

1 DG Formulation

The shallow water equations in conservative form:

$$\frac{\partial H}{\partial t} + \frac{\partial(Hu)}{\partial x} + \frac{\partial(Hv)}{\partial y} = 0, \quad (1)$$

$$\frac{\partial(Hu)}{\partial t} + \frac{\partial}{\partial x} \left(Hu^2 + \frac{1}{2}gH^2 \right) + \frac{\partial}{\partial y} (Huv) = gH \frac{\partial b}{\partial x} - \tau Hu, \quad (2)$$

$$\frac{\partial(Hv)}{\partial t} + \frac{\partial}{\partial x} (Huv) + \frac{\partial}{\partial y} \left(Hv^2 + \frac{1}{2}gH^2 \right) = gH \frac{\partial b}{\partial y} - \tau Hv. \quad (3)$$

Define

$$U \equiv Hu, \quad V \equiv Hv, \quad (4)$$

so the system can be written in terms of 3 variables

$$\frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (5)$$

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} \left(\frac{U^2}{H} + \frac{1}{2}gH^2 \right) + \frac{\partial}{\partial y} \left(\frac{UV}{H} \right) = gH \frac{\partial b}{\partial x} - \tau U, \quad (6)$$

$$\frac{\partial V}{\partial t} + \frac{\partial}{\partial x} \left(\frac{UV}{H} \right) + \frac{\partial}{\partial y} \left(\frac{V^2}{H} + \frac{1}{2}gH^2 \right) = gH \frac{\partial b}{\partial y} - \tau V. \quad (7)$$

This is written in vector form as

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}, \quad (8)$$

where,

$$\mathbf{Q} = \begin{bmatrix} H \\ U \\ V \end{bmatrix}, \quad \mathbf{F} = \mathbf{F}(\mathbf{Q}) = \begin{bmatrix} U \\ \frac{U^2}{H} + \frac{1}{2}gH^2 \\ \frac{UV}{H} \end{bmatrix}, \quad \mathbf{G} = \mathbf{G}(\mathbf{Q}) = \begin{bmatrix} V \\ \frac{UV}{H} \\ \frac{V^2}{H} + \frac{1}{2}gH^2 \end{bmatrix}, \quad \mathbf{S} = \mathbf{S}(\mathbf{Q}) = \begin{bmatrix} 0 \\ gH \frac{\partial b}{\partial x} - \tau U \\ gH \frac{\partial b}{\partial y} - \tau V \end{bmatrix}. \quad (9)$$

The finite element space is $\mathcal{V}_h^p = \{w_h(x, y) : w_h(x, y)|_{\Omega_i} \in \mathcal{P}^p(\Omega_i)\}$. Multiply by a test function $w_h \in \mathcal{V}_h^p$ and integrate over an arbitrary element domain, Ω_i

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i + \int_{\Omega_i} \frac{\partial \mathbf{F}}{\partial x} w_h d\Omega_i + \int_{\Omega_i} \frac{\partial \mathbf{G}}{\partial y} w_h d\Omega_i = \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \quad (10)$$

Integrate the flux terms by parts

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i - \int_{\Omega_i} \mathbf{F} \frac{\partial w_h}{\partial x} d\Omega_i - \int_{\Omega_i} \mathbf{G} \frac{\partial w_h}{\partial y} d\Omega_i + \int_{\partial\Omega_i} n_x \mathbf{F} w_h d\partial\Omega_i + \int_{\partial\Omega_i} n_y \mathbf{G} w_h d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \quad (11)$$

This results in integrals over the element edge $\partial\Omega_i$ where n_x and n_y are the edge normals in the x and y directions. Grouping terms

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}}{\partial t} w_h d\Omega_i - \int_{\Omega_i} \left(\mathbf{F} \frac{\partial w_h}{\partial x} + \mathbf{G} \frac{\partial w_h}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F} + n_y \mathbf{G}) w_h d\partial\Omega_i = \int_{\Omega_i} \mathbf{S} w_h d\Omega_i. \quad (12)$$

The exact solution variables in \mathbf{Q} are replaced with an approximation \mathbf{Q}_h where $H_h, U_h, V_h \in \mathcal{V}_h^p$

