1 DG Formulation

The shallow water equations in conservative form:

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} = 0, \tag{1}$$

$$\frac{\partial (Hu)}{\partial t} + \frac{\partial}{\partial x}\left(Hu^2 + \frac{1}{2}gH^2\right) + \frac{\partial}{\partial y}(Huv) = gH\frac{\partial b}{\partial x} - \tau Hu, \tag{2}$$

$$\frac{\partial (Hv)}{\partial t} + \frac{\partial}{\partial x}(Huv) + \frac{\partial}{\partial y}\left(Hv^2 + \frac{1}{2}gH^2\right) = gH\frac{\partial b}{\partial y} - \tau Hv. \tag{3}$$

Define

$$U \equiv Hu, \qquad V \equiv Hv, \tag{4}$$

so the system can be written in terms of 3 variables

$$\frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{5}$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{U^2}{H} + \frac{1}{2}gH^2\right) + \frac{\partial}{\partial y}\left(\frac{UV}{H}\right) = gH\frac{\partial b}{\partial x} - \tau U, \tag{6}$$

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{UV}{H}\right) + \frac{\partial}{\partial y}\left(\frac{V^2}{H} + \frac{1}{2}gH^2\right) = gH\frac{\partial b}{\partial y} - \tau V. \tag{7}$$

This is written in vector form as

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}, \tag{8}$$

where,

$$\mathbf{Q} = \begin{bmatrix} H \\ U \\ V \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} U \\ \frac{U^2}{H} + \frac{1}{2}gH^2 \\ \frac{UV}{H} \end{bmatrix}, \quad \mathbf{G} = \mathbf{G}(\mathbf{Q}) = \begin{bmatrix} V \\ \frac{UV}{H} \\ \frac{V^2}{H} + \frac{1}{2}gH^2 \end{bmatrix}, \quad \mathbf{S} = \mathbf{S}(\mathbf{Q}) = \begin{bmatrix} 0 \\ gH\frac{\partial b}{\partial x} - \tau U \\ gH\frac{\partial b}{\partial y} - \tau V \end{bmatrix}. \tag{9}$$

The finite element space is $\mathcal{V}_h^p = \left\{ w_h(x,y) : w_h(x,y)\big|_{\Omega_i} \in \mathcal{P}^p(\Omega_i) \right\}$. Multiply by a test function $w_h \in \mathcal{V}_h^p$ and integrate over an arbitrary element domain, Ω_i

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h \, d\Omega_i + \int_{\Omega_i} \frac{\partial \mathbf{F}}{\partial x} w_h \, d\Omega_i + \int_{\Omega_i} \frac{\partial \mathbf{G}}{\partial y} w_h \, d\Omega_i = \int_{\Omega_i} \mathbf{S} w_h \, d\Omega_i. \tag{10}$$

Integrate the flux terms by parts

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h \, d\Omega_i - \int_{\Omega_i} \mathbf{F} \frac{\partial w_h}{\partial x} \, d\Omega_i - \int_{\Omega_i} \mathbf{G} \frac{\partial w_h}{\partial y} \, d\Omega_i + \int_{\partial\Omega_i} n_x \mathbf{F} w_h \, d\partial\Omega_i + \int_{\partial\Omega_i} n_y \mathbf{G} w_h \, d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} w_h \, d\Omega_i. \tag{11}$$

This results in integrals over the element edge $\partial\Omega_i$ where n_x and n_y are the edge normals in the x and y directions. Grouping terms

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h \, d\Omega_i - \int_{\Omega_i} \left(\mathbf{F} \frac{\partial w_h}{\partial x} + \mathbf{G} \frac{\partial w_h}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F} + n_y \mathbf{G}) \, w_h \, d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} w_h \, d\Omega_i. \tag{12}$$

The exact solution variables in $\mathbf{Q}$ are replaced with an approximation $\mathbf{Q}_h$ where $H_h, U_h, V_h \in \mathcal{V}_h^p$

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} w_h \, d\Omega_i - \int_{\Omega_i} \left(\mathbf{F}_h \frac{\partial w_h}{\partial x} + \mathbf{G}_h \frac{\partial w_h}{\partial y} \right) \, d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* \, w_h \, d\partial\Omega_i = \int_{\Omega_i} \mathbf{S}_h w_h \, d\Omega_i. \tag{13}$$

The quantity $(n_x \mathbf{F}_h + n_y \mathbf{G}_h)^*$ is the numerical flux and is needed to determine a single value of the edge integral given the two solutions that exist at the element interface due to the discontinuous approximation. The approximate solutions are expressed as linear combinations of unknown, time-dependent coefficients $\left(H_i^{(m)}, U_i^{(m)}, V_i^{(m)}\right)$ and spatially dependent functions $\left\{\phi_i^{(m)}\right\}_{m=1}^{K}$ that form a basis for $\mathcal{P}^p(\Omega_i)$

$$\mathbf{Q}_h = \begin{bmatrix} H_h \\ U_h \\ V_h \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{K} H_i^{(m)}(t) \phi_i^{(m)}(x,y) \\ \sum_{m=1}^{K} U_i^{(m)}(t) \phi_i^{(m)}(x,y) \\ \sum_{m=1}^{K} V_i^{(m)}(t) \phi_i^{(m)}(x,y) \end{bmatrix}, \qquad (x,y), \in \Omega_i \tag{14}$$

$$\mathbf{F}_h = \mathbf{F}(\mathbf{Q}_h), \qquad \mathbf{G}_h = \mathbf{G}(\mathbf{Q}_h), \qquad \mathbf{S}_h = \mathbf{S}(\mathbf{Q}_h), \tag{15}$$

The number of degrees of freedom K, depends on the order of the polynomial basis, p

$$K = \frac{(p+1)(p+2)}{2}. \tag{16}$$

The test functions are chosen as

$$w_h = \phi_i^{(l)}(x,y), \qquad l = 1, \ldots, K. \tag{17}$$

Substituting for w_h

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} \phi_i^{(l)} d\Omega_i - \int_{\Omega_i} \left(\mathbf{F}_h \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* \, \phi_i^{(l)} d\partial\Omega_i = \int_{\Omega_i} \mathbf{S}_h \phi_i^{(l)} d\Omega_i$$
$$l = 1, \ldots, K. \tag{18}$$

The Lax Friedrichs numerical flux is used in evaluating the edge integral

$$(n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* = n_x \left(\frac{\mathbf{F}(\mathbf{Q}_h^{ex}) + \mathbf{F}(\mathbf{Q}_h^{in})}{2} \right) + n_y \left(\frac{\mathbf{G}(\mathbf{Q}_h^{ex}) + \mathbf{G}(\mathbf{Q}_h^{in})}{2} \right) - \frac{\alpha}{2} \left(\mathbf{Q}_h^{ex} - \mathbf{Q}_h^{in} \right), \tag{19}$$

where

$$\alpha = \max_{\mathbf{Q}_h^{in}, \mathbf{Q}_h^{ex}} \left| u n_x + v n_y + \sqrt{gH} \right|. \tag{20}$$

