UNIVERSITÄT
BIELEFELD

Faculty of Business Administration
and Economics

Lagrange multiplier type test to detect structural intra-personal heterogeneity in Composite Marginal Likelihood estimation of discrete choice models

Talk at the International Choice Modelling Conference 2024
Puerto Varas, 02.04.2024

Sebastian Büscher, Dietmar Bauer

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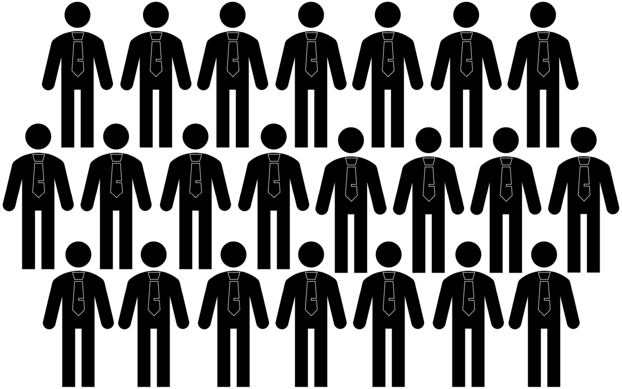
1. Heterogeneity

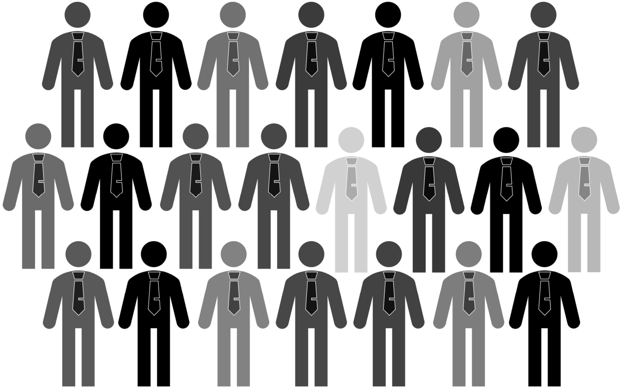


Homogeneous population

⇒ One Model with one true parameter vector

$$\theta_0$$

1. Heterogeneity

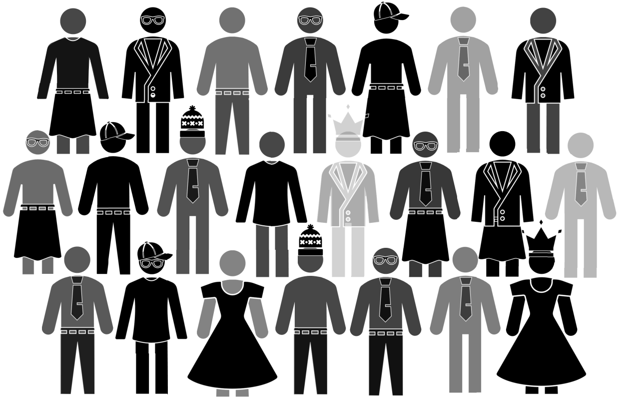


Heterogeneity in population

⇒ Mixed effects parameter from joint distribution

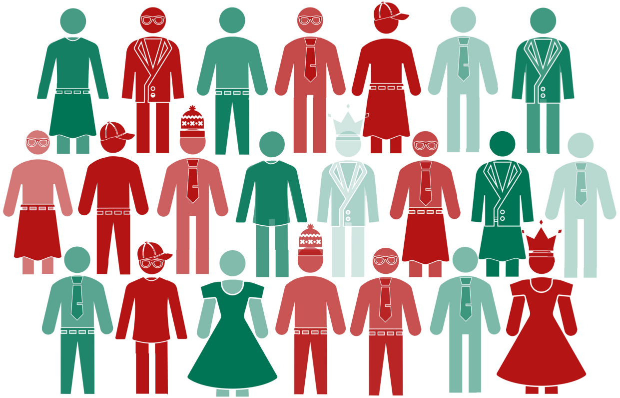
$$\theta_n \sim \mathcal{N}(\theta_0, \sigma^2)$$

1. Heterogeneity



More individuality than covered by mixed effects

1. Heterogeneity



More individuality than covered by mixed effects

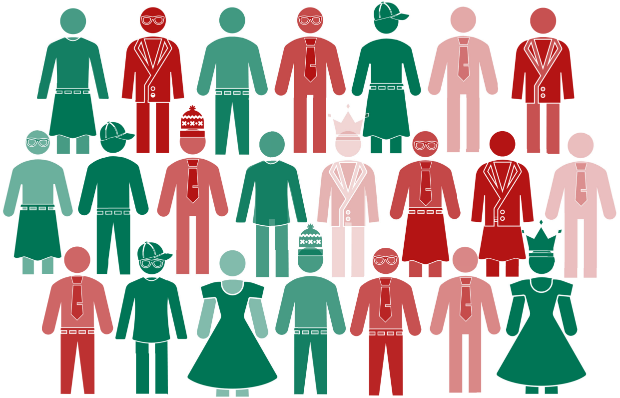
$\Rightarrow$ Potentially groups (G_1 and G_2) of individuals with structural differences

$$\theta_n \sim \mathcal{N}(\theta_{0,1}, \sigma_1^2) \quad \text{and} \quad \theta_n \sim \mathcal{N}(\theta_{0,2}, \sigma_2^2)$$

$n \in G_1$ — no headgear

$n \in G_2$ — headgear

1. Heterogeneity



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$n \in G_1$ $\quad$ $n \in G_2$

casual $\quad$ formal

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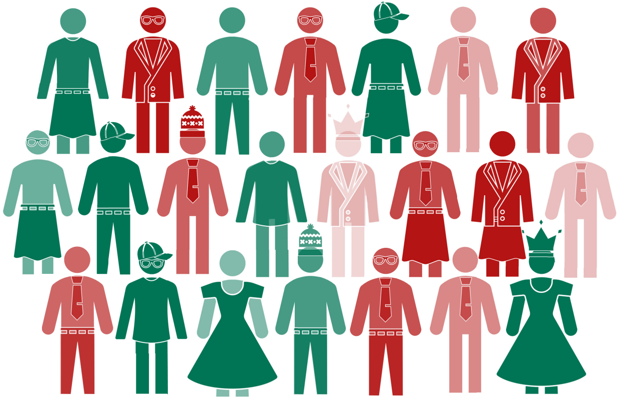

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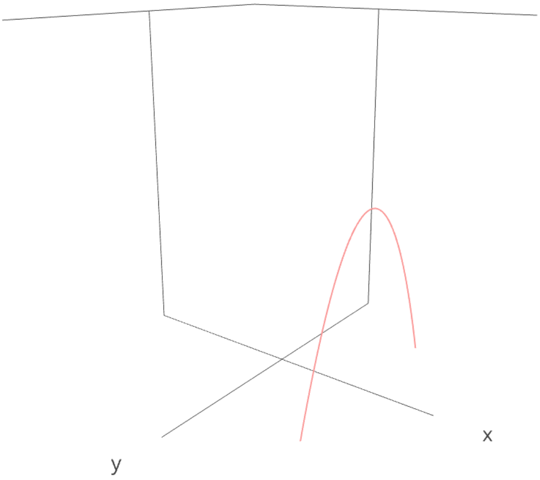
casual formal