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} w_h d\Omega_i - \int_{\Omega_i} \left(\mathbf{F}_h \frac{\partial w_h}{\partial x} + \mathbf{G}_h \frac{\partial w_h}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* w_h d\partial\Omega_i = \int_{\Omega_i} \mathbf{S}_h w_h d\Omega_i. \quad (13)$$

The quantity $(n_x \mathbf{F}_h + n_y \mathbf{G}_h)^*$ is the numerical flux and is needed to determine a single value of the edge integral given the two solutions that exist at the element interface due to the discontinuous approximation. The approximate solutions are expressed as linear combinations of unknown, time-dependent coefficients $(H_i^{(m)}, U_i^{(m)}, V_i^{(m)})$ and spatially dependent functions $\{\phi_i^{(m)}\}_{m=1}^K$ that form a basis for $\mathcal{P}^p(\Omega_i)$

$$\mathbf{Q}_h = \begin{bmatrix} H_h \\ U_h \\ V_h \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^K H_i^{(m)}(t) \phi_i^{(m)}(x, y) \\ \sum_{m=1}^K U_i^{(m)}(t) \phi_i^{(m)}(x, y) \\ \sum_{m=1}^K V_i^{(m)}(t) \phi_i^{(m)}(x, y) \end{bmatrix}, \quad (x, y) \in \Omega_i \quad (14)$$

$$\mathbf{F}_h = \mathbf{F}(\mathbf{Q}_h), \quad \mathbf{G}_h = \mathbf{G}(\mathbf{Q}_h), \quad \mathbf{S}_h = \mathbf{S}(\mathbf{Q}_h), \quad (15)$$

The number of degrees of freedom K , depends on the order of the polynomial basis, p

$$K = \frac{(p+1)(p+2)}{2}. \quad (16)$$

The test functions are chosen as

$$w_h = \phi_i^{(l)}(x, y), \quad l = 1, \dots, K. \quad (17)$$

Substituting for w_h

$$\int_{\Omega_i} \frac{\partial \mathbf{Q}_h}{\partial t} \phi_i^{(l)} d\Omega_i - \int_{\Omega_i} \left(\mathbf{F}_h \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h \frac{\partial \phi_i^{(l)}}{\partial y} \right) d\Omega_i + \int_{\partial\Omega_i} (n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* \phi_i^{(l)} d\partial\Omega_i = \int_{\Omega_i} \mathbf{S}_h \phi_i^{(l)} d\Omega_i \quad l = 1, \dots, K. \quad (18)$$

The Lax Friedrichs numerical flux is used in evaluating the edge integral

$$(n_x \mathbf{F}_h + n_y \mathbf{G}_h)^* = n_x \left(\frac{\mathbf{F}(\mathbf{Q}_h^{ex}) + \mathbf{F}(\mathbf{Q}_h^{in})}{2} \right) + n_y \left(\frac{\mathbf{G}(\mathbf{Q}_h^{ex}) + \mathbf{G}(\mathbf{Q}_h^{in})}{2} \right) - \frac{\alpha}{2} (\mathbf{Q}_h^{ex} - \mathbf{Q}_h^{in}), \quad (19)$$

where

$$\alpha = \max_{\mathbf{Q}_h^{in}, \mathbf{Q}_h^{ex}} |un_x + vn_y + \sqrt{gH}|. \quad (20)$$

Substituting for the components of the \mathbf{Q}_h vector gives

$$\begin{aligned} \sum_{m=1}^K \frac{\partial H_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i = \\ \int_{\Omega_i} \left(\mathbf{F}_h^{(1)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(1)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(1)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} (n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)})^* \phi_i^{(l)} d\partial\Omega_i \end{aligned} \quad l = 1, \dots, K \quad (21)$$

$$\begin{aligned} \sum_{m=1}^K \frac{\partial U_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i = \\ \int_{\Omega_i} \left(\mathbf{F}_h^{(2)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(2)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(2)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} \left(n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)} \right)^* \phi_i^{(l)} d\partial\Omega_i \end{aligned} \quad l = 1, \dots, K \quad (22)$$

$$\begin{aligned} \sum_{m=1}^K \frac{\partial V_i^{(m)}}{\partial t} \int_{\Omega_i} \phi_i^{(m)} \phi_i^{(l)} d\Omega_i = \\ \int_{\Omega_i} \left(\mathbf{F}_h^{(3)} \frac{\partial \phi_i^{(l)}}{\partial x} + \mathbf{G}_h^{(3)} \frac{\partial \phi_i^{(l)}}{\partial y} + \mathbf{S}_h^{(3)} \phi_i^{(l)} \right) d\Omega_i - \int_{\partial\Omega_i} \left(n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)} \right)^* \phi_i^{(l)} d\partial\Omega_i \end{aligned} \quad l = 1, \dots, K \quad (23)$$

The each element is mapped to a reference element (Δ) with the transformation

$$x = -\frac{1}{2} \left((r+s)x_i^{(1)} - (1+r)x_i^{(2)} - (1+s)x_i^{(3)} \right) \quad (24)$$

$$y = -\frac{1}{2} \left((r+s)y_i^{(1)} - (1+r)y_i^{(2)} - (1+s)y_i^{(3)} \right) \quad (25)$$

$$r = \frac{1}{A_i} \left[\left(y_i^{(3)} - y_i^{(1)} \right) \left(x - \frac{1}{2} \left(x_i^{(2)} + x_i^{(3)} \right) \right) + \left(x_i^{(1)} - x_i^{(3)} \right) \left(y - \frac{1}{2} \left(y_i^{(2)} - y_i^{(3)} \right) \right) \right], \quad (26)$$

$$s = \frac{1}{A_i} \left[\left(y_i^{(1)} - y_i^{(2)} \right) \left(x - \frac{1}{2} \left(x_i^{(2)} + x_i^{(3)} \right) \right) + \left(x_i^{(2)} - x_i^{(1)} \right) \left(y - \frac{1}{2} \left(y_i^{(2)} - y_i^{(3)} \right) \right) \right], \quad (27)$$

where

$$A_i = \frac{1}{2} \left(\left(x_i^{(2)} y_i^{(3)} - x_i^{(3)} y_i^{(2)} \right) + \left(x_i^{(3)} y_i^{(1)} - x_i^{(1)} y_i^{(3)} \right) + \left(x_i^{(1)} y_i^{(2)} - x_i^{(2)} y_i^{(1)} \right) \right). \quad (28)$$

Therefore, the area and edge integrals can be preformed numerically by

$$\int_{\Omega_i} f(x, y) dx dy = \frac{A_i}{2} \int_{\Delta} f(r, s) dr ds, \quad (29)$$

$$= \frac{A_i}{2} \sum_{j=1}^{q(a)} \tilde{w}_j^{(a)} f \left(\tilde{r}_j^{(a)}, \tilde{s}_j^{(a)} \right), \quad (30)$$

$$\int_{\partial\Omega_i} f(x, y) d\partial\Omega_i = \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 f(\xi) d\xi, \quad (31)$$

$$= \sum_{k=1}^3 \frac{l_{i,k}}{2} \sum_{j=1}^{q(e)} \tilde{w}_j^{(e)} f \left(\tilde{\xi}_j^{(e)} \right). \quad (32)$$

The derivatives must also be transformed to the reference element coordinates

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x}, \quad (33)$$

$$= \frac{1}{A_i} \left(\frac{\partial f}{\partial r} \left(y_i^{(3)} - y_i^{(1)} \right) + \frac{\partial f}{\partial s} \left(y_i^{(1)} - y_i^{(2)} \right) \right), \quad (34)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial y}, \quad (35)$$

$$= \frac{1}{A_i} \left(\frac{\partial f}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial f}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right). \quad (36)$$

$$(37)$$

All integrals are now written in terms of the reference element coordinates

$$\begin{aligned} & \frac{A_i}{2} \sum_{m=1}^K \frac{\partial H_i^{(m)}}{\partial t} \int_{\Delta} \phi_i^{(m)} \phi_i^{(l)} d\Delta = \\ & \frac{A_i}{2} \int_{\Delta} \left(\mathbf{F}_h^{(1)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(1)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + \mathbf{S}_h^{(1)} \phi_i^{(l)} \right) d\Delta \\ & \quad - \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 \left(n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)} \right)^* \phi_i^{(l)} d\xi \\ & \quad l = 1, \dots, K \end{aligned} \quad (38)$$

$$\begin{aligned} & \frac{A_i}{2} \sum_{m=1}^K \frac{\partial U_i^{(m)}}{\partial t} \int_{\Delta} \phi_i^{(m)} \phi_i^{(l)} d\Delta = \\ & \frac{A_i}{2} \int_{\Delta} \left(\mathbf{F}_h^{(2)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(2)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + \mathbf{S}_h^{(2)} \phi_i^{(l)} \right) d\Delta \\ & \quad - \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 \left(n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)} \right)^* \phi_i^{(l)} d\xi \\ & \quad l = 1, \dots, K \end{aligned} \quad (39)$$

$$\begin{aligned} & \frac{A_i}{2} \sum_{m=1}^K \frac{\partial V_i^{(m)}}{\partial t} \int_{\Delta} \phi_i^{(m)} \phi_i^{(l)} d\Delta = \\ & \frac{A_i}{2} \int_{\Delta} \left(\mathbf{F}_h^{(3)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(3)} \frac{1}{A_i} \left(\frac{\partial \phi_i^{(l)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(l)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + \mathbf{S}_h^{(3)} \phi_i^{(l)} \right) d\Delta \\ & \quad - \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 \left(n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)} \right)^* \phi_i^{(l)} d\Omega_i \\ & \quad l = 1, \dots, K \end{aligned} \quad (40)$$

Expanding these equations into vector form and simplifying the area integral transformation constant leads to

$$\begin{aligned}
& \frac{A_i}{2} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} = \\
& \left(\begin{bmatrix} \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(1)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(1)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(1)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(1)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(1)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(1)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
& \quad \left. - \begin{bmatrix} \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (41)
\end{aligned}$$

$$\begin{aligned}
& \frac{A_i}{2} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} = \\
& \left(\begin{bmatrix} \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(2)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(2)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(2)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(2)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(2)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(2)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
& \quad \left. - \begin{bmatrix} \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (42)
\end{aligned}$$

$$\begin{aligned}
& \frac{A_i}{2} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} = \\
& \left(\begin{bmatrix} \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(3)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(3)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(3)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \frac{1}{2} \int_{\Delta} \left(\mathbf{F}_h^{(3)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(3)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(3)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
& \quad \left. - \begin{bmatrix} \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 \frac{l_{i,k}}{2} \int_{-1}^1 (n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (43)
\end{aligned}$$

Inverting the mass matrix leads to a system of ordinary differential equations

$$\begin{aligned}
& \frac{\partial}{\partial t} \begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} = \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
& \left(\begin{bmatrix} \int_{\Delta} \left(\mathbf{F}_h^{(1)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(1)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(1)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\mathbf{F}_h^{(1)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(1)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(1)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
& \quad \left. - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(1)} + n_y \mathbf{G}_h^{(1)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (44)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\left(\begin{bmatrix} \int_{\Delta} \left(\mathbf{F}_h^{(2)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(2)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(2)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\mathbf{F}_h^{(2)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(2)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(2)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
&\quad \left. - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(2)} + n_y \mathbf{G}_h^{(2)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (45)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\left(\begin{bmatrix} \int_{\Delta} \left(\mathbf{F}_h^{(3)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(3)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(3)} \phi_i^{(1)} \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\mathbf{F}_h^{(3)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + \mathbf{G}_h^{(3)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) + A_i \mathbf{S}_h^{(3)} \phi_i^{(K)} \right) d\Delta \end{bmatrix} \right. \\
&\quad \left. - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)})^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x \mathbf{F}_h^{(3)} + n_y \mathbf{G}_h^{(3)})^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (46)
\end{aligned}$$

The components of the **F**, **G** and **S** vectors are substituted to give

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\ &\left(\begin{bmatrix} \int_{\Delta} \left(U_h \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + V_h \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(U_h \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) + V_h \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) \right) d\Delta \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x U_h + n_y V_h)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 (n_x U_h + n_y V_h)^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\ &\left(\begin{bmatrix} \int_{\Delta} \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \end{bmatrix} \right. \\ &\quad + \begin{bmatrix} \int_{\Delta} \frac{U_h V_h}{H_h} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \frac{U_h V_h}{H_h} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \end{bmatrix} + \begin{bmatrix} A_i \int_{\Delta} \left(g H_h \frac{\partial b}{\partial x} - \tau U_h \right) \phi_i^{(1)} d\Delta \\ \vdots \\ A_i \int_{\Delta} \left(g H_h \frac{\partial b}{\partial x} - \tau U_h \right) \phi_i^{(K)} d\Delta \end{bmatrix} \\ &\quad \left. - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) + n_y \frac{U_h V_h}{H_h} \right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\frac{U_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) + n_y \frac{U_h V_h}{H_h} \right)^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \end{aligned} \quad (48)$$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \int_{\Delta} \frac{U_h V_h}{H_h} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \frac{U_h V_h}{H_h} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \end{bmatrix} \\
&+ \begin{bmatrix} \int_{\Delta} \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \end{bmatrix} \\
&\quad + \begin{bmatrix} A_i \int_{\Delta} \left(g H_h \frac{\partial b}{\partial y} - \tau V_h \right) \phi_i^{(1)} d\Delta \\ \vdots \\ A_i \int_{\Delta} \left(g H_h \frac{\partial b}{\partial y} - \tau V_h \right) \phi_i^{(K)} d\Delta \end{bmatrix} \\
&\quad - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \frac{U_h V_h}{H_h} + n_y \left(\frac{V_h^2}{H_h} + \frac{1}{2} g H_h^2 \right) \right)^* \phi_i^{(K)} d\xi \end{bmatrix} \quad (49)
\end{aligned}$$

Substituting the approximate expansions for H_h , U_h , and V_h

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} H_i^{(1)} \\ \vdots \\ H_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \int_{\Delta} \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \end{bmatrix} \\
&+ \begin{bmatrix} \int_{\Delta} \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \end{bmatrix} \\
&\quad - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) + n_y \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) + n_y \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \right)^* \phi_i^{(K)} d\xi \end{bmatrix} \quad (50)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} U_i^{(1)} \\ \vdots \\ U_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\quad \left(\begin{bmatrix} \int_{\Delta} \left(\frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \end{bmatrix} \right. \\
&\quad + \begin{bmatrix} \int_{\Delta} \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \end{bmatrix} \\
&\quad + \begin{bmatrix} A_i \int_{\Delta} \left(g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r} \frac{1}{A_i} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(m)}}{\partial s} \frac{1}{A_i} (y_i^{(1)} - y_i^{(2)}) \right) \right) - \tau \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \right) \phi_i^{(1)} d\Delta \\ \vdots \\ A_i \int_{\Delta} \left(g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r} \frac{1}{A_i} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(m)}}{\partial s} \frac{1}{A_i} (y_i^{(1)} - y_i^{(2)}) \right) \right) - \tau \left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \right) \phi_i^{(K)} d\Delta \end{bmatrix} \\
&\quad - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) + n_y \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \left(\frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) + n_y \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \right)^* \phi_i^{(K)} d\xi \end{bmatrix} \right) \quad (51)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \begin{bmatrix} V_i^{(1)} \\ \vdots \\ V_i^{(K)} \end{bmatrix} &= \frac{1}{A_i} \begin{bmatrix} \int_{\Delta} \phi_i^{(1)} \phi_i^{(1)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(1)} d\Delta \\ \vdots & \ddots & \vdots \\ \int_{\Delta} \phi_i^{(1)} \phi_i^{(K)} d\Delta & \dots & \int_{\Delta} \phi_i^{(K)} \phi_i^{(K)} d\Delta \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \int_{\Delta} \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \left(\frac{\partial \phi_i^{(1)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} \left(\frac{\partial \phi_i^{(K)}}{\partial r} (y_i^{(3)} - y_i^{(1)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (y_i^{(1)} - y_i^{(2)}) \right) d\Delta \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_{\Delta} \left(\frac{\left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) \left(\frac{\partial \phi_i^{(1)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(1)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \\ \vdots \\ \int_{\Delta} \left(\frac{\left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)^2 \right) \left(\frac{\partial \phi_i^{(K)}}{\partial r} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(K)}}{\partial s} (x_i^{(2)} - x_i^{(1)}) \right) d\Delta \end{bmatrix} \\
&\quad + \begin{bmatrix} A_i \int_{\Delta} \left(g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r} \frac{1}{A_i} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(m)}}{\partial s} \frac{1}{A_i} (x_i^{(2)} - x_i^{(1)}) \right) \right) - \tau \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \right) \phi_i^{(1)} d\Delta \\ \vdots \\ A_i \int_{\Delta} \left(g \left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K b_i^{(m)} \left(\frac{\partial \phi_i^{(m)}}{\partial r} \frac{1}{A_i} (x_i^{(1)} - x_i^{(3)}) + \frac{\partial \phi_i^{(m)}}{\partial s} \frac{1}{A_i} (x_i^{(2)} - x_i^{(1)}) \right) \right) - \tau \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right) \right) \phi_i^{(K)} d\Delta \end{bmatrix} \\
&\quad - \begin{bmatrix} \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + n_y \left(\frac{\left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g H_h^2 \right) \right)^* \phi_i^{(1)} d\xi \\ \vdots \\ \sum_{k=1}^3 l_{i,k} \int_{-1}^1 \left(n_x \frac{\left(\sum_{m=1}^K U_i^{(m)} \phi_i^{(m)} \right) \left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + n_y \left(\frac{\left(\sum_{m=1}^K V_i^{(m)} \phi_i^{(m)} \right)^2}{\left(\sum_{m=1}^K H_i^{(m)} \phi_i^{(m)} \right)} + \frac{1}{2} g H_h^2 \right) \right)^* \phi_i^{(K)} d\xi \end{bmatrix} \quad (52)
\end{aligned}$$

2 Basis Functions

$$\phi^{(m)}(r, s) = \sqrt{2} P_i^{(0,0)}(a) P_j^{(2i+1,0)}(b) (1-b)^i \quad (53)$$

$$m = j + (K+1)i + 1 - \frac{i}{2}(i-1) \quad (54)$$

$$a = 2 \frac{1+r}{1-s} - 1, \quad b = s \quad (55)$$

$$\frac{\partial \phi^{(m)}}{\partial r} = \frac{\partial \phi^{(m)}}{\partial a} \frac{\partial a}{\partial r} + \frac{\partial \phi^{(m)}}{\partial b} \frac{\partial b}{\partial r} \quad (56)$$

$$\frac{\partial \phi^{(m)}}{\partial s} = \frac{\partial \phi^{(m)}}{\partial a} \frac{\partial a}{\partial s} + \frac{\partial \phi^{(m)}}{\partial b} \frac{\partial b}{\partial s} \quad (57)$$

$$\frac{\partial \phi^{(m)}}{\partial a} = \sqrt{2} \frac{dP_i^{(0,0)}(a)}{da} P_j^{(2i+1,0)}(b) (1-b)^i \quad (58)$$

$$\frac{\partial \phi^{(m)}}{\partial b} = \sqrt{2} P_i^{(0,0)}(a) \left(\frac{dP_j^{(2i+1,0)}(b)}{db} (1-b)^i - i(1-b)^{i-1} P_j^{(2i+1,0)}(b) \right) \quad (59)$$

$$\frac{\partial a}{\partial r} = \frac{2}{1-s} \quad (60)$$

$$\frac{\partial b}{\partial r} = 0 \quad (61)$$

$$\frac{\partial a}{\partial s} = \frac{2(1+r)}{(1-s)^2} \quad (62)$$

$$\frac{\partial b}{\partial s} = 1 \quad (63)$$

$$\frac{\partial \phi^{(m)}}{\partial r} = \sqrt{2} \frac{dP_i^{(0,0)}(a)}{da} P_j^{(2i+1,0)}(b) (1-b)^i \left(\frac{2}{1-s} \right) \quad (64)$$

$$\begin{aligned} \frac{\partial \phi^{(m)}}{\partial s} &= \sqrt{2} \frac{dP_i^{(0,0)}(a)}{da} P_j^{(2i+1,0)}(b) (1-b)^i \left(\frac{2(1+r)}{(1-s)^2} \right) \\ &\quad + \sqrt{2} P_i^{(0,0)}(a) \left(\frac{dP_j^{(2i+1,0)}(b)}{db} (1-b)^i - i(1-b)^{i-1} P_j^{(2i+1,0)}(b) \right) \end{aligned} \quad (65)$$

3 Curvilinear Elements

$$\Phi_i(r, s) = \sum_{k=1}^N a_{i,k} \varphi_k(r, s) \quad (66)$$

$$(67)$$

$$\begin{aligned}
\varphi_1 &= 1 \\
\varphi_2 &= r \quad \varphi_3 = s \\
\varphi_4 &= r^2 \quad \varphi_5 = rs \quad \varphi_6 = s^2 \\
\varphi_7 &= r^3 \quad \varphi_8 = r^2s \quad \varphi_9 = rs^2 \quad \varphi_{10} = s^3 \\
\varphi_{11} &= r^4 \quad \varphi_{12} = r^3s \quad \varphi_{13} = r^2s^2 \quad \varphi_{14} = rs^3 \quad \varphi_{15} = s^4 \\
&\ddots \qquad \qquad \qquad \ddots
\end{aligned}$$

For nodes (r_i, s_i) , $i = 1, \dots, N$ solve for coefficients $a_{i,k}$, $k = 1, \dots, N$ such that $\Phi_i(r_j, s_j) = \delta_{ij}$

$$\frac{\partial \Phi_i(r, s)}{\partial r} = \sum_{k=1}^N a_{i,k} \frac{\partial \varphi_k(r, s)}{\partial r} \quad (68)$$

$$\frac{\partial \Phi_i(r, s)}{\partial s} = \sum_{k=1}^N a_{i,k} \frac{\partial \varphi_k(r, s)}{\partial s} \quad (69)$$

$$x = \sum_{j=1}^N \Phi_j(r, s) x_j \quad (70)$$

$$y = \sum_{j=1}^N \Phi_j(r, s) y_j \quad (71)$$

$$\frac{\partial x}{\partial r} = \sum_{j=1}^N \frac{\partial \Phi_j(r, s)}{\partial r} x_j \quad (72)$$

$$\frac{\partial y}{\partial r} = \sum_{j=1}^N \frac{\partial \Phi_j(r, s)}{\partial r} y_j \quad (73)$$

$$\frac{\partial x}{\partial s} = \sum_{j=1}^N \frac{\partial \Phi_j(r, s)}{\partial s} x_j \quad (74)$$

$$\frac{\partial y}{\partial s} = \sum_{j=1}^N \frac{\partial \Phi_j(r, s)}{\partial s} y_j \quad (75)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \quad (76)$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad (77)$$

$$\underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_J \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \quad (78)$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{\underbrace{\frac{\partial x}{\partial r} \frac{\partial y}{\partial s} - \frac{\partial y}{\partial r} \frac{\partial x}{\partial s}}_{|J|}} \begin{bmatrix} \frac{\partial y}{\partial s} & -\frac{\partial y}{\partial r} \\ -\frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \quad (79)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x} \quad (80)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial y} \quad (81)$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{bmatrix} \quad (82)$$

$$\frac{\partial r}{\partial x} = \frac{1}{|J|} \frac{\partial y}{\partial s} \quad (83)$$

$$\frac{\partial s}{\partial x} = -\frac{1}{|J|} \frac{\partial y}{\partial r} \quad (84)$$

$$\frac{\partial r}{\partial y} = -\frac{1}{|J|} \frac{\partial x}{\partial s} \quad (85)$$

$$\frac{\partial s}{\partial y} = \frac{1}{|J|} \frac{\partial x}{\partial r} \quad (86)$$

$$\frac{\partial f}{\partial x} = \frac{1}{|J|} \left(\frac{\partial y}{\partial s} \frac{\partial f}{\partial r} - \frac{\partial y}{\partial r} \frac{\partial f}{\partial s} \right) \quad (87)$$

$$\frac{\partial f}{\partial y} = \frac{1}{|J|} \left(\frac{\partial x}{\partial r} \frac{\partial f}{\partial s} - \frac{\partial x}{\partial s} \frac{\partial f}{\partial r} \right) \quad (88)$$

$$dxdy = |J|drds \quad (89)$$