Substituting for the components of the $\mathbf{Q}_h$ vector gives

$$\sum_{m=1}^{K} \frac{\partial H_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i =$$
$$\int_{\Omega_i} \left(\mathbf{F}_h^{(1)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(1)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(1)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} \left(n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)} \right)^* \phi_i^{(l)} d\partial\Omega_i$$
$$l = 1, \ldots, K \tag{21}$$

$$\sum_{m=1}^{K} \frac{\partial U_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i =$$
$$\int_{\Omega_i} \left(\mathbf{F}_h^{(2)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(2)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(2)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} \left(n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)} \right)^* \phi_i^{(l)} d\partial\Omega_i$$
$$l = 1, \ldots, K \quad (22)$$

$$\sum_{m=1}^{K} \frac{\partial V_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i =$$
$$\int_{\Omega_i} \left(\mathbf{F}_h^{(3)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(3)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(3)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} \left(n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)} \right)^* \phi_i^{(l)} d\partial\Omega_i$$
$$l = 1, \ldots, K \quad (23)$$

The each element is mapped to a reference element (Δ) with the transformation

$$x = -\frac{1}{2} \left((r+s) x_i^{(1)} - (1+r) x_i^{(2)} - (1+s) x_i^{(3)} \right) \quad (24)$$
$$y = -\frac{1}{2} \left((r+s) y_i^{(1)} - (1+r) y_i^{(2)} - (1+s) y_i^{(3)} \right) \quad (25)$$

$$r = \frac{1}{A_i} \left[\left(y_i^{(3)} - y_i^{(1)} \right) \left(x - \frac{1}{2} \left(x_i^{(2)} + x_i^{(3)} \right) \right) + \left(x_i^{(1)} - x_i^{(3)} \right) \left(y - \frac{1}{2} \left(y_i^{(2)} - y_i^{(3)} \right) \right) \right], \quad (26)$$
$$s = \frac{1}{A_i} \left[\left(y_i^{(1)} - y_i^{(2)} \right) \left(x - \frac{1}{2} \left(x_i^{(2)} + x_i^{(3)} \right) \right) + \left(x_i^{(2)} - x_i^{(1)} \right) \left(y - \frac{1}{2} \left(y_i^{(2)} - y_i^{(3)} \right) \right) \right], \quad (27)$$

where

$$A_i = \frac{1}{2} \left(\left(x_i^{(2)} y_i^{(3)} - x_i^{(3)} y_i^{(2)} \right) + \left(x_i^{(3)} y_i^{(1)} - x_i^{(1)} y_i^{(3)} \right) + \left(x_i^{(1)} y_i^{(2)} - x_i^{(2)} y_i^{(1)} \right) \right). \quad (28)$$

Therefore, the area and edge integrals can be preformed numerically by

$$\int_{\Omega_i} f(x,y) dx dy = \frac{A_i}{2} \int_{\Delta} f(r,s) dr ds, \quad (29)$$
$$= \frac{A_i}{2} \sum_{j=1}^{q^{(a)}} \tilde{w}_j^{(a)} f\left(\tilde{r}_j^{(a)}, \tilde{s}_j^{(a)} \right), \quad (30)$$

$$\int_{\partial\Omega_i} f(x,y) d\partial\Omega_i = \sum_{k=1}^{3} \frac{l_{i,k}}{2} \int_{-1}^{1} f(\xi) d\xi, \quad (31)$$
$$= \sum_{k=1}^{3} \frac{l_{i,k}}{2} \sum_{j=1}^{q^{(e)}} \tilde{w}_j^{(e)} f\left(\tilde{\xi}_j^{(e)} \right). \quad (32)$$

The derivatives must also be transformed to the reference element coordinates

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x}, \quad (33)$$
$$= \frac{1}{A_i} \left(\frac{\partial f}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial f}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right), \quad (34)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y}, \tag{35}$$

$$= \frac{1}{A_i}\left(\frac{\partial f}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial f}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right). \tag{36}$$

$$\tag{37}$$

All integrals are now written in terms of the reference element coordinates

$$\frac{A_i}{2}\sum_{m=1}^{K}\frac{\partial H_i^{(m)}}{\partial t}\int_\Delta \phi_i^{(m)}\phi_i^{(l)}d\Delta =$$
$$\frac{A_i}{2}\int_\Delta\left(\mathbf{F}_h^{(1)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) + \mathbf{G}_h^{(1)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) + \mathbf{S}_h^{(1)}\phi_i^{(l)}\right)d\Delta$$
$$-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(1)} + n_y\mathbf{G}_h^{(1)}\right)^*\phi_i^{(l)}d\xi$$
$$l = 1,\ldots,K \tag{38}$$

$$\frac{A_i}{2}\sum_{m=1}^{K}\frac{\partial U_i^{(m)}}{\partial t}\int_\Delta \phi_i^{(m)}\phi_i^{(l)}d\Delta =$$
$$\frac{A_i}{2}\int_\Delta\left(\mathbf{F}_h^{(2)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) + \mathbf{G}_h^{(2)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) + \mathbf{S}_h^{(2)}\phi_i^{(l)}\right)d\Delta$$
$$-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(2)} + n_y\mathbf{G}_h^{(2)}\right)^*\phi_i^{(l)}d\xi$$
$$l = 1,\ldots,K \tag{39}$$

$$\frac{A_i}{2}\sum_{m=1}^{K}\frac{\partial V_i^{(m)}}{\partial t}\int_\Delta \phi_i^{(m)}\phi_i^{(l)}d\Delta =$$
$$\frac{A_i}{2}\int_\Delta\left(\mathbf{F}_h^{(3)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) + \mathbf{G}_h^{(3)}\frac{1}{A_i}\left(\frac{\partial\phi_i^{(l)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial\phi_i^{(l)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) + \mathbf{S}_h^{(3)}\phi_i^{(l)}\right)d\Delta$$
$$-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(3)} + n_y\mathbf{G}_h^{(3)}\right)^*\phi_i^{(l)}d\partial\Omega_i$$
$$l = 1,\ldots,K \tag{40}$$