$$\theta_{0,1} \stackrel{?}{=} \theta_{0,2}$$

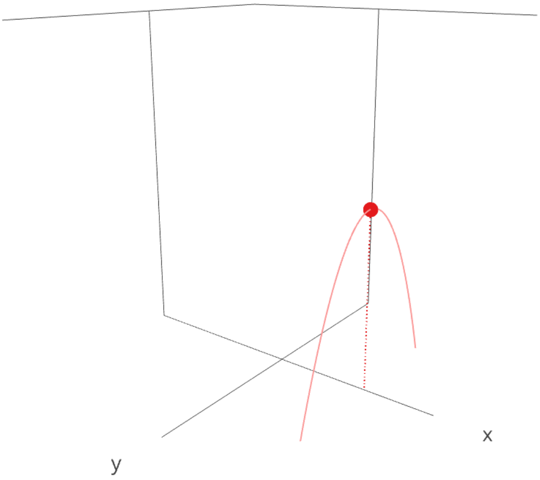
2. Discriminating between nested models



Restricted model

$ll(\theta; R(\theta) = 0)$

2. Discriminating between nested models

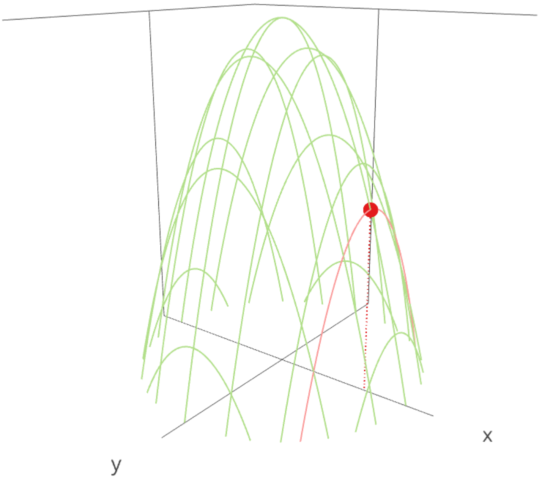


Restricted model

$$ll(\theta; R(\theta) = 0)$$

$$\hat{\theta}_r = \arg\max_\theta ll(\theta; R(\theta) = 0)$$

2. Discriminating between nested models



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Unrestricted model

$ll(\theta)$

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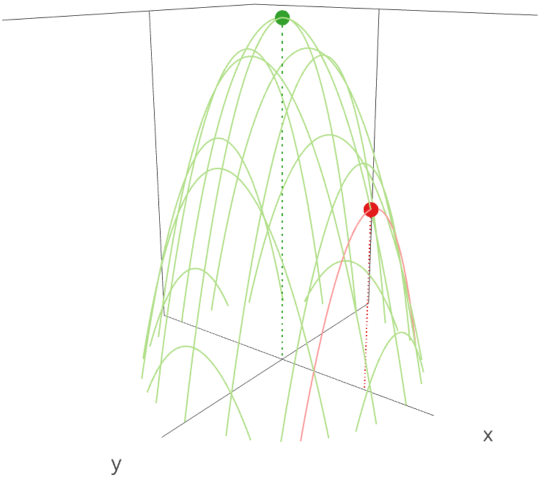

Restricted model

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$\hat{\theta}_{\mathrm{r}} = \arg\max_\theta ll(\theta; R(\theta) = 0)$

Unrestricted model

$ll(\theta)$

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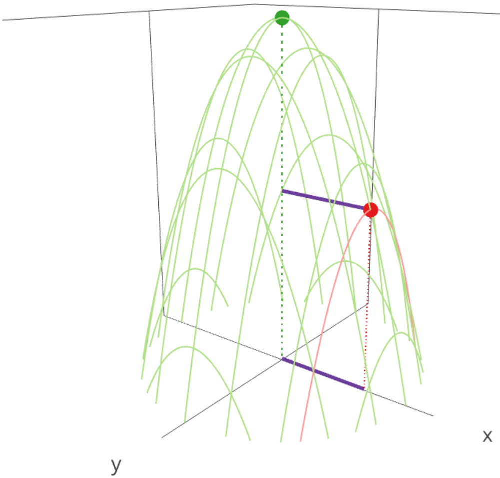


Restricted model

$$ll(\theta; R(\theta) = 0)$$

$$\hat{\theta}_r = \arg\max_\theta ll(\theta; R(\theta) = 0)$$

Unrestricted model

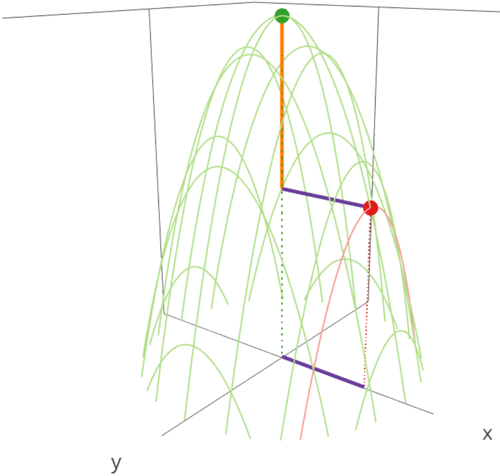
$$ll(\theta)$$

$$\hat{\theta}_{ur} = \arg\max_\theta ll(\theta)$$

Wald test

$$(\hat{\theta}_{ur} - \hat{\theta}_r)' \widehat{Var}(ll(\hat{\theta}))^{-1} (\hat{\theta}_{ur} - \hat{\theta}_r)$$

2. Discriminating between nested models



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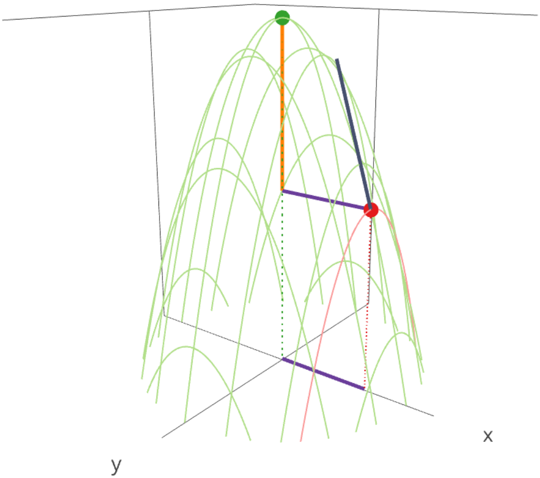
Likelihood ratio test

$-2(ll(\hat{\theta}_r) - ll(\hat{\theta}_{ur}))$

2. Discriminating between nested models



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$$ll(\theta; R(\theta) = 0)$$

$$\hat{\theta}_r = \arg\max_\theta ll(\theta; R(\theta) = 0)$$

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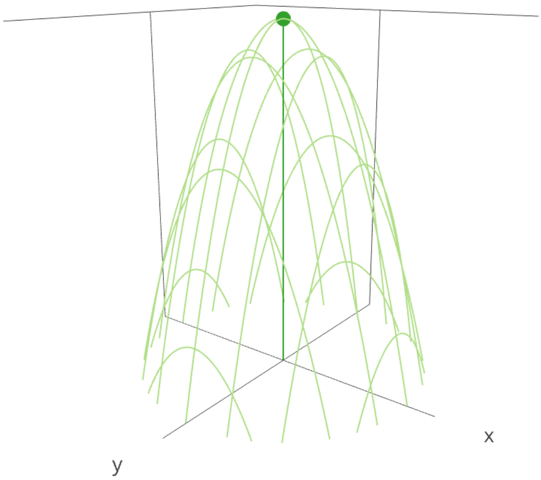
Likelihood ratio test

$$-2(ll(\hat{\theta}_r) - ll(\hat{\theta}_{ur}))$$

Lagrange multiplier test

$$(\partial_{\theta_{ur}} ll(\hat{\theta}_r))' \left(\mathrm{E}(\partial^2_{\theta_{ur}} ll(\hat{\theta}_r))\right)^{-1} (\partial_{\theta_{ur}} ll(\hat{\theta}_r))$$

2. 1. Testing for Pooling



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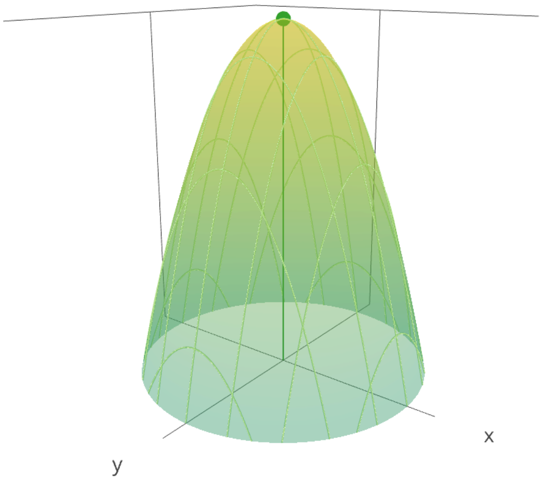

