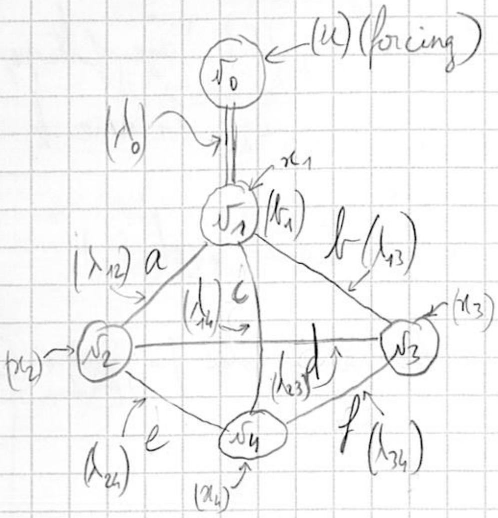
Network of magma reservoirs



$$\begin{cases} b_1 \dfrac{dx_1}{dt} = \lambda_0 (u - x_1) - \lambda_{12}(x_1 - x_2) - \lambda_{13}(x_1 - x_3) - \lambda_{14}(x_1 - x_4) \\ b_2 \dfrac{dx_2}{dt} = \qquad - \lambda_{12}(x_2 - x_1) - \lambda_{23}(x_2 - x_3) - \lambda_{24}(x_2 - x_4) \\ b_3 \dfrac{dx_3}{dt} = \qquad - \lambda_{13}(x_3 - x_1) - \lambda_{23}(x_3 - x_2) - \lambda_{34}(x_3 - x_4) \\ b_4 \dfrac{dx_4}{dt} = \qquad - \lambda_{14}(x_4 - x_1) - \lambda_{24}(x_4 - x_2) - \lambda_{34}(x_4 - x_3) \end{cases}$$

let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \Rightarrow \dot{x} = Ax + Bu$ - Ignoring λ_0 (if $\lambda_0 \to 0$) $\Rightarrow \dot{x} = Ax$

$$A = \begin{pmatrix} -\dfrac{\lambda_{12}+\lambda_{13}+\lambda_{14}}{b_1} & \lambda_{12}/b_1 & \lambda_{13}/b_1 & \lambda_{14}/b_1 \\ \lambda_{12}/b_2 & -\dfrac{\lambda_{12}+\lambda_{23}+\lambda_{24}}{b_2} & \lambda_{23}/b_2 & \lambda_{24}/b_2 \\ \lambda_{13}/b_3 & \lambda_{23}/b_3 & -\dfrac{\lambda_{13}+\lambda_{23}+\lambda_{34}}{b_3} & \lambda_{34}/b_3 \\ \lambda_{14}/b_4 & \lambda_{24}/b_4 & \lambda_{34}/b_4 & -\dfrac{\lambda_{14}+\lambda_{24}+\lambda_{34}}{b_4} \end{pmatrix}$$

diagonal　　symmetric

$A = V L$ with

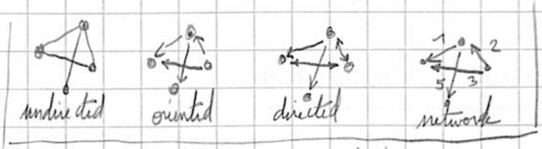
$$B = \begin{pmatrix} -(\lambda_{12}+\lambda_{13}+\lambda_{14}) & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{12} & -(\lambda_{12}+\lambda_{23}+\lambda_{24}) & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & -(\lambda_{13}+\lambda_{23}+\lambda_{34}) & \lambda_{34} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & -(\lambda_{14}+\lambda_{24}+\lambda_{34}) \end{pmatrix}$$

$\hookrightarrow L = L^T$ (symmetric)

and $V = \begin{pmatrix} 1/b_1 & & & 0 \\ & 1/b_2 & & \\ & & 1/b_3 & \\ 0 & & & 1/b_4 \end{pmatrix}$

(diagonal)

Graphs



$G = (V, E)$ — V: vertices or nodes; E: edges

$|V|$ = "order of the graph" ; $|E|$ = "size of the graph"

Incidence matrix (undirected graph)

$B(G) = (b_{ij})$ / $b_{ij} = \begin{cases} +1 & \text{if } e_j = \{v_i ; v_k\} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$

→ or ± 1 for oriented graph

→ row sum of B = "degree"
→ col " " B = 2.