Expanding these equations into vector form and simplifying the area integral transformation constant leads to

$$\frac{A_i}{2}\begin{bmatrix}\int_\Delta \phi_i^{(1)}\phi_i^{(1)}d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)}d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)}d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)}d\Delta\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}H_i^{(1)} \\ \vdots \\ H_i^{(K)}\end{bmatrix} =$$
$$\left(\begin{bmatrix}\frac{1}{2}\int_\Delta\left(\mathbf{F}_h^{(1)}\left(\frac{\partial\phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(1)}\left(\frac{\partial\phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(1)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \frac{1}{2}\int_\Delta\left(\mathbf{F}_h^{(1)}\left(\frac{\partial\phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(1)}\left(\frac{\partial\phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(1)}\phi_i^{(K)}\right)d\Delta\end{bmatrix}\right.$$
$$\left.-\begin{bmatrix}\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(1)}+n_y\mathbf{G}_h^{(1)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(1)}+n_y\mathbf{G}_h^{(1)}\right)^*\phi_i^{(K)}d\xi\end{bmatrix}\right) \quad (41)$$

$$\frac{A_i}{2}\begin{bmatrix}\int_\Delta \phi_i^{(1)}\phi_i^{(1)}d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)}d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)}d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)}d\Delta\end{bmatrix}\frac{\partial}{\partial t}\begin{bmatrix}U_i^{(1)} \\ \vdots \\ U_i^{(K)}\end{bmatrix} =$$
$$\left(\begin{bmatrix}\frac{1}{2}\int_\Delta\left(\mathbf{F}_h^{(2)}\left(\frac{\partial\phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(2)}\left(\frac{\partial\phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(2)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \frac{1}{2}\int_\Delta\left(\mathbf{F}_h^{(2)}\left(\frac{\partial\phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(2)}\left(\frac{\partial\phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(2)}\phi_i^{(K)}\right)d\Delta\end{bmatrix}\right.$$
$$\left.-\begin{bmatrix}\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(2)}+n_y\mathbf{G}_h^{(2)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(2)}+n_y\mathbf{G}_h^{(2)}\right)^*\phi_i^{(K)}d\xi\end{bmatrix}\right) \quad (42)$$

$$
\frac{A_i}{2}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix} \frac{\partial}{\partial t}\begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} =
$$

$$
\left(\begin{bmatrix} \frac{1}{2}\int_\Delta \left(\mathbf{F}_h^{(3)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(3)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(3)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \frac{1}{2}\int_\Delta \left(\mathbf{F}_h^{(3)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(3)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(3)}\phi_i^{(K)}\right)d\Delta \end{bmatrix} - \begin{bmatrix} \sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(3)}+n_y\mathbf{G}_h^{(3)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(3)}+n_y\mathbf{G}_h^{(3)}\right)^*\phi_i^{(K)}d\xi \end{bmatrix}\right) \tag{43}
$$

Inverting the mass matrix leads to a system of ordinary differential equations

$$
\frac{\partial}{\partial t}\begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}
$$

$$
\left(\begin{bmatrix} \int_\Delta \left(\mathbf{F}_h^{(1)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(1)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(1)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \int_\Delta \left(\mathbf{F}_h^{(1)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(1)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(1)}\phi_i^{(K)}\right)d\Delta \end{bmatrix} - \begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(1)}+n_y\mathbf{G}_h^{(1)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(1)}+n_y\mathbf{G}_h^{(1)}\right)^*\phi_i^{(K)}d\xi \end{bmatrix}\right) \tag{44}
$$

$$
\frac{\partial}{\partial t}\begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}
$$

$$
\left(\begin{bmatrix} \int_\Delta \left(\mathbf{F}_h^{(2)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(2)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(2)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \int_\Delta \left(\mathbf{F}_h^{(2)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(2)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(2)}\phi_i^{(K)}\right)d\Delta \end{bmatrix}\right.
$$

$$
\left. - \begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(2)}+n_y\mathbf{G}_h^{(2)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(2)}+n_y\mathbf{G}_h^{(2)}\right)^*\phi_i^{(K)}d\xi \end{bmatrix}\right) \tag{45}
$$

$$
\frac{\partial}{\partial t}\begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}
$$

$$
\left(\begin{bmatrix} \int_\Delta \left(\mathbf{F}_h^{(3)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(3)}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(3)}\phi_i^{(1)}\right)d\Delta \\ \vdots \\ \int_\Delta \left(\mathbf{F}_h^{(3)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+\mathbf{G}_h^{(3)}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)+A_i\mathbf{S}_h^{(3)}\phi_i^{(K)}\right)d\Delta \end{bmatrix}\right.
$$

$$
\left. - \begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(3)}+n_y\mathbf{G}_h^{(3)}\right)^*\phi_i^{(1)}d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\mathbf{F}_h^{(3)}+n_y\mathbf{G}_h^{(3)}\right)^*\phi_i^{(K)}d\xi \end{bmatrix}\right) \tag{46}
$$

The components of the **F**, **G** and **S** vectors are substituted to give

$$\frac{\partial}{\partial t}\begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}$$

$$\left(\begin{bmatrix} \int_\Delta \left(U_h\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+V_h\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)d\Delta \\ \vdots \\ \int_\Delta \left(U_h\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)+V_h\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)d\Delta \end{bmatrix}\right.$$

$$\left.-\begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x U_h + n_y V_h\right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x U_h + n_y V_h\right)^* \phi_i^{(K)} d\xi \end{bmatrix}\right) \tag{47}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}$$

$$\left(\begin{bmatrix} \int_\Delta \left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2\right)\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)d\Delta \\ \vdots \\ \int_\Delta \left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2\right)\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)d\Delta \end{bmatrix}\right.$$

$$+\begin{bmatrix} \int_\Delta \frac{U_h V_h}{H_h}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)d\Delta \\ \vdots \\ \int_\Delta \frac{U_h V_h}{H_h}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)d\Delta \end{bmatrix}+\begin{bmatrix} A_i\int_\Delta \left(gH_h\frac{\partial b}{\partial x}-\tau U_h\right)\phi_i^{(1)} d\Delta \\ \vdots \\ A_i\int_\Delta \left(gH_h\frac{\partial b}{\partial x}-\tau U_h\right)\phi_i^{(K)} d\Delta \end{bmatrix}$$

$$\left.-\begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2\right)+n_y\frac{U_h V_h}{H_h}\right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2\right)+n_y\frac{U_h V_h}{H_h}\right)^* \phi_i^{(K)} d\xi \end{bmatrix}\right) \tag{48}$$

$$
\frac{\partial}{\partial t}\begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}
\left(\begin{bmatrix} \int_\Delta \frac{U_h V_h}{H_h}\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \frac{U_h V_h}{H_h}\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \end{bmatrix} \right.
$$

$$
+ \begin{bmatrix} \int_\Delta \left(\frac{V_h^2}{H_h} + \frac{1}{2}gH_h^2\right)\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \left(\frac{V_h^2}{H_h} + \frac{1}{2}gH_h^2\right)\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \end{bmatrix} + \begin{bmatrix} A_i \int_\Delta \left(gH_h \frac{\partial b}{\partial y} - \tau V_h\right)\phi_i^{(1)} d\Delta \\ \vdots \\ A_i \int_\Delta \left(gH_h \frac{\partial b}{\partial y} - \tau V_h\right)\phi_i^{(K)} d\Delta \end{bmatrix}
$$

$$
\left. - \begin{bmatrix} \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2}gH_h^2\right)\right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2}gH_h^2\right)\right)^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \tag{49}
$$