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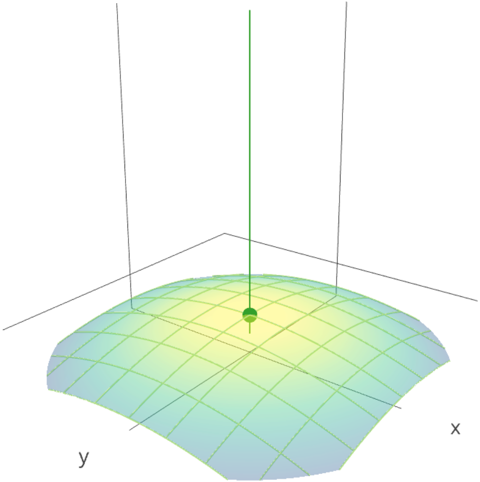
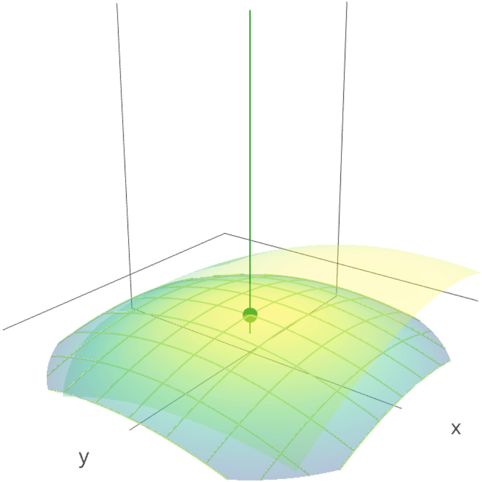


Individual contributions:

$$ll(\theta) =$$

2. 1. Testing for Pooling



Individual contributions:

$$ll(\theta) = ll_1(\theta)$$

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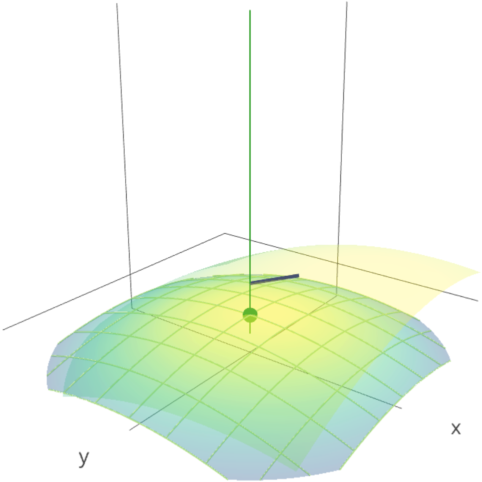
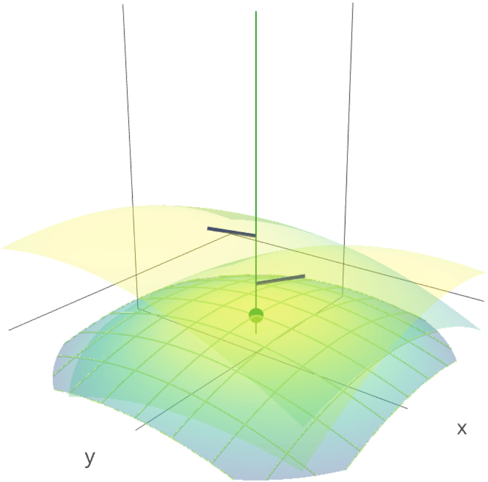


Individual contributions:

$$ll(\theta) = ll_1(\theta)$$

$$\partial_\theta ll(\hat{\theta}) = \partial_\theta ll_1(\hat{\theta})$$

2. 1. Testing for Pooling



Individual contributions:

$$ll(\theta) = ll_1(\theta) + ll_2(\theta)$$

$$\partial_\theta ll(\hat{\theta}) = \partial_\theta ll_1(\hat{\theta}) + \partial_\theta ll_2(\hat{\theta})$$

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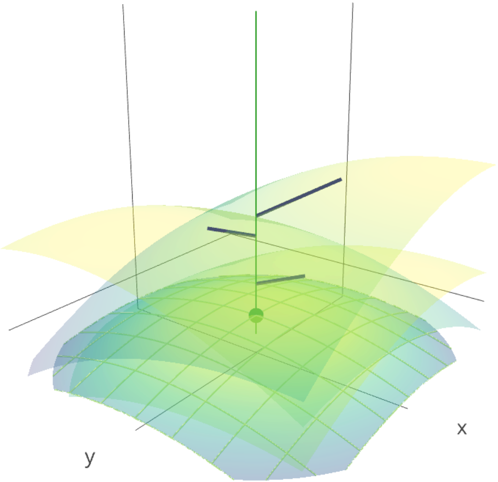

Individual contributions:

$$ll(\theta) = ll_1(\theta) + ll_2(\theta) + ll_3(\theta)$$

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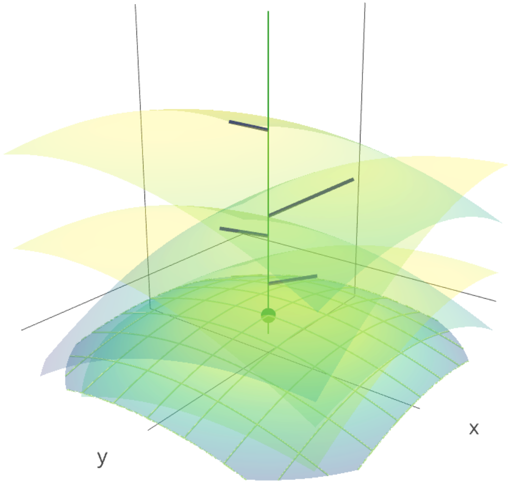

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$$ll(\theta) = ll_1(\theta) + ll_2(\theta) + ll_3(\theta) + ll_4(\theta)$$

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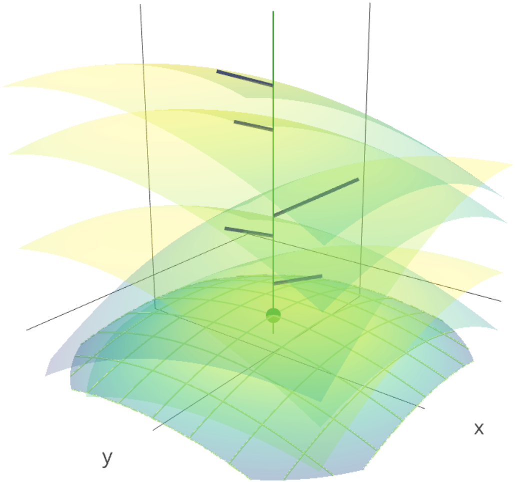

Individual contributions:

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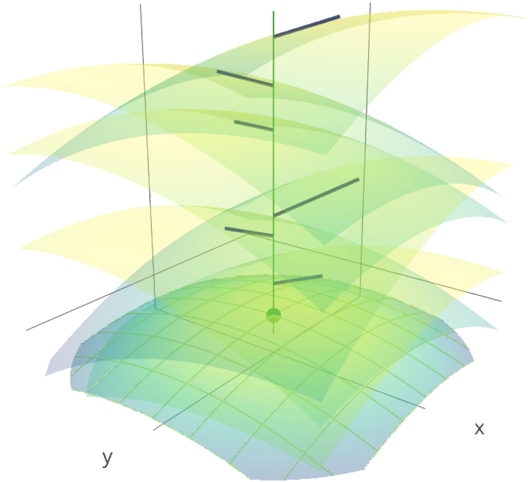

Individual contributions:

$$\mathit{ll}(\theta) = \mathit{ll}_1(\theta) + \mathit{ll}_2(\theta) + \mathit{ll}_3(\theta) + \mathit{ll}_4(\theta) + \mathit{ll}_5(\theta) + \mathit{ll}_6(\theta) + \dots$$

$$\partial_\theta \mathit{ll}(\hat{\theta}) = \partial_\theta \mathit{ll}_1(\hat{\theta}) + \partial_\theta \mathit{ll}_2(\hat{\theta}) + \partial_\theta \mathit{ll}_3(\hat{\theta}) + \partial_\theta \mathit{ll}_4(\hat{\theta}) + \partial_\theta \mathit{ll}_5(\hat{\theta}) + \partial_\theta \mathit{ll}_6(\hat{\theta}) + \dots$$