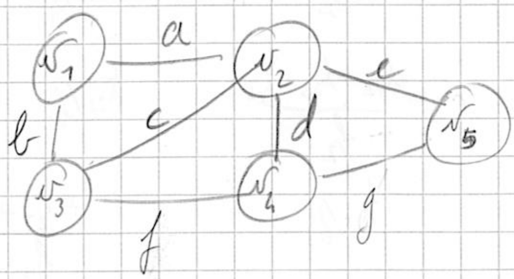
Adjacency matrix (unweighted)

$A(G) = (a_{ij})$ / $a_{ij} = \begin{cases} 1 & \text{if there is some edge } \{v_i ; v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$

Degree matrix

$D(G) = (d_{ij}) = \mathrm{diag}(d(v_i))$, $d(v_i) = \left| \{ u \in V / \{u, v_i\} \in E \} \right|$

Example



$$D = \begin{array}{c} \begin{pmatrix} 2 & & & & 0 \\ & 4 & & & \\ & & 3 & & \\ & & & 3 & \\ 0 & & & & 2 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \\ \leftarrow v_4 \\ \leftarrow v_5 \end{array}$$

$$B_g = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a & b & c & d & e & f & g \end{array} \end{array} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \\ \leftarrow v_4 \\ \leftarrow v_5 \\ \\ \\ \end{array}$$

(graph)

$$A = \begin{array}{c} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \end{array} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \\ \leftarrow v_4 \\ \leftarrow v_5 \\ \\ \\ \end{array}$$

diagonal always 0 for simple graph (no self-loop)

$$B_o = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

(oriented)

Laplacian matrix (oriented unweighted)

$$L = B_o B_o^T = D - A$$

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}$$

L is $\begin{cases} \text{symmetric} \\ \text{positive} \\ \text{semidefinite} \\ \text{eigenvalues} \in \mathbb{R}^+ \end{cases}$

Smallest eigenvalue = 0, with eigenvector = $(1, 1, 1, \dots, 1)$

Normalized ← $L_{sym} = D^{-1/2} L D^{1/2} = I - D^{-1/2} A D^{1/2}$

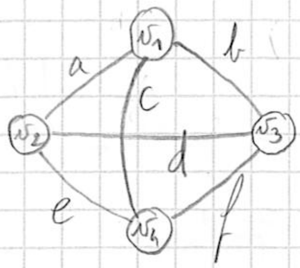
Random walk ← $L_{rw} = D^{-1} L = I - D^{-1} A$

Weights (= edge weights)

$$W = \mathrm{diag}(w_{ij}) \;/\; w_{ij} = \text{weight of edge } e_{ij} = \{v_i, v_j\}$$

$\rightarrow$ Laplacian matrix : $L = B W B^T$ (with $B = B_o$)
(weighted, oriented)

Example



$$B = \begin{array}{c} \\ \left(\begin{array}{cccccc} -1 & -1 & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & -1 & -1 & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot & 1 & 1 \end{array}\right) \\ \begin{array}{cccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a & b & c & d & e & f \\ \times\lambda_{12} & \times\lambda_{13} & \times\lambda_{14} & \times\lambda_{23} & \times\lambda_{24} & \times\lambda_{34} \end{array} \end{array} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \\ \leftarrow v_4 \\ \\ \\ \\ \end{array}$$

$$W = \begin{pmatrix} \lambda_{12} & & & & & \\ & \lambda_{13} & & & \bigcirc & \\ & & \lambda_{14} & & & \\ & & & \lambda_{23} & & \\ & \bigcirc & & & \lambda_{24} & \\ & & & & & \lambda_{34} \end{pmatrix}$$

$$BW = \begin{pmatrix} -\lambda_{12} & -\lambda_{13} & -\lambda_{14} & \cdot & \cdot & \cdot \\ \lambda_{12} & \cdot & \cdot & -\lambda_{23} & -\lambda_{24} & \cdot \\ \cdot & \lambda_{13} & \cdot & \lambda_{23} & \cdot & -\lambda_{34} \\ \cdot & \cdot & \lambda_{14} & \cdot & \lambda_{24} & \lambda_{34} \end{pmatrix}$$

$$L = BWB^T = \begin{pmatrix} \lambda_{12}+\lambda_{13}+\lambda_{14} & -\lambda_{12} & -\lambda_{13} & -\lambda_{14} \\ -\lambda_{12} & \lambda_{12}+\lambda_{23}+\lambda_{24} & -\lambda_{23} & -\lambda_{24} \\ -\lambda_{13} & -\lambda_{23} & \lambda_{13}+\lambda_{23}+\lambda_{34} & -\lambda_{34} \\ -\lambda_{14} & -\lambda_{24} & -\lambda_{34} & \lambda_{14}+\lambda_{24}+\lambda_{34} \end{pmatrix} \rightarrow \text{symmetric}$$

Eigenvalue approach to solving $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$

* case 1: $\underline{\underline{A}}$ is diagonal : $A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$

  ↳ decoupled

initial condition

$$\Rightarrow \dot{x}_i(t) = a_{ii} \cdot x_i(t) \Rightarrow x_i(t) = e^{a_{ii} t} \times x_i(0)$$

* case 2: $\underline{\underline{A}}$ is not diag (and non-singular)

$\Rightarrow$ eigendecomposition of A : $AT = TD$ with: $\left\{ \begin{array}{l} T = \begin{bmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{bmatrix} \\ \text{↳ eigenvectors} \\ \text{eigenvalues} \\ D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \end{array} \right.$

change of coordinates: $x = Tz \Leftrightarrow z = T^{-1} x$

$$T^{-1}AT = T^{-1}TD = D \text{ or } A = TDT^{-1}$$

$$\dot{x} = Ax \xRightarrow{T^{-1}\times} T^{-1}\dot{x} = T^{-1}.A.x$$

$$= T^{-1}.(TDT^{-1}).x$$

$$\underbrace{T^{-1}\dot{x}}_{\dot{z}} = D\underbrace{T^{-1}x}_{z} \Rightarrow \dot{z} = \underset{\text{diag.}}{D} z$$

$$\Downarrow$$

$$z(t) = e^{Dt} \cdot z(0)$$

with $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}$

$$A = TDT^{-1}$$
$$A^2 = (TDT^{-1})(TDT^{-1}) = TD^2T^{-1}$$
$$A^3 = - - - - - = TD^3T^{-1}$$
$$\vdots$$
$$A^n = TD^nT^{-1}$$

Taylor expansion: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

substitution: $x \to At$

$$\hookrightarrow e^{At} = 1 + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$De^{Dt} = e^{Dt}D = \frac{d}{dt}\left(e^{Dt}\right)$$

$$= TT^{-1} + TDT^{-1}t + TD^2T^{-1}\frac{t^2}{2!} + TD^3T^{-1}\frac{t^3}{3!} + \cdots$$

$$= T.\left(I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots\right).T^{-1}$$

$$e^{At} = T.e^{Dt}.T^{-1} \quad \text{with } e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}$$

$$\Rightarrow x(t) = T\underbrace{e^{Dt}\underbrace{T^{-1}x(0)}_{z(0)}}_{z(t)} \text{ is solution of } \dot{x} = Ax$$

Generalization to $\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u}$ (with forcing)

$$\Rightarrow x(t) = \underbrace{e^{\underline{\underline{A}}t}.x(0)}_{\text{initial condition response}} + \int_0^t e^{A(t-\tau)}.\underline{\underline{B}}.u(\tau)\,d\tau$$

$$\dot{x} = Ax + Bu \Leftrightarrow \dot{x} + Px = Bu \quad (\text{with } A = -P)$$

$e^{Pt}\times$

$$e^{Pt}.\dot{x} + e^{Pt}.P.x = e^{Pt}.Bu$$

but: $Pe^{tP} = P(I + Pt + \frac{P^2t^2}{2!} + \frac{P^3t^3}{3!} + \dots) = P + P^2t + \frac{P^3t^2}{2!} + \dots$

$$= (I + Pt + \frac{P^2t^2}{2!} + \frac{P^3t^3}{3!})P = e^{tP}.P$$

and: $\frac{d}{dt}(e^{Pt}) = P.e^{Pt} = e^{Pt}.P$

$$\Rightarrow e^{Pt}.\frac{d}{dt}(x) + \frac{d}{dt}(e^{Pt}).x = \frac{d}{dt}(e^{Pt}.x) = e^{Pt}.Bu$$

$\int_{t_0}^{t}(\ )d\tau$

$$e^{Pt}x(t) = \int_{t_0}^{t} e^{P\tau}.B.u(\tau)\,d\tau + c$$

$t = t_0$

$$e^{Pt_0}x(t_0) = \underbrace{\int_{t_0}^{t_0} e^{P\tau}.B.u(\tau)\,d\tau}_{=0} + c \Rightarrow e^{Pt_0}x(t_0) = c$$

$$e^{Pt}x(t) = e^{Pt_0}x(t_0) + \int_{t_0}^{t} e^{P\tau}.B.u(\tau)\,d\tau$$

$\times e^{-Pt} = \times(e^{Pt})^{-1} \Rightarrow$

$$x(t) = e^{P(t_0-t)}.x(t_0) + e^{-Pt}\int_{t_0}^{t} e^{P\tau}.B.u(\tau).d\tau$$

$$= e^{P(t_0-t)}x(t_0) + \int_{t_0}^{t} e^{P(\tau-t)}B.u(\tau)\,d\tau$$

$$x(t) = e^{A(t-t_0)}.x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B.u(\tau)\,d\tau$$

(Duhamel)

Special case: $x(t_0) = x(0) = 0$ and $u(t) = \delta(0)$ and $B = x_0$

$\Rightarrow$ equivalent to homogeneous case $\Rightarrow h(t) = e^{At}.B$ "impulse response"

$$x(t) = e^{At}.x(0) + \underbrace{\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau}_{h * u}$$