Substituting the approximate expansions for H_h, U_h, and V_h

$$
\frac{\partial}{\partial t}\begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} = \frac{1}{A_i}\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)} d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)} d\Delta \end{bmatrix}^{-1}
\left(\begin{bmatrix} \int_\Delta \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \end{bmatrix} \right.
$$

$$
+ \begin{bmatrix} \int_\Delta \left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \end{bmatrix}
$$

$$
\left. - \begin{bmatrix} \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right) + n_y \left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right) + n_y \left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\right)^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \tag{50}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} = \frac{1}{A_i}
&\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)}\,d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)}\,d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)}\,d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)}\,d\Delta \end{bmatrix}^{-1} \\
&\Bigg(\Bigg(\begin{bmatrix} \int_\Delta \left(\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} + \frac{1}{2} g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2 \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \left(\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} + \frac{1}{2} g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2 \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)} - y_i^{(2)}\right)\right) d\Delta \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_\Delta \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)} - x_i^{(3)}\right) + \frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)} - x_i^{(1)}\right)\right) d\Delta \end{bmatrix} \\
+ &\begin{bmatrix} A_i \int_\Delta \left(g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right) \left(\sum_{m=1}^{K} b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r}\frac{1}{A_i}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(m)}}{\partial s}\frac{1}{A_i}\left(y_i^{(1)} - y_i^{(2)}\right)\right)\right) - \tau \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\right) \phi_i^{(1)}\,d\Delta \\ \vdots \\ A_i \int_\Delta \left(g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right) \left(\sum_{m=1}^{K} b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r}\frac{1}{A_i}\left(y_i^{(3)} - y_i^{(1)}\right) + \frac{\partial \phi_i^{(m)}}{\partial s}\frac{1}{A_i}\left(y_i^{(1)} - y_i^{(2)}\right)\right)\right) - \tau \left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\right) \phi_i^{(K)}\,d\Delta \end{bmatrix} \\
- &\begin{bmatrix} \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \left(\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} + \frac{1}{2} g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2 \right) + n_y \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \right)^{*} \phi_i^{(1)}\,d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k} \int_{-1}^{1} \left(n_x \left(\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} + \frac{1}{2} g \left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2 \right) + n_y \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \right)^{*} \phi_i^{(K)}\,d\xi \end{bmatrix}\Bigg)\Bigg)
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} = \frac{1}{A_i}&\begin{bmatrix} \int_\Delta \phi_i^{(1)}\phi_i^{(1)}\, d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(1)}\, d\Delta \\ \vdots & \ddots & \vdots \\ \int_\Delta \phi_i^{(1)}\phi_i^{(K)}\, d\Delta & \dots & \int_\Delta \phi_i^{(K)}\phi_i^{(K)}\, d\Delta \end{bmatrix}^{-1} \\
&\left(\begin{bmatrix} \int_\Delta \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)} \left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right) d\Delta \end{bmatrix}\right. \\
&+\begin{bmatrix} \int_\Delta \left(\frac{\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+\frac{1}{2}g\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2\right)\left(\frac{\partial \phi_i^{(1)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(1)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right) d\Delta \\ \vdots \\ \int_\Delta \left(\frac{\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+\frac{1}{2}g\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)^2\right)\left(\frac{\partial \phi_i^{(K)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(K)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right) d\Delta \end{bmatrix} \\
&+\begin{bmatrix} A_i\int_\Delta \left(g\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} b_i^{(m)}\left(\frac{\partial \phi_i^{(m)}}{\partial r}\frac{1}{A_i}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(m)}}{\partial s}\frac{1}{A_i}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)-\tau\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\right)\phi_i^{(1)}\, d\Delta \\ \vdots \\ A_i\int_\Delta \left(g\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} b_i^{(m)}\left(\frac{\partial \phi_i^{(m)}}{\partial r}\frac{1}{A_i}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial \phi_i^{(m)}}{\partial s}\frac{1}{A_i}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)-\tau\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)\right)\phi_i^{(K)}\, d\Delta \end{bmatrix} \\
&\left.-\begin{bmatrix} \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+n_y\left(\frac{\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+\frac{1}{2}gH_h^2\right)\right)^{*}\phi_i^{(1)}\, d\xi \\ \vdots \\ \sum_{k=1}^{3} l_{i,k}\int_{-1}^{1}\left(n_x\frac{\left(\sum_{m=1}^{K} U_i^{(m)}\phi_i^{(m)}\right)\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+n_y\left(\frac{\left(\sum_{m=1}^{K} V_i^{(m)}\phi_i^{(m)}\right)^2}{\left(\sum_{m=1}^{K} H_i^{(m)}\phi_i^{(m)}\right)}+\frac{1}{2}gH_h^2\right)\right)^{*}\phi_i^{(K)}\, d\xi \end{bmatrix}\right)
\end{aligned}
\tag{52}
$$

2 Basis Functions

$$\phi^{(m)}(r,s) = \sqrt{2}P_i^{(0,0)}(a)P_j^{(2i+1,0)}(b)(1-b)^i \tag{53}$$
$$m = j + (K+1)i + 1 - \frac{i}{2}(i-1) \tag{54}$$

$$a = 2\frac{1+r}{1-s} - 1, \qquad b = s \tag{55}$$

$$\frac{\partial \phi^{(m)}}{\partial r} = \frac{\partial \phi^{(m)}}{\partial a}\frac{\partial a}{\partial r} + \frac{\partial \phi^{(m)}}{\partial b}\frac{\partial b}{\partial r} \tag{56}$$
$$\frac{\partial \phi^{(m)}}{\partial s} = \frac{\partial \phi^{(m)}}{\partial a}\frac{\partial a}{\partial s} + \frac{\partial \phi^{(m)}}{\partial b}\frac{\partial b}{\partial s} \tag{57}$$

$$\frac{\partial \phi^{(m)}}{\partial a} = \sqrt{2}\frac{dP_i^{(0,0)}(a)}{da}P_j^{(2i+1,0)}(b)(1-b)^i \tag{58}$$
$$\frac{\partial \phi^{(m)}}{\partial b} = \sqrt{2}P_i^{(0,0)}(a)\left(\frac{dP_j^{(2i+1,0)}(b)}{db}(1-b)^i - i(1-b)^{i-1}P_j^{(2i+1,0)}(b)\right) \tag{59}$$
$$\frac{\partial a}{\partial r} = \frac{2}{1-s} \tag{60}$$
$$\frac{\partial b}{\partial r} = 0 \tag{61}$$
$$\frac{\partial a}{\partial s} = \frac{2(1+r)}{(1-s)^2} \tag{62}$$
$$\frac{\partial b}{\partial s} = 1 \tag{63}$$