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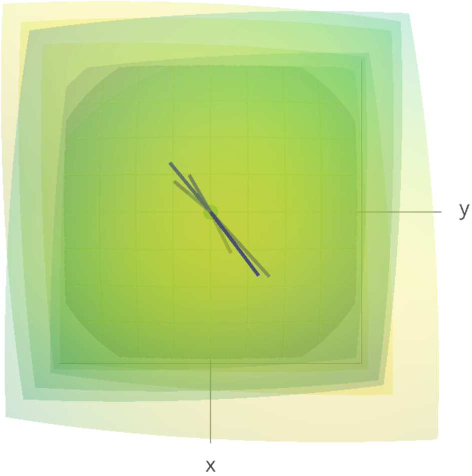

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$$ll(\theta) = ll_1(\theta) + ll_2(\theta) + ll_3(\theta) + ll_4(\theta)$$
$$+ ll_5(\theta) + ll_6(\theta) + \dots$$

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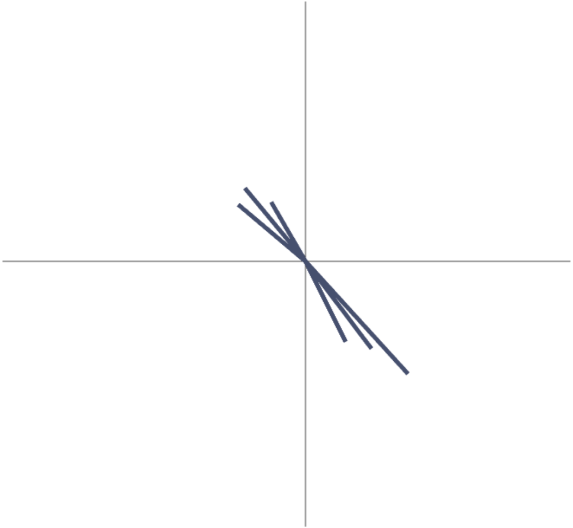

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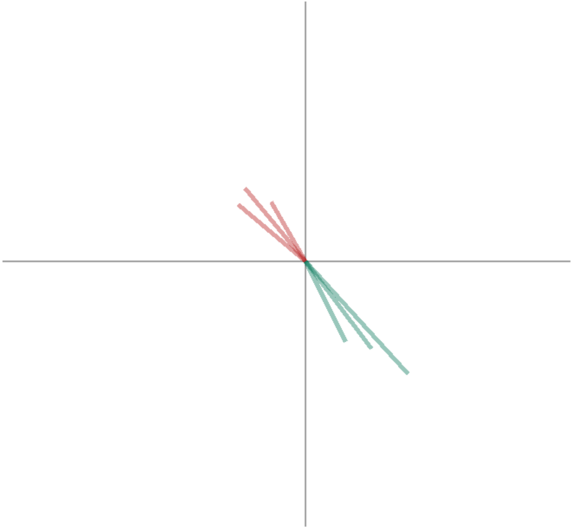

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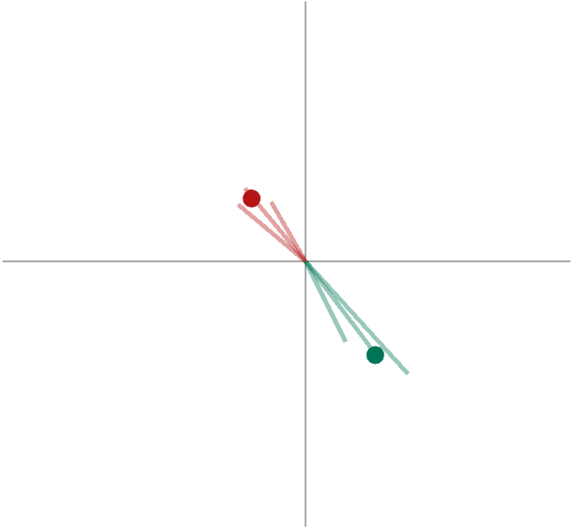

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Individual contributions:

$$g_1(\hat{\theta}) = \sum_{n \in G_1} \partial_\theta ll_n(\hat{\theta})$$

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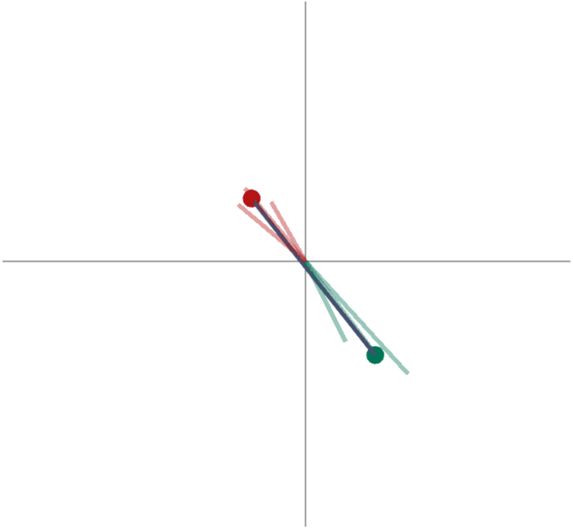

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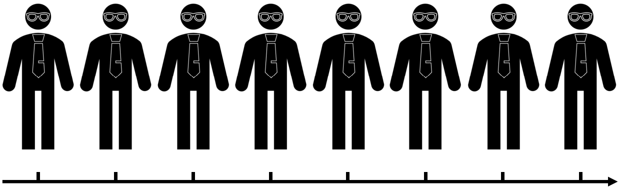
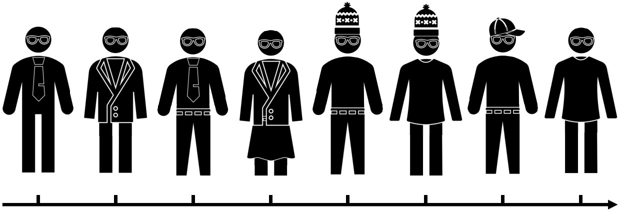
$$\Delta g_{G_1,G_2}(\hat{\theta}) = g_2(\hat{\theta}) - g_1(\hat{\theta})$$

2. 1. Testing for Pooling



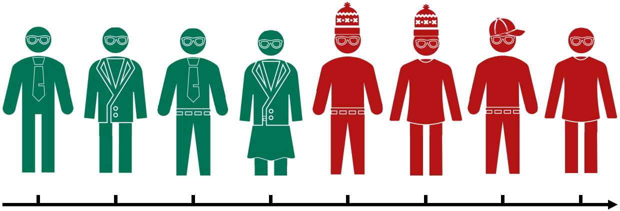
Individual is homogeneous across time

2. 1. Testing for Pooling



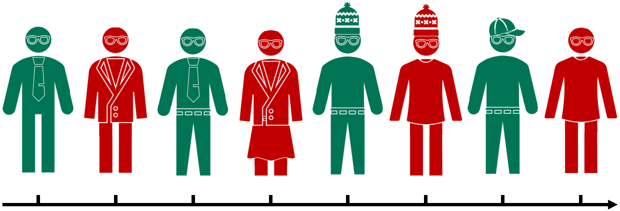
Heterogeneity across time

2. 1. Testing for Pooling



Groups of observations with structural differences

2. 1. Testing for Pooling



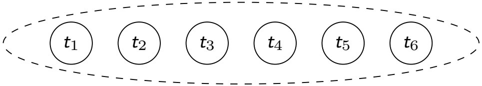
Groups of observations with structural differences

Maximum Likelihood

t_1 t_2 t_3 t_4 t_5 t_6

3. ML vs CML

Maximum Likelihood



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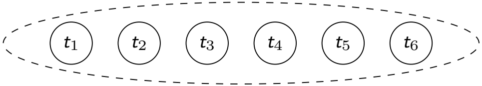

$$ll_{n,\mathrm{ML}}(\theta) = \log L_n(\theta)$$

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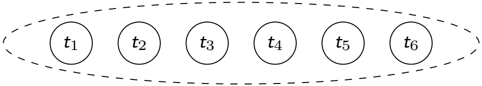
Maximum Likelihood



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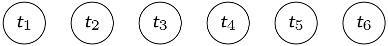
Composite Marginal Likelihood



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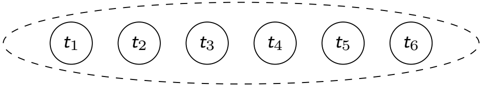

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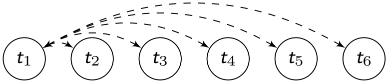

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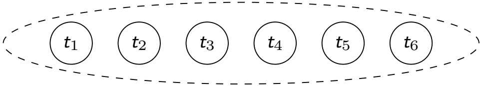

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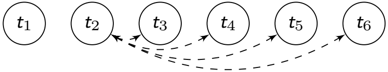

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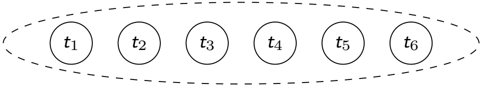

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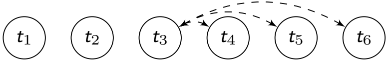
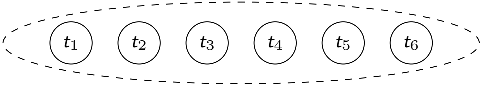
Composite Marginal Likelihood



3. ML vs CML

Maximum Likelihood



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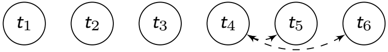
Composite Marginal Likelihood



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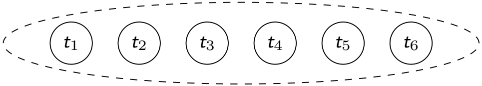

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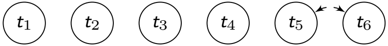
Composite Marginal Likelihood



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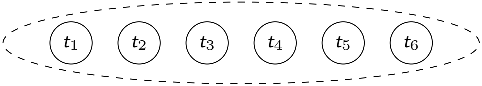
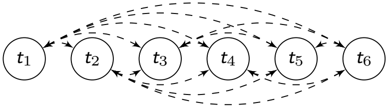


$$ll_{n,\mathrm{ML}}(\theta) = \log L_n(\theta)$$

Composite Marginal Likelihood

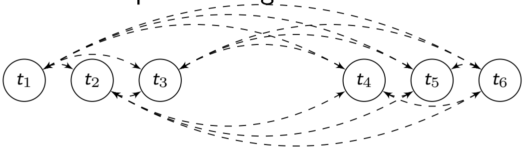


$$ll_{n,\mathrm{CML}}(\theta) = \sum_{q=1}^{T_n-1} \sum_{p=q+1}^{T_n} w_{n,q,p} \log L_{n,q,p}(\theta)$$

Maximum Likelihood

$$ll_{n,\mathrm{ML}}(\theta) = \log L_n(\theta)$$

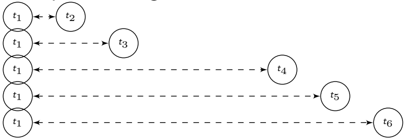
Composite Marginal Likelihood



$$ll_{n,\mathrm{CML}}(\theta) = \sum_{q=1}^{T_n-1} \sum_{p=q+1}^{T_n} w_{n,q,p} \log L_{n,q,p}(\theta)$$

3. 2. CML in Detail

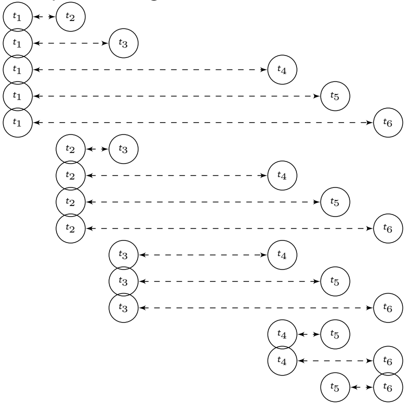
Composite Marginal Likelihood



$$ll_{n,\mathrm{CML}}(\theta) = \sum_{p=q+1}^{T_n} w_{n,q,p} \log L_{n,q,p}(\theta)$$

3. 2. CML in Detail

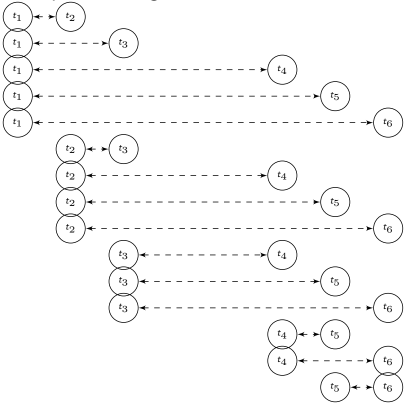
Composite Marginal Likelihood



$$ll_{n,\mathrm{CML}}(\theta) = \sum_{q=1}^{T_n-1} \sum_{p=q+1}^{T_n} w_{n,q,p} \log L_{n,q,p}(\theta)$$

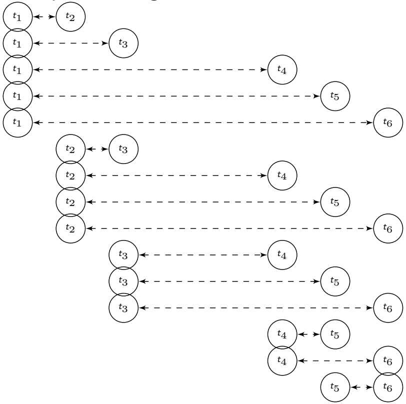
3. 3. Gradients of CML

Composite Marginal Likelihood



3. 3. Gradients of CML

Composite Marginal Likelihood

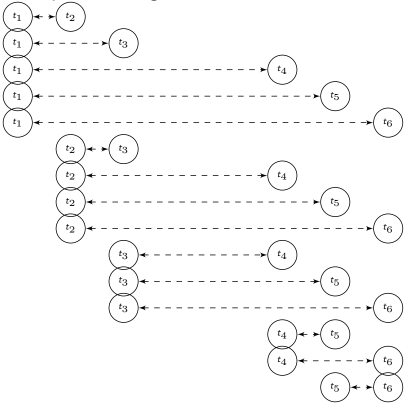


Gradient

$$\partial_\theta ll_{n,\mathrm{CML}}(\theta) =$$

3. 3. Gradients of CML

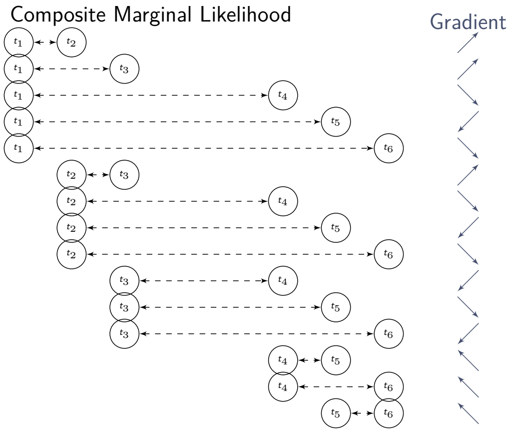
Composite Marginal Likelihood



Gradient

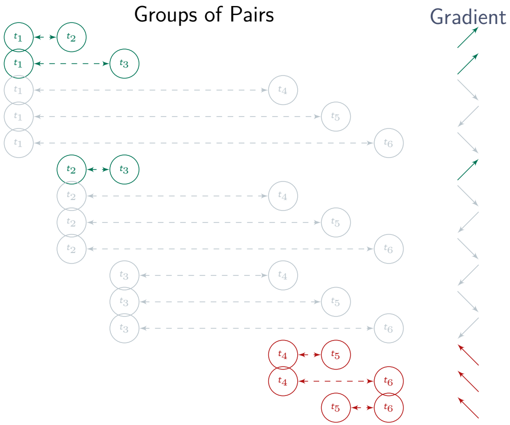
$$\partial_\theta ll_{n,\mathrm{CML}}(\theta) = \qquad w_{n,q,p}\partial_\theta \log L_{n,q,p}(\theta)$$

3. 3. Gradients of CML

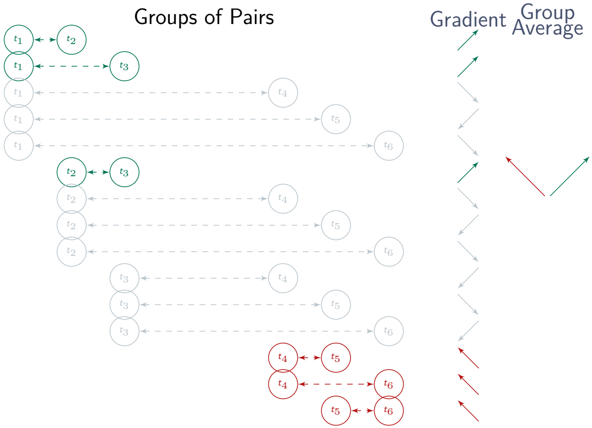


$$\partial_\theta ll_{n,\mathrm{CML}}(\theta) = \sum_{q=1}^{T_n-1} \sum_{p=q+1}^{T_n} w_{n,q,p} \partial_\theta \log L_{n,q,p}(\theta)$$

3. 4. Groups of Pairs



3. 4. Groups of Pairs



$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\, g_{n,a,b}$$

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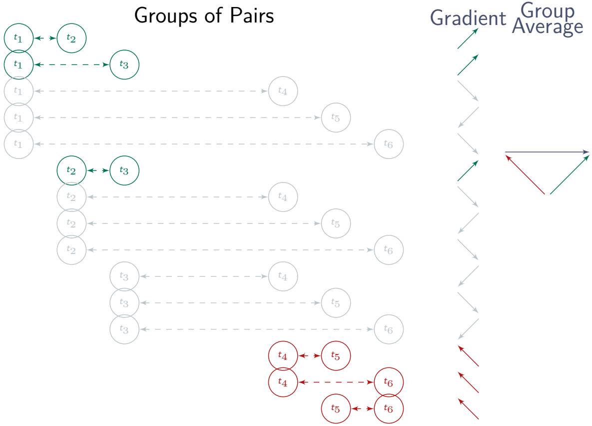
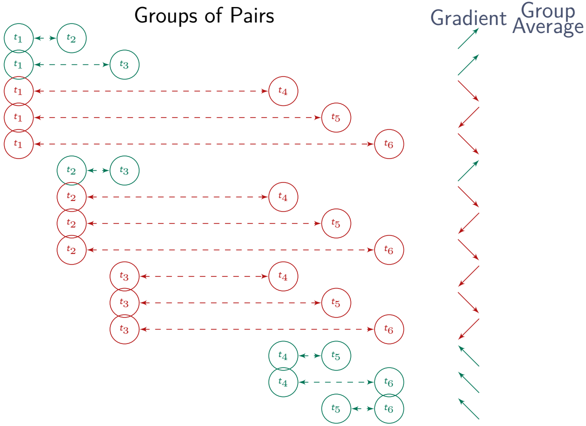


$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\, g_{n,a,b}$$

$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\, g_{n,a,b}$$

$$\Delta\bar{g}_{G_1,G_2,n} = \bar{g}_{G_1,n} - \bar{g}_{G_2,n}$$

3. 4. Groups of Pairs

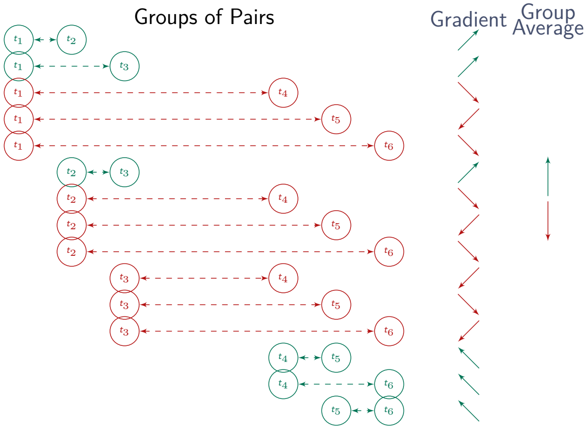


$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\, g_{n,a,b}$$

$$\bar{g}_{G_2,n} = \frac{1}{|G_{2,n}|} \sum_{a,b:(a,b)\in G_2} w_{n,a,b}\, g_{n,a,b}$$

$$\Delta\bar{g}_{G_1,G_2,n} = \bar{g}_{G_1,n} - \bar{g}_{G_2,n}$$

3. 4. Groups of Pairs

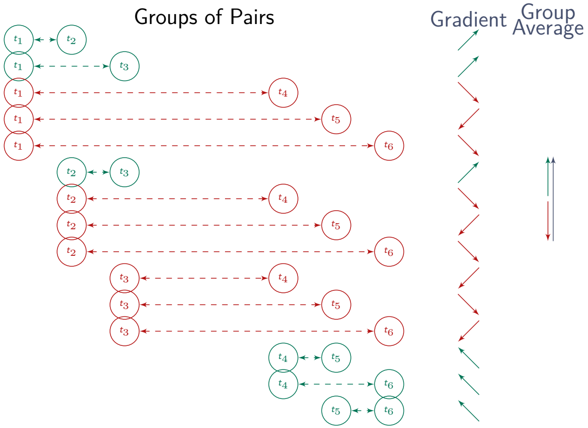


$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\; g_{n,a,b}$$

$$\bar{g}_{G_2,n} = \frac{1}{|G_{2,n}|} \sum_{a,b:(a,b)\in G_2} w_{n,a,b}\; g_{n,a,b}$$

$$\Delta\bar{g}_{G_1,G_2,n} = \bar{g}_{G_1,n} - \bar{g}_{G_2,n}$$

3. 4. Groups of Pairs



$$\bar{g}_{G_1,n} = \frac{1}{|G_{1,n}|} \sum_{a,b:(a,b)\in G_1} w_{n,a,b}\, g_{n,a,b}$$

$$\bar{g}_{G_2,n} = \frac{1}{|G_{2,n}|} \sum_{a,b:(a,b)\in G_2} w_{n,a,b}\, g_{n,a,b}$$

$$\Delta\bar{g}_{G_1,G_2,n} = \bar{g}_{G_1,n} - \bar{g}_{G_2,n}$$

4. CML Gradient Test

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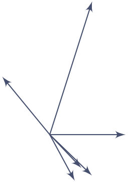
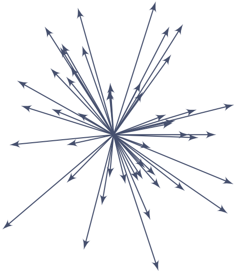
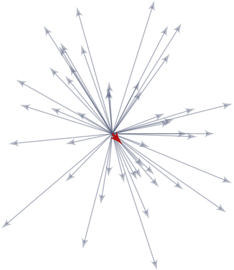


4. CML Gradient Test

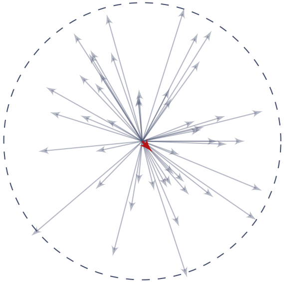


4. CML Gradient Test



$$\Delta \bar{g}_{G_1,G_2} = \frac{1}{N} \sum_{n=1}^{N} \Delta \bar{g}_{G_1,G_2,n}$$

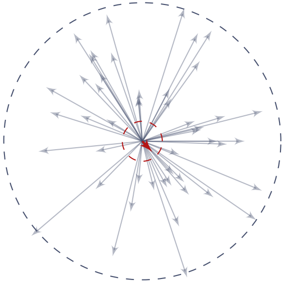
4. CML Gradient Test



$$\Delta\bar{g}_{G_1,G_2} = \frac{1}{N}\sum_{n=1}^{N}\Delta\bar{g}_{G_1,G_2,n}$$

$$\widehat{\text{Var}}(\Delta\bar{g}_{G_1,G_2,n}) = \frac{1}{N-1}\sum_{n=1}^{N}(\Delta\bar{g}_{G_1,G_2,n} - \Delta\bar{g}_{G_1,G_2})(\Delta\bar{g}_{G_1,G_2,n} - \Delta\bar{g}_{G_1,G_2})'$$

4. CML Gradient Test

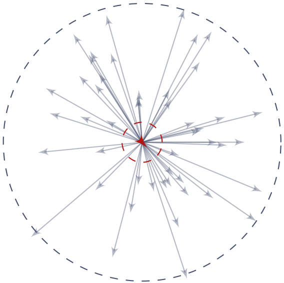


$$\Delta\bar{g}_{G_1,G_2} = \frac{1}{N}\sum_{n=1}^{N}\Delta\bar{g}_{G_1,G_2,n}$$

$$\widehat{\mathrm{Var}}(\Delta\bar{g}_{G_1,G_2,n}) = \frac{1}{N-1}\sum_{n=1}^{N}(\Delta\bar{g}_{G_1,G_2,n} - \Delta\bar{g}_{G_1,G_2})(\Delta\bar{g}_{G_1,G_2,n} - \Delta\bar{g}_{G_1,G_2})'$$

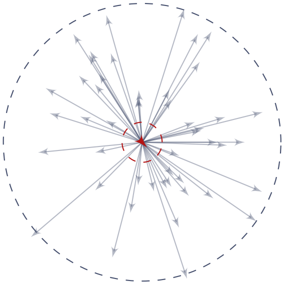
$$\widehat{\mathrm{Var}}(\Delta\bar{g}_{G_1,G_2}) = \frac{1}{N}\widehat{\mathrm{Var}}(\Delta\bar{g}_{G_1,G_2,n})$$

Null Hypothesis

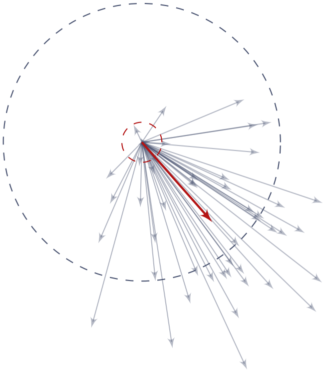


4. 1. Hypothesis and Alternative

Null Hypothesis



Alternative



4. 2. Test Statistic

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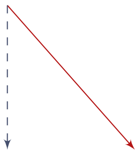
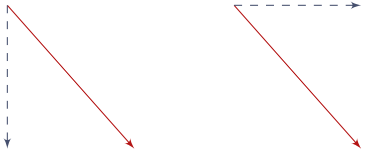


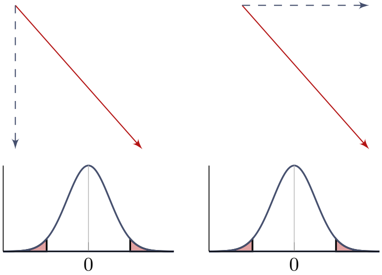
Componentwise

4. 2. Test Statistic



Componentwise

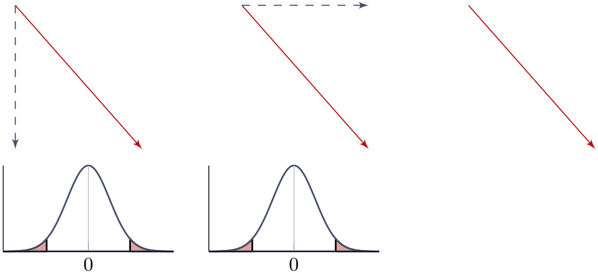
4. 2. Test Statistic



Componentwise

$$\Delta\bar{g}^{(p)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\mathsf{Var}}(\Delta\bar{g}_{G_1,G_2})^{(p,p)})$$

4. 2. Test Statistic

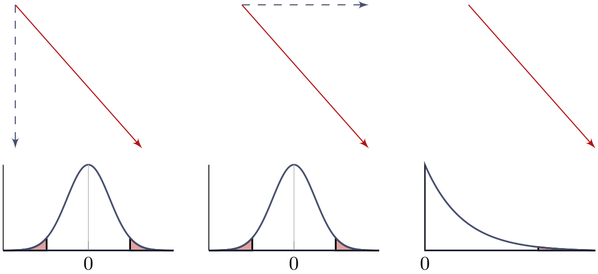


Componentwise

$$\Delta\bar{g}^{(p)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\mathsf{Var}}(\Delta\bar{g}_{G_1,G_2})^{(p,p)})$$

Joint

4. 2. Test Statistic



Componentwise

$$\Delta\bar{g}^{(p)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\text{Var}}(\Delta\bar{g}_{G_1,G_2})^{(p,p)})$$

Joint

$$t^2 = \Delta\bar{g}'_{G_1,G_2}\widehat{\text{Var}}(\Delta\bar{g}_{G_1,G_2})^{-1}\Delta\bar{g}_{G_1,G_2}$$

$$\frac{P(N-1)}{N-P}t^2 \overset{a}{\sim} F_{P,N-P}$$

Simulations

5. 1. Model and Data

Discrete choice model with two alternatives:

$$
\begin{aligned}
y^*_{n,t,1} &= \varepsilon_{n,t,1}, \\
y^*_{n,t,2} &= \beta_0 + x_{n,t}\beta_1 + \gamma_n + \varepsilon_{n,t,2}, \\
\varepsilon_{n,t} &= (\varepsilon_{n,t,1}, \varepsilon_{n,t,2})' \sim \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & {\sigma_\varepsilon}^2 \end{pmatrix}\right) \\
\gamma_n &\sim \mathcal{N}(0, {\sigma_\gamma}^2)
\end{aligned}
$$

5. 1. Model and Data

Discrete choice model with two alternatives:

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\end{aligned}
$$

- $t \in \{1, \dots, T/2, 366, \dots, 366 + T/2\}$ $\Rightarrow$ two panel waves
- $T \in \{10, 14, 20, 30\}$ for different simulations

5. 1. Model and Data

Discrete choice model with two alternatives:

$$
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y^*_{n,t,1} &= \varepsilon_{n,t,1}, \\
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\varepsilon_{n,t} &= (\varepsilon_{n,t,1}, \varepsilon_{n,t,2})' \sim \mathcal{N}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & {\sigma_\varepsilon}^2 \end{pmatrix}\right) \\
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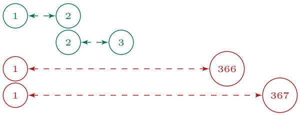
\end{aligned}
$$

- $t \in \{1, \ldots, T/2, 366, \ldots, 366 + T/2\}$ $\Rightarrow$ two panel waves
- $T \in \{10, 14, 20, 30\}$ for different simulations
- $n \in \{1, \ldots, N\}$
- $N \in \{100, 500, 1000\}$ for different simulations

5. 2. Tested Groupings

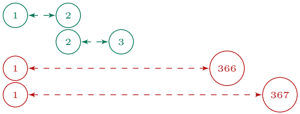


NearFar

$|t_a - t_b|$ small

$|t_a - t_b|$ large

5. 2. Tested Groupings



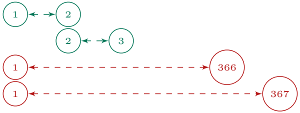
NearFar

$|t_a - t_b|$ small

$|t_a - t_b|$ large

Different correlation between close and distant pairs

5. 2. Tested Groupings



NearFar

$|t_a - t_b|$ small

$|t_a - t_b|$ large

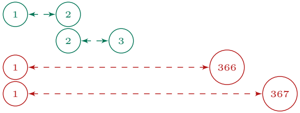
Different correlation between close and distant pairs

FirstLast

$(t_a < 366) \wedge (t_b < 366)$

$(t_a \geq 366) \wedge (t_b \geq 366)$

5. 2. Tested Groupings



NearFar

$|t_a - t_b|$ small

$|t_a - t_b|$ large

Different correlation between close and distant pairs

FirstLast

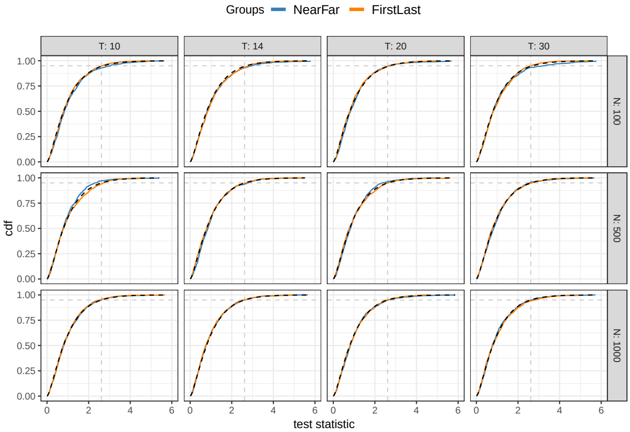
$(t_a < 366) \wedge (t_b < 366)$

$(t_a \geq 366) \wedge (t_b \geq 366)$

Structural change between panel waves

5. 3. Test Distribution under Null Hypothesis

Joint test



Parameter:

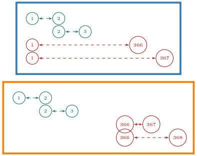
$$\theta = (\beta_0, \sigma_\gamma, \sigma_\varepsilon)$$

Statistic:

$$t^2 = \Delta \bar{g}'_{G_1,G_2} \widehat{\text{Var}}(\Delta \bar{g}_{G_1,G_2})^{-1} \Delta \bar{g}_{G_1,G_2}$$

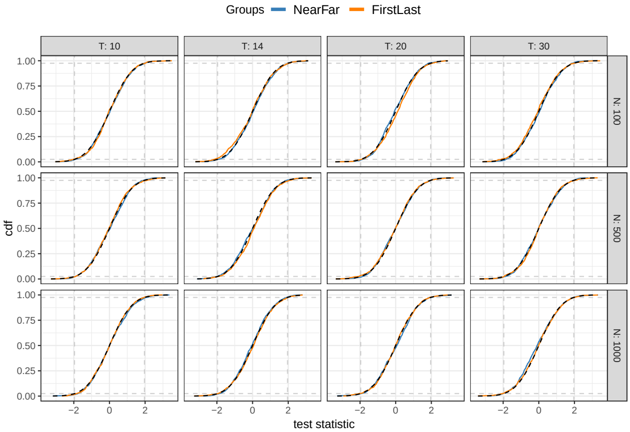
$$\frac{P(N-1)}{N-P} t^2 \overset{a}{\sim} F_{P,N-P}$$

Groupings:



5. 3. Test Distribution under Null Hypothesis

Testing β_0



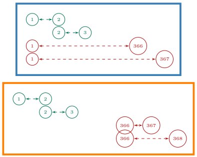
Parameter:

$$\theta = (\beta_0, \sigma_\gamma, \sigma_\varepsilon)$$

Statistic:

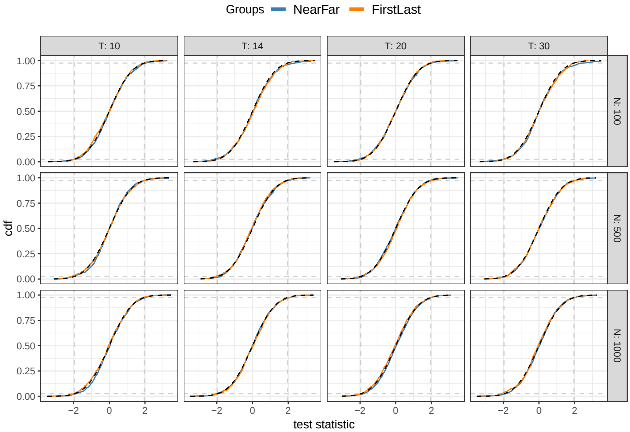
$$\Delta\bar{g}^{(1)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\mathsf{Var}}(\Delta\bar{g}_{G_1,G_2})^{(1,1)})$$

Groupings:



5. 3. Test Distribution under Null Hypothesis

Testing σ_γ



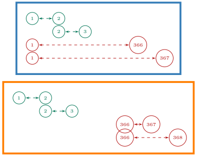
Parameter:

$$\theta = (\beta_0, \sigma_\gamma, \sigma_\varepsilon)$$

Statistic:

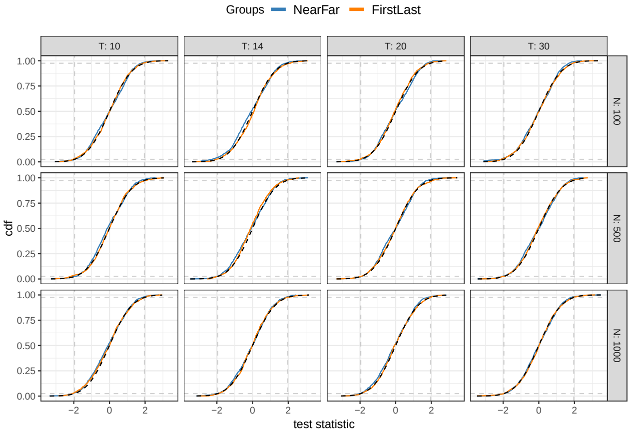
$$\Delta\bar{g}^{(2)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\text{Var}}(\Delta\bar{g}_{G_1,G_2})^{(2,2)})$$

Groupings:



5. 3. Test Distribution under Null Hypothesis

Testing σ_ε



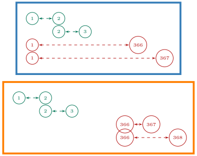
Parameter:

$$\theta = (\beta_0, \sigma_\gamma, \sigma_\varepsilon)$$

Statistic:

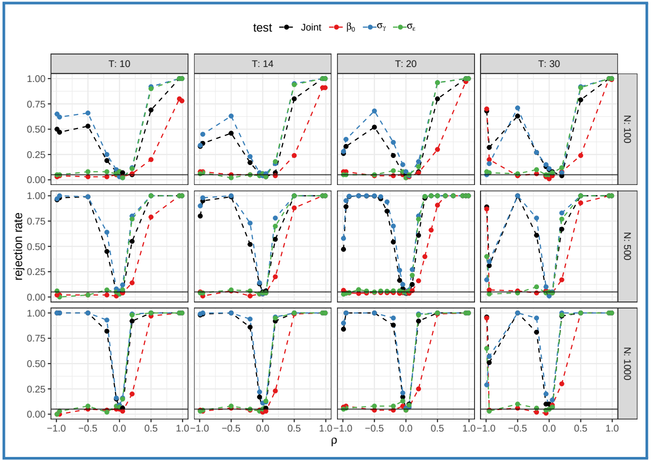
$$\Delta\bar{g}^{(3)}_{G_1,G_2} \overset{a}{\sim} \mathcal{N}(0, \widehat{\mathrm{Var}}(\Delta\bar{g}_{G_1,G_2})^{(3,3)})$$

Groupings:



5. 4. Rejection Rate Autoregressive Errors

NearFar



DGP:

$$\varepsilon_{n,t} = \rho\varepsilon_{n,t-1} + \tilde{\varepsilon}_{n,t}, \quad \tilde{\varepsilon}_{n,t} \sim \mathcal{N}\left(0, \begin{pmatrix} 1-\rho^2 & 0 \\ 0 & 1-\rho^2 \end{pmatrix}\right)$$

$$\gamma_n = 0$$

$$\text{Cor}(\varepsilon_{n,t_a,1}, \varepsilon_{n,t_b,1}) = \rho^{|t_a-t_b|}$$

$$\text{Cor}(\varepsilon_{n,t_a,2}, \varepsilon_{n,t_b,2}) = \rho^{|t_a-t_b|}$$

Model:

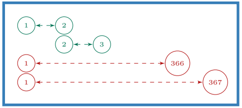
$$\text{Cor}(\varepsilon_{n,t_a,1}, \varepsilon_{n,t_b,1}) = \frac{\sigma_\gamma^2}{1+\sigma_\gamma^2}$$

$$\text{Cor}(\varepsilon_{n,t_a,2}, \varepsilon_{n,t_b,2}) = \frac{\sigma_\gamma^2}{\sigma_\varepsilon^2+\sigma_\gamma^2}$$

Parameter:

