainty}) < \text{true value} < (\text{result} + \text{uncertainty})$$

at $\text{confidence} = 99.99\%.$

Let C, D be the Ramp Parameters to be estimated, and C_{GT}, D_{GT} their Ground Truth values. The estimation in a Noisy Ramp experiment returns C_e, D_e, one result for each experiment e. Let C_e be normal distributed with standard deviation σ_C and let the uncertainty U_C of C_e be chosen to equal 3 times σ_C:

σ_C	standard deviation in C
$U_C = 3 \cdot \sigma_C$	the uncertainty at C_e

The uncertainty U_C of C_e then satisfies the following inequalities:

$C_e - U_C < C_{GT} < C_e + U_C$	holds true, with
$fd \geq 99.73\%$	the Confidence level

By repeating the experiment N_e times with the same Ground Truth Ramp but each time with different Gaussian noise random number series, the estimates C_e, D_e jitter somewhat close to their respective Ground Truth values. The errors in N_e such experiments can be represented by mean value and standard deviation in C_e, D_e. Mean errors over N_e experiments with same Ground Truth and same noise level, will be referred to as “Bias” and the variable $\tilde{\mu}$. The standard deviation over N_e experiments will be referred to as “Noise”, variable $\tilde{\sigma}$, see Fig. 2-5, top left diagram.

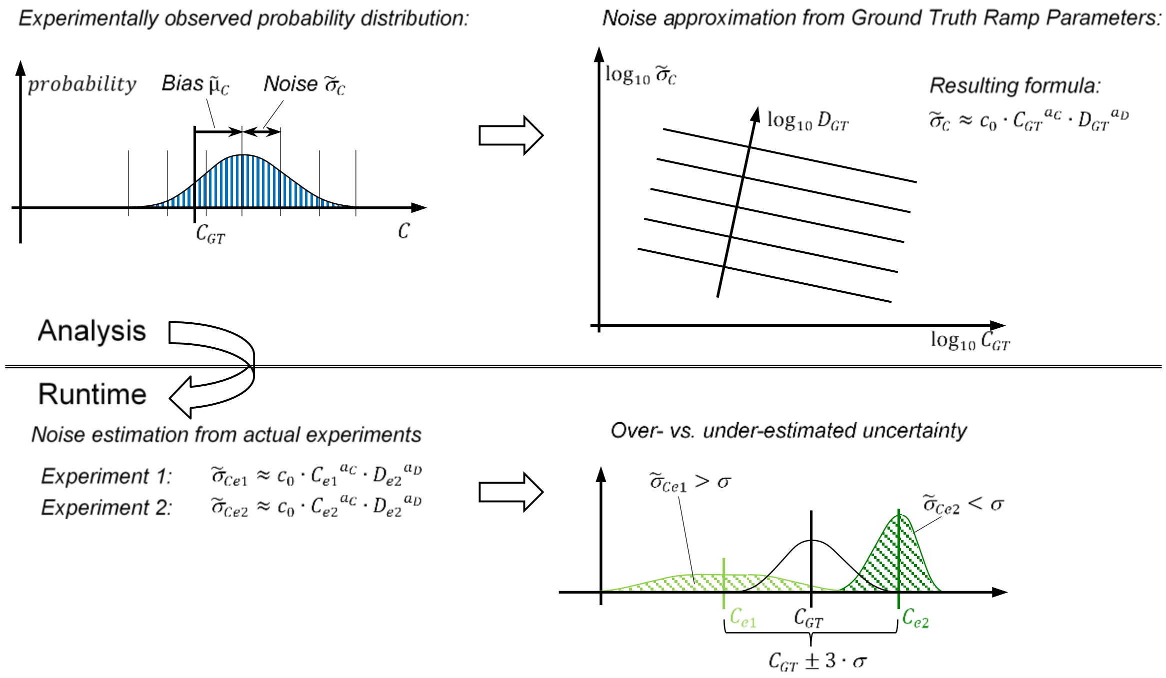


Fig. 2-5: From Noise estimation based on Ground Truth to result Uncertainty.

As will be shown further down, the noise in C_e, D_e is approximately normal distributed and correlates to the true Ramp parameters C_{GT}, D_{GT}. Thus, good approximation formulae $\tilde{\sigma}_C = f_C(C_{GT}, D_{GT})$ and $\tilde{\sigma}_D = f_D(C_{GT}, D_{GT})$ can be found, see Fig. 2-5, right top diagram.

Unfortunately, Ground Truth values C_{GT}, D_{GT} are not available in a Runtime experiment. What if approximation formulae for $\tilde{\sigma}_C, \tilde{\sigma}_D$ were just calculated with the available experimental results C_e, D_e instead of C_{GT}, D_{GT}, Fig. 2-5 left bottom? As C_e, D_e differ very little from C_{GT}, D_{GT}, approximation formulae for $\tilde{\sigma}_{Ce}, \tilde{\sigma}_{De}$ deliver very similar values. However, if the experimental distributions of C_e, D_e are not perfectly Gaussian, see Fig. 2-5, bottom right diagram, the validity of $\tilde{\sigma}_{Ce}, \tilde{\sigma}_{De}$ deteriorates. Two possible scenarii are shown:

- Let $\tilde{\sigma}_{C,e1}$ overestimate the true Noise $\tilde{\sigma}_C$ in experiment 1 and let $U_{C1} = 3 \cdot \tilde{\sigma}_{C,e1}$ be the uncertainty prediction. $C_{e1} \pm U_{C1}$ overestimates the error range of C_{e1}, thus U_{C1} is too careful. This case, in the end, is detrimental to process performance.
- Result C_{e2}, D_{e2} under-estimates the Noise $\tilde{\sigma}_{C,e2} < \tilde{\sigma}_C$. With $U_{C2} = 3 \cdot \tilde{\sigma}_{C,e2}$, the uncertainty inequality $C_{e2} - U_{C2} < C_{GT} < C_{e2} + U_{C2}$ can not be trusted any longer.

To avoid such effects, the uncertainty could adjusted by $U_{C1} = m_C \cdot \tilde{\sigma}_{C,e1}$, with m_C chosen such, that the required confidence level is met.

3 Objectives

3.1 Top level objectives

Reminding Fig. 1-2, Ramp Identification should be thought of as a block in a real-time signal flow diagram, see Fig. 3-1. Once per Cycle, the RI block is fed with the newest sensor signal and is expected to identify a Ramp within the latest 100 signal values. Identification means both, estimate Ramp parameters and estimate uncertainties in Ramp parameters.

Any Ramp Identification method has to find satisfying answers to three basic questions:

- $Q(1)$ Ramp detection: Is it possible to detect the occurrence of a Ramp in a Noisy Signal at a guaranteed confidence level?
- $Q(2)$ Bias & Noise modelling: Is it possible to represent the Bias and Noise in Ramp parameters σ, H, L, D as functions of the very same Ramp parameters σ, H, L, D?
- $Q(3)$ Uncertainty & Confidence gauging: Is it possible to estimate the uncertainties of estimated Ramp parameters σ, H, L, D at a guaranteed confidence level?

Note that $Q(1)$ is associated to a confidence level, which quantifies the probability of false-positive results. The presence of a Ramp is a necessary condition for step $Q(2)$, i.e., gathering enough signal data to estimate Bias and Noise of Ramp parameters σ, H, L, D. With $Q(2)$ accomplished, Uncertainties in σ, H, L, D are determined for the required Confidence level.

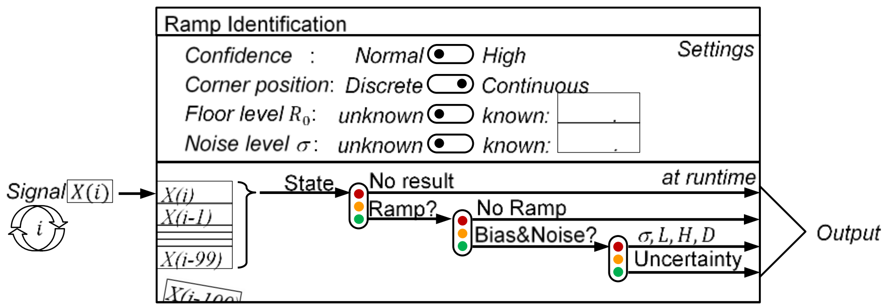


Fig. 3-1: Variants and runtime data flow of Ramp Identification

3.2 Problem Variants

3.2.1 Incremental vs. continuous Length resolution

Ramp Corner position is either given by parameters i_C, C or indirectly by the Ramp Length L. In a continuous process, the exact position of the Ramp Corner along the i-axis is a real number while increment numbers are integers. For those controllers, for which incremental resolution is sufficient, the integer part of the Corner position should be provided.

The incremental Corner position will be denoted with floor brackets $\lfloor i_C \rfloor$, $\lfloor C \rfloor$ the continuous position i_C or C.

3.2.2 Ground Truth and Requirements from the Process

External knowledge about the expected Ramp and Noise, if available, should be considered by RI, to help improve its computational speed or robustness against outliers. This applies to, compare Fig. 3-2:

- The Floor level of the Ramp Signal: R_0
- The Noise level of the Signal: σ_{GT}
- Increment range i_{1E} to i_E where Ramp Detection is expected to occur.
- Minimum and Maximum expected Ramp Gradients: D_{min} and D_{max}

Process requirements influencing on RI results:

- The Confidence level required, is $fd > 99.73\%$ or $fd > 99.99\%$
- Breaking condition on the signal level: $X(i) < X_{break}$

To avoid premature firing of the breaking condition, consider

$X_{Break} > R_0 + 5 \cdot \sigma_{max}$ with σ_{max}, the maximum expected standard deviation

3.2.3 Observation Window

In a real-time application, the Ramp Identification method is applied to signal data that spans from the present time into the past. For RI development, a simulation is used instead: an observation Window shifts increment by increment over a previously known, synthetic Noisy Ramp. This Window observes $N_W = 100$ increments numbered from i_{1W} to i_W. Index 'W' can be read either as 'Window' or 'noW' and denotes the most recently received increment.

3.2.4 Parameter overview

Relevant parameters and basic formulae are given in Fig. 3-2.

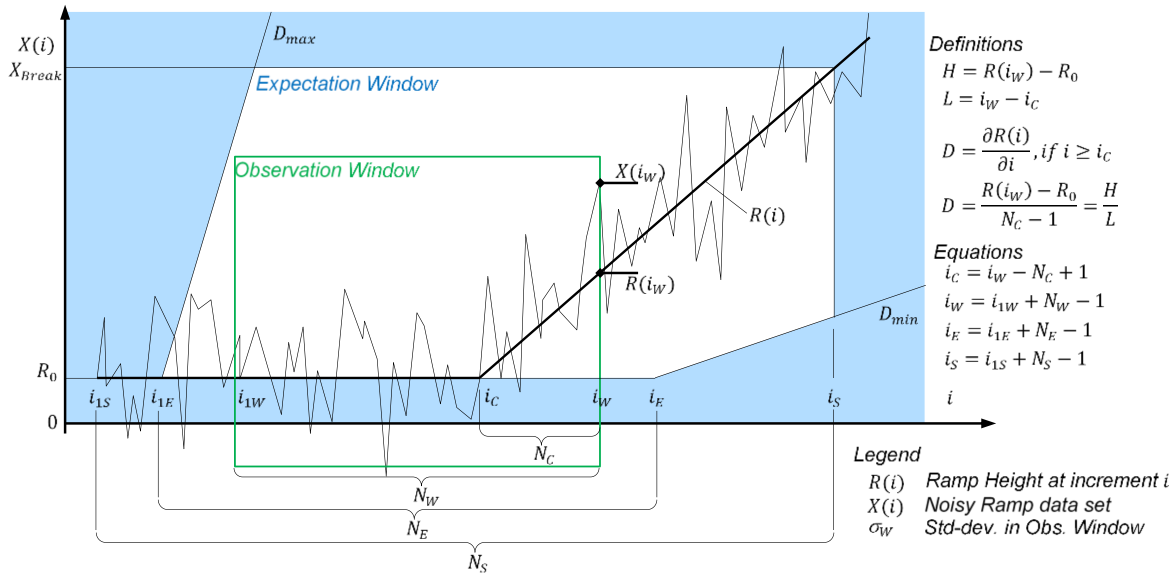


Fig. 3-2: Definitions and equations for Ramp Identification

Note: parameters i_C, C, L, N_C are equivalent and interchangeable ways to specify the Ramp Corner position. The Ramp Height is equivalently denoted as $R(i_W), R_W, H$ for RTRamps and r_W, h for NNRamps.

3.3 *Experiments and implementation*

A great number of synthetic Normal Noisy Ramps are to be analyzed to identify patterns in Bias and Noise of Ramp parameters R_0, σ, H, L, D. In the absence of a theoretical understanding, empirical modelling of such patterns is a valid method. For comparability between competing model variants, they must be run on the very same set of synthetic Normal Noisy Ramps.

A wide range of Ramp Gradients should be covered by RI. In case of small Gradients, the process designer should be warned about Ramp detection limits of RI. A real-time implementation of Ramp Identification must be developed which guarantees an upper limit of computation cost per Cycle. Computation effort is to be minimized, evaluated, and further reduced if simplifications are possible, e.g., for incremental resolution of the Corner position.

4 Experimentation concept

4.1 Reminder

Given 100 data points from a noisy signal source, the Least Squares method will always find estimates for Ramp parameters σ, H, L, D that best fit the data. These values alone give no clue about how close they come to the actual Ramp hidden under the noise. 'Ramp Identification' (RI) aims at overcoming this deficit.

As previously stated, 'Ramp Identification' must find satisfying answers to three questions:

- $Q(1)$ When and How is Ramp detection possible within a noisy signal at a desired confidence level?
- $Q(2)$ When and How is it possible to represent the errors in Ramp parameter estimates as functions of the values of Ramp parameter estimates?
- $Q(3)$ How is it possible to adjust the uncertainties of estimated Ramp parameters R_0, σ, H, L, D to a particular confidence level?

If Ramp parameter estimation is executed over many Normal Noisy Ramps (NNRamp), random effects should cancel out and patterns become visible that could allow to find the answers to $Q(1)$, $Q(2)$ and $Q(3)$.

4.2 NNRamp setup

4.2.1 Observation Window shifting over NNRamps

Without loss of generality, Normal Noisy Ramps are defined over a given range of $N_S = 250$ increments, see Fig. 4-1. Let the Ramp Corner be placed exactly at increment point $i_{C,GT} = 131$. By convention, this Increment point is considered to be part of the Ramp rather than of the Floor. Increment interval 131 fully belongs to the Ramp. Starting from the left, an observation window is placed over the data set and shifted by f increments. For shift $f = 2 \dots 29$ the Ground Truth Corner is not yet inside the Window. When the shift reaches $f = 30$ than $i_{C,GT}$ equals i_W, i.e., the Ground Truth Corner is at the brink of the Window. For $f = 31 \dots 130$ the Ground Truth Corner lies within Window borders, for $f = 131$ at its left border. At shift $f = 100$ the observation window contains 30 increments with Ground Truth Floor value $r(i) = r_0 = 0$, including the Corner, and 70 increments with Ground Truth Ramp values $r(i) > 0$.

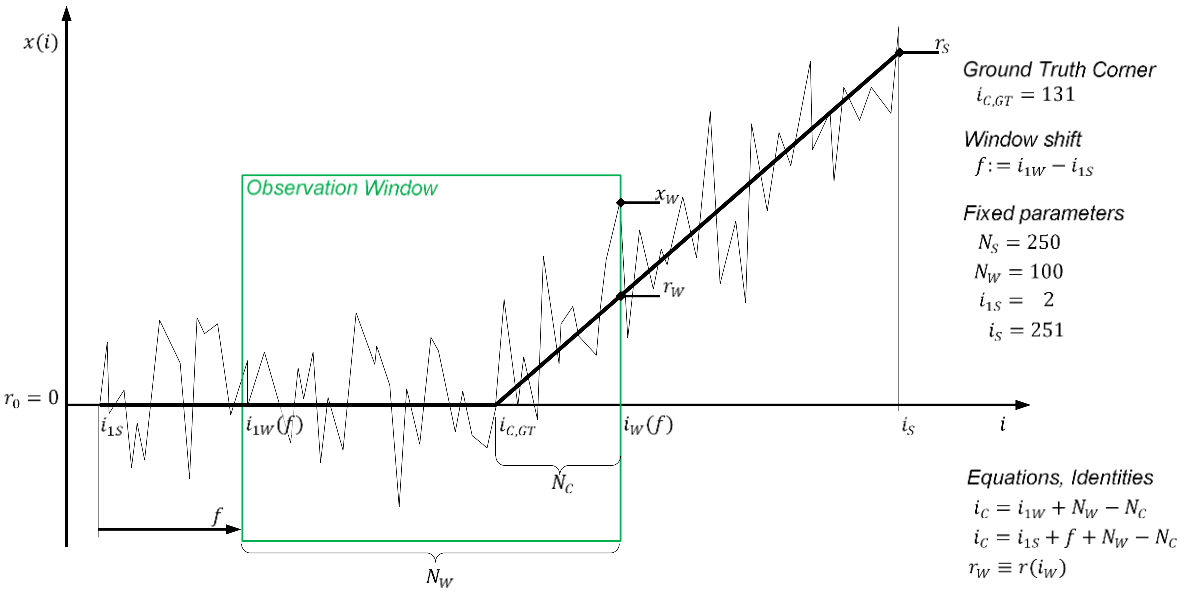


Fig. 4-1: Parameter for ML Ramp sensitivity analysis

Note: N_S, N_W, N_C count the number of Increment points within a range. N_C equals the number of Increment points inside the Observation Window belonging to the Ramp and $N_W - N_C$

Increment points belonging to the Floor. In case of $i_{C,GT}$ being an integer, $N_C - 1$ equals the Ramp Length. Ramp function $r(i)$ is dimensionless due to normalization to $\sigma = 1$.

4.2.2 Gradient range

The uncertainties in estimating Corner increment i_C and Ramp gradient d will certainly depend on the number N_c of actual Ramp Increment points inside the Window, but also on the magnitude of Ramp gradient d itself.

RI is designed to work within a range of Gradients $d_{min} < d < d_{max}$ which should be chosen as broad as possible to cover all potential problems such as contact mechanics and robotics.

The minimum detectable ramp gradient d_{min} is estimated by a combination of two rules

- Ramp values should grow significantly over the Noise in the Window:
  $r_{min}(i_W) > 3$
- Ramp detection should occur long before the Corner leaves the shifting Window. At $N_C = 70$, 69 out of 100 Increment intervals are influenced by the Ground Truth Ramp.
- Thus
  $d_{min} = r_{min}/(N_C - 1) > 3/69$

The maximum reasonable ramp gradient is generously estimated by

- a growth rate of $3 \cdot \sigma$ per increment, i.e., $r_i - r_{i-1} < 3$. If this growth begins at the Corner increment i_C and lasts until i_W, the ramp value reaches
  $r_{max}(i_W) < r_0 + 3 \cdot (i_W - i_C) = 3 \cdot (N_C - 1)$.
- Thus
  $d_{max} = r_{max}/(N_C - 1) < 3$

To summarize, Ramp Gradients are expected within the range

$$\frac{3}{69} < d < 3$$

For convenient NNRamp synthesizing, the Ground Truth Ramp Gradient range was widened and finally discretized into a geometric series with 9 entries:

$$d_{GT} \in \left\{\frac{3}{69} \dots 3\right\} \approx \{0.043 \dots 3\} \subset \{2^{-6} \dots 2^{2}\}$$

$$\Rightarrow \qquad d_{GT} \in \{2^{-6}, 2^{-5}, 2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 2^{0}, 2^{1}, 2^{2}\}$$

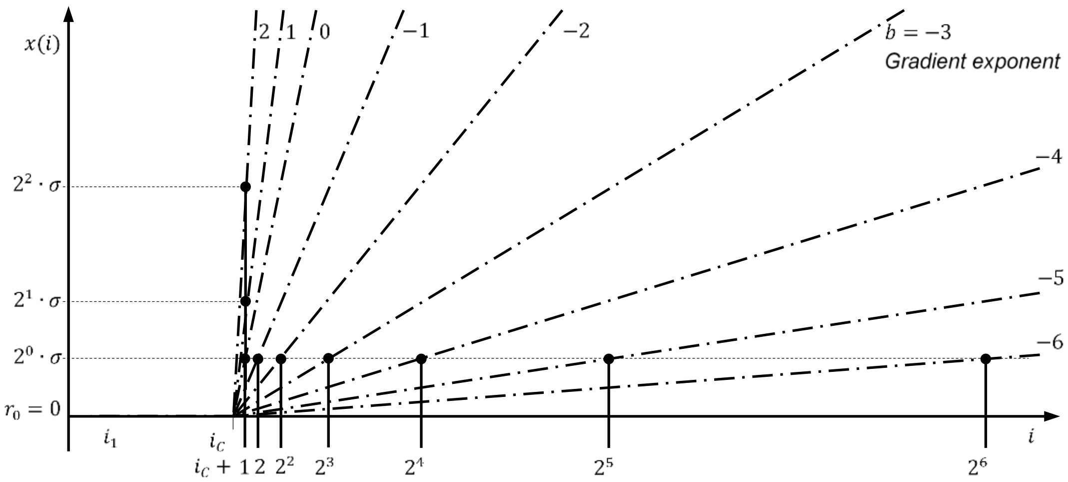


Fig. 4-2: Ramp gradients defined by the Gradient exponent b: 2^b growth per increment for a Ramp scaled to $\sigma = 1$

The Gradient exponent $b_{GT} = \log_2 d_{GT}$ simplifies the handling of Gradients d_{GT} over several orders of magnitude.

4.2.3 Schedule

A total of 16k random number series with $\sigma = 1$ and length $N_S = 250$ is generated at first. Ground Truth Ramp Parameters for Corner position $i_{C,GT}$ and 9 different Ramp Gradients d_{GT} are defined and each random number series added to them, making up $144'000$ NNRamps. An Observation Window shifts over each NNRamp:

- The observation window has width of $N_W = 100$ increments.
- For shift values f= $2 \ldots 30$, only Ground Truth Floor increments are inside the Window.
- For higher shift values $f = 31 \ldots 130$, data points from Floor and Ramp are inside the Window except for $f = 130$.
- When $f = 131 \ldots 150$, the Corner has left the Window.

In total, the ML solver will estimate Ramp parameters σ, H, L, D in $21.456\,10^6$ Observation Windows. ML solves a best-fit problem for Ramp parameters i_C and d which allows to separate the Signal into a Ramp and ML-filtered Noise. The standard deviation of the ML-filtered Noise estimates the standard deviation of the generating random number series.

4.3 Finding answers to Q1,Q2,Q3

4.3.1 About Q1: Ramp detection

Binary Yes-No decisions cannot be made over random data since even the utmost unlikely cases might occur sooner or later. In process control it must be accepted that false-positives are just 'unlikely enough' to a significant source of failure. For an interval of $[r_0 - 3 \cdot \sigma, r_0 + 3 \cdot \sigma]$ and a normal distributed data set, the likelihood for a data sample to lie inside the interval is 99.73%. This should be 'likely enough' if the underlying process is able to cope with 2.7 failures in 1000 attempts. Otherwise, the interval $[r_0 - 5 \cdot \sigma, r_0 + 5 \cdot \sigma]$ reasonably excludes any false-positive Ramp detection at a failure rate of $0.6 \cdot 10^{-6}$.

Ground Truth data is available for synthetic Normal Noisy Ramps. However, when ML is applied to Runtime Noisy Ramps, this is not the case. What is more: ML answers in reel numbers but Q(1): "Does a Ramp Corner exist inside the Window?" requires a binary true-or-false answer.

When ML runs over that window, it will often find results $r_W > 0$ and $N_C > 0$. This seems to indicate that a Corner has been found, but 'very small' values for r_W should be discarded. Thus, a threshold for Corner detection is needed which allows to obtain *True/False* results from ML that most likely are true-positives. The variable $Q1$ is defined as:

$$Q1 \coloneqq (r_{W,ML} >^? threshold)$$

Read the sign $>^?$ as a condition to be checked: "truly bigger than?" with the answer being *True* or *False*. The threshold could be expressed as a multiple m of the signal noise level σ_W. The best estimation for σ_W available so far is $\sigma_{W,ML}$, the ML-filtered standard deviation over the Window, thus

$$Q1 = \left(r_{W,ML} >^? m \cdot \sigma_{W,ML}\right) = \begin{cases} True\!: the\ Window\ contains\ a\ Corner \\ False\!: there\ is\ no\ Corner\ in\ the\ Window \end{cases} \tag{4-1}$$

This formulation generates three new problems:

- What could be a meaningful value of m ?
- how many increments N_C are necessary for true-positive results being most likely?
- How should "most likely" be defined?

4.3.2 About Q2: Bias & Noise modelling

Question $Q2$ asks for the possibility of modelling the estimation errors made by ML. Estimation models must be based on nothing else but the ML results themselves and on error patterns found in big Noisy Ramp data sets. Error estimation splits into Bias & Noise estimation, none of them must depend on Ground Truth knowledge. It is somewhat intuitive that ML's estimation errors decrease when more Ramp increments become visible in the shifting Observation Window. With a steeper Gradient, the Ramp-Noise-ratio increases and ML's estimation errors should decrease. Thus, question $Q2$ can be split into two subsequent phases:

a) Bias & Noise horizon: How much of the Ramp needs to be observed so that Bias and Noise become predictable? Or: when do statistical patterns in ML errors emerge out of the noise?
b) Bias & Noise model: How are these ML errors formally related to ML estimates of these very same parameter?

4.3.3 About Q3: Uncertainty & Confidence gauging

Uncertainty and Confidence are well understood for Gaussian noise. However, the noise in ML errors might not be normal distributed. By counting outliers within the big collection of NNRamps, uncertainty intervals could be adjusted to the confidence level requested by the application developer.

4.4 ML prototype in MS EXCEL®

A prototype ML solver was implemented in MS Excel as benchmark for the real-time solver to be developed thereafter. Due to file size and computation time limits, only 1000 random number series were manageable. The Ramp function is formulated using N_C, r_W instead of i_C, d:

$$r(i) = \begin{cases} i < i_C: & 0 \\ i \geq i_C: & \mathrm{r_W} \cdot (1 - (i_W - i)/(N_C - 1)) \end{cases}$$

with equivalences for Corner increment and Gradient:

$$i_C = i_W - N_C + 1$$
$$r_W = d \cdot (i_W - i_C)$$

The Least Squares problem, equation (2-7) is:

$$\min_{N_C, r_W} \sum_{i=\mathrm{i_{1W}}}^{\mathrm{i_W}} \begin{cases} i < i_C: & \left(x(i)\right)^2 \\ i \geq i_C: & \left(x(i) - \mathrm{r_W} \cdot \left(1 - \frac{i_W - i}{N_C - 1}\right)\right)^2 \end{cases} \tag{4-2}$$

The ML problem is solved by a general-purpose solver, the 'GRG-nonlinear' implemented in MS Excel. Start values of minimization parameters N_C, r_W, denoted by a bracketed exponent $^{(0)}$, are set by:

$${r_W}^{(0)} = x_W$$
$${N_C}^{(0)} = 10$$

Random series were generated with EXCEL function $\mathrm{random()}$. Box-Muller method transforms the equally distributed random series from $\mathrm{random()}$ into normal distributed series with $\sigma = 1$.

Fig. 4-3 shows a typical result of the ML solver. At shift $f = 60$, 30 out of 100 increments of the Window, marked with a green rectangle, contain Ground Truth Ramp increments. The rest is just noise without any Ramp growth. ML estimates the Corner increment $i_{C,ML}$ with an error of $i_{C,ML} - i_{C,GT} = -3.417$ increments. The ML filtered standard deviation within the Window is overestimated $\sigma_{W,ML} > \sigma_{GT}$, the Gradient underestimates its Ground Truth value $d_{ML} < d_{GT}$. The estimated Ramp Height error $r_{W,ML} - r_{W,GT}$ is quite small for this particular signal noise at that particular increment. For any other random number set combined with the same Ground Truth Ramp, all errors were different. If the uncertainties in ML results were known, the depending Controller could predict its accuracy and safely anticipate critical events.

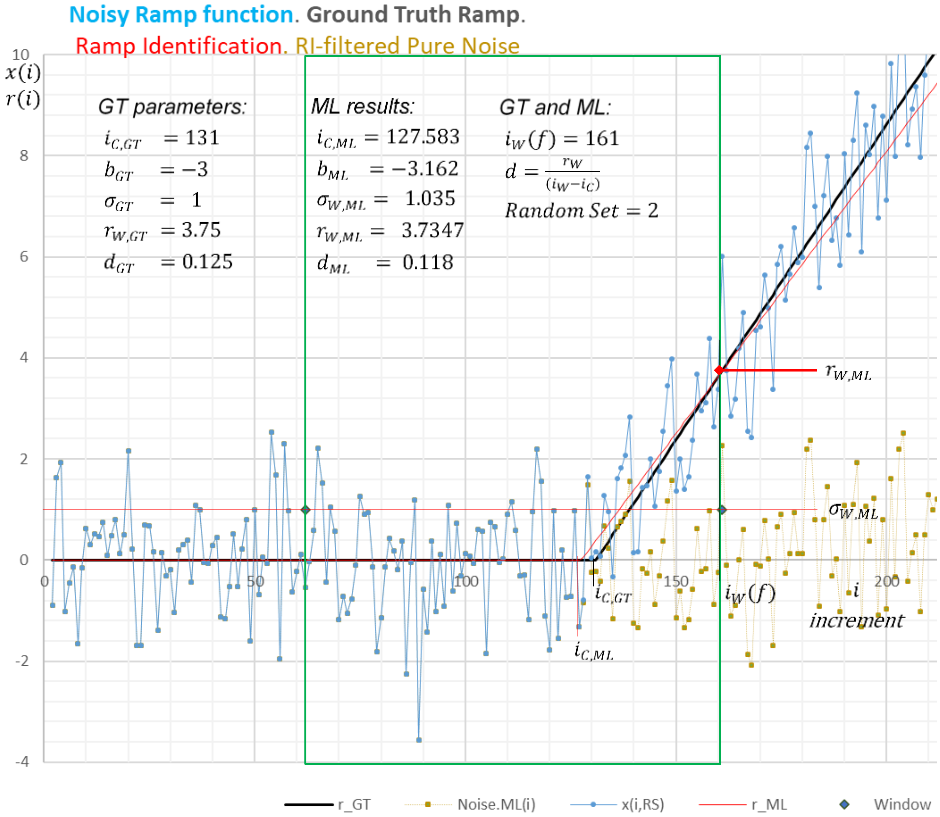


Fig. 4-3: ML solver for Ramp Identification using MS EXCEL.

5 Real-time Ramp Identification Algorithm

5.1 Validation by cross-check

Calculations and analysis were first conducted using MS Excel: generate random series and noisy ramps, solve the ML minimization problem, and store the results. Further graphical representations, regressions, and result analysis were executed within MS Excel as well.

The solutions found with Excel served as reference for result cross-checking, error search and as a benchmark for the real-time algorithm to be developed.

5.2 Continuous Formulation

In a runtime robotics process, safety of humans and damage avoidance to physical equipment is primordial. Sensor signals that might contain the signature of a mechanical collision must be checked every controller cycle. The Ramp Identification algorithm, which was developed for this purpose previously, takes the most recent $N_W = 100$ sensor signals $x(i)$, where increment i is an integer number ranging from i_{1W} to i_W, i.e., $i_W = i_{1W} + N_W - 1$.

The problem to be solved was described above as: Find the actual ramp height r_W and the number N_C of increments that are belonging to the ramp by finding the minimum (5-1), referred to as $F(i_C, r_W)$, see Fig. 5-1:

$$F(i_C, r_W) := \min_{r_W, N_C} \sum_{i=\mathrm{i}_{1\mathrm{W}}}^{\mathrm{i}_\mathrm{W}} \begin{cases} i < i_C\colon & \left(x(i)\right)^2 \\ i \geq i_C\colon & \left(x(i) - \mathrm{r_W} \cdot \left(1 - \frac{i_W - i}{N_C - 1}\right)\right)^2 \end{cases}, \text{ with } i_C = i_W - N_C + 1 \qquad (5\text{-}1)$$

The parameter N_C locates the Ramp Corner by measuring the number of increments that belong to the Ramp within the Observation Window. Since N_C is an integer number, the resolution of Corner location equals the width of one increment. Hence, in case of large increments e.g., of a fast-moving Robot, or steep Ramps, higher, sub-incremental resolution is required. Thus, the Corner location i_C is defined as a real number on the i-axis. Other variables e.g., i_{1W} or i_W remain integers which induces some confusion. As consequence, the Corner location will be referred to as C throughout this chapter. For compatibility with previous chapters, the following identity holds at any time.

$$C \equiv i_C$$

The integer numbers surrounding a Corner C are denoted with floor brackets $\lfloor C \rfloor = floor(C)$:

$$\lfloor C \rfloor \leq C \leq \lfloor C \rfloor + 1$$

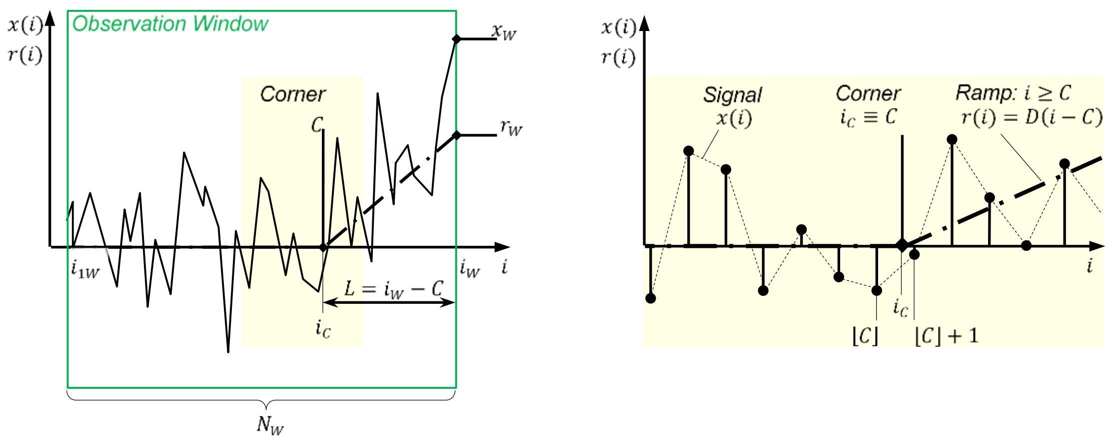


Fig. 5-1: Continuous Corner position amongst discrete, equidistant signals

Function (5-1) is now expanded into a continuous function $F(C,D)$ by applying the following modifications:

- Introduce the continuous Corner position C which is nothing else than i_C.
- Note: increment number i remains an integer. $\lfloor C \rfloor$ denotes the integer part of C.
- The continuous Length of the Ramp is $L = i_W - C$.
- The slope of the ramp is defined as $D = r_W / L = r_W / (i_W - C)$.
- The minimum of $F(C,D)$ is referred to as M_{CD}

With these modifications, the problem to be solved becomes:

$$M_{CD} = \min_{C,D}\big(F(C,D)\big) = \min_{C,D} \sum_{i=\mathrm{i}_{1\mathrm{W}}}^{\mathrm{i}_\mathrm{W}} \begin{cases} i < \mathrm{C}: & \big(x(i)\big)^2 \\ i \geq \mathrm{C}: & \big(x(i) - D\cdot(i-C)\big)^2 \end{cases} \tag{5-2}$$

The sum $F(C,D)$ can be split into two separate sums and rewritten with simplified notations. The summation boundaries are derived from the integer part of the Corner position: $\lfloor C \rfloor$.

$$F(C,D) = \textstyle\sum_{i=i_{1W}}^{\lfloor C \rfloor}(x)^2 + \sum_{i=\lfloor C \rfloor+1}^{i_W}\big(x - D\cdot(i-C)\big)^2$$

The summands in the second sum are developed into:

$$\begin{aligned}\big(x - D\cdot(i-C)\big)^2 & \\ &= x^2 - 2xD(i-C) + D^2(i^2 - 2i\cdot C + C^2) \\ &= x^2 - xi\cdot 2D + x\cdot 2DC + D^2 i^2 - i\cdot 2D^2 C + D^2 C^2\end{aligned}$$

Now, the sum $F(C,D)$ becomes:

$$\begin{aligned}F(C,D) = &\textstyle\sum_{i=i_{1W}}^{\lfloor C \rfloor}(x)^2 \ldots \\ &+\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(x)^2 + 2DC\sum_{i=\lfloor C \rfloor+1}^{i_W}(x) - 2D\sum_{i=\lfloor C \rfloor+1}^{i_W}(xi) \ldots \\ &+D^2\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(i)^2 - 2D^2C\sum_{i=\lfloor C \rfloor+1}^{i_W}(i) + D^2C^2\sum_{i=\lfloor C \rfloor+1}^{i_W}(1)\end{aligned}$$

Sums containing $1, i$ or i^2 are evaluated directly. Note that the summation formulae only work for positive, non-zero, integer values of i. To avoid collisions between increment numbers, array indices should start at 1 instead of 0.

$$\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(1) = i_W - \lfloor C \rfloor$$

$$\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(i) = \sum_{i=1}^{i_W}(i) - \sum_{i=1}^{\lfloor C \rfloor}(i) = \frac{i_W^2 + i_W}{2} - \frac{\lfloor C \rfloor^2 + \lfloor C \rfloor}{2} = \frac{1}{2}\left(i_W^2 + i_W - \lfloor C \rfloor^2 - \lfloor C \rfloor\right)$$

$$\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(i)^2 = \sum_{i=1}^{i_W}(i)^2 - \sum_{i=1}^{\lfloor C \rfloor}(i)^2 = \frac{i_W(i_W+1)(2i_W+1)}{6} - \frac{\lfloor C \rfloor(\lfloor C \rfloor+1)(2\lfloor C \rfloor+1)}{6}$$

Eventually, the continuous version of the goal function F over increments $i_{1W} \ldots i_W$ for a Corner position C and gradient D becomes:

$$\begin{aligned}F(C,D) = &\textstyle\sum_{i=i_{1W}}^{i_W}(x)^2 + \cdots \\ &\ldots + 2D\left[C\textstyle\sum_{i=\lfloor C \rfloor+1}^{i_W}(x) - \sum_{i=\lfloor C \rfloor+1}^{i_W}(xi)\right] + \cdots \\ &\ldots + D^2\left[\left(\tfrac{i_W(i_W+1)(2i_W+1)}{6} - \tfrac{\lfloor C \rfloor(\lfloor C \rfloor+1)(2\lfloor C \rfloor+1)}{6}\right) - C\left(i_W^2 + i_W - \lfloor C \rfloor^2 - \lfloor C \rfloor\right) + C^2\left(i_W - \lfloor C \rfloor\right)\right]\end{aligned} \tag{5-3}$$

Note that only $\sum xi$ and $\sum x$ depend on $\lfloor C \rfloor$. These sums must be evaluated once for each increment of interest i where $i = \lfloor C \rfloor$. Another form of (5-3) is given below.

$$F(C,D) = \textstyle\sum_{i=i_{1W}}^{i_W}(x)^2 \ldots \tag{5-4}$$

$$-D \cdot 2\sum_{i=\lfloor C\rfloor+1}^{i_W}(xi) + D^2 \cdot \left(\frac{i_W(i_W+1)(2i_W+1)}{6} - \frac{\lfloor C\rfloor(\lfloor C\rfloor+1)(2\lfloor C\rfloor+1)}{6}\right) \dots$$
$$+CD \cdot 2\sum_{i=\lfloor C\rfloor+1}^{i_W}(x) - CD^2 \cdot (i_W^2 + i_W - \lfloor C\rfloor^2 - \lfloor C\rfloor) \dots$$
$$+C^2D^2 \cdot (i_W - \lfloor C\rfloor)$$

Equation (5-4) shows a polynomial structure. Coefficients $F_0, F_D, F_{D2}, F_{CD}, F_{CD2}, F_{C2D2}$ are introduced for compactness.

$$F(C,D) = F_0 + D \cdot F_D + D^2 \cdot F_{D2} + CD \cdot F_{CD} + CD^2 \cdot F_{CD2} + C^2D^2 \cdot F_{C2D2} \tag{5-5}$$

The coefficients $F_0, F_D, F_{D2}, F_{CD}, F_{CD2}, F_{C2D2}$ expand into:

$$F_0 = \sum_{i=i_{1W}}^{i_W}(x)^2$$
$$F_D = -2\sum_{i=\lfloor C\rfloor+1}^{i_W}(xi)$$
$$F_{D2} = \frac{i_W(i_W+1)(2i_W+1)}{6} - \frac{\lfloor C\rfloor(\lfloor C\rfloor+1)(2\lfloor C\rfloor+1)}{6}$$
$$F_{CD} = 2\sum_{i=\lfloor C\rfloor+1}^{i_W}(x)$$
$$F_{CD2} = -(i_W^2 + i_W - \lfloor C\rfloor^2 - \lfloor C\rfloor)$$
$$F_{C2D2} = i_W - \lfloor C\rfloor$$

5.3 Four-step Minimum estimation procedure

5.3.1 Outline

The minimum of the goal function $F(C,D)$ will be estimated in four steps:

a) Place a candidate Corner C at every integer increment point within the Window $\lfloor C\rfloor \in \{i_{1W} \le \lfloor C\rfloor < i_W\}$ and find for each a D that minimizes the goal function $F(\lfloor C\rfloor, D)$:
$M_D(\lfloor C\rfloor) = \min_D F(\lfloor C\rfloor, D)$ Minimum for a candidate Corner at increment $\lfloor C\rfloor$

b) Sort minima $M_D(\lfloor C\rfloor)$ in ascending order. Among a short list of the 10 smallest minima search for direct neighbors, i.e., $M_D(\lfloor C\rfloor)$ and $M_D(\lfloor C\rfloor+1)$ which are both on the short list.

c) For each $M_D(\lfloor C\rfloor)$ having a neighbor $M_D(\lfloor C\rfloor+1)$ on the short list, estimate the sub-incremental minimum between the neighbors: inside the interval $\lfloor C\rfloor < C < \lfloor C\rfloor+1$:
$M_D(C_{Mi}) \approx \min_{C,D} F(C,D)$ where $\lfloor C\rfloor < C_{Mi} < \lfloor C\rfloor+1$.

d) Find the global minimum M_{CD} as the smallest of all incremental and sub-incremental minima:
$M_{CD} \approx \min(M_D(\lfloor C\rfloor), M_D(C_{Mi})\ \forall\ i_C \in \{i_{1W} \dots i_W\}, i \in \{i_{1W} \dots i_W - 1\})$

In doing so, the two-dimensional minimization problem is split into two one-dimensional problems that take advantage of the very different nature of $F(C,D)$ along the dimensions C and D.

5.3.2 Step a)

$F(C,D)$ is a quadratic function over D, and thus can be rewritten in the form:

$$F(C,D) = (F_{D2} + CF_{CD2} + C^2F_{C2D2}) \cdot D^2 + (F_D + CF_{CD}) \cdot D + F_0$$

The goal function $F(C,D)$ has a minimum over D at

$$D_M(C) = -\frac{(F_D + CF_{CD})}{2(F_{D2} + CF_{CD2} + C^2 F_{C2D2})}$$

The value of the goal function at its minimum D_M at a give Corner position C is:

$$M_D(C) = \min_D F(C,D) = F(C, D_M(C))$$

A typical graph of $F(C,D)$ close to its minimum is shown in Fig. 5-2.

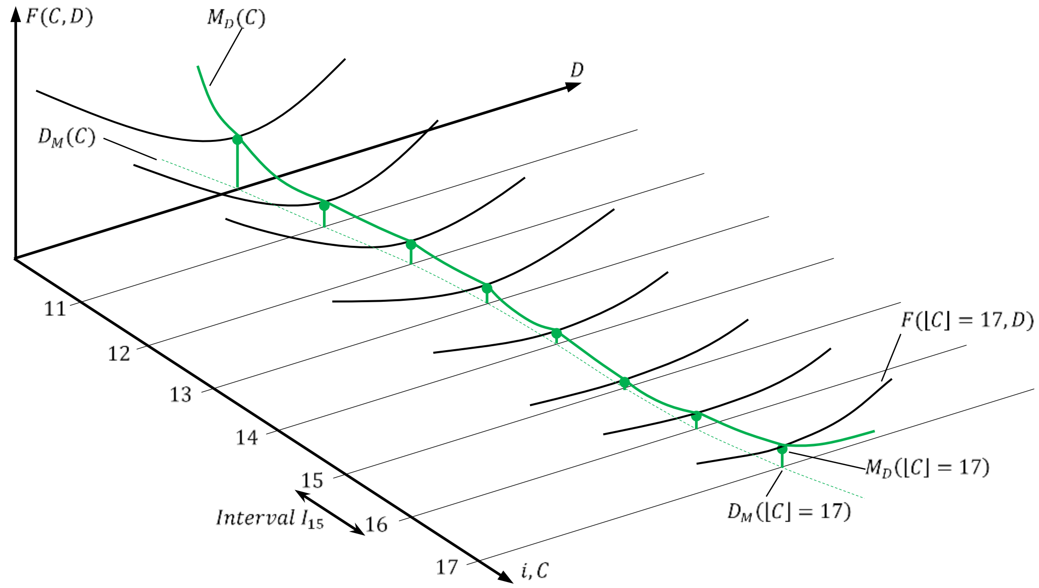


Fig. 5-2: Typical behavior of the Objective function $F(\lfloor C \rfloor, D)$

$D_M(C)$ describes a curve in the $D - C$ –plane, the corresponding goal function values $M_D(C)$ resembles an overhead wire crossing a valley. Green blobs mark $F(\lfloor C \rfloor, D_M(\lfloor C \rfloor))$, i.e., goal function values minimized in D direction with C's having integer values $\lfloor C \rfloor$.

Black curves, such as $F(\lfloor C \rfloor = 17, D)$, draw the goal function F for the Corner position C being fixed to integer value $i = 17$ and a continuous range in D. Hence, green dots $M_D(\lfloor C \rfloor)$ mark the minima of $F(\lfloor C \rfloor, D)$ w.r.t D. $D_M(\lfloor C \rfloor)$ denotes the value of D where the minimum $M_D(\lfloor C \rfloor)$ occured.

The curve $M_D(C)$ is susceptible to have more than one local minimum, when the ramp signal is still small compared to the standard deviation of signal noise. Setting C to each increment number $\lfloor C \rfloor$ in the current Observation Window and finding all $M_D(\lfloor C \rfloor)$'s gives an overview over $F(\lfloor C \rfloor, D)$ that allows to find all candidates for local minima.

If RI is set to 'incremental resolution', it is sufficient to find the smallest $M_D(\lfloor C \rfloor)$ amongst all increment numbers $\lfloor C \rfloor \in \{i_{1W} \leq \lfloor C \rfloor < i_W\}$ and the following steps can be skipped.

5.3.3 Step b)

The line of minima $M_D(C)$ in Fig. 5-2 is now drawn from two new perspectives. In Fig. 5-3 on top the line is seen in $+D$ direction. In the bottom diagram it is looked at from above in $-F(C,D)$ direction.

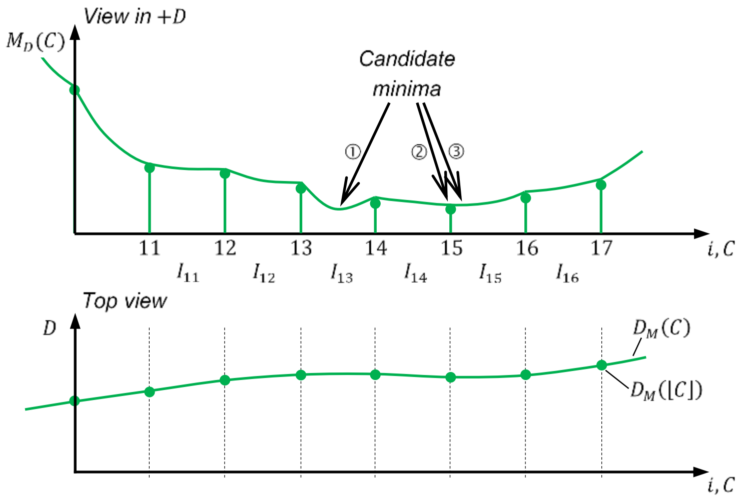


Fig. 5-3: Top: Minima M_D over C, Bottom: Minima M_D seen from above in the $D - C$ –plane.

Due to the noise in signal $x(i)$, the global minimum cannot be determined analytically but needs to be looked for in a candidate list. Fig. 5-3 on top shows that some Intervals, such as I_{13} and I_{15}, contain a local minimum, but others do not, e.g., I_{12}, I_{14}. The global minimum unpredictably either coincides with an increment border or lies somewhere in-between two borders.

The algorithm presented in this paper assumes that the global minimum of $F(C, D)$ can be found at-or-around the 10 smallest $M_D(\lfloor C \rfloor)$ values within the Window of 100 increments observed. The ascending order of minima $M_D(\lfloor C \rfloor)$ for Fig. 5-3-top looks like

$$M_D(15) < M_D(14) < M_D(16) < M_D(17) < M_D(13) < M_D(12) < M_D(11) < (the\ rest)$$

Further on, the algorithm searches among the 10 smallest $M_D(\lfloor C \rfloor)$'s for direct neighbors, as is the case in Fig. 5-3-top for Intervals I_{13} and I_{15}. It suspects that, for small neighboring $M_D(\lfloor C \rfloor)$'s, the continuous function $M_D(C)$ might have an even lower minimum in-between, denoted as $M_D(C_{Mi})$, where $i < C_{Mi} < i + 1$ is the C coordinate of that minimum.

The global minimum of $F(C, D)$ is eventually chosen among the minimum $M_D(\lfloor C \rfloor)$ and all minima $M_D(C_{Mi})$. For Fig. 5-3-top, numbers ①,②,③ mark the three minimum candidates found by the algorithm:

$$\min_{C,D} F(C, D) = \min\left({}^{②}M_D(15), {}^{①}M_D(C_{M13}), {}^{③}M_D(C_{M15})\right)$$

5.3.4 Step c)

Function $M_D(C)$ is a sequence of curves between increment numbers. The global minimum of $M_D(C)$ lies either exactly on an increment i, see Fig. 5-3, candidate ②, or somewhere between two i's, see candidates ① and ③.

Analysis of equation (5-3) reveals that $F(C, D)$ is a quadratic function in C, if D was fixed, as long as C stays within the same increment. It could be written as:

$$F(C, D) = b_2(D) \cdot C^2 + b_1(D) \cdot C + b_0$$

Unfortunately, the minima $M_D(C)$ do not line up with a constant D_M, see Fig. 5-3, Top view. Fig. 5-4 shows the same situation in detail for one interval. $D_M(C)$ follows some smooth function where it changes a little with the Corner position. Hence, $D_M(C)$ is approximated as a straight line between increments, see grey line in Fig. 5-4, Top view. The course of $M_D(C)$ above this straight line in the $C - D$ –plane will be approximated by a parabola.

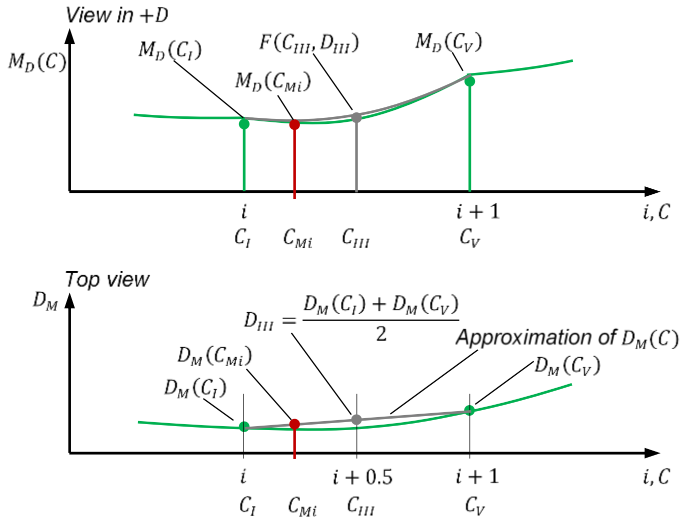


Fig. 5-4: Approximate Minimum of $M_D(C)$ inside the interval $\{I_j \dots (I_j+1)\}$.

To determine that parabola and its minimum, three triplets $\{C, D, F(C,D)\}$ are needed, where $\{C, D\}$ denotes a point in the $C-D$ –plane and $F(C,D)$ the goal function values at this point.

Two triplet pairs are already known from the boundaries of interval I_i, i and $i+1$:

Triplet I:	$C_I = i$	$D = D_M(C_I)$	$F(C,D) = M_D(C_I)$
Triplet V:	$C_V = i+1$	$D = D_M(C_V)$	$F(C,D) = M_D(C_V)$

The third value pair is chosen in the middle for both C and D:

Triplet III: $C_{III} = i + \frac{1}{2}$ $\quad D_{III} = \frac{D_M(C_I)+D_M(C_V)}{2}$ $\quad F(C_{III}, D_{III})$

With 3 points above the $C-D$ –plane, a parabola is defined, and its minimum can be determined. The estimated minimum of the goal function inside interval i is referred to by

Triplet Mi: C_{Mi} from minimum search along 3-point parabola

$$D_{Mi} = D_M(C_I) + \frac{C_{Mi}-C_I}{C_V-C_I}\left(D_M(C_V) - D_M(C_I)\right)$$

$$M_D(C_{Mi}) = F(C_{Mi}, D_{Mi})$$

Triplet Mi is valid only if the sub-incremental minimum occurs inside its own interval:

$$i \le C_{Mi} < i+1$$

The remaining error in the minimum found so far turned out to be too big to be ignored. For steep Ramps and very few Ramp increments, the gradient D is significantly overestimated, see Fig. 5-5. Therefore, Newton steps are added until reasonable convergence is reached.

A Newton step requires the gradient and Hessian of the goal function at the starting point. The goal function $F(C,D)$ is a polynomial of degree two, as long as it is evaluated inside the interval between adjacent increments.

$$F(C,D) = D^2F_{D2} + CD^2F_{CD2} + C^2D^2F_{C2D2} + DF_D + CDF_{CD} + F_0$$

The gradient in a point (C,D) is:

$$\nabla F(C,D) =: g(C,D) = \begin{pmatrix} \frac{\partial F}{\partial C} \\ \frac{\partial F}{\partial D} \end{pmatrix} = \begin{pmatrix} D^2F_{CD2} + 2CD^2F_{C2D2} + DF_{CD} \\ 2DF_{D2} + 2CDF_{CD2} + 2C^2DF_{C2D2} + F_D + CF_{CD} \end{pmatrix}$$

The Hessian $H(C,D)$

$$\nabla^2 F(C,D) =: H(C,D) = \begin{pmatrix} \frac{\partial^2 F}{\partial C^2} & \frac{\partial^2 F}{\partial C \partial D} \\ \frac{\partial^2 F}{\partial D \partial C} & \frac{\partial^2 F}{\partial D^2} \end{pmatrix}$$

$$= \begin{pmatrix} 2D^2 F_{C2D2} & 2DF_{CD2} + 4CDF_{C2D2} + F_{CD} \\ 2DF_{CD2} + 4CDF_{C2D2} + F_{CD} & 2F_{D2} + 2CF_{CD2} + 2C^2 F_{C2D2} \end{pmatrix}$$

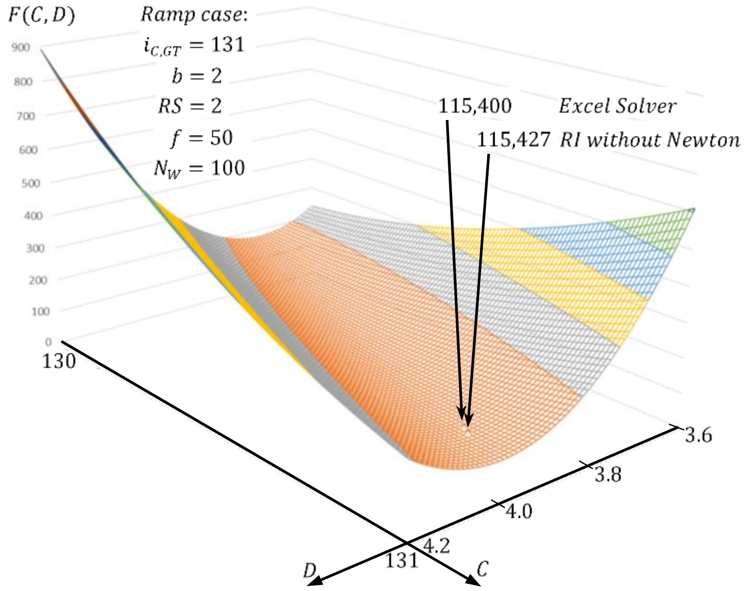


Fig. 5-5: Ramp case with insufficient RI-solution from parabolic approximation

The Taylor series of a general quadratic polynomial at multi-dimensional point p for a step q can be written as:

$$F(p+q) = \frac{1}{2} q^T H(p) q + g(p)^T q + F(p)$$

The gradient $g(p)$ and Hessian $H(p)$ derive from applying the Nabla operator to $F(p)$:

$$\nabla F(p) = H(p)q + g(p)$$
$$\nabla^2 F(p) = H(p)$$

The minimum condition for a quadratic polynomial delivers a system of linear equations which can be solved directly for the step q:

$\nabla F(p+q) = \vec{0}$ $\quad\quad \rightarrow p+q$ is a local minimum to F

$$\nabla F(p+q) = Hq + g = \vec{0}$$
$$\Rightarrow \quad Hq = -g$$

The iteration process starts at p_1, where $F(p_1), g(p_1)$ and $H(p_1)$ are determined. An optimization step q_1 is obtained by solving $H(p_1) \cdot q_1 = g(p_1)$. The step adds to the actual function point p_1 and a new iteration cycle begins.

$$p_{i+1} = p_i + q_i$$

For the present problem, the following substitutions hold true:

$$p_1 = \begin{pmatrix} C_{Mi} \\ D_{Mi} \end{pmatrix}$$
$$H(p_i) \cdot q_i = g(p_i)$$
$$q_i = \begin{pmatrix} C_{qi} \\ D_{qi} \end{pmatrix}$$

Using the following notations for H and g, a direct solution is given below:

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},\ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$q_{i,1} = -\frac{h_{22}g_1 - h_{12}g_2}{h_{11}h_{22} - h_{12}h_{21}},\ q_{i,2} = \frac{h_{21}g_1 - h_{11}g_2}{h_{11}h_{22} - h_{12}h_{21}}$$

It turned out that a single Newton step converges to $\min_{C,D} F(C,D)$.

5.3.5 Step d)

Finally, the global minimum M_{CD} of goal function $F(C,D)$ within Observation Window $i \in \{i_{1W} \dots i_W\}$ is estimated as the smallest of all incremental and sub-incremental minima:

$$M_{CD} \approx \min \begin{cases} M_D(\lfloor C \rceil) & \forall \lfloor C \rceil \in \{i_{1W} \dots i_W\} \\ M_D(C_{Mi})\ \forall & i \in \{i_{1W} \dots i_W - 1\} \end{cases}$$

The minimum is expressed by its coordinates C_M, D_M and the goal function value:

$$F_M = F(C_M, D_M) = M_{CD}$$

5.3.6 Minimum of a parabola from three equidistant points

Given a general second order polynomial, the minimum can be found from three known, equidistant evaluation points, for convenience referred to as $\{u_I, v_I\}, \{u_{III}, v_{III}\}, \{u_V, v_V\}$, see Fig. 5-6. The first derivative of $v(u)$ in the middle of each interval is:

$$u_{II} = \frac{u_I + u_{III}}{2},\ u_{IV} = \frac{u_{III} + u_V}{2}$$

$$v'_{II} = \frac{v_{III} - v_I}{u_{III} - u_I},\ v'_{IV} = \frac{v_V - v_{III}}{u_V - u_{III}},\ v'_{III} = \frac{v'_{II} + v'_{IV}}{2}$$

The second derivative and $v'' = \frac{v'_{IV} - v'_{II}}{u_{IV} - u_{II}} = const$.

The derivative function

$$v'(u) = v'_{II} + (u - u_{II}) \cdot v''(u)$$

The minimum of $v(u)$, referred to as $(u_M, v(u_M))$ coincides with $v'(u_M) = 0$, thus:

$$u_M = u_{II} - \frac{v'_{II}}{v''}$$

The parabola function $v(u)$ is derived by integrating $v'' = const$ twice and using known values $\{u_{III}, v_{III}\}$, and v'_{III}:

$$v(u) = v'' \cdot (u - u_{III})^2 + v'_{III} \cdot (u - u_{III}) + v_{III}$$

Finally, the function value at its minimum:

$$v_M = v'' \cdot u_M \cdot (u_M - u_{III}) + v'_{III} \cdot (u_M - u_{III}) + v_{III}$$

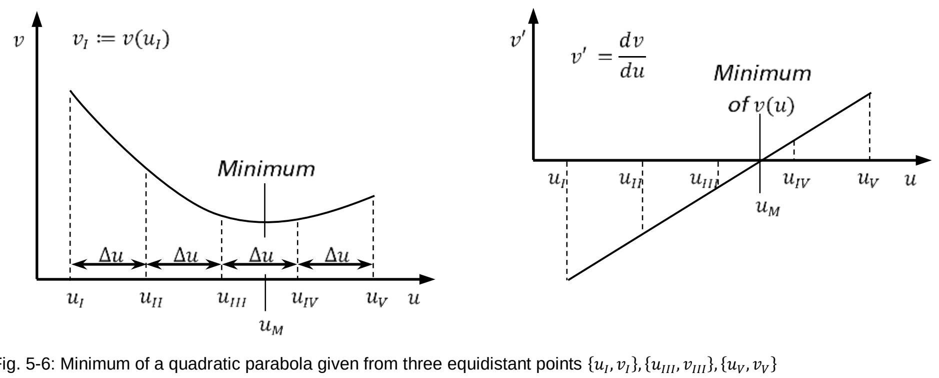


Fig. 5-6: Minimum of a quadratic parabola given from three equidistant points $\{u_I, v_I\}, \{u_{III}, v_{III}\}, \{u_V, v_V\}$

5.4 Implementation notes

5.4.1 Notation

The number of increments to be observed is referred to as N which abbreviates N_W.

Incoming increments are mapped onto a cyclic array that has dimension $2 \cdot N$ and array index u. For compatibility with investigations done within Matlab and/or MS Excel, array indices start with 2 for the first element.

Increments i and u are distant by shift f_{iu}, with

$$i = u + f_{iu}$$

The actual Observation Window spans over increments $u \in [u1W, uW]$.

Sensor data in the Observation Window is tested with Corner positions uCt that takes all 100 integer increment values $u1W \dots uW$, thus:

$$\lfloor uCt \rfloor = uCt$$

Results from test Corner positions are stored in arrays at the index equal to the Corner position. E.g.: the value of error function $F(C,D)$ at a test Corner position uCt and a test Gradient Dt is stored in array $F[]$ at index uCt:

$$F[uCt] = F(uCt, Dt) = \sum_{u=u_{1W}}^{u_W} \begin{cases} u < uCt: & \big(x(u)\big)^2 \\ u \geq uCt: & \big(x(u) - Dt \cdot (u - uCt)\big)^2 \end{cases}$$

5.4.2 Fast calculation of standard deviations

If sums of incremental data are already available, the following identities can be used.

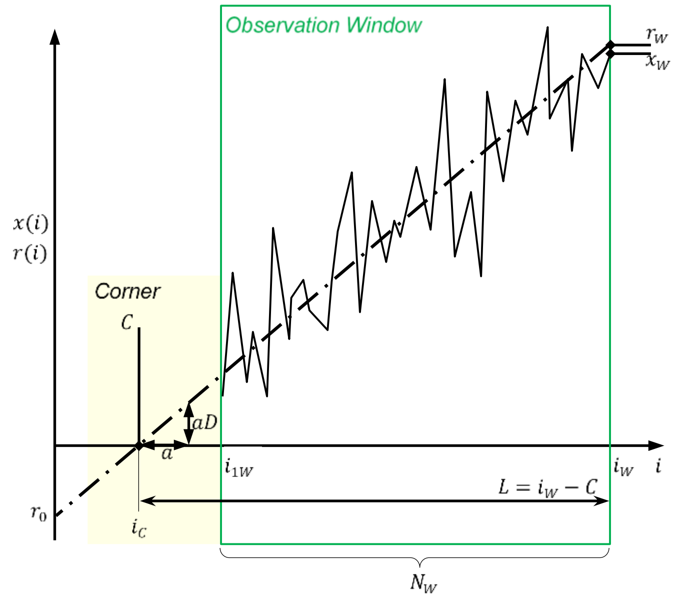
$$\sigma_y = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (y(i) - \bar{y})^2} \quad \text{with } \bar{y} = \frac{1}{N} \sum_{i=1}^{N} y(i)$$

$$\Rightarrow \quad \sigma_y = \sqrt{\frac{1}{N-1} \left[\sum_{i=1}^{N} y(i)^2 - 2\bar{y} \cdot \sum_{i=1}^{N} y(i) + N \cdot \bar{y}^2 \right]} \tag{5-6}$$

Version (5-6) allows to simultaneously sum $y(i)$ and $y(i)^2$ rather than having two subsequent sums to calculate.

5.4.3 Ramp Corner in the past

If the Corner had occurred more than N increments in the past, the Observation Window shows a noisy ascending straight line, see Fig. 5-7.



Ramp:

$$r(i) = D \cdot (i - C)$$

Linear Regression:

$$\bar{\iota} = \frac{1}{N_W} \sum_{i=i_{1W}}^{i_W} i$$

$$\bar{x} = \frac{1}{N_W} \sum_{i=i_{1W}}^{i_W} x_i$$

$$D = \frac{\sum_{i=i_{1W}}^{i_W} (i - \bar{\iota})(x_i - \bar{x})}{\sum_{i=i_{1W}}^{i_W} (i - \bar{\iota})^2}$$

$$r_0 = \bar{x} - D \cdot \bar{\iota}$$

$$C = \frac{-r_0}{D} = \bar{\iota} - \frac{\bar{x}}{D}$$

Fig. 5-7: Observation Window has passed the Corner

The situation is characterized by:

$$\lfloor C \rfloor + 1 \leq i_{1W}$$

The minimization problem simplifies to

$$\min_{C,D}\big(F(C,D)\big) = \min_{C,D} \sum_{i=\mathrm{i_{1W}}}^{\mathrm{i_W}} \big(x(i) - D\cdot(i - C)\big)^2 \tag{5-7}$$

Developing the sum leads to the polynomial:

$$F(C,D) = F_0 + D \cdot F_D + D^2 \cdot F_{D2} + CD \cdot F_{CD} + CD^2 \cdot F_{CD2} + C^2D^2 \cdot F_{C2D2} \tag{5-8}$$

The Ramp Identification Algorithm already calculates coefficients $F_0, F_D, F_{D2}, F_{CD}, F_{CD2}, F_{C2D2}$ for test Corners inside the Window. By placing the test Corner to the left, outside the Window, the very same coefficients can be used to find the linear regression inside the Window. To include all Sensor data, the lower summation limit simply becomes i_{1W}:

$$F_{0\_} = \sum_{i=i_{1W}}^{i_W} (x)^2$$

$$F_{D\_} = -2 \sum_{i=i_{1W}}^{i_W} (xi)$$

$$F_{D2\_} = \sum_{i=i_{1W}}^{i_W} (i)^2 = \frac{i_W(i_W+1)(2i_W+1)}{6} - \frac{(i_{1W}-1)\cdot i_{1W}\cdot(i_{1W}+1)}{6}$$

$$F_{CD\_} = 2 \sum_{i=i_{1W}}^{i_W} (x)$$

$$F_{CD2\_} = -2 \sum_{i_{1W}}^{i_W} (i) = -\big(i_W^2 + i_W - (i_{1W} - 1)^2 - (i_{1W} - 1)\big)$$

$$F_{C2D2\_} = \sum_{i_{1W}}^{i_W} (1) = i_W - (i_{1W} - 1)$$

The formula for the Gradient D develops into similar sums. The Window spans over N_W increments.

$$N_W = i_W - i_{1W} + 1$$

Mean values can be expressed with the above coefficients:

$$\bar{x} = \frac{1}{N_W} \sum_{i=i_{1W}}^{i_W} (x) = \frac{F_{CD}}{2N_W} \qquad \bar{\iota} = \frac{1}{N_W} \sum_{i=i_{1W}}^{i_W} (i) = \frac{F_{CD2}}{-2N_W}$$

Finally, the Gradient and Corner position are:

$$D = \frac{\sum_{i=i_{1W}}^{i_W} (i-\bar{\iota})(x_i-\bar{x})}{\sum_{i=i_{1W}}^{i_W} (i-\bar{\iota})^2} = \frac{\sum_{i=i_{1W}}^{i_W} (xi) - \bar{x}\sum_{i=i_{1W}}^{i_W} (i) - \bar{\iota}\sum_{i=i_{1W}}^{i_W} (x) + N_W\cdot\bar{\iota}\bar{x}}{\sum_{i=i_{1W}}^{i_W} (i^2) - 2\bar{\iota}\sum_{i=i_{1W}}^{i_W} (i) + N_W\cdot\bar{\iota}^2} = \frac{\sum_{i=i_{1W}}^{i_W} (xi) - \bar{x}\cdot\frac{F_{CD2}}{-2} - \bar{\iota}\sum_{i=i_{1W}}^{i_W} (x) + N_W\cdot\bar{\iota}\bar{x}}{F_{D2} - 2\bar{\iota}\cdot\frac{F_{CD2}}{-2} + N_W\cdot\bar{\iota}^2}$$

$$C = \bar{\iota} - \frac{\bar{x}}{D}$$

5.4.4 Usage of RI implemented as C++ class

RI is implemented as a C++ class `SR__RI`. The caller of its methods is referred to as Controller. A sample declaration of an instance of SR__RI and related input and output data is:

```
SR__RI theRI;                  // the RI object instance
RI_expectations RIexpects; // struct containing expectations to RI
RI_results RIresults;      // struct containing RI results
enumRIstates RIstate;      // Functional state of theRI object
```

The Controllers expectations about the observed Ramp Process are transmitted by calling

```
theRI.RI_set(*RIexpects);
```

Whenever a new data point $X(i)$ is available, the Controller sends it to the RI instance by calling

```
RIstate = theRI.RI_update(i,X, *RIresults);
```

The function returns `RIstate,` which equals one of the RI states listed below. Further results return via a `struct RIresults`, see documentation for details.

`RI_new`	RI object is created. Signal data buffer is empty.
`RI_set`	Expectations are set or modified. No change to Signal data buffer.
`RI_reset`	Expectations are set. Signal data buffer is emptied.
`RI_collects`	RI object has less than 100 Signal data collected
`RI_noRamp`	no Ramp detected: either “not yet” or “not anymore”
`RI_detected`	Ramp detected: iC, sigma,L,D,H,F_CD,
`RI_identified`	Ramp Parameters and Uncertainties are determined
`RI_unexpected`	Ramp outside expectations D, iE, X>X_Break
`RI_stopped`	i<i1S, i>iS
`RI_error`	Erroneous input data

5.4.5 Performances of RI

An instance of class SR__RI requires less than 2 kB of RAM.

Function `theRI.RI_update()` executes roughly

24	function calls: 1x `sqrt()` , 6x `pow()` ,17x methods of `theRI`
2 k	`for` loops
2 k	`if` conditions
7 k	basic arithmetic operations on `double` and `long long` variables
9 k	‘`=`’ assignments
13 k	`array[index]` element accesses

6 Research on Q1, Q2 and Q3

6.1 Terminology

This chapter defines terms that are needed to analyze the results of Ramp Identification solver ML on synthesized Normal Noisy Ramps. Each NNRamp spans over 250 consecutive controller increments allowing to analyze 100 Data points as a shifting observation window, indexed ‘W’. Normal Noisy Ramps are synthesized as input data to ML. While ‘Ground Truth’ information is not accessible to ML, it is available in this chapter to analyze the results of ML.

As expected, the accuracy of Ramp Identification is degraded if Random Noise was added to a Ramp signal. To understand that degradation, the very same Ground Truth Ramp has been Ramp-Identified $N_{rs} = 16000$ times with as many different Random Sets. Whenever comparing all these N_{rs} results to each other, the tilde sign ‘~’ will be used as accent.

Random Set	Recorded set of random numbers with normal distribution μ=0, σ=1.
Ramp Set	Set of N_{rs} Ramps with same Ground Truth. Each Ramp in the set is altered by a different Random Set.
Bias	mean error of ML-results in a Ramp Set w.r.t Ground Truth
Noise	standard deviation of ML-results in a Ramp Set
σ	Ramp Noise
$\sigma_{p,q}$	Ramp Noise estimation - over a certain data set ‘p’ - derived by some method ‘q’
$\sim$	refers to all N_{rs} different random number sets, short: ‘for a Ramp Set’
e_σ	absolute error of a single ML result $e_\sigma = \sigma_{ML} - \sigma_{GT}$.
ε_σ	relative error of a single ML result $\varepsilon_\sigma = (\sigma_{ML} - \sigma_{GT})/\sigma_{GT}$
$\tilde{\mu}_\sigma$	mean error (“Bias”) of σ over a Ramp Set. $\tilde{\mu}_\sigma$ is written in ASCII characters: ‘mrs_sig’
$\tilde{\sigma}_\sigma$	standard deviation (“Noise”) of σ over a Ramp Set. $\tilde{\sigma}_\sigma$ is written in ASCII characters: ‘srs_sig’
U_σ	uncertainty in the estimation of σ: $\sigma_{ML} - U_\sigma < \sigma_{GT} < \sigma_{ML} + U_\sigma$ holds true with confidence fd
fd_σ	Confidence in the estimation of σ
σ_{ML}	Ramp Noise derived by ‘ML-filtering’
σ_W	Ramp Noise for the data set in a Window of observation
σ_{GT}	Ground Truth Ramp Noise,
$\sigma_{GT,W}$	standard deviation of the Random Set inside the Window
$\tilde{\mu}_\sigma, \tilde{\mu}_L, \tilde{\mu}_D, \tilde{\mu}_H$	Bias in ML estimations of σ, L, D, H for a Ramp Set
$\tilde{\sigma}_\sigma, \tilde{\sigma}_L, \tilde{\sigma}_D, \tilde{\sigma}_H$	Noise in ML estimations of σ, L, D, H for a Ramp Set

For comparability, a fixed color code is being used within all diagrams and images in this chapter to distinguish between Ramp Gradient exponents $b = \log_2(d)$:

b= -6 b= -5 b= -4 b= -3 b= -2 b= -1 b= 0 b= 1 b= 2

Ramp Identification solver’s direct results are:

ML-filtered standard deviation	σ_{ML}
Ramp Corner increment	$i_{C,ML}$
Ramp Gradient	D_{ML}
Goal Function value	$F_{C,D}$

With i_W, the most recent data increment in the Window, Length and Height of the Ramp are

Ramp Length equals $L_{ML} = i_W - i_{C,ML}$

Ramp Height equals $H_{ML} = L_{ML} \cdot D_{ML}$

Conversion from ML results to Normal Noisy Ramp dimensions:

Normal Ramp Height $h_{ML} = H_{ML}/\sigma_{ML}$

Normal Ramp Gradient $d_{ML} = D_{ML}/\sigma_{ML}$

6.2 *Q1: Ramp Detection*

6.2.1 Signal-to-Noise ratio

If H_{ML} is the Ramp Signal at the current point in time estimated by ML, and σ_{ML} measures the standard deviation of Signal Noise, the Signal-to-Noise ratio can be defined as

$\frac{H_{ML}}{\sigma_{ML}}$ Signal-to-Noise ratio

It is determined from output values of ML without having any Ground Truth knowledge about the Ramp or the sensor. With growing Signal-to-Noise ratio, the occurrence of a Ramp becomes increasingly significant.

6.2.2 False-positives threshold

The dimensionless quotient H_{ML}/σ_{ML} expresses ML's estimation of the Ramp Height as a multiple of the estimated signal noise. Positive Height estimations H_{ML} may indicate the presence of a Ramp or they may just occur due to signal noise. To discriminate between outliers and the occurrence of a Ramp, a threshold for H_{ML}/σ_{ML} needs to be found. Note: this has already been stated in (4-1), since H_{ML} equals $r_{W,ML}$. The threshold must provide a maximum of true-positive test results and a minimum, or better: zero false-positive results.

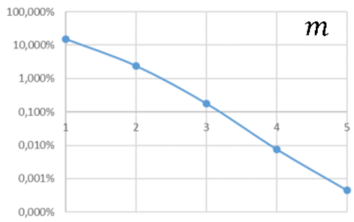
RI tests the dimensionless threshold m against its estimation for the Signal-to-Noise ratio:

If $\frac{H_{ML}}{\sigma_{ML}} > m$ then RI discovered the presence of a Ramp in a data set

Different values for m were applied to the experimental results and compared with Ground Truth data. The number of false-positive detections was counted and expressed as percentage of the total number of ML runs:

Table 1: Percentage of false-positives for different ramp detection thresholds m

m	false-positives
1	15.3603 %
2	2.4009 %
3	0.1786 %
4	0.0004 %
5	0.0000 %



For $m = 3$, the number of false-positive Ramp discoveries is $< 0.2\%$. When $m \geq 5$ is reached, Ramp discoveries can be trusted blindly. Thus, critical processes should use $m = 5$ as Signal-to-Noise threshold for Ramp Identification, whereas fault tolerant processes can be built on threshold $m = 3$.

$\frac{H_{ML}}{\sigma_{ML}} > 3$ Ramp Detection with $>$ 99.8% confidence (6-1)

$\frac{H_{ML}}{\sigma_{ML}} > 5$ Ramp Detection with $\approx$ 100.00% confidence (6-2)

6.2.3 Ramp Detection Length using Ground Truth

With each new controller cycle, the Observation Windows shifts by one increment to the right. Ground Truth data tells whether the Ramp Corner lies inside the Window. The arising question is: how many Ramp increments must lie inside the Window before RI reliably detects a Ramp? In other words, which Ground Truth Ramp Length is necessary for the Ramp Detection test to hold true?

For a steeper ramp, this Length is expected to be smaller. Yet, for higher noise levels, the Length will increase. Fig. 6-1 shows the number of false-negatives Ramp detections for different Ramp Gradients over the Ground Truth Ramp Length.

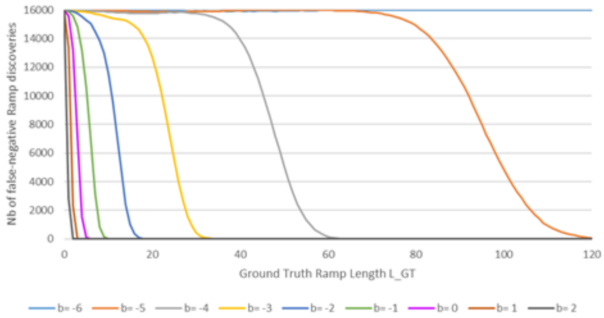


Fig. 6-1: False-negative Ramp discoveries with $m = 3$ decrease with growing Ramp Length.

Note that the Window is $N_W = 100$ increments wide for all data sets. For a very flat ramp such as $b = -5$, ML still delivers false-negative results even if the Window is entirely spanning over Ramp increments, see example in Fig. 6-2.

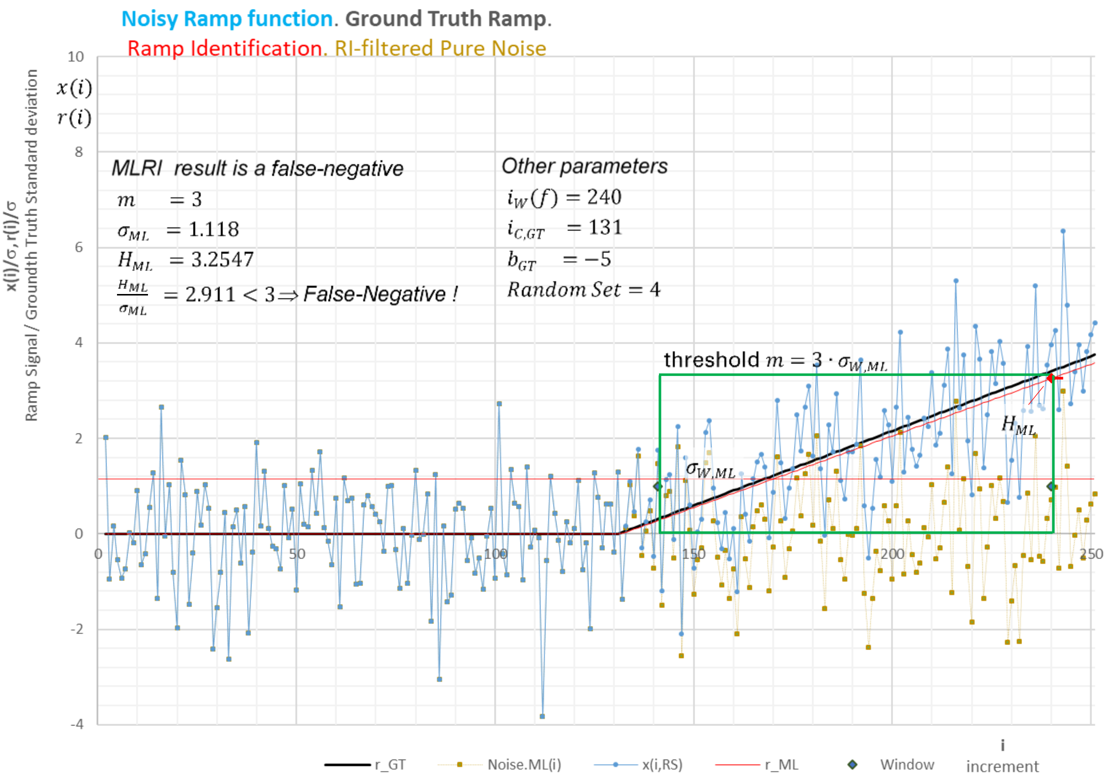


Fig. 6-2: False-negative result for a ramp that is fully inside the window

The minimum number of Ramp increments inside the Window necessary to produce no false-negatives was determined for different thresholds m and different Ramp Gradients d, see Fig. 6-3. Remind: “The number of Ramp increments inside the Window” is just another way to express “Ramp Length”, the Ramp Length necessary to detect the Ramp is referred to as “Ramp Detection Length” L_{RD}:

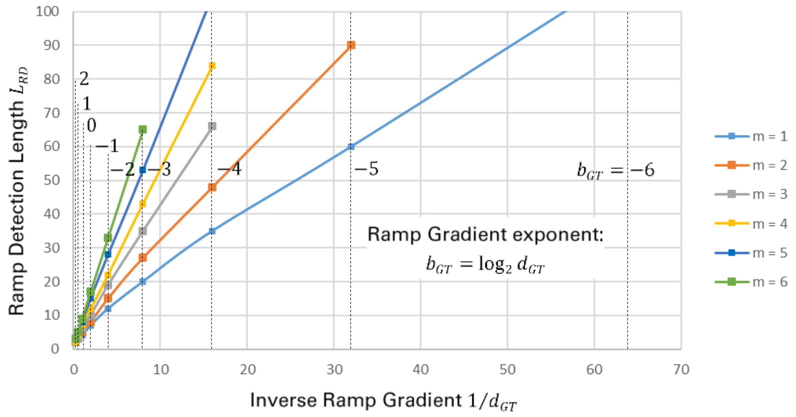


Fig. 6-3: Necessary Ramp Length for reliable Ramp detection as a function of Ramp Gradient.

The case “Ramp Detection criterion $m = 3$”

$$m = 3 < \frac{H_{ML}}{\sigma_{ML}}$$

is now being analyzed. Table 2 shows experimental results for the Ramp Detection Length L_{RD} with threshold $m = 3$. The acceptable rate of false-negatives was set to $< 0.2\%$. Furthermore, a linear regression of L_{RD} with confidence >99.8% was found, used round-up brackets ⌈ ⌉, by:

$$L_{RD}(fd > 99.8\%) = \left\lceil \frac{4}{d_{GT}} + 2 \right\rceil \tag{6-3}$$

Table 2: Ramp Detection Length L_{RD} for *m*=3

b	${}^{1}/_{d_{GT}} = 2^{-b}$	$L_{RD}(experiment)$	$L_{RD}(regression)$	False negatives for $m = 3$
-4	16	66	66	0.1 %
-3	8	35	34	0.1 %
-2	4	19	18	0.1 %
-1	2	10	10	0.1 %
0	1	6	6	0.0 %
1	0.5	3	4	0.1 %
2	0.25	2	3	0.0 %

If threshold $m = 5$ is being used, there is almost absolute certainty about Ramp Detection. However, it takes longer for all Noisy Ramps to pass over that threshold:

$$L_{RD}(fd > 99.99\%) = \left\lceil \frac{6.5}{d_{GT}} + 1.5 \right\rceil \tag{6-4}$$

Table 3: Ramp Detection Length L_{RD} for *m*=5.

b	${}^{1}/_{d_{GT}} = 2^{-d}$	$L_{RD}(experiment)$	$L_{RD}(regression)$	False negatives for $m = 5$
-4	16	105	106	0.01 %
-3	8	53	54	0.00 %
-2	4	28	28	0.00 %
-1	2	15	15	0.00 %
0	1	8	8	0.00 %
1	0.5	5	5	0.00 %
2	0.25	3	4	0.00 %

Row $b = -2$ of Table 3 should be read like this: If Ramp Detection threshold is chosen to be $m = 5$ and the true Ramp Gradient equals $d_{GT} = 2^b = 0.25 \cdot \sigma_{ML}\ per\ increment$, than it takes a true Ramp Length $L_{GT} = 28$ until that threshold is reached for 99.99% of Noisy Ramps. Note, this statement refers to Ground Truth, both for Gradient and Ramp Length. Thus, Table 3 is of no practical use at runtime, but at design-time (6-3) resp. (6-4) allow forecasting whether Ramp Detection could work in a supposed application scenario.

6.3 Q2-a: Bias & Noise Horizon

6.3.1 Ramp Detection randomness

Applying different Random Sets to the same Ground Truth Ramp causes the Ramp Detection criterion *Q1* to be randomized itself. Consider Fig. 6-4, grey dots of Noisy Ramps with $b = -4$. Some of them see *Q1* firing as early as $L_{GT} = 30$. At $L_{GT} = 51$, 75% of all Noisy Ramps have *Q1*=1 while at $L_{GT} = 60$ the latest Noisy Ramps finally signal *Q1=true*.

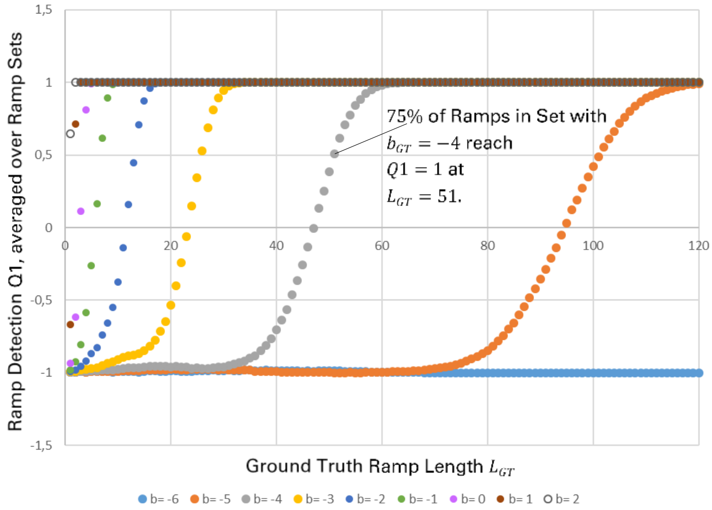


Fig. 6-4: Ramp Identification result *Q1*, averaged over all Random Sets, plotted against Ground Truth Ramp Length L_{GT}

6.3.2 Incremental Ramp Detection versus Ground Truth

Variable $Q1$, such as defined in (4-1), can now be written as

$$Q1 = \left(H_{ML} >^{?} m \cdot \sigma_{ML}\right)$$

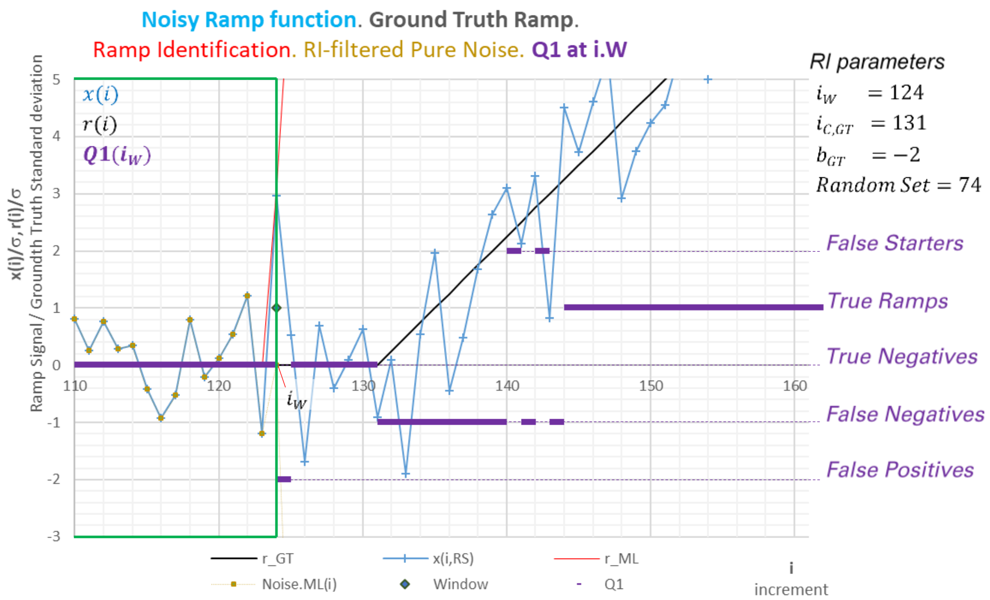


Fig. 6-5: Sequence of Q1 results while the Observation Window shifts to the right

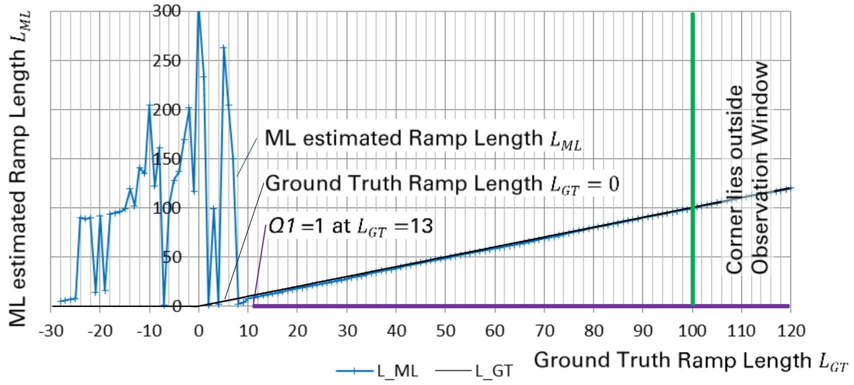
By comparing the result of $Q1$ to Ground Truth, five cases can be distinguished, see purple dashes in Fig. 6-5. When the Observation Window, green line in Fig. 6-5, reaches increment $i_W = 124$, ML causes a *False Positive Q1*, i.e. *Q1* fires *True* even though the Corner $i_C = 131$ lies still outside. At increment 132, $Q1$ fires *False* even though the Corner is now inside the Window. At increments 140 and 142, $Q1$ correctly fires *True* for having detected a Corner in the Window, but just one increment later retracts to a *False*. This behavior reminds of a false start in athletic competitions. Only beginning with increment 145, $Q1$ is continuously sure about having found the Ramp Corner inside the Observation Window. This allows for a modified definition of $Q1$ results:

$$Q1 = \begin{cases} 2: False\ Starter\ \ (FS) \\ 1: True\ Ramp\ \ \ \ \ (TR) \\ 0: True\ Negative (TN) \\ -1: False\ Negative\ (FN) \\ -2: False\ Positive\ \ (FP) \end{cases}$$

6.3.3 ML's incremental estimation of Ramp Length L

ML estimates the Ramp Length L_{ML} from data points inside the actual Observation Window, i.e., from the 100 most recent data points received from the sensor. Fig. 6-6, top diagram, shows a typical curve of L_{ML} plotted against the Ground Truth Ramp Length L_{GT} for a sample Ramp. Note that the Ramp Corner coincides with $L_{GT} = 0$. At $L_{GT} = 0$, the Observation Window contains data from increments $-99 \dots 0$, i.e., only Floor and no Ramp data points are inside the Window. By consequence ML delivers an arbitrary result, as for any other $L_{GT} \le 0$.

When the Observation Window has passed by some increments over the Corner, the estimated Ramp Length L_{ML} becomes much more stable. Beginning at $L_{GT} = 13$, *Q1* fires '$1: True\ Ramp\ (TR)$', marked in purple color. The estimation error $e_L = L_{ML} - L_{GT}$ gradually approaches *zero* over the following increments until $L_{GT} = 99$, see Fig. 6-6, lower diagram. At $L_{GT} = 100$, marked with a green vertical line, the Observation Window contains data points from increments $1 \dots 100$. The Corner is already outside the Window and no data point represents the Floor level. The error e_L then gradually regains in magnitude and in chaotic behavior until the last $L_{GT} = 120$.



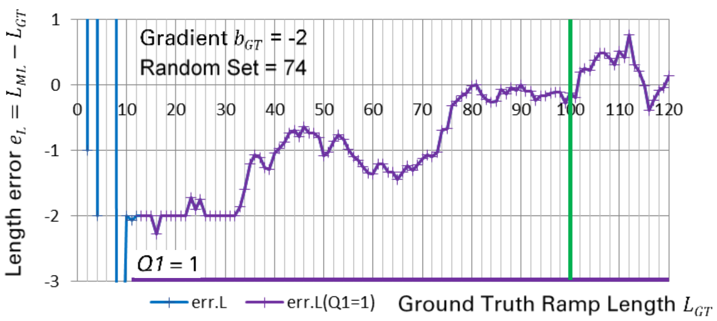


Fig. 6-6: ML's Ramp Length L_{ML} estimation and its error $e_L = L_{ML} - L_{GT}$, drawn against Ground Truth Ramp Length L_{GT} for a Ramp sample.

The estimation error e_L tends towards *zero* which means that the ML estimated Ramp Length L_{ML} comes quite close to Ground Truth Ramp Length L_{GT}. Thus, plotting e_L against $\lfloor L_{ML} \rfloor$ instead of L_{GT} shows the same tendency, see Fig. 6-7. This operation will be referred to as '*L_{ML}-alignment*'. The main advantage of *L_{ML}-alignment* is that it can be done without having

Ground Truth knowledge about the Ramp Length. Its major drawback is poor accuracy for small L_{ML}. To give an example: where ML estimated $\lfloor L_{ML} \rfloor = 11$, it actually was $L_{GT} = 13$. Another case: a gap at $\lfloor L_{ML} \rfloor = 14$ indicates that ML never estimated this particular Length.

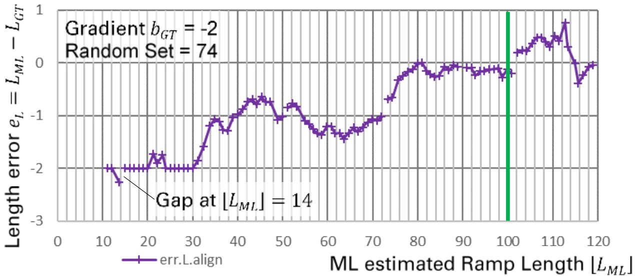


Fig. 6-7: Error in ML's Ramp Length estimation for the Ramp sample, plotted against the Ground Truth Ramp Length.

6.3.4 L_{ML}-error patterns in Ramp Sets

6.3.4.1 L_{ML}-alignment on a Ramp Set

When ML is executed incrementally on a Noisy Ramp, it delivers $\sigma_{ML}, L_{ML}, D_{ML}, H_{ML}, Q1$, or in short: ML-results. In case of synthetic Noise Ramps, errors in ML-results are known. A Ramp Set is a set of synthetic Noisy Ramps which all have the same Ground Truth Ramp in common but with different Noise added to them.

Ramp Sets were introduced to cancel out the effect of individual Ramp Noise and to discover statistically stable patterns in Ramp Identification results using ML. For that purpose, ML-results from all Ramps in the Set have to be collected and related to each other. Since Ground Truth information, e.g. Ramp Length L_{GT}, is not available for Runtime Noisy Ramps, the relating of ML-results shall be based on the ML-estimated Ramp Length L_{ML}. ML-results of different Ramps that occurred at the same $\lfloor L_{ML} \rfloor$ are collected into a sample 'basket'. This operation is referred to as L_{ML}-alignment. Once L_{ML}*-alignment* is accomplished on the entire Ramp Set, statistical analysis may begin.

ML delivers rational numbers for L_{ML} which means that "the same" L_{ML} never occurs twice. Instead, to relate Ramps to each other based on "similar" L_{ML}, L_{ML}-baskets were defined and labeled with integer numbers. A ML-result is then collected into a L_{ML}-basket labeled with that number that equals $\lfloor L_{ML} \rfloor$ of the ML-result, which is L_{ML} rounded down to the nearest lower integer, also expressed by the function call $floor(L_{ML})$.

After L_{ML}-alignment on a Ramp Set, all ML-estimation errors $e_L = L_{ML} - L_{GT}$ with same $\lfloor L_{ML} \rfloor$ are collected into baskets. The distribution of errors in L_{ML} is expressed by

$e_{L.a}\lfloor L_{ML} \rfloor$	error of an individual Ramp related to its ML-result $floor(L_{ML})$
$\tilde{\mu}_{L.a}\lfloor L_{ML} \rfloor$	mean value of all $e_{L.a}\lfloor L_{ML} \rfloor$ in a L_{ML}-aligned Ramp Set, i.e.: at $\lfloor L_{ML} \rfloor$
$\tilde{\sigma}_{L.a}\lfloor L_{ML} \rfloor$	standard deviation of all $e_{L.a}\lfloor L_{ML} \rfloor$ in a L_{ML}-aligned Ramp Set

6.3.4.2 Q1 discontinuity after L_{ML}-alignment

A ML-result $Q1 = 1$ signals that a True Ramp is being detected. Based on the Ramp sample depicted in paragraph 6.3.3, L_{ML}-alignment steps applied to ML-results $Q1 = 1$ are represented in Table 4.

Table 4: Alignment of ML results to L_{ML}

L_ML aligned to Ground Truth

L_GT	L_ML	Q1.m3
8	2.394886	-1
9	3.740394	2
10	8	-1
11	8.926433	2
12	10	-1
13	11	1
14	12	1
15	13	1
16	13.731162	1
17	15	1
18	16	1
19	17	1
20	18	1

L_{ML}-alignment to L_{ML}-baskets $\lfloor L_{ML} \rfloor$

Basket n°	L_ML	Q1.aln.m3
8		
9		
10		
11	11	1
12	12	1
13	13.731162	1
14	(none)	set to 1
15	15	1
16	16	1
17	17	1
18	18	1
:	:	:
:	:	:

In case that two ML-estimations $\lfloor L_{ML} \rfloor$ are falling into the same L_{ML}-basket, the bigger L_{ML} result is kept, see Table 4, L_{ML}-basket n° 13: result $L_{ML} = 13$ is overwritten by the next incremental result $L_{ML} = 13.731162$.

L_{ML}-alignment sometimes leaves baskets empty even though shorter L_{ML}'s have been found previously, see Table 4 for L_{ML}-basket 14. In this case the Ramp Detection criterion is set to 1 directly to keep $Q1 = 1$ continuously.

6.3.4.3 Length error and counting Ramps with Q1=1

Fig. 6-8 shows ML-results for a sample Ramp Set. The results were L_{ML}-aligned and plotted against L_{ML}-baskets in ascending order, to make it short: "plotted against L_{ML}". Blue dots on blue lines indicate:

- mean L_{ML}-aligned error $\tilde{\mu}_{L.a}$, averaged over the Ramp Set, plotted against L_{ML}
- mean error $\tilde{\mu}_{L.a}$ plus / minus standard deviation $\tilde{\sigma}_{L.a}$ in the Ramp Set at $\lfloor L_{ML} \rfloor$
- the maximum positive and maximum negative error in the Ramp Set.

The mean error $\tilde{\mu}_{L.a}$ is negative at the beginning and close to *zero* for $L_{ML} \geq 21$. The standard deviation $\tilde{\sigma}_{L.a}$ has its maximum near $L_{ML} = 21$ and decreases to a minimum at $L_{ML} = 100$. Error extrema are mostly lying within $\tilde{\mu}_L \pm 3 \cdot \tilde{\sigma}_L$ with a certain asymmetry.

Fig. 6-8, as a reminder, includes in purple color the error behavior of the Ramp sample described previously. For this particular Ramp, $Q1 = 1$ fired true for the first time when Ramp Length was ML-estimated $L_{ML} = 11$.

$Q1 = 1$ is now going to be used as a criterion to divide a Ramp Set into Ramps with $Q1 = 1$ and those with $Q1 \neq 1$. Remind that $Q1 = 1$ indicates "True Ramp detected". Some Ramps are reaching true detection sooner, others later. Some Ramps are truly detected at $\lfloor L_{ML} \rfloor = 11$ but other Ramps have not yet reached $Q1 = 1$ even until $\lfloor L_{ML} \rfloor = 24$.

The Ramp Set given in Fig. 6-8 consists of N_{rs} =16000 synthetic Noisy Ramps. 53.5% of them reach $Q1 = 1$ at ML-estimated Ramp Length $\lfloor L_{ML} \rfloor = 11$. A brown triangle in Fig. 6-8 marks this finding. To name it shortly, the "Q1-tile" at $L_{ML} = 11$ equals 53.5%. At $L_{ML} = 21$, the "Q1-tile" has become >99%.

The term "Q1-tile" was inspired by "quantile" and is defined as follows:

$$Q1tile\lfloor L_{ML} \rfloor = \frac{Number\ of\ Ramps\ in\ a\ \ L_{ML}{-}aligned\ Set\ with\ Q1{=}1\ at\ \lfloor L_{ML} \rfloor}{Total\ number\ of\ Ramps\ in\ the\ Set}$$

A most interesting coincidence can be observed in Fig. 6-8: Q1-tile and $\tilde{\mu}_{L.a}$ reach their final value virtually at the same $\lfloor L_{ML} \rfloor$. At $\lfloor L_{ML} \rfloor = 16$ the Q1-tile equals 90% while $\tilde{\mu}_{L.a}$ reaches $zero$ but keeps growing. From $\lfloor L_{ML} \rfloor = 21$ and Q1-tile >99% onwards, the mean error asymptotically decreases against $\tilde{\mu}_{L.a} = 0$. This approximate coincidence is now expressed as an equivalence between two conditions:

$$Q1tile(L_{ML}) >^{?} 0.99 \quad \Leftrightarrow \quad \tilde{\mu}_{L.a}(L_{ML}) \approx^{?} 0$$

The explanation to this observation is: criterion $Q1 = 1$ excludes Ramps that take more time to reach True Ramp detection. Length grows over time, hence Q1-tile is biased towards shorter $\lfloor L_{ML} \rfloor$. In the beginning phase there is no way to find out whether a Ramp parameter estimation is accurate or biased by the noise. However, this bias can be overcome simply by waiting until $\lfloor L_{ML} \rfloor = 21$, when ML-results, on average, come close enough to Ground Truth.

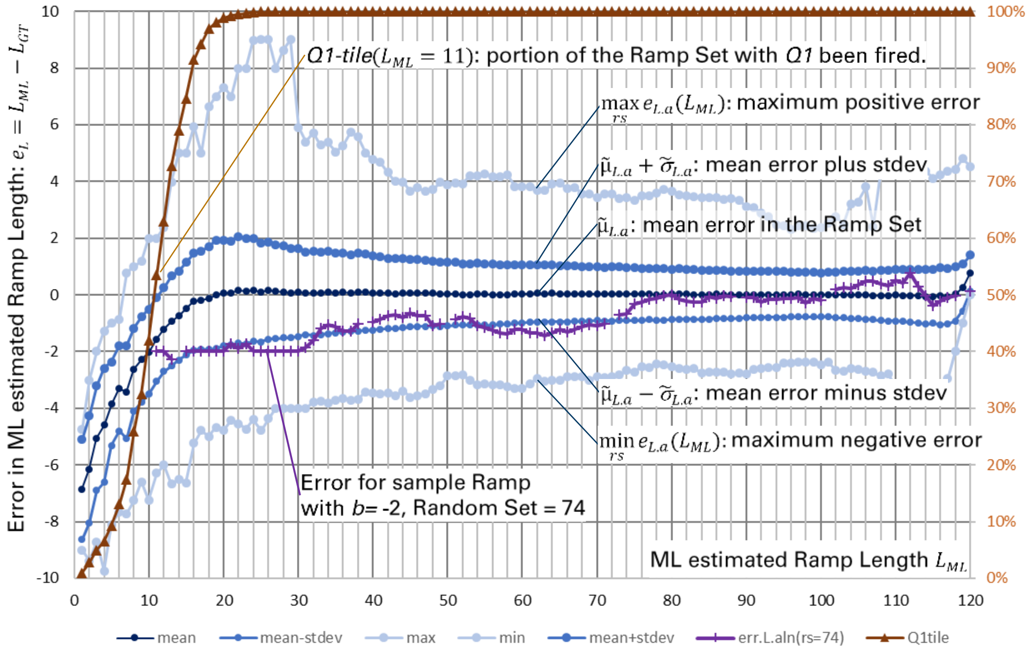


Fig. 6-8: Sample error. L_{ML}-*aligned* error in L_{ML}: mean ±standard deviation and extrema, over the Random Set, plotted against L_{ML}. *Q1–tile*.

6.3.4.4 Influence of the Ramp Gradient on Q1-tile

A simulated Ramp Set of $N_{rs} = 16000$ Random Series per Ground Truth Ramp and 9 different Ramp Gradients was analyzed to find patterns in Q1-tile, see Fig. 6-9. The brown Q1-tile curve for $b = -2$ is a identical to the one in Fig. 6-8.

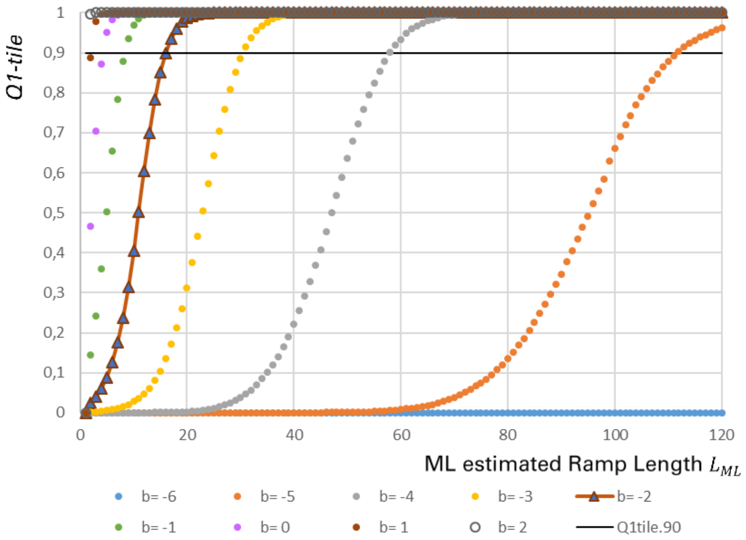


Fig. 6-9: *Q1-tile* for different Ramp Gradients Exponents b, plotted against the ML estimated Ramp Length L_{ML}

To compare results for different Ramp Gradients, three test thresholds for *Q1-tile* were defined at 90%, 99% and 100%, referred to as $Q1tile.90$, $Q1tile.99$ and $Q1tile100$. The Ramp Length where the Q1-tile of a Ramp Set crosses over the 90% threshold is referred to as $L_{ML}(Q1tl90)$, see Table 5 and Fig. 6-10. Remind that:

Q1-tile(L_{ML})	percentage of Ramps in a Ramp Set which have *Q1=1* at L_{ML}
$L_{ML}(Q1tl99)$	ML-estimated Ramp Length required for a Ramp Set to cross over threshold $Q1tile.99$, i.e., *Q1-tile>99%.*

Table 5: Ramp Length L_{ML} required by a Ramp Set with known Ground Truth Ramp Gradient to cross a given *Q1-tile* threshold.

Ground Truth Ramp Gradient			Length L_ML for three Q1-tile-thresholds			Linearization
b_{GT}	d_{GT}	$1/d_{GT}$	L_ML(Q1tl90)	**L_ML(Q1tl99)**	L_ML(Q1tl100)	L_{aQt}
-6	0,015625	64				
-5	0,03125	32	112			
-4	0,0625	16	58	68	76	73
-3	0,125	8	31	38	42	38
-2	0,25	4	17	21	24	21
-1	0,5	2	9	12	14	12
0	1	1	5	7	9	7
1	2	0,5	3	4	6	5
2	4	0,25	2	2	3	4

For small Ramp Gradients d_{GT}, the crossing of a *Q1-tile*-threshold occurs at higher Ramp Length. Steep Ramps already cross over *Q1-tile*-thresholds when they are still short. Fig. 6-10 represents Table 5 graphically. All three $L_{ML}(Q1tl)$'s behave linearly, except for $L_{ML} < 10$. A regression for $L_{ML}(Q1tl99)$ was found and is referred to as L_{aQt}:

$$L_{ML}(Q1tl99) \approx L_{aQt} = \left\lceil \frac{4.4}{d_{GT}} + 2.5 \right\rceil$$

Note that the Ramp Set with Gradient $1/d_{GT} = 32$ crosses $Q1tile.99$ for $L_{ML} = 112$, i.e. when the ML estimation error has begun degrading due to the Corner lying outside of the Observation Window. Hence, this Ramp Set is excluded from error analysis. The flattest Ramp Set with $b_{GT} = -6 \Leftrightarrow 1/d_{GT} = 64$ never crosses any Q1-tile threshold within the analyzed experimental data and can not be detected by RI.

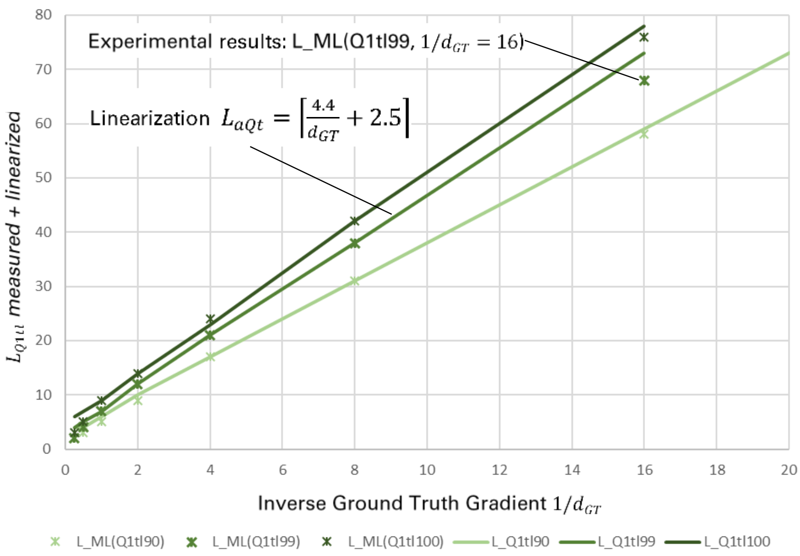


Fig. 6-10: L_Q1tl90 ,99 and 100 plotted against inverse Ramp Gradient

To achieve the objective of "uncertainty estimation", any dependency on Ground Truth knowledge must be eliminated. Whether it is possible to replace d_{GT} by d_{ML} will be investigated further down.

6.3.4.5 Method 'aQt'

The previous sub-chapters describe a series of steps that evaluate the significance of ML-results on Ramp Sets. It consists of L_{ML}-alignment followed by Q1tile-thresholding and will by referred to by the abbreviation "aQt", pronounceable as 'acute'.

This following list summarizes the steps of the aQt-method:

- Given a Ramp Set of N_{rs} Noisy Ramps with the same, known Ground Truth
- Apply ML-estimation with an Observation Window of $N_W = 100$ increments.
- Determine Ramp parameters $\sigma_{ML}, L_{ML}, D_{ML}, H_{ML}$ and derive $Q1$ for each increment.
- Determine further results involving Ground Truth data: e_L, e_D, e_H, etc.
- Execute L_{ML}-alignment and discontinuity treatment as described above on the results.
- Determine the L_{ML} threshold for statistical significance by
  - On a Ramp Set: determine the Q1-tile for each increment and determine the Ramp Length $L_{ML.aQt}$, where Q1-tile$>99\%$ is true for the first time.
  - On a Runtime Signal: determine an approximate L_{aQt} by
    $$L_{aQt} = \frac{4.4}{d_{ML}} + 2.5$$
- Take ML-results into further consideration, if and only if $L_{aQt} \le L_{ML} < N_W$

Except for the fact that $L_{ML}(Q1tl99)$ depends on Ground Truth Gradient d_{GT}, this finding gives hope, that the answer to question "Q2-a: Bias & Noise horizon" might be a "Yes", at least for the Ramp Length L_{ML}. Remind Q2-a= $true$ means that the error in L_{ML} follows some simple pattern within clear ranges for all estimated Ramp parameters.

6.4 Q2-b: Bias & Noise model for L_{ML}

6.4.1 Bias model for "aQt'd" mean error of Ramp Length L_{ML}

In chapter 6.3.4 the observation was made that the mean error $\tilde{\mu}_L$ asymptotically reaches $zero$ at the same Length L_{ML}, where the *Q1-tile* crosses threshold 99%. This observation led to the definition of aQt-method. If the finding was true for all Ramp Sets, L_{aQt} could be used to eliminate biased mean errors $\tilde{\mu}_L$.

Fig. 6-11, left diagram, shows $\tilde{\mu}_L$: the $\_ML$ -aligned mean errors in L_{ML} for all Ramp Sets. Blue dots are identical to the dark blue curve $\tilde{\mu}_{L.a}$ in Fig. 6-8. Error $\tilde{\mu}_L$ depends on L_{ML} and the Ramp Gradient $d_{GT} = 2^{(b_{GT})}$. Each incremental series of $\tilde{\mu}_L$ starts with a big negative bias, i.e., the ML estimated Ramp Length L_{ML} is up to 20 increments too short, in average.

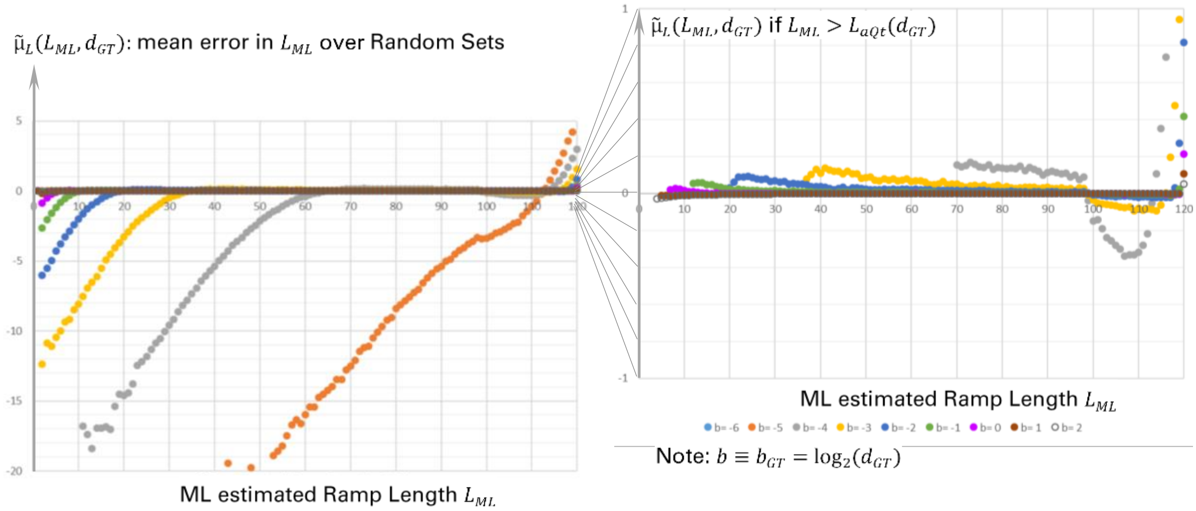


Fig. 6-11: L_{ML}-aligned error in L_{ML}, averaged over Random Sets, plotted against L_{ML}. Right: only results that hold true $L_{ML} >^? L_{ML}(Q1tl99)$

The diagram Fig. 6-11, right side, shows the same data points, excluding those that haven't reached L_{aQt}. The size of $\tilde{\mu}_L$ decreases considerably from the left to the right diagram. The remaining mean error in ML-estimated Ramp Length L_{ML} remains below 0.2 increments. The mean relative error in L_{ML} is far below 1%:

$$\tilde{\mu}_L < 0.2 \text{ and } \frac{\tilde{\mu}_L}{L_{ML}} \ll 1\% \qquad \text{if } L_{ML} > L_{aQt}$$

For $L_{ML} \geq 100$, however, the mean error $\tilde{\mu}_L$ increases again. The interesting part of $\tilde{\mu}_L$ lies between $L_{ML} = L_{aQt}$ and $L_{ML} = 99$ and will be referred to as $\tilde{\mu}_{L.aQt}$. Note: L_{aQt} depends on d_{GT} as well. Formal definition:

$$\tilde{\mu}_{L.aQT} = \{\tilde{\mu}_L(L_{ML}, d_{GT}) | L_{aQt} < L_{ML} \leq 99\}$$

From the above said, an upper limit of the mean error in L_{ML} can be given

$$|\tilde{\mu}_{L.aQt}| < 0.2$$

In many practical applications, an accuracy of $L_{ML} \pm 0.5$ is already sufficient, in particular in case of flat Ramps. Thus, considering $\tilde{\mu}_{L.aQt} = 0$ would be a good enough approximation of the Bias $\tilde{\mu}_L$ in ML-estimated Ramp Length L_{ML}. Except for the fact that $\tilde{\mu}_L(L_{ML}, d_{GT})$ is still dependent on Ground Truth Gradient d_{GT}, this finding gives hope, that the answer to question “Q2-b: Bias & Noise modelling” might be a “Yes”, at least for $\tilde{\mu}_L$. Remind: Q2-b means that ML estimation-errors can be modelled as functions of ML estimations.

“Ramp Identification” is modelling the Bias in ML-estimated Ramp Length L_{ML} by equation:

$$\tilde{\mu}_{L.RI} = 0 \tag{6-5}$$

6.4.2 Noise model in aQt'd L_{ML}

ML-estimations of Ramp Length L_{ML} deteriorate under Signal Noise. The standard deviation over all ML-results from a Ramp Set reflects the noisiness of ML estimations. Fig. 6-12 shows the Noise $\tilde{\sigma}_{L.a}$ of ML-estimated Ramp Length L_{ML} over Ramp Sets, i.e., the Noise $\tilde{\sigma}_L$ was L_{ML}-aligned and plotted against $\lfloor L_{ML} \rfloor$.

At higher Ramp Gradients d_{GT}, i.e., at smaller Ramp Gradient exponents b_{GT}, L_{ML} estimations are more accurate and less noisy. For small L_{ML}, the Noise doesn't show an easy pattern. This is also true for $L_{ML} > N_W$. After applying the aQt-method, Fig. 6-13, all confusing behavior has vanished. Obviously $\tilde{\sigma}_{L.aQt}$ can be approximated by linear regression over ML-estimated Ramp Length L_{ML} and Ground Truth Ramp Gradient exponent b_{GT}.

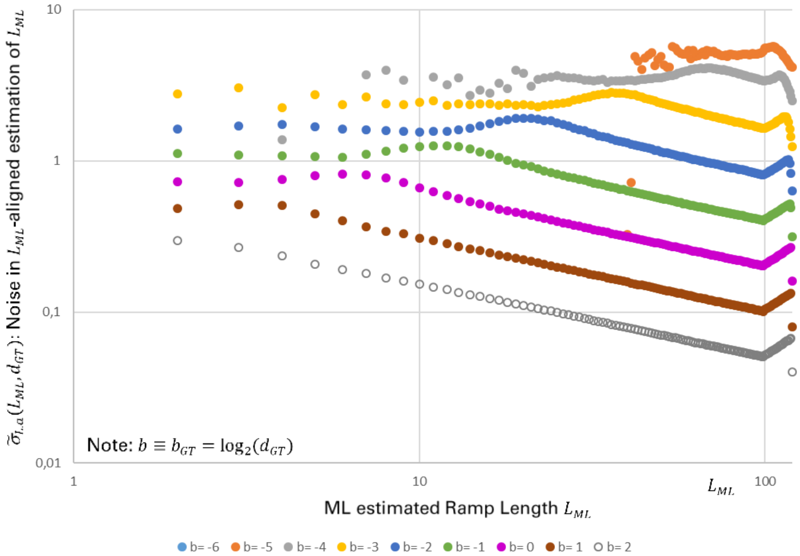


Fig. 6-12: Noise in ML-estimated Ramp Length L_ML over Ramp Sets, L_ML-aligned and plotted against L_ML

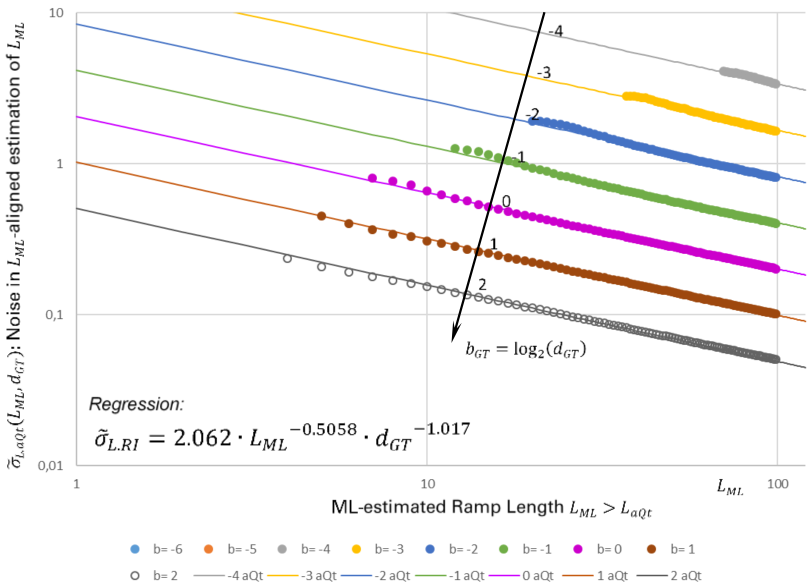


Fig. 6-13: Noise in L_{ML}-estimations of Ramp Sets, aQt-ed and plotted against L_{ML}

The diagram Fig. 6-13 suggests linear dependency of $\log_{10}(\tilde{\sigma}_{L.aQt})$ from $\log_{10}(L_{ML})$ and $b_{GT} = \log_2(d_{GT})$. Bilinear regression solves for its three coefficients.

$$\log_{10}(\tilde{\sigma}_{L.aQt}) \approx a_{1,\sigma L} \cdot \log_{10}(L_{ML}) + a_{2,\sigma L} \cdot \log_2(d_{GT}) + c_{0,\sigma L}$$

$$a_{1,\sigma L} \approx -0.5058, \qquad a_{2,\sigma L} \approx -0.3062, \qquad c_{0,\sigma L} \approx 0.3143$$

$$\Rightarrow \quad \tilde{\sigma}_{L.aQt} \approx 10^{c_{0,\sigma L}} \cdot (L_{ML})^{a_{1,\sigma L}} \cdot (d_{GT})^{\frac{a_{2,\sigma L}}{\log_{10}(2)}}$$
$$\Rightarrow \quad \tilde{\sigma}_{L.aQt} \approx 2.062 \cdot L_{ML}{}^{-0.5058} \cdot d_{GT}{}^{-1.017}$$

This regression estimates the Noise $\tilde{\sigma}_{L.aQt}$ from the ML-estimated Ramp Length L_{ML} and Ground Truth Ramp Gradient d_{GT}. However, d_{GT} is not available for a Runtime Signal and must be replaced by the ML-estimated Ramp Gradient d_{ML} at some point. This exchange probably adds more uncertainty to the estimation of $\tilde{\sigma}_L$.

Ramp Identification estimates the Noise in L_{ML} by the exponential function:

$$\tilde{\sigma}_{L.RI} = 2.062 \cdot L_{ML}{}^{-0.5058} \cdot d_{ML}{}^{-1.017} \tag{6-6}$$

6.5 Q2-b on ML-filtered standard deviation σ_{ML}

6.5.1 Definitions

ML was run as a shifting Window with width $N_W = 100$, over Synthetic Noisy Ramps with different Ground Truth Gradients. A Ramp Set is N_{rs} times the same Ramp but with N_{rs} different Random Series being added. The resulting standard deviation σ_{ML} of each Noisy Ramp contains an error e_σ that can be determined since Ground Truth is known. To achieve comparability with Runtime Signals where Ground Truth is not available, the error is divided by σ_{ML}. This step is referred to as "normalizing" und will be applied to many other ML results. "Normal Bias" $\tilde{\mu}_{n\sigma}$ is then defined as the mean value over a Ramp Set of $e_{n\sigma}$, the normal error in σ_{ML}. "Normal Noise" $\tilde{\sigma}_{n\sigma}$ stands for the standard deviation in $e_{n\sigma}$ over a Ramp Set.

$e_\sigma = \sigma_{ML} - \sigma_{GT}$	Ground Truth error in σ_{ML}
$e_{\sigma\sim}$	set of Ground Truth errors e_σ of a Ramp Set
$\tilde{\mu}_\sigma = \tilde{\mu}(e_{\sigma\sim})$	Ground Truth Bias in σ_{ML}: mean of e_σ of a Ramp Set
$e_{n\sigma} = \frac{\sigma_{ML} - \sigma_{GT}}{\sigma_{ML}}$	normal error in σ_{ML}
$e_{n\sigma\sim}$	set of normal errors $e_{n\sigma}$ of a Ramp Set
$\tilde{\mu}_{n\sigma} = \tilde{\mu}(e_{n\sigma\sim})$	Normal Bias in σ_{ML}: mean of $e_{n\sigma\sim}$ of a Ramp Set
$\tilde{\mu}_{n\sigma.aQt}$	aQt'd version of $\tilde{\mu}_{n\sigma}$
$\tilde{\mu}_{n\sigma.RI}$	RI's approximation model of $\tilde{\mu}_{n\sigma.aQt}$
$\tilde{\sigma}_{n\sigma} = \tilde{\sigma}(e_{n\sigma\sim})$	Normal Noise in σ_{ML}: standard deviation of $\varepsilon_{n\sigma\sim}$
$\tilde{\sigma}_{n\sigma.aQt} \approx \tilde{\sigma}_{n\sigma}$	aQt'd version of $\tilde{\sigma}_{n\sigma}$
$\tilde{\sigma}_{n\sigma.RI} \approx \tilde{\sigma}_{n\sigma}$	RI's approximation model of $\tilde{\sigma}_{n\sigma.aQt}$

Back-scaling of RI Bias and Noise models to the size of the Sample Ramp

$\tilde{\sigma}_{\sigma.RI} \approx \tilde{\sigma}_{n\sigma.RI} \cdot \sigma_{ML}$	RI-estimated Noise in σ_{ML}
$\tilde{\mu}_{\sigma.RI} \approx \tilde{\mu}_{n\sigma.RI} \cdot \sigma_{ML}$	RI-estimated Bias error in σ_{ML}

6.5.2 Normal Bias in σ_{ML}

Fig. 6-14 shows $\tilde{\mu}_\sigma$: the Ground Truth Bias in σ_{ML} of all Ramp Sets, plotted against the Ground Truth Ramp Length. The fluctuation in y-axis reflects the influence of the Ramp Gradient on ML's minimum search. The fluctuation in x-axis shows that the noises averaged over N_{rs} Random Series do not fully cancel out.

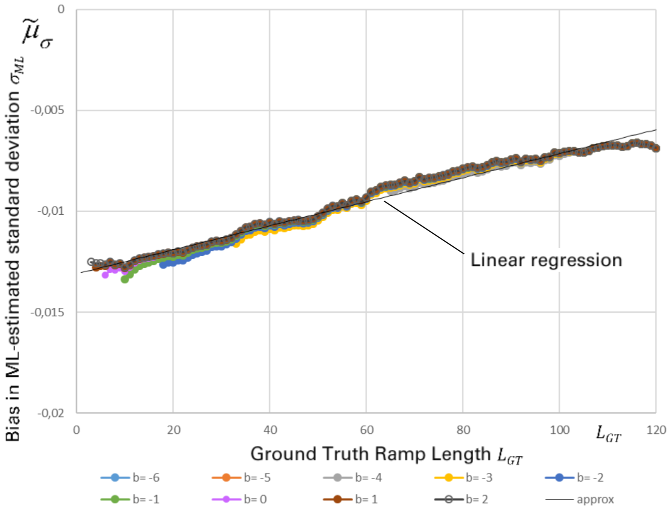


Fig. 6-14: Bias of σ_{ML} plotted against Ground Truth Ramp Length L_{GT}

Data in Fig. 6-14 shows results only if $L_{GT} > L_{RD}(conf > 99.8\%)$, i.e., long enough Ramps so that Ramp Detection criterion $Q1 = true$ delivers 99.8% true-positive results. Fig. 6-15 has 3 modifications w.r.t. Fig. 6-14:

- Normal Bias instead of Ground Truth Bias is shown. Thus, L_{GT} is no longer needed to generate the diagram.
- All results with $Q1 = true$ are represented, including false-positives.
- Results are grouped by Ground Truth Ramp Gradient exponents b_{GT}, and then aligned and averaged by their ML-estimated Ramp Length L_{ML}.

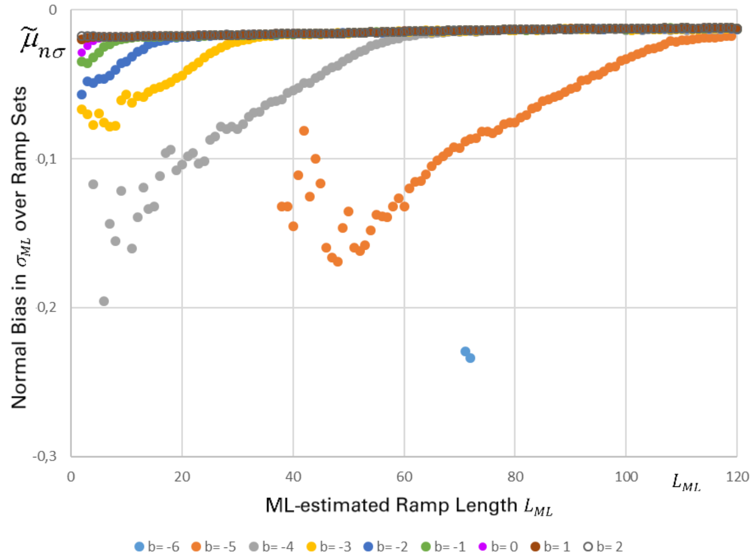


Fig. 6-15: Normal Bias in σ_{ML}, L_{ML}-aligned and plotted against L_{ML}

Normal Bias $\widetilde{\mu}_{n\sigma}$ degrades heavily due to the fact that also false-positives of $Q1$, i.e. false Ramp Detections, are now included in the diagram. No simple pattern can be found any more.

With aQt-method used as a filter, see Fig. 6-16, the pattern of Fig. 6-14 re-emerges.

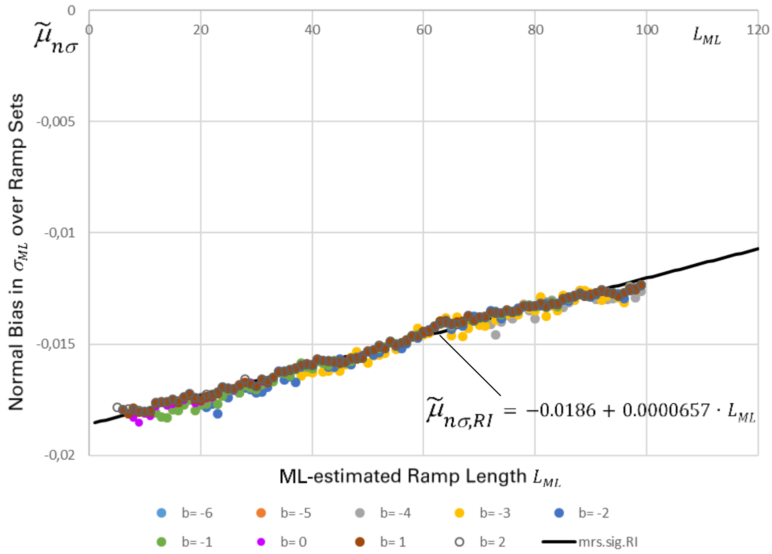


Fig. 6-16: Normal Bias in σ_{ML}, aQt-filtered and plotted against L_{ML}. RI-estimation formula.

A linear regression over this pattern is selected by the Ramp Identification method as an estimation of the Normal Bias in σ_{ML}. In short: the RI-estimated Normal Bias $\widetilde{\mu}_{n\sigma}$ is:

$$\widetilde{\mu}_{n\sigma.RI} = -0.0186 + 0.0000657 \cdot L_{ML} \tag{6-7}$$

6.5.3 Noise in σ_{ML}

The same steps were applied to the Normal Noise in σ_{ML}, see Fig. 6-17. The RI-estimated Normal Noise $\widetilde{\sigma}_{n\sigma}$ is determined by:

$$\widetilde{\sigma}_{n\sigma.RI} = 0.0731 + 5.57 \cdot 10^{-6} \cdot L_{ML} \tag{6-8}$$

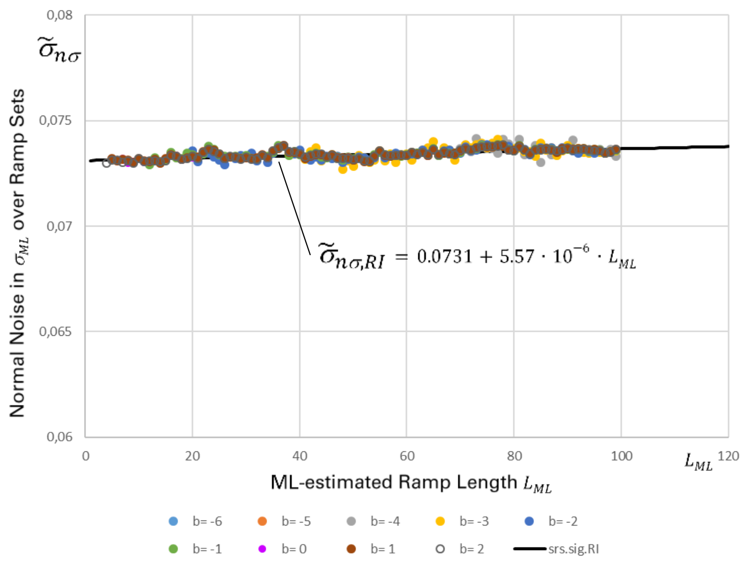


Fig. 6-17: Normal Noise in σ_{ML} of Ramp Sets, aQt-filtered and plotted against L_{ML}. RI-estimation formula.

6.6 *Q2-b on Ramp Gradient D_{ML}*

6.6.1 Error estimation from Normal Noisy Ramp Sets

Remind: ML-results are available for Normal Noisy Ramp Sets, i.e. for a synthetic set of Noisy Ramps with $\mu = 0, \sigma = 1$ and with Ground Truth D_{GT} being available. A Runtime Noisy Ramp is an unknown, non-identified Noisy Ramp that is ML-estimated at each increment.

The unknown error e_D in ML-estimated Ramp Gradient D_{ML} of a RTRamp shall be estimated by comparing it with error results from NNRamp Sets.

$e_D = D_{ML} - D_{GT}$	error in ML-estimated Ramp Gradient
$e_{D\sim}$	set of errors e_D of all Noisy Ramps in a Ramp Set
$\tilde{\mu}_D = \tilde{\mu}(e_{D\sim})$	mean value of $e_D\sim$
$\tilde{\sigma}_D = \tilde{\sigma}(e_{D\sim})$	standard deviation of $e_{D\sim}$

Normalizing by ML-estimated standard deviation of the RTRamp σ_{ML}

$e_{nD} = \frac{D_{ML}-D_{GT}}{\sigma_{ML}}$	normal error in ML-estimated Ramp Gradient
$e_{nD\sim} = \frac{D_{ML\sim}-D_{GT}}{\sigma_{ML\sim}}$	set of normal errors e_{nD} in a Ramp Set
$\tilde{\mu}_{nD} = \tilde{\mu}(e_{nD\sim})$	Normal Bias in D_{ML}: mean value of $e_{nD\sim}$
$\tilde{\mu}_{nD.aQt}$	aQt'd version of $\tilde{\mu}_{nD}$
$\tilde{\mu}_{nD.RI}$	RI's model of $\tilde{\mu}_{nD.aQt}$ developed in this paper
$\tilde{\sigma}_{nD} = \tilde{\sigma}(e_{nD\sim})$	Normal Noise in D_{ML}: standard deviation of $e_{nD\sim}$
$\tilde{\sigma}_{nD.aQt}$	aQt'd version of $\tilde{\sigma}_{nD}$
$\tilde{\sigma}_{nD.RI}$	RI's model of $\tilde{\sigma}_{nD.aQt}$ developed in this paper

Back-scaling of Bias and Noise to the size of the Sample Ramp

$\tilde{\mu}_{D.RI} \approx \tilde{\mu}_{nD.RI} \cdot \sigma_{ML}$	RI-estimated Bias in D_{ML}
$\tilde{\sigma}_{D.RI} \approx \tilde{\sigma}_{nD.RI} \cdot \sigma_{ML}$	RI-estimated Noise in D_{ML}

6.6.2 Normal Bias in D_{ML}

The Normal Bias in D_{ML}, Fig. 6-18, decreases over L_{ML} but no simple relation can be found.

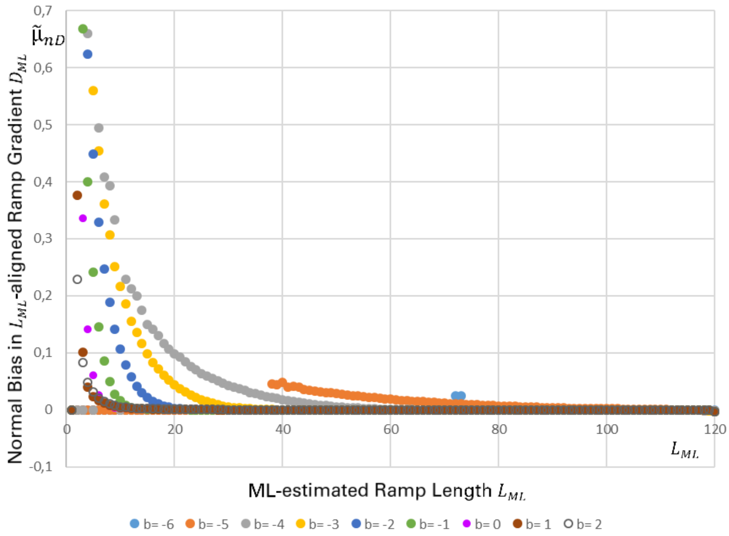


Fig. 6-18: Normal Bias in ML-estimated Ramp Gradient D_{ML} of Ramp Sets, L_{ML}-aligned and plotted against L_{ML}

Re-ordering and filtering by aQt-method eliminates big errors and also eliminates dependencies on the actual Gradient magnitude.

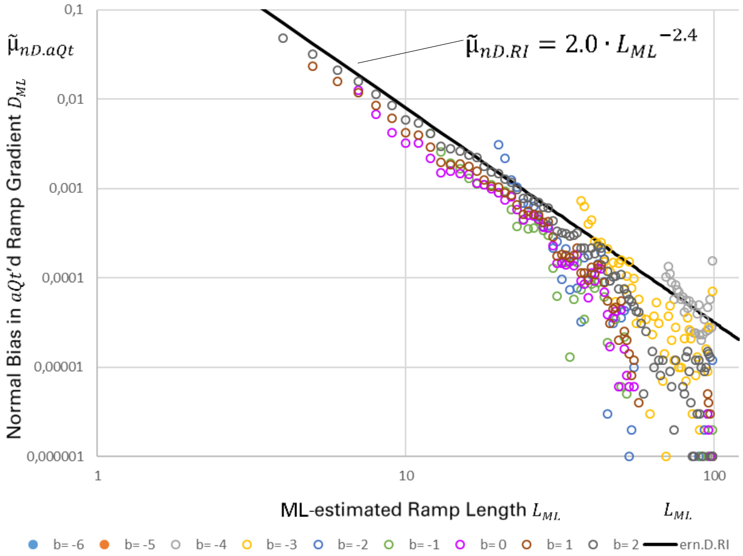


Fig. 6-19: RI-Regression of Normal Bias in Ramp Gradient D_{ML} of Ramp Sets, aQt-filtered and plotted against L_{ML}

Fig. 6-19 suggests an exponential dependency of $\tilde{\mu}_{nD.aQt}$ from L_{ML}:

$$\tilde{\mu}_{nD.aQt} \approx c_{0,\mu nD} \cdot L_{ML}{}^{a_{0,\mu nD}}$$

The Bias in D_{ML} tends towards zero with growing L_{ML} and randomly becomes negative. Zero and small negative values were manually deleted from the data set to allow for logarithmic representation. Linear regression thereafter solves for coefficients $a_{0,\mu nD}$ and $c_{0,\mu nD}$. The RI-estimated Normal Bias in D_{ML} is:

$$\tilde{\mu}_{nD.RI} = 2.0 \cdot L_{ML}{}^{-2.4} \qquad (6\text{-}9)$$

6.6.3 Normal Noise in D_{ML}

Fig. 6-20 shows $\tilde{\sigma}_{nD.aQt}$, the Normal Noise in the ML-estimated Gradient D_{ML} after being aQt'd. Its linear regression has coefficients $a_{0,\sigma nD} \approx -1.45$ and $c_{0,\sigma nD} \approx 2.8$.

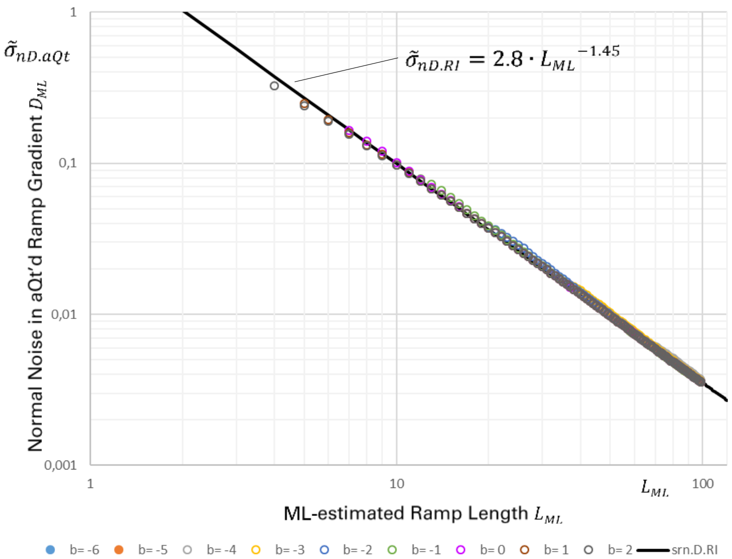


Fig. 6-20: RI-Regression of Normal Noise in Ramp Gradient D_{ML} of Ramp Sets, aQt-filtered and plotted against L_{ML}

The RI-estimated Normal Noise in D_{ML} is:

$$\tilde{\sigma}_{nD.RI} = 2.8 \cdot L_{ML}{}^{-1.45} \qquad (6\text{-}10)$$

6.7 *Q2-b on Ramp Height H_{ML}*

6.7.1 Normal Bias in H_{ML}

As for the Gradient, the Normal Bias in Ramp Height H_{ML} decreases with no significant pattern if plotted against L_{ML}, Fig. 6-21:

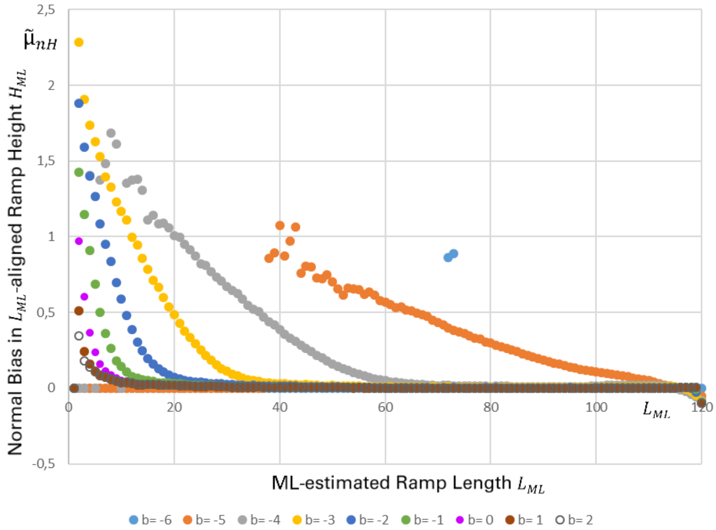


Fig. 6-21: Normal Bias in ML-estimated Ramp Height D_{ML} of Ramp Sets, L_{ML}-aligned and plotted against L_{ML}

Re-ordering, filtering by aQt-method and deleting negative results allows for exponential regression of the Bias in H_{ML}, see Fig. 6-22.

$$\tilde{\mu}_{nH.RI} = 0.6 \cdot L_{ML}{}^{-0.8} \tag{6-11}$$

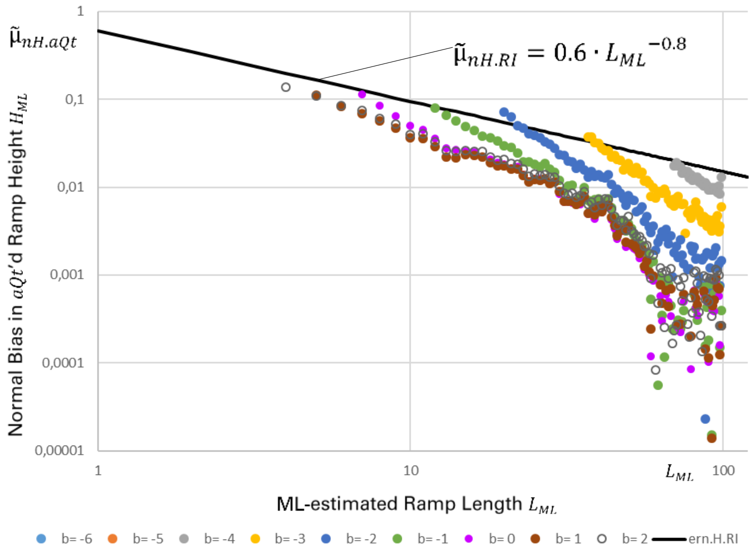


Fig. 6-22: RI-Regression of Normal Bias in Ramp Height H_{ML} of Ramp Sets, aQt-filtered and plotted against L_{ML}

6.7.2 Normal Noise in H_{ML}

Fig. 6-23 shows $\tilde{\sigma}_{nH.aQt}$, the Normal Noise in the ML-estimated Height H_{ML} after being aQt'd.

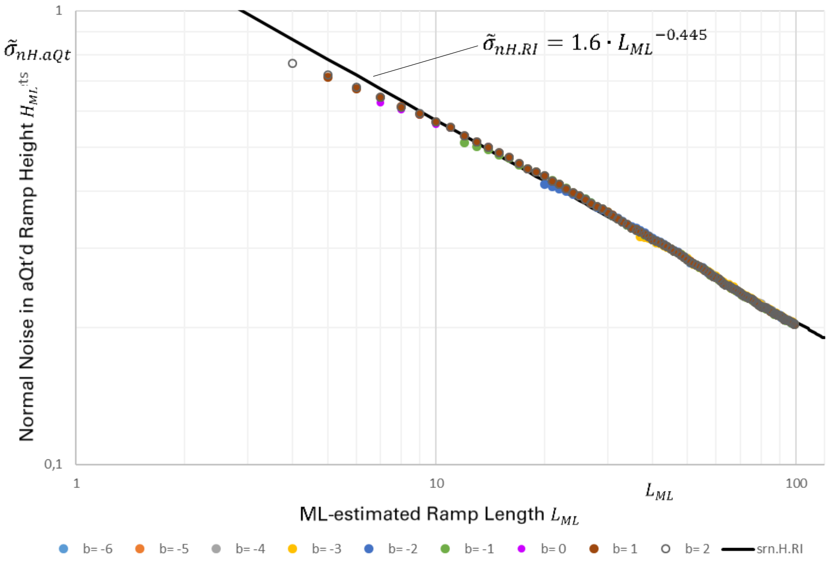


Fig. 6-23: RI-Regression of Normal Noise in Ramp Height H_{ML} of Ramp Sets, aQt-filtered and plotted against L_{ML}

The RI-estimated Normal Noise in H_{ML} is:

$$\tilde{\sigma}_{nH.RI} = 1.6 \cdot L_{ML}{}^{-0.445} \tag{6-12}$$

6.8 Q3: Uncertainty and Confidence Gauging

6.8.1 Problem statement

Thus far, Ramp Parameters of many Normal Noisy Ramps were estimated by ML. Bias and Noise of Ramp Parameters over a big number of NNRamps were modeled by RI as functions of L_{ML} and d_{GT}. Resulting parameters are:

ML estimated Ramp Parameters (P_{ML})	$\sigma_{ML}, L_{ML}, D_{ML}, H_{ML}$
Ramp Parameters normalized by σ_{ML}	d_{ML}, h_{ML}
Normal Bias models of P_{ML}'s defined by RI	$\tilde{\mu}_{n\sigma.RI}, \tilde{\mu}_{L,RI}, \tilde{\mu}_{nD.RI}, \tilde{\mu}_{nH.RI}$
Normal Noise models in P_{ML}'s defined by RI	$\tilde{\sigma}_{n\sigma.RI}, \tilde{\sigma}_{L.RI}, \tilde{\sigma}_{nD.RI}, \tilde{\sigma}_{nH.RI}$

Let

P	any of the Ramp Parameters σ, L, D, H
P_{ML}	ML estimation of parameter P
$\tilde{\mu}_{P,RI}$	the RI modeled Bias in P_{ML}. Normal Bias is $\tilde{\mu}_{nP,RI}$
$\tilde{\sigma}_{P,RI}$	the RI modeled Noise in P_{ML}. Normal Noise is $\tilde{\sigma}_{nP,RI}$

There are several aspects that need validation or adjustment.

(1) Scale back Normal Bias and Noise models to real Ramp size

Except for $\tilde{\mu}_{L,RI}$ and $\tilde{\sigma}_{L.RI}$, Bias and Noise models were derived from normalized errors, i.e., from errors that were divided by σ_{ML}. Multiplication with σ_{ML} scales RI Bias and Noise estimates back to the dimension of the real Ramp:

$$\tilde{\mu}_{P,RI} = \tilde{\mu}_{nP,RI} \cdot \sigma_{ML}$$
$$\tilde{\sigma}_{P,RI} = \tilde{\sigma}_{nP,RI} \cdot \sigma_{ML}$$

(2) Replacement of d_{GT} by d_{ML} in the Bias model for L_{ML}.

RI's Noise model for the Ramp Length $\tilde{\sigma}_{L.RI}$ uses ML-estimated Normal Ramp Gradient d_{ML}, but the Noise model was found with d_{GT}, the Ground Truth Gradient in Ramps Sets instead. The error introduced by this replacement increases the Uncertainty of RI's Noise model for L_{ML}, which must be measured by Uncertainty and Confidence Gauging below.

(3) Un-bias ML estimates using RI Bias models.

RI provides Bias models $\tilde{\mu}_{P,RI}$ for ML estimations of Ramp Parameters P_{ML}. By subtracting the Bias from P_{ML}, RI makes an improved estimation P_{RI} which is centered around Ground Truth P_{GT}, see Fig. 6-24:

$$P_{RI} = P_{ML} - \tilde{\mu}_{P,RI}$$

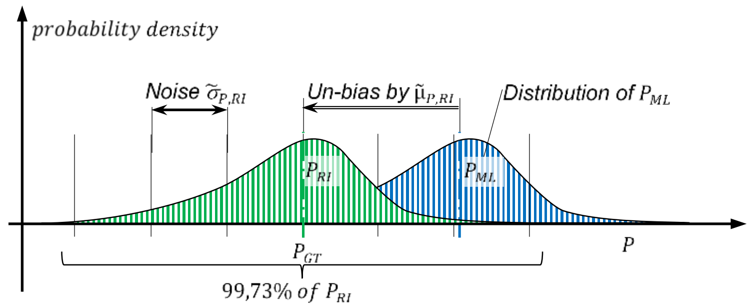


Fig. 6-24: un-biased Ramp Parameter P_{RI} with non-normal distribution

(4) Modelling Uncertainties of Ramp Parameters from RI Bias and Noise models

As Fig. 6-24 shows, the un-biased parameter P_{RI} of a big Ramp Set has a certain probability density distribution, which results from the distribution of P_{ML} and the errors made by RI Bias model. As RI applied multiple sorting and filtering steps, the probability density of P_{RI} cannot be assumed to be normal distributed.

In a normal distribution, Uncertainty and Confidence are directly related to multiples of the standard deviation. E.g., the Confidence equals $fd \approx 99{,}73\%$ if the Uncertainty equals 3 times the standard deviation. For non-normal distributions, the Noise-multiplier m_P deviates from 3 to reach that same Confidence. The RI Uncertainty model becomes:

$$U_P = m_P \cdot \tilde{\sigma}_{P,RI}$$

(5) Gauge the Uncertainty to the desired Confidence.

To find the Confidence-Uncertainty-relation in such a case the inliers for a test-Uncertainty could simply be counted and the test-Uncertainty be adjusted until a required Confidence is reached. Finally, the test-Uncertainty is expressed as a multiple of the standard deviation, referred to as Noise-multiplier. The procedure described above is referred to as Confidence and Uncertainty Gauging.

Confidence $$fd(P, U_P) = \frac{Number\ of\ Inlier\ Samples\ in\ \{P_{RI} \pm U_P\}}{Total\ Number\ of\ Samples}$$

Typical Confidence levels in engineering applications are:

Basic Confidence: $fd_2 = 95.00\%$
Fair Confidence: $fd_3 = 99.73\%$
High Confidence: $fd_4 = 99.99\%$

Fair Confidence is used in processes that can recover from outlier detection at acceptable costs. With Uncertainty equal to $\pm 4 \cdot \sigma$ the Confidence raises above 99,99%, which reduces the costs of outliers and of outlier avoidance. The number of samples available to this paper does not allow to investigate for higher Confidence levels.

Aiming at finding Noise-multipliers m, the term “Deviation-quotient” $devq$ is now being defined for each Ramp parameter as the error in RI estimation divided by RI-Noise. $devq$ is evaluated for every single Ramp Identification experiment and its distribution over a Ramp Set analyzed. To put it formally:

$$devq = \frac{P_{RI} - P_{GT}}{\tilde{\sigma}_{P,RI}}$$

The $deviation\text{-}quotient$ in ML-estimated Ramp Length L_{ML} becomes:

$$devq_L = \frac{L_{RI} - L_{GT}}{\tilde{\sigma}_{L,RI}} = \frac{L_{ML} - \tilde{\mu}_{L.RI} - L_{GT}}{\tilde{\sigma}_{L.RI}}$$

The $deviation\text{-}quotient$ in σ_{ML}:

$$devq_\sigma = \frac{\sigma_{RI} - \sigma_{GT}}{\tilde{\sigma}_{\sigma,RI}} = \frac{\sigma_{ML} - \tilde{\mu}_{\sigma.RI} - \sigma_{GT}}{\tilde{\sigma}_{\sigma.RI}} = \frac{\left(\frac{\sigma_{ML} - \sigma_{GT}}{\sigma_{ML}}\right) - \tilde{\mu}_{n\sigma.RI}}{\tilde{\sigma}_{n\sigma.RI}}$$

The $deviation\text{-}quotient$ in D_{ML} and H_{ML}:

$$devq_D = \frac{\left(\frac{D_{ML} - D_{GT}}{\sigma_{ML}}\right) - \tilde{\mu}_{nD.RI}}{\tilde{\sigma}_{nD.RI}}$$

$$devq_H = \frac{\left(\frac{H_{ML} - H_{GT}}{\sigma_{ML}}\right) - \tilde{\mu}_{nH.RI}}{\tilde{\sigma}_{nH.RI}}$$

6.8.2 Noise-multiplier that gauge RI Uncertainty to Confidence level

RI Uncertainty needs to be gauged to a requested Confidence. This is done by analyzing all available $21.456 \cdot 10^6$ Synthetic Noisy Ramps that were run through the RI-procedure as described above. More than $8.7 \cdot 10^6$ passed Ramp detection criterion Q1 and also complied with the Ramp Length range $L_{aQt} < L_{ML} < N_W = 100$.

6.8.2.1 Uncertainty in σ_{ML}

Adjusted Noise-multipliers m_σ for RI-estimation of GT-error in ML-estimated standard deviation σ_{ML} were found for confidence-levels $fd_3 = 99.73\%$ and $fd_4 = 99.99\%$, see Fig. 6-25 and Table 6. Column heights represent the number of NNRamps that comply with RI filters and have a Deviation-quotient within ± 0.25 of the range-center as drawn in the diagram.

$$m_{\sigma 3} = m_\sigma(fd_3) = 3.4$$
$$m_{\sigma 4} = m_\sigma(fd_4) = 5$$

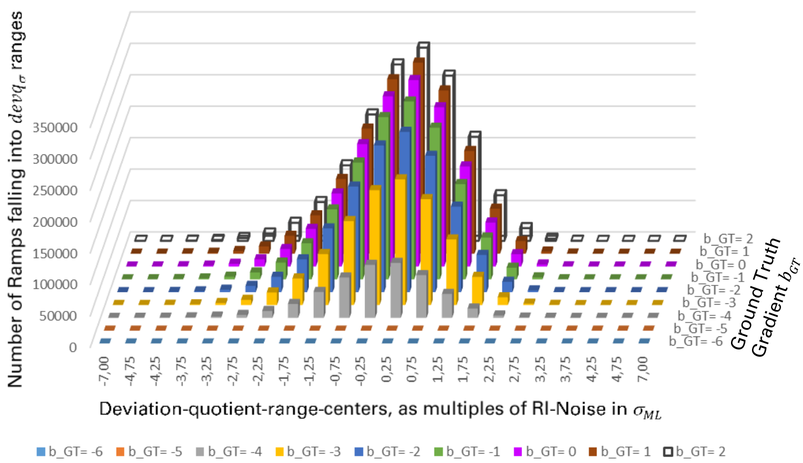


Fig. 6-25: Distribution of $devq_\sigma$, scaled by RI-Noise $\tilde{\sigma}_{\sigma RI}$, plotted against Ramp Gradient b_{GT}

Table 6: Confidence-levels by Ramp Gradient for characteristic Noise-multipliers m_σ

b_{GT}	N_{Ramps}	$m_\sigma = 3$ confidence	$m_\sigma = 3.4$ $\geq fd_3$	$m_\sigma = 5$ $\geq fd_4$
-6	0	0,00%	0,00%	0,00%
-5	121	63,64%	76,86%	99,17%
-4	443231	99,18%	99,65%	99,99%
-3	1004909	99,44%	99,77%	99,99%
-2	1282935	99,48%	99,79%	99,99%
-1	1421079	99,50%	99,80%	100,00%
0	1488949	99,51%	99,80%	100,00%
1	1521862	99,51%	99,80%	100,00%
2	1536437	99,52%	99,80%	100,00%
Total/Average	8699523	99,48%	99,79%	99,99%

6.8.2.2 Uncertainty in L_{ML}

The distribution of the deviation-quotient for L_{ML} is plotted against Ground Truth Ramp Length and Gradient, see Fig. 6-26. The number of Ramps was counted with $devq_L$ falling into the same range. $devq_L$ ranges are $0.5 \cdot \tilde{\sigma}_{L.RI}$ wide and marked by their center value. E.g., the range $-0.5 \cdot \tilde{\sigma}_{L.RI} \leq devq_L < 0$ is marked -0.25. The next range to the right is marked 0.25 and is counts deviation-quotients $0 < devq_L < 0.5 \cdot \tilde{\sigma}_{L.RI}$.

Integer values shouldn't be expected for deviation-quotients, but there exists one significant exception for $devq_L$. In fact, $devq_L = 0$ is a frequent result due to the design of Synthetic Noisy Ramps in this paper since L_{GT} was chosen to coincide with exact increment borders. If ML correctly estimates L_{GT} as an integer number, $devq_L = 0$ occurs. For this particularity in the deviation-quotient for L_{ML}, the distribution becomes asymmetric in Fig. 6-26.

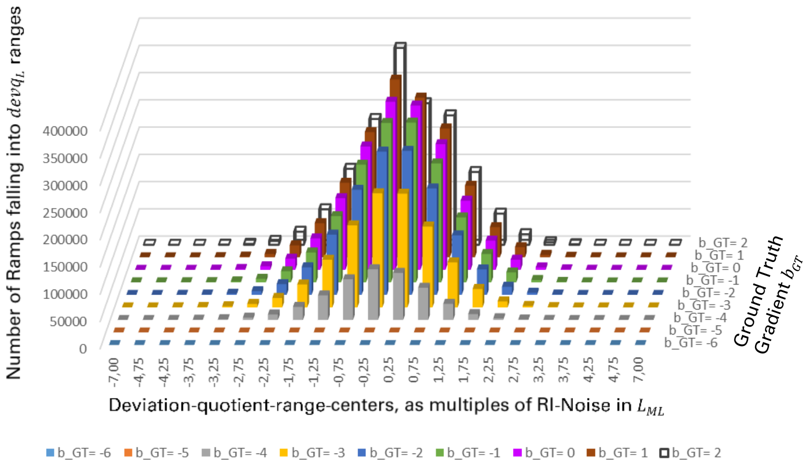


Fig. 6-26: Distribution of $devq_L$, scaled by RI-Noise $\tilde{\sigma}_{L.RI}$, plotted against Ground Truth Gradient Exponent b_{GT}

For a given Noise-multiplier m_L, Ramps with $-m_L < devq_L < +m_L$ count as inliers. The number of inlier Ramps divided by the total number of Ramps gives the confidence-level. Table 7 shows confidence-levels for different Noise-multipliers m_L, both, per Ramp Gradient and in total.

Table 7: Confidence-levels by Ramp Gradient for characteristic Noise-multipliers m_L

b_{GT}	N_{Ramps}	$m_\sigma = 3$ confidence	$m_\sigma = 3.4$ $\geq fd_3$	$m_\sigma = 5$ $\geq fd_4$
-6	0	0,00%	0,00%	0,00%
-5	121	42,15%	57,02%	85,12%
-4	443231	99,17%	99,68%	99,99%
-3	1004909	99,35%	99,75%	99,99%
-2	1282935	99,47%	99,80%	99,99%

-1	1421079	99,55%	99,84%	100,00%
0	1488949	99,58%	99,87%	100,00%
1	1521862	99,62%	99,89%	100,00%
2	1536437	99,57%	99,87%	100,00%
Total/Average	8699523	99,52%	99,83%	100,00%

Adjusted Noise-multipliers m_L for RI-estimation of GT-error in ML-estimated Ramp Length L_{ML} were found for confidence-levels fd_3 and fd_4:

$$m_{L3} = m_L(fd_3) \;=\; 3.4$$
$$m_{L4} = m_L(fd_4) \;=\; 5$$

The noise $\tilde{\sigma}_{L.RI}$ in L_{ML} was derived from equation (6-6) using d_{ML} instead of d_{GT}. The resulting deviation quotient $devq_L$ is nearly normal distributed. Thus, finally, it is correct to state that Bias and Noise of Ramp Parameters can be modelled without any Ground Truth of the underlying Ramp.

6.8.2.3 Uncertainty in D_{ML}

Adjusted Noise-multipliers m_D for RI-estimation of GT-error in ML-estimated Ramp Gradient D_{ML} were found for confidence-levels fd_3 and fd_4, see Fig. 6-27 and Table 8:

$$m_{D3} = m_D(fd_3) \;=\; 3.4$$
$$m_{D4} = m_D(fd_4) \;=\; 5.0$$

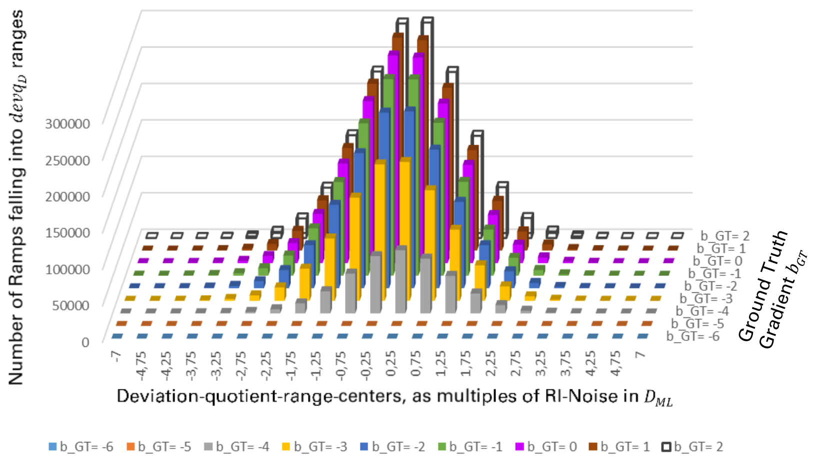


Fig. 6-27: Distribution of $devq_D$, scaled by RI-Noise $\tilde{\sigma}_{D.RI}$, plotted against Ramp Gradient b_{GT}

Table 8: Confidence-levels by Ramp Gradient for characteristic Noise-multipliers m_D

b_{GT}	N_{Ramps}	$m_\sigma = 3$ confidence	$m_\sigma = 3.4$ $\geq fd_3$	$m_\sigma = 5$ $\geq fd_4$
-6	0	0,00%	0,00%	0,00%
-5	121	49,59%	65,29%	99,17%
-4	443231	99,50%	99,84%	100,00%
-3	1004909	99,42%	99,79%	99,99%
-2	1282935	99,43%	99,80%	100,00%
-1	1421079	99,47%	99,80%	99,99%
0	1488949	99,51%	99,83%	99,99%
1	1521862	99,57%	99,86%	100,00%
2	1536437	99,61%	99,89%	100,00%
Total/Average	8699523	99,51%	99,83%	100,00%

6.8.2.4 *Uncertainty in H_{ML}*

Adjusted Noise-multipliers m_H for RI-estimation of GT-error in ML-estimated Ramp Gradient H_{ML} were found for confidence-levels fd_3 and fd_4, see Fig. 6-28 and Table 9:

$$m_{H3} = m_H(fd_3) \; = \; 3.2$$
$$m_{H4} = m_H(fd_4) \; = \; 4.4$$

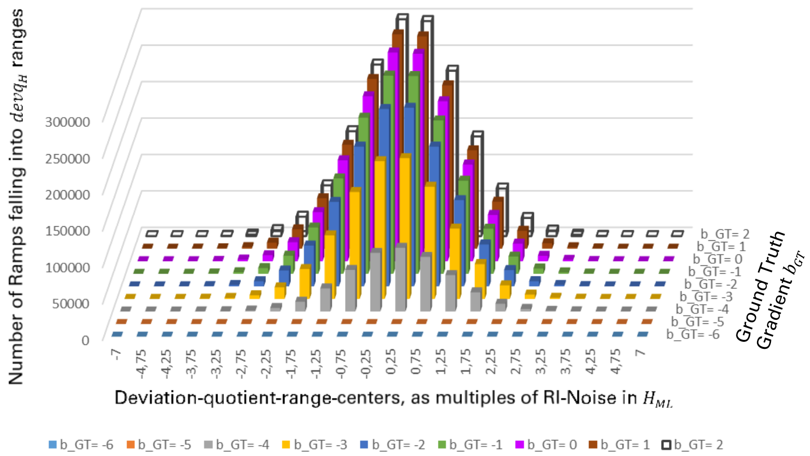


Fig. 6-28: Distribution of $devq_H$, scaled by RI-Noise $\tilde{\sigma}_{H.RI}$, plotted against Ramp Gradient b_{GT}

Table 9: Confidence-levels by Ramp Gradient for characteristic Noise-multipliers m_H

b_{GT}	N_{Ramps}	$m_\sigma = 3$ Confidence	$m_\sigma = 3.2$ $\geq fd_3$	$m_\sigma = 4.2$ $\geq fd_4$
-6	0	0,00%	0,00%	0,00%
-5	121	63,64%	72,73%	94,21%
-4	443231	99,67%	99,83%	100,00%
-3	1004909	99,63%	99,80%	99,99%
-2	1282935	99,62%	99,79%	99,99%
-1	1421079	99,63%	99,79%	99,99%
0	1488949	99,63%	99,80%	99,99%
1	1521862	99,64%	99,81%	99,99%
2	1536437	99,65%	99,81%	99,99%
Total/Average	8699523	99,64%	99,80%	99,99%

6.8.3 Checks and Problem variants

6.8.3.1 *Case: Controller provides Ground Truth of R_0*

The results achieved so far were compared with variants of the Ramp Identification problem. In the case presented above, the Controller has a good idea about the Floor level R_0 but doesn't know the standard deviation σ of Signal Noise. Table 10 comprises the Noise-Multipliers found for different Confidence levels and for Ramp Gradient exponents $b_{ML} \geq -4$.

Table 10: Noise-multipliers of Reference case: R_0 is known, σ is unknown by the Controller

Confidence levels		fd_3: normal $\geq 99.73\%$			fd_4: high $\geq 99.99\%$			b_{ML}
Noise Multipliers	m_σ fd_3	m_σ fd_4	m_L fd_3	m_L fd_4	m_D fd_3	m_D fd_4	m_H fd_3	m_H fd_4	$\geq$
$GT: R_0$	3.4	5.0	3.4	5.0	3.4	5.0	3.2	4.2	-4

6.8.3.2 Invariance to Corner position i_C

Invariance of Bias and Noise models against the Corner position i_C were checked by analyzing with $i_C = 131, 131.1, 130.9$, and $133.\ddot{3}$. Compared to the Reference case, the same Noise-multipliers deliver the same confidence levels.

6.8.3.3 Incremental Corner resolution

Finding the Corner with sub-incremental resolution costs some extra computation time, see chapter 5. There are two good reasons to avoid this cost. The first one being that Corners of the particular Process occur exactly on increment borders. The second reason being that integer resolution of the Corner position simply is sufficient for the Controller. However, it was found that the cost-benefit-relation for incremental Corner resolution is almost always negative.

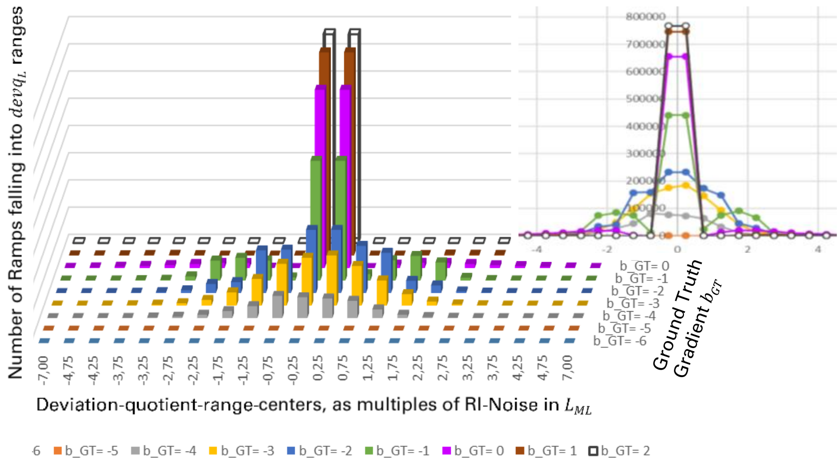


Fig. 6-29: Distribution of $devq_L$, scaled by RI-Noise $\tilde{\sigma}_{L.RI}$, for incremental Corner resolution. Compare with Fig. 6-26.

It turns out that the computational cost of sub-incremental minimum search is less than 20% of total Ramp Identification cost. Uncertainties do not lower compared to sub-incremental Corner resolution. Results for high Ramp Gradients even become much worse. Corner positions close to the middle of an increment are rounded either to the correct or the incorrect increment border. By consequence, the distribution of the Length errors is no longer a normal distribution but looks more like a sequence of ripples, see purple, green and blue distributions in Fig. 6-29. Hence, incremental resolution of the Corner position will not anymore be provided by Ramp Identification.

6.8.3.4 Controller provides Ground Truth Signal Noise σ_{GT} and Floor $R_{0,GT}$

RI estimates the standard deviation of the Signal Noise from σ_{ML} and its Bias $\tilde{\mu}_{\sigma.RI}$, both determined within the Observation Window. If the Controller knows a better estimation, it can force RI to use that value as Ground Truth σ_{GT}.

All NNRamp sets were analyzed with and without $known\ \sigma_{GT}$, but Bias and Noise model coefficients were kept unchanged. Slightly smaller Noise-Multipliers were found for Length, Gradient and Height estimation, i.e., RI results become a little more certain.

Table 11: Noise-multiplier if Controller provides Floor level $R_{0,GT}$ and Signal Noise σ_{GT}

Noise-multipliers	m_σ		m_L		m_D		m_H	
Confidence	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4
$GT: R_0\ and\ \sigma$	0.0	0.0	3.3	4.8	3.3	4.7	3.1	4.1

This means, that in case of very stable or predictable Gaussian Signal Noise, the Controller can improve RI results by providing Ground Truth for σ_{GT}. In general, RI's method to determine the standard deviation should also be sufficient. The influence of non-Gaussian noise on RI's results has not been studied.

6.8.3.5 Controller provides no Ground Truth

If the Ground Truth Floor level $R_{0,GT}$ is not available, RI estimates a value from $N_W = 100$ increments. The Controller has to make sure that no Ramp or other perturbation occur during this period.

As a result, some Noise model coefficients for Length L_{ML} and Height H_{ML} need to be adjusted for this case, referred to as $unknown\ R_{0,GT}$:

$$\tilde{\sigma}_{L.RI} = 1.89 \cdot L_{ML}{}^{-0.466} \cdot (D_{ML}/\sigma_{ML})^{-1.02}$$
$$\tilde{\sigma}_{nH.RI} = 1.47 \cdot L_{ML}{}^{-0.406}$$

Confidence level fd_3 is achievable with identical Noise-Multipliers. For fd_4 Noise-multipliers for Length and Height change a little.

Table 12: Noise-multipliers in case that Controller provides neither Floor level $R_{0,GT}$ nor Signal Noise σ_{GT}

Noise-multipliers	m_σ		m_L		m_D		m_H	
Confidence	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4
$GT: none$	3.4	5.0	3.4	4.8	3.4	5.0	3.2	4.3

Uncertainties in Ramp Parameters slightly decrease for shorter Ramps and increase for longer Ramps. All in all, RI maintains its accuracy even if the Floor $R_{0,GT}$ level is not provided by the Controller. However, if the Controller cannot guarantee $N_W = 100$ Ramp free increments after start, or cannot afford to wait that long, the Ground Truth Floor level $R_{0,GT}$ must be provided.

6.8.3.6 Controller provides Ground Truth Signal Noise σ_{GT}

If the Controller provides an estimation of the Signal Noise σ but wants RI to find a Floor level R_0, the results were the same as if no Ground Truth was provided at all.

Table 13: Noise-multipliers in case that Controller provides Signal Noise σ_{GT} only

Noise-multipliers	m_σ		m_L		m_D		m_H	
Confidence	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4
$GT: \sigma$	3.4	5.0	3.4	4.8	3.4	5.0	3.2	4.3

7 Instructions for use

7.1 Preparation

7.1.1 Process, Ramp and Controller

Given a physical process quantity which, after a constant state, inadvertently begins linearly ramping up. A Controller observes the process in real-time via a sensor signal that is subject to noise. The Controller intends to react as fast, accurately, and safely as possible to the process states Floor and Ramp. The Controller may have some expectations or some ground truth knowledge about the Ramp to occur, both in quality and quantity. These expectations, knowledge and the actual sensor signal are transmitted to the "Ramp Identification" (short: RI) procedure. In return, the Controller is informed by RI about Ramp appearances, Ramp parameters and related Uncertainties and Confidence levels.

Ramp Identification applies to an incremental Runtime signal of at least 100 equally spaced data points $i, X(i)$, that show constant Gaussian noise and possibly a Ramp $R(i)$, see Fig. 7-1. Increment numbers i must be equally spaced, positive integers. In a setup state, RI gathers 100 data points to estimate the Floor level R_0 and the standard deviation of signal Noise σ, if no Ground Truth knowledge was provided.

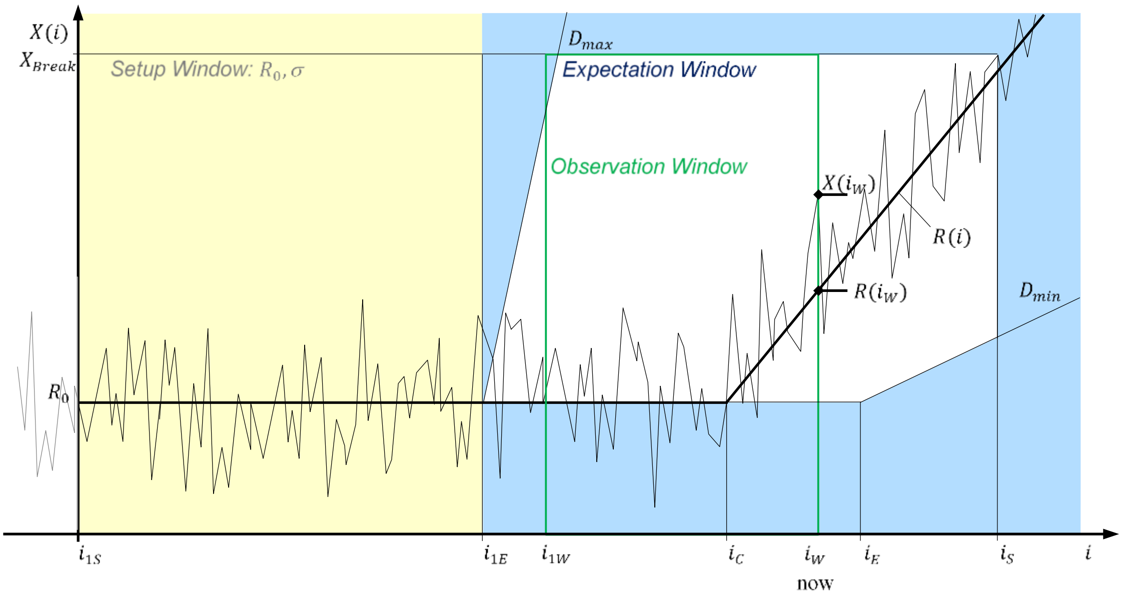


Fig. 7-1: Noisy Ramp Signal

7.1.2 Applicability of Ramp Identification

Beyond the conditions mentioned above, a Noisy Ramp signal must comply with some other design rules for RI to be successfully applicable. To check these rules, the following parameters are required:

- Floor level R_0 and standard deviation σ of the signal $X(i)$
- Upper limit of signal value X_{Break} for Process safety.
- Minimum and maximum expected Ramp Gradient $D_{min}; D_{max}$ with $D(i) = \partial R(i)/\partial i$.
- Confidence level for RI results, as required by the Controller.

The breaking signal value X_{Break} is defined such that the Process might cease to be a Ramp if $X(i) > X_{Break}$, hence RI would not deliver meaningful results. Design rule "Minimum Signal value" states that $X_{Break} > R_0 + 5 \cdot \sigma$, otherwise RI might not be able to achieve all its goals.

"Ramp Detection Length" L_{RD} is defined as the distance between the actual Ramp Corner and the increment when RI is certain about its detection. This length depends on the noise level, but an upper limit can be found using the normalized Gradient $d_{GT} = D_{min}/\sigma$ and the required confidence level, see below. Design rule “Maximum Ramp Detection latency” states: the Controller must cope with the fact that it takes L_{RD} increments before RI will detect the appearance of a Ramp with the confidence required.

Confidence level	normal $> 99.8\%$	high $> 99.99\%$
Ramp Detection Length L_{RD}	$L_{RD,3} = \left\lceil \frac{4}{d_{GT}} + 2 \right\rceil$	$L_{RD,5} = \left\lceil \frac{6.5}{d_{GT}} + 1.5 \right\rceil$

Design rule “Minimum detectable Gradient” states that for detectability within a limited Observation Window, the Height-Noise-ratio $h_{min,RD} = (R - R_0)/\sigma$ must exceed a certain limit, depending on the required confidence. If a Ramp grows across the entire Observation Window width of $N_W = 100$ data points, $(R - R_0)$ equals $N_W \cdot D_{min}$, and the minimum Ramp Gradient requirement becomes

$D_{min} > \frac{3 \cdot \sigma}{100}$ for Ramp Detection with $> 99.8\%$ confidence

$D_{min} > \frac{5 \cdot \sigma}{100}$ for Ramp Detection with $> 99.99\%$ confidence

Design rule “Maximum detectable Gradient” is due another limitation of this paper which is restricted to normalized Gradients $d_{max} = D_{max}/\sigma < 2^2$, thus $D_{max} < 4 \cdot \sigma$.

7.1.3 Process Knowledge and Expectations

The Controller transmits process knowledge and expectations to RI:

- At increment i_{1S} the Ramp Process is in Floor state and stable.
- The Ramp Corner is expected to occur not earlier than increment i_{1E}. For RI to be operational, $i_{1E} \geq i_S + 100$ must hold true.
- Ground Truth Floor level $R_{0,GT}$ and Ground Truth standard deviation σ_{GT}, if known.
- If $i \geq i_S$, RI will stop running, whether i_S was set by the Controller or by RI itself.
- The Ramp Corner is expected to occur not later than increment i_E, if at all.
- Stop condition X_{Break}. If $X(i) > X_{Break}$, RI will set $i_S = i$ on return.
- Minimum and maximum expected Ramp Gradient D_{min}, D_{max}, if known.
- Required Confidence level for Ramp Detection and Uncertainty estimation:
  $fd = 3$ if Confidence required is $> 99.73\%$
  $fd = 4$ if Confidence required is $> 99.99\%$

7.2 Initialization and continuous tasks

7.2.1 Gathering 100 equidistant data points

Data gathering begins at increment i_{1S} and ends at i_S. In each cycle, the Controller updates RI with the actual increment number i_W and signal value $X(i_W)$, i.e., a data point $\{i_W, X(i_W)\}$. This state lasts until RI has collected $N_W = 100$ data points. Without loss of generality, increment numbers are integers that grow by a constant positive integer.

7.2.2 Setup Ramp Floor level

An initial data set of $N_W = 100$ equidistant data points $\{i, X(i)\}$ with $i = i_{1W} \dots i_W$ and $i_{1W} = i_W - N_W + 1$ has been gathered. The data set represents RI’s Observation Window onto Signal X and associated increment numbers i. If $i_W < i_{1E}$, RI stays on hold. When i_W enters the Expectation Window, Ramp search finally begins. If Ground Truth value $R_{0,GT}$ was not provided by the Controller, RI estimates it from the data received:

- Set or estimate the Ramp Floor level R_0:

$$R_0 = \begin{cases} R_{0,GT}: if\ provided \\ R_{0,W}: otherwise \end{cases} \qquad \text{with } R_{0,W} = \frac{1}{N_W} \sum_{i=i_{1W}}^{i_W} X(i)$$

7.2.3 RIs task: Ramp Parameter estimation by ML

While $i_W \leq i_S$, add data point $\{i_W, X(i_W)\}$ to the Observation Window and discard the most ancient data point $\{i_W - N_W, X(i_W - N_W)\}$ from it. Set $i_{1W} = i_W - N_W + 1$.
RI solves the ML problem:

- ML estimated Ramp function

$$R_{ML}(i) = \begin{cases} i < i_{C,ML}: & R_0 \\ i \geq i_{C,ML}: & R_0 + D_{ML} \cdot (i - i_C) \end{cases}$$

- Solve the ML Least Squares problem over the Observation Window:

$$\min_{i_C, D_{ML}} \sum_{i=i_{1W}}^{i_W} (X(i) - R_{ML}(i))^2$$

- Set or determine the ML-filtered standard deviation σ_{ML}.

$$\sigma_{ML} = \begin{cases} \sigma_{GT}: if\ provided \\ \sigma_{ML,W}: otherwise \end{cases} \qquad \text{with } \sigma_{ML,W} = \sqrt{\frac{1}{N_W - 1} \cdot \sum_{i=i_{1W}}^{i_W} (X(i) - R_{ML}(i))^2}$$

- Set Ramp Length and Height:

$$L_{ML} = i_W - i_C$$
$$H_{ML} = L_{ML} \cdot D_{ML}$$

7.2.4 Controller tasks

At the present state of runtime process observation, the Ramp has not yet occurred. The Controller monitors and reacts to three potential events:

- RI has stopped processing data points: $i_W > i_S$
- Ramp Detection is being signaled by RI
- Signal level exceeds breaking condition: $X(i) > X_{Break}$

7.3 Q1: Ramp Detection

RI checks the occurrence of a Ramp using the logical condition:

$$Q1 = \left(H_{ML} >^? m \cdot \sigma_{ML}\right) = \begin{cases} true \\ false \end{cases}$$

The Ramp Detection criterion m depends on the confidence level for Ramp Detection:

Confidence level	$> 99.8\%$ “very likely”	$> 99.99\%$ “almost certain”
Ramp Detection criterion	$m = 3$	$m = 5$

In case of Ramp Detection flag being raised, Ramp Parameters L, D, H are not immediately available and the flag might even drop back. Thus, $Q1 = true$ only informs the Controller about the possibility of a Ramp. The Controller monitors these RI return states:

- RI has stopped processing data points: $i > i_S$
- Signal level exceeds breaking condition: $X(i) > X_{Break}$
- Ramp Detection is lost $Q1 = false$.
- RI transmits Ramp Parameters and Uncertainties

7.4 Q2: Bias & Noise modelling

Bias and Noise model validity for Ramp Parameters

aQt Length $L_{aQt} = \left\lceil \frac{4.4}{d_{ML}} + 2.5 \right\rceil$

Validity range of Bias and Noise models $L_{aQt} \leq L_{ML} < 100 = N_W$

Bias and Noise estimation for Ramp Parameters

Bias	RI Bias estimation	with
Bias in σ_{ML}	$\tilde{\mu}_{\sigma,RI} = \sigma_{ML} \cdot \tilde{\mu}_{n\sigma.RI}$	$\tilde{\mu}_{n\sigma.RI} = -0.0186 + 0.0000657 \cdot L_{ML}$
Bias in L_{ML}	$\tilde{\mu}_{L,RI} = 0$	$\tilde{\mu}_{L,RI} = 0$
Bias in D_{ML}	$\tilde{\mu}_{D,RI} = \sigma_{ML} \cdot \tilde{\mu}_{nD.RI}$	$\tilde{\mu}_{nD.RI} = 2.0 \cdot {L_{ML}}^{-2.4}$
Bias in H_{ML}	$\tilde{\mu}_{H,RI} = \sigma_{ML} \cdot \tilde{\mu}_{nH.RI}$	$\tilde{\mu}_{nH.RI} = 0.6 \cdot {L_{ML}}^{-0.8}$

Noise	RI Noise estimation	with
Noise in σ_{ML}	$\tilde{\sigma}_{\sigma,RI} = \sigma_{ML} \cdot \tilde{\sigma}_{n\sigma.RI}$	$\tilde{\sigma}_{n\sigma.RI} = 0.0731 + 5.57 \cdot 10^{-6} \cdot L_{ML}$
Noise in L_{ML}	$\tilde{\sigma}_{L.RI} = 2.062 \cdot {L_{ML}}^{-0.5058} \cdot (D_{ML}/\sigma_{ML})^{-1.017}$ with $R_{0,GT}$ $\tilde{\sigma}_{L.RI} = 1.89 \cdot {L_{ML}}^{-0.466} \cdot (D_{ML}/\sigma_{ML})^{-1.02}$ unknown $R_{0,GT}$	
Noise in D_{ML}	$\tilde{\sigma}_{D,RI} = \sigma_{ML} \cdot \tilde{\sigma}_{nD.RI}$	$\tilde{\sigma}_{nD.RI} = 2.8 \cdot {L_{ML}}^{-1.45}$
Noise in H_{ML}	$\tilde{\sigma}_{H,RI} = \sigma_{ML} \cdot \tilde{\sigma}_{nH.RI}$	$\tilde{\sigma}_{nH.RI} = 1.6 \cdot {L_{ML}}^{-0.445}$ with $R_{0,GT}$ $\tilde{\sigma}_{nH.RI} = 1.47 \cdot {L_{ML}}^{-0.406}$ unknown $R_{0,GT}$

Improve Ramp Parameters by using RI Bias models:

Parameter	RI estimation
Ramp Noise	$\sigma_{RI} = \sigma_{ML} - \tilde{\mu}_{\sigma.RI}$
Ramp Length	$L_{RI} = L_{ML} - \tilde{\mu}_{L.RI}$
Ramp Gradient	$D_{RI} = D_{ML} - \tilde{\mu}_{D.RI}$
Ramp Height	$H_{RI} = H_{ML} - \tilde{\mu}_{H.RI}$

7.5 Q3: Uncertainty & Confidence Gauging

Based on the above findings, RI estimates Noisy Ramp Parameters $\sigma_{RI}, L_{RI}, D_{RI}, H_{RI}$ are available. Respective Uncertainties $U_{\sigma x}, U_{Lx}, U_{Dx}, U_{Hx}$ were gauged for two confidence levels fd_x, defined as:

Index x	Symbol	Confidence Level
3	fd_3	$> 99.73\%$
4	fd_4	$> 99.99\%$

Uncertainties span a Confidence Interval that contains Ground Truth values at an indexed Confidence level fd_x:

Parameter	RI Uncertainty	Confidence Interval	contains Ground Truth
Noise	$U_{\sigma x} = m_{\sigma x} \cdot \tilde{\sigma}_{\sigma.RI}$	$\{\sigma_{RI} \pm U_{\sigma x}\}$	$\ni \sigma_{GT}$
Length	$U_{Lx} = m_{Lx} \cdot \tilde{\sigma}_{L.RI}$	$\{L_{RI} \pm U_{Lx}\}$	$\ni L_{GT}$
Gradient	$U_{Dx} = m_{Dx} \cdot \tilde{\sigma}_{D.RI}$	$\{D_{RI} \pm U_{Dx}\}$	$\ni D_{GT}$
Height	$U_{Hx} = m_{Hx} \cdot \tilde{\sigma}_{H.RI}$	$\{H_{RI} \pm U_{Hx}\}$	$\ni H_{GT}$

RI Noise-multipliers for problem variants where Ground Truth knowledge is available to the Controller:

Table 14: Noise-multipliers for Ground Truth variants

Confidence levels	fd_3: normal $\geq 99.73\%$				fd_4: high $\geq 99.99\%$				b_{ML}	σ_{GT}	$R_{0,GT}$
Noise Multipliers	m_σ		m_L		m_D		m_H		$\geq$	Known: ✓ Unknown: ×	
	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4	fd_3	fd_4			
$GT: none$	3.4	5.0	3.4	4.8	3.4	5.0	3.2	4.3	-4	×	×
$GT: \sigma$	0.0	0.0	3.4	4.8	3.4	5.0	3.2	4.3	-4	✓	×
$GT: R_0$	3.4	5.0	3.4	5.0	3.4	5.0	3.2	4.2	-4	×	✓
$GT: R_0$ and σ	0.0	0.0	3.3	4.8	3.3	4.7	3.1	4.1	-4	✓	✓

8 Summary

This paper presents a method that determines the Uncertainties of Ramp Identification, referred to as RI, in 100 Equidistant Data Points which suffer Gaussian Noise, and it develops an efficient algorithm for real-time execution of RI. The basic approach to this method consists of synthesizing a great number of Noisy Ramps and searching for patterns that do not depend on any Ground Truth knowledge. After an initial best-fit step which estimates Ramp Parameters L, D, H, σ, the Length, Gradient, Height and standard deviation, three principal question arise:

- $Q(1)$ Is there enough evidence for a Ramp occurrence, despite the Noise?
- $Q(2)$ Is it possible to express the estimation errors in L, D, H, σ as functions of the estimated magnitudes of L, D, H, σ?
- $Q(3)$ If outliers are acceptable up to a given limit, how much of Uncertainty lies between true and estimated Ramp?

Answers to all three questions were found. A sample Noisy Ramp with Ramp Identification results $L_{RI}, D_{RI}, H_{RI}, \sigma_{RI}$ is shown in Fig. 8-1. Ground Truth Length, Gradient and Height are well contained inside their respective Uncertainty intervals $\{L_{RI} \pm U_L\}, \{D_{RI} + U_D\}, \{H_{RI} \pm U_H\}$.

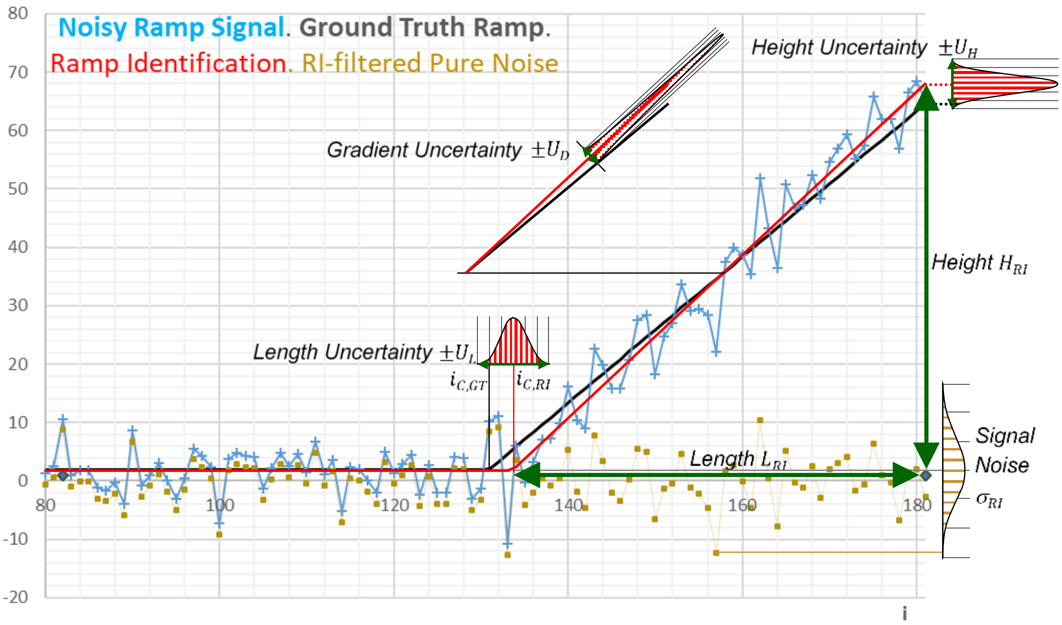


Fig. 8-1: Sample Noisy Ramp with Ramp Identification results.

RI method works for 100 equidistant data points, for perfect Ground Truth Ramps and pure Gaussian Noise. RI was derived from 16000 synthetic Noisy Ramps. Overcoming these limitations is desirable but requires considerably increasing efforts.

The motivation to this paper stems from precise collision detection and control in robotics using Force-Torque-Sensor signals. Linear-elastic mechanics and process-independent noise would then result into a Noisy Ramp signal as specified by RI. However, linear-elasticity in mechanisms is often spoiled by effects such as clearance, settling, friction, buckling and so forth. Gaussian noise may be superposed by mechanical shocks, vibrations, electro-magnetic interference, inaccurate motion control etc.. No practical experiments were conducted with RI so far, but it appears to be highly desirable to extend RI by methods that could estimate degradation effects caused by non-ideal Ramp Processes and Signal Noise.

9 Sources

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chat.openai.com — Response to the question: What could the term “Ramp Identification” mean? Used as confirmation to coin the term “Ramp Identification”.

dict.leo.org — Translations between German and English.

10 List of Abbreviations

aQt	pronounce ‘acute’: getting rid of Ground Truth Length dependency
Bias	mean error in a signal. Average of signal and its ground truth.
Gradient	Height to Length ratio of a Ramp
Length, Height	Distance of the actual point on the Ramp to the Ramp Corner
ML	Solver of RI initially based on Maximum Likelihood
NNRamp	Normal Noisy Ramp created from synthetic random numbers
Noise	Standard deviation of a signal
Ramp Corner	where the Floor level ends and the Ramp starts ascending
RI	Ramp Identification: method developed in this paper
RTRamp	Runtime Ramp: Signal that might show a Ramp
Runtime Ramp	a noisy signal being tested positive by RI
Runtime Signal	a noisy signal to be tested by Ramp Identification
Signal	Pulsed output at regular intervals from a sensor with white noise