$$\frac{\partial \phi^{(m)}}{\partial r} = \sqrt{2}\frac{dP_i^{(0,0)}(a)}{da}P_j^{(2i+1,0)}(b)(1-b)^i\left(\frac{2}{1-s}\right) \tag{64}$$

$$\begin{aligned}\frac{\partial \phi^{(m)}}{\partial s} = \sqrt{2}\frac{dP_i^{(0,0)}(a)}{da}P_j^{(2i+1,0)}(b)(1-b)^i\left(\frac{2(1+r)}{(1-s)^2}\right) \\ + \sqrt{2}P_i^{(0,0)}(a)\left(\frac{dP_j^{(2i+1,0)}(b)}{db}(1-b)^i - i(1-b)^{i-1}P_j^{(2i+1,0)}(b)\right)\end{aligned} \tag{65}$$

3 Curvilinear Elements

$$\Phi_i(r,s) = \sum_{k=1}^{N} a_{i,k}\varphi_k(r,s) \tag{66}$$
$$\tag{67}$$

$$\varphi_1 = 1$$

$$\varphi_2 = r \quad \varphi_3 = s$$

$$\varphi_4 = r^2 \quad \varphi_5 = rs \quad \varphi_6 = s^2$$

$$\varphi_7 = r^3 \quad \varphi_8 = r^2 s \quad \varphi_9 = rs^2 \quad \varphi_{10} = s^3$$

$$\varphi_{11} = r^4 \quad \varphi_{12} = r^3 s \varphi_{13} = r^2 s^2 \varphi_{14} = rs^3 \quad \varphi_{15} = s^4$$

$$\iddots \qquad\qquad\qquad\qquad \ddots$$

For nodes (r_i, s_i), $i = 1, \ldots, N$ solve for coefficients $a_{i,k}$, $k = 1, \ldots, N$ such that $\Phi_i(r_j, s_j) = \delta_{ij}$

$$\frac{\partial \Phi_i(r,s)}{\partial r} = \sum_{k=1}^{N} a_{i,k} \frac{\partial \varphi_k(r,s)}{\partial r} \tag{68}$$

$$\frac{\partial \Phi_i(r,s)}{\partial s} = \sum_{k=1}^{N} a_{i,k} \frac{\partial \varphi_k(r,s)}{\partial s} \tag{69}$$

$$x = \sum_{j=1}^{N} \Phi_j(r,s) x_j \tag{70}$$

$$y = \sum_{j=1}^{N} \Phi_j(r,s) y_j \tag{71}$$

$$\frac{\partial x}{\partial r} = \sum_{j=1}^{N} \frac{\partial \Phi_j(r,s)}{\partial r} x_j \tag{72}$$

$$\frac{\partial y}{\partial r} = \sum_{j=1}^{N} \frac{\partial \Phi_j(r,s)}{\partial r} y_j \tag{73}$$

$$\frac{\partial x}{\partial s} = \sum_{j=1}^{N} \frac{\partial \Phi_j(r,s)}{\partial s} x_j \tag{74}$$

$$\frac{\partial y}{\partial s} = \sum_{j=1}^{N} \frac{\partial \Phi_j(r,s)}{\partial s} y_j \tag{75}$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \tag{76}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \tag{77}$$

$$\underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \tag{78}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{\underbrace{\frac{\partial x}{\partial r}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial s}}_{|J|}} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \tag{79}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} \tag{80}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} \tag{81}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \tag{82}$$

$$\frac{\partial r}{\partial x} = \frac{1}{|J|}\frac{\partial y}{\partial s} \tag{83}$$

$$\frac{\partial s}{\partial x} = -\frac{1}{|J|}\frac{\partial y}{\partial r} \tag{84}$$

$$\frac{\partial r}{\partial y} = -\frac{1}{|J|}\frac{\partial x}{\partial s} \tag{85}$$

$$\frac{\partial s}{\partial y} = \frac{1}{|J|}\frac{\partial x}{\partial r} \tag{86}$$

$$\frac{\partial f}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial s}\frac{\partial f}{\partial r} - \frac{\partial y}{\partial r}\frac{\partial f}{\partial s}\right) \tag{87}$$

$$\frac{\partial f}{\partial y} = \frac{1}{|J|}\left(\frac{\partial x}{\partial r}\frac{\partial f}{\partial s} - \frac{\partial x}{\partial s}\frac{\partial f}{\partial r}\right) \tag{88}$$

$$dxdy = |J|drds \tag{89}$$

4 Eddy Viscosity Terms (LDG)

With the addition of the eddy viscosity terms, the shallow water equations become

$$\frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{90}$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{U^2}{H} + \frac{1}{2}gH^2\right) + \frac{\partial}{\partial y}\left(\frac{UV}{H}\right) = gH\frac{\partial b}{\partial x} - \tau U + \frac{\partial H\tau_{xx}}{\partial x} + \frac{\partial H\tau_{yx}}{\partial y}, \tag{91}$$

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{UV}{H}\right) + \frac{\partial}{\partial y}\left(\frac{V^2}{H} + \frac{1}{2}gH^2\right) = gH\frac{\partial b}{\partial y} - \tau V + \frac{\partial H\tau_{xy}}{\partial x} + \frac{\partial H\tau_{yy}}{\partial y}. \tag{92}$$

There are two different formulations used for $H\tau_{xx}$, $H\tau_{yx}$, $H\tau_{xy}$ and $H\tau_{yy}$. The following is currently implemented in the CG version of ADCIRC

$$H\tau_{xx} = 2e_v \frac{\partial U}{\partial x}, \tag{93}$$
$$H\tau_{yx} = e_v \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right), \tag{94}$$
$$H\tau_{xy} = e_v \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right), \tag{95}$$
$$H\tau_{yy} = 2e_v \frac{\partial V}{\partial y}. \tag{96}$$

The second is the old formulation used in CG ADCIRC and currently implemented in DG ADCIRC

$$H\tau_{xx} = e_v \frac{\partial U}{\partial x}, \tag{97}$$
$$H\tau_{yx} = e_v \frac{\partial U}{\partial y}, \tag{98}$$
$$H\tau_{xy} = e_v \frac{\partial V}{\partial x}, \tag{99}$$
$$H\tau_{yy} = e_v \frac{\partial V}{\partial y}. \tag{100}$$

Auxiliary variables are defined so the system can be written in terms of first derivatives only. (note that the order of subscripts indicates the momentum variable direction and then the independent variable it is differentiated with respect to)

$$E_{xx} = \frac{\partial U}{\partial x}, \tag{101}$$
$$E_{xy} = \frac{\partial U}{\partial y}, \tag{102}$$
$$E_{yx} = \frac{\partial V}{\partial x}, \tag{103}$$
$$E_{yy} = \frac{\partial V}{\partial y}. \tag{104}$$

The vector form the shallow water equations is

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S} + \frac{\partial \mathbf{D}}{\partial x} + \frac{\partial \mathbf{E}}{\partial y}, \tag{105}$$
$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{F} - \mathbf{D}) + \frac{\partial}{\partial y}(\mathbf{G} - \mathbf{E}) = \mathbf{S}. \tag{106}$$

where

$$\mathbf{D} = \begin{bmatrix} 0 \\ 2e_v \frac{\partial U}{\partial x} \\ e_v \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 \\ e_v \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\ 2e_v \frac{\partial V}{\partial y} \end{bmatrix}. \tag{107}$$

or after substituting the auxiliary variables

$$\mathbf{D} = \begin{bmatrix} 0 \\ 2e_v E_{xx} \\ e_v \left(E_{xy} + E_{yx} \right) \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 \\ e_v \left(E_{xy} + E_{yx} \right) \\ 2e_v E_{yy} \end{bmatrix}. \tag{108}$$

Multiply by a test function, $w_h \in \mathcal{V}_h^p$ and integrate over an arbitrary element domain, Ω_i

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i + \int_{\Omega_i} \frac{\partial}{\partial x} (\mathbf{F} - \mathbf{D}) w_h d\Omega_i + \int_{\Omega_i} \frac{\partial}{\partial y} (\mathbf{G} - \mathbf{E}) w_h d\Omega_i = \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \tag{109}$$

Integrate the spatial derivative terms by parts

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i - \int_{\Omega_i} (\mathbf{F} - \mathbf{D}) \frac{\partial w_h}{\partial x} d\Omega_i - \int_{\Omega_i} (\mathbf{G} - \mathbf{E}) \frac{\partial w_h}{\partial y} d\Omega_i$$
$$+ \int_{\partial\Omega_i} n_x(\mathbf{F} - \mathbf{D}) w_h d\partial\Omega_i + \int_{\partial\Omega_i} n_y(\mathbf{G} - \mathbf{E}) w_h d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \tag{110}$$

Grouping terms

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i - \int_{\Omega_i} \left((\mathbf{F} - \mathbf{D}) \frac{\partial w_h}{\partial x} + (\mathbf{G} - \mathbf{E}) \frac{\partial w_h}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} \left(n_x(\mathbf{F} - \mathbf{D}) + n_y(\mathbf{G} - \mathbf{E}) \right) w_h d\partial\Omega_i$$
$$= \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \tag{111}$$

Substitute approximate solutions $H_h,\ U_h,\ V_h,\ E_{xx,h},\ E_{xy,h},\ E_{yx,h},\ E_{yy,h} \in \mathcal{V}_h^p$ and choose $w_h = \phi_i^{(l)}, \quad l = 1, \ldots, K$

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} \phi_i^{(l)} d\Omega_i - \int_{\Omega_i} \left((\mathbf{F}_h - \mathbf{D}_h) \frac{\partial \phi_i^{(l)}}{\partial x} + (\mathbf{G}_h - \mathbf{E}_h) \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i$$
$$+ \int_{\partial\Omega_i} \left(n_x(\widehat{\mathbf{F}}_h - \widehat{\mathbf{D}}_h) + n_y(\widehat{\mathbf{G}}_h - \widehat{\mathbf{E}}_h) \right) \phi_i^{(l)} d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} \phi_i^{(l)} d\Omega_i$$
$$l = 1, \ldots, K. \tag{112}$$

where $\widehat{\mathbf{F}}_h$, $\widehat{\mathbf{G}}_h$, $\widehat{\mathbf{D}}_h$, and $\widehat{\mathbf{E}}_h$ are the numerical fluxes. As before the Local Lax-Friedrichs flux is used for $(n_x \mathbf{F}_h + n_y \mathbf{G}_h)^*$ while $\widehat{\mathbf{D}}_h = \mathbf{D}_h^{(in)}$ and $\widehat{\mathbf{E}}_h = \mathbf{E}_h^{(in)}$

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} \phi_i^{(l)} d\Omega_i - \int_{\Omega_i} \left((\mathbf{F}_h - \mathbf{D}_h) \frac{\partial \phi_i^{(l)}}{\partial x} + (\mathbf{G}_h - \mathbf{E}_h) \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i$$
$$+ \int_{\partial\Omega_i} \left((n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* - n_x \mathbf{D}_h^{(in)} - n_y \mathbf{E}_h^{(in)} \right) \phi_i^{(l)} d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} \phi_i^{(l)} d\Omega_i$$
$$l = 1, \ldots, K. \tag{113}$$

To discretize the auxiliary variable equations, multiply by w_h and integrate over an arbitrary element Ω_i

$$\int_{\Omega_i} E_{xx} w_h d\Omega_i - \int_{\Omega_i} \frac{\partial U}{\partial x} w_h d\Omega_i = 0, \tag{114}$$
$$\int_{\Omega_i} E_{xy} w_h d\Omega_i - \int_{\Omega_i} \frac{\partial U}{\partial y} w_h d\Omega_i = 0, \tag{115}$$
$$\int_{\Omega_i} E_{yx} w_h d\Omega_i - \int_{\Omega_i} \frac{\partial V}{\partial x} w_h d\Omega_i = 0, \tag{116}$$
$$\int_{\Omega_i} E_{yy} w_h d\Omega_i - \int_{\Omega_i} \frac{\partial V}{\partial y} w_h d\Omega_i = 0. \tag{117}$$

Integrate the spatial derivative terms by parts

$$\int_{\Omega_i} E_{xx} w_h d\Omega_i + \int_{\Omega_i} U \frac{\partial w_h}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x U w_h d\partial\Omega_i = 0, \tag{118}$$

$$\int_{\Omega_i} E_{xy} w_h d\Omega_i + \int_{\Omega_i} U \frac{\partial w_h}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y U w_h d\partial\Omega_i = 0, \tag{119}$$

$$\int_{\Omega_i} E_{yx} w_h d\Omega_i + \int_{\Omega_i} V \frac{\partial w_h}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x V w_h d\partial\Omega_i = 0, \tag{120}$$

$$\int_{\Omega_i} E_{yy} w_h d\Omega_i + \int_{\Omega_i} V \frac{\partial w_h}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y v w_h d\partial\Omega_i = 0. \tag{121}$$

Substitute approximate solutions $U_h,\ V_h,\ E_{xx,h},\ E_{xy,h},\ E_{yx,h},\ E_{yy,h} \in \mathcal{V}_h^p$

$$\int_{\Omega_i} E_{xx,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x \widehat{U}_h \phi_i^{(l)} d\partial\Omega_i = 0, \tag{122}$$

$$\int_{\Omega_i} E_{xy,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y \widehat{U}_h \phi_i^{(l)} d\partial\Omega_i = 0, \tag{123}$$

$$\int_{\Omega_i} E_{yx,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x \widehat{V}_h \phi_i^{(l)} d\partial\Omega_i = 0, \tag{124}$$

$$\int_{\Omega_i} E_{yy,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y \widehat{V}_h \phi_i^{(l)} d\partial\Omega_i = 0, \tag{125}$$

$$l = 1, \ldots, K.$$

Again, $\widehat{U}_h$ and $\widehat{V}_h$ are numerical fluxes which are chosen as $\widehat{U}_h = U_h^{(ex)}$ and $\widehat{V}_h = V_h^{(ex)}$

$$\int_{\Omega_i} E_{xx,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x U_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{126}$$

$$\int_{\Omega_i} E_{xy,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y U_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{127}$$

$$\int_{\Omega_i} E_{yx,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x V_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{128}$$

$$\int_{\Omega_i} E_{yy,h} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y V_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{129}$$

$$l = 1, \ldots, K.$$

Substituting the expansions for the components of the $\mathbf{Q}_h$ vector and auxiliary variables

$$\sum_{m=1}^{K} \frac{\partial H_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i - \int_{\Omega_i} \left(U_h \frac{\partial \phi_i^{(l)}}{\partial x} + V_h \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x U_h + n_y V_h)^* \phi_i^{(l)} d\partial\Omega_i = 0$$

$$l = 1, \ldots, K. \tag{130}$$

$$
\begin{aligned}
&\sum_{m=1}^{K} \frac{\partial U_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i \\
&\qquad - \int_{\Omega_i} \left(\left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 - 2 e_v E_{xx,h} \right) \frac{\partial \phi_i^{(l)}}{\partial x} + \left(\frac{U_h V_h}{H_h} - e_v \left(E_{xy,h} + E_{yx,h} \right) \right) \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i \\
&+ \int_{\partial\Omega_i} \left(\left(n_x \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) + n_y \frac{U_h V_h}{H_h} \right)^* - n_x \left(2 e_v E_{xx,h}^{(in)} \right) - n_y \left(e_v \left(E_{xy,h}^{(in)} + E_{yx,h}^{(in)} \right) \right) \right) \phi_i^{(l)} d\partial\Omega_i \\
&\qquad\qquad = \int_{\Omega_i} \left(g H_h \frac{\partial b_h}{\partial x} - \tau U_h \right) \phi_i^{(l)} d\Omega_i \\
&\qquad\qquad\qquad l = 1, \ldots, K.
\end{aligned} \tag{131}
$$

$$
\begin{aligned}
&\sum_{m=1}^{K} \frac{\partial V_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i \\
&\qquad - \int_{\Omega_i} \left(\left(\frac{U_h V_h}{H_h} - e_v \left(E_{xy,h} + E_{yx,h} \right) \right) \frac{\partial \phi_i^{(l)}}{\partial x} + \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 - 2 e_v E_{yy,h} \right) \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i \\
&+ \int_{\partial\Omega_i} \left(\left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \right)^* - n_x \left(e_v \left(E_{xy,h}^{(in)} + E_{yx,h}^{(in)} \right) \right) - n_y \left(2 e_v E_{yy,h}^{(in)} \right) \right) \phi_i^{(l)} d\partial\Omega_i \\
&\qquad\qquad = \int_{\Omega_i} \left(g H_h \frac{\partial b_h}{\partial y} - \tau V_h \right) \phi_i^{(l)} d\Omega_i \\
&\qquad\qquad\qquad l = 1, \ldots, K.
\end{aligned} \tag{132}
$$

$$
\sum_{m=1}^{K} E_{xx,i}^{(m)} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x U_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{133}
$$

$$
\sum_{m=1}^{K} E_{xy,i}^{(m)} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} U_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y U_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{134}
$$

$$
\sum_{m=1}^{K} E_{yx,i}^{(m)} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial x} d\Omega_i - \int_{\partial\Omega_i} n_x V_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{135}
$$

$$
\sum_{m=1}^{K} E_{yy,i}^{(m)} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i + \int_{\Omega_i} V_h \frac{\partial \phi_i^{(l)}}{\partial y} d\Omega_i - \int_{\partial\Omega_i} n_y V_h^{(ex)} \phi_i^{(l)} d\partial\Omega_i = 0, \tag{136}
$$

$$
l = 1, \ldots, K.
$$

Transforming to local coordinates

$$
\begin{aligned}
&\frac{A_i}{2} \sum_{m=1}^{K} \frac{\partial H_i^{(m)}}{\partial t} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta \\
&-\frac{A_i}{2} \int_{\Delta} \left(U_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) + V_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) \right) d\Delta \\
&\qquad\qquad + \sum_{k=1}^{3} \frac{l_{i,k}}{2} \int_{-1}^{1} \left(n_x U_h + n_y V_h \right)^* \phi^{(l)} d\xi = 0 \\
&\qquad\qquad\qquad l = 1, \ldots, K.
\end{aligned} \tag{137}
$$

$$
\begin{aligned}
&\frac{A_i}{2}\sum_{m=1}^{K}\frac{\partial U_i^{(m)}}{\partial t}\int_\Delta \phi^{(m)}\phi^{(l)}d\Delta \\
&\qquad -\frac{A_i}{2}\int_\Delta\left(\left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2-2e_vE_{xx,h}\right)\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)\right. \\
&\qquad \left.+\left(\frac{U_hV_h}{H_h}-e_v\left(E_{xy,h}+E_{yx,h}\right)\right)\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)d\Delta \\
&\quad +\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(\left(n_x\left(\frac{U_h^2}{H_h}+\frac{1}{2}gH_h^2\right)+n_y\frac{U_hV_h}{H_h}\right)^*-n_x\left(2e_vE_{xx,h}^{(in)}\right)-n_y\left(e_v\left(E_{xy,h}^{(in)}+E_{yx,h}^{(in)}\right)\right)\right)\phi^{(l)}d\xi \\
&\qquad\qquad =\frac{A_i}{2}\int_\Delta\left(gH_h\frac{\partial b_h}{\partial x}-\tau U_h\right)\phi^{(l)}d\Delta \\
&\qquad\qquad\qquad l=1,\ldots,K. \qquad (138)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{A_i}{2}\sum_{m=1}^{K}\frac{\partial V_i^{(m)}}{\partial t}\int_\Delta \phi^{(m)}\phi^{(l)}d\Delta \\
&\qquad -\frac{A_i}{2}\int_\Delta\left(\left(\frac{U_hV_h}{H_h}-e_v\left(E_{xy,h}+E_{yx,h}\right)\right)\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)\right. \\
&\qquad \left.+\left(\frac{V_h^2}{H_h}+\frac{1}{2}gH_h^2-2e_vE_{yy,h}\right)\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)\right)d\Delta \\
&\quad +\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}\left(\left(n_x\frac{U_hV_h}{H_h}+n_y\left(\frac{V_h^2}{H_h}+\frac{1}{2}gH_h^2\right)\right)^*-n_x\left(e_v\left(E_{xy,h}^{(in)}+E_{yx,h}^{(in)}\right)\right)-n_y\left(2e_vE_{yy,h}^{(in)}\right)\right)\phi^{(l)}d\xi \\
&\qquad\qquad =\frac{A_i}{2}\int_\Delta\left(gH_h\frac{\partial b_h}{\partial y}-\tau V_h\right)\phi^{(l)}d\Delta \\
&\qquad\qquad\qquad l=1,\ldots,K. \qquad (139)
\end{aligned}
$$

$$
\frac{A_i}{2}\sum_{m=1}^{K}E_{xx,i}^{(m)}\int_\Delta\phi^{(m)}\phi^{(l)}d\Delta+\frac{A_i}{2}\int_\Delta U_h\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)d\Delta-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}n_xU_h^{(ex)}\phi^{(l)}d\xi=0, \qquad (140)
$$

$$
\frac{A_i}{2}\sum_{m=1}^{K}E_{xy,i}^{(m)}\int_\Delta\phi^{(m)}\phi^{(l)}d\Delta+\frac{A_i}{2}\int_\Delta U_h\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)d\Delta-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}n_yU_h^{(ex)}\phi^{(l)}d\xi=0, \qquad (141)
$$

$$
\frac{A_i}{2}\sum_{m=1}^{K}E_{yx,i}^{(m)}\int_\Delta\phi^{(m)}\phi^{(l)}d\Delta+\frac{A_i}{2}\int_\Delta V_h\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(y_i^{(3)}-y_i^{(1)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(y_i^{(1)}-y_i^{(2)}\right)\right)d\Delta-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}n_xV_h^{(ex)}\phi^{(l)}d\xi=0, \qquad (142)
$$

$$
\frac{A_i}{2}\sum_{m=1}^{K}E_{yy,i}^{(m)}\int_\Delta\phi^{(m)}\phi^{(l)}d\Delta+\frac{A_i}{2}\int_\Delta V_h\frac{1}{A_i}\left(\frac{\partial\phi^{(l)}}{\partial r}\left(x_i^{(1)}-x_i^{(3)}\right)+\frac{\partial\phi^{(l)}}{\partial s}\left(x_i^{(2)}-x_i^{(1)}\right)\right)d\Delta-\sum_{k=1}^{3}\frac{l_{i,k}}{2}\int_{-1}^{1}n_yV_h^{(ex)}\phi^{(l)}d\xi=0 \qquad (143)
$$

$$
l=1,\ldots,K.
$$

Rearranging

$$\sum_{m=1}^{K} \frac{\partial H_i^{(m)}}{\partial t} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta$$
$$= \int_{\Delta} \left(U_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) + V_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) \right) d\Delta$$
$$- \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} \left(n_x U_h + n_y V_h \right)^* \phi^{(l)} d\xi$$
$$l = 1, \ldots, K. \quad (144)$$

$$\sum_{m=1}^{K} \frac{\partial U_i^{(m)}}{\partial t} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta$$
$$= \int_{\Delta} \left(\left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 - 2 e_v E_{xx,h} \right) \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) \right.$$
$$\left. + \left(\frac{U_h V_h}{H_h} - e_v \left(E_{xy,h} + E_{yx,h} \right) \right) \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) + \left(g H_h \frac{\partial b_h}{\partial x} - \tau U_h \right) \phi^{(l)} \right) d\Delta$$
$$- \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} \left(\left(n_x \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) + n_y \frac{U_h V_h}{H_h} \right)^* - n_x \left(2 e_v E_{xx,h}^{(in)} \right) - n_y \left(e_v \left(E_{xy,h}^{(in)} + E_{yx,h}^{(in)} \right) \right) \right) \phi^{(l)} d\xi$$
$$l = 1, \ldots, K. \quad (145)$$

$$\sum_{m=1}^{K} \frac{\partial V_i^{(m)}}{\partial t} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta$$
$$= \int_{\Delta} \left(\left(\frac{U_h V_h}{H_h} - e_v \left(E_{xy,h} + E_{yx,h} \right) \right) \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) \right.$$
$$\left. + \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 - 2 e_v E_{yy,h} \right) \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) + \left(g H_h \frac{\partial b_h}{\partial y} - \tau V_h \right) \phi^{(l)} \right) d\Delta$$
$$- \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} \left(\left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \right)^* - n_x \left(e_v \left(E_{xy,h}^{(in)} + E_{yx,h}^{(in)} \right) \right) - n_y \left(2 e_v E_{yy,h}^{(in)} \right) \right) \phi^{(l)} d\xi$$
$$l = 1, \ldots, K. \quad (146)$$

$$\sum_{m=1}^{K} E_{xx,i}^{(m)} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta = - \int_{\Delta} U_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) d\Delta + \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} n_x U_h^{(ex)} \phi^{(l)} d\xi, \quad (147)$$

$$\sum_{m=1}^{K} E_{xy,i}^{(m)} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta = - \int_{\Delta} U_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) d\Delta + \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} n_y U_h^{(ex)} \phi^{(l)} d\xi, \quad (148)$$

$$\sum_{m=1}^{K} E_{yx,i}^{(m)} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta = - \int_{\Delta} V_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right) d\Delta + \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} n_x V_h^{(ex)} \phi^{(l)} d\xi, \quad (149)$$

$$\sum_{m=1}^{K} E_{yy,i}^{(m)} \int_{\Delta} \phi^{(m)} \phi^{(l)} d\Delta = - \int_{\Delta} V_h \frac{1}{A_i} \left(\frac{\partial \phi^{(l)}}{\partial r} \left(x_i^{(1)} - x_i^{(3)} \right) + \frac{\partial \phi^{(l)}}{\partial s} \left(x_i^{(2)} - x_i^{(1)} \right) \right) d\Delta + \sum_{k=1}^{3} \frac{l_{i,k}}{A_i} \int_{-1}^{1} n_y V_h^{(ex)} \phi^{(l)} d\xi, \quad (150)$$

$$l = 1, \ldots, K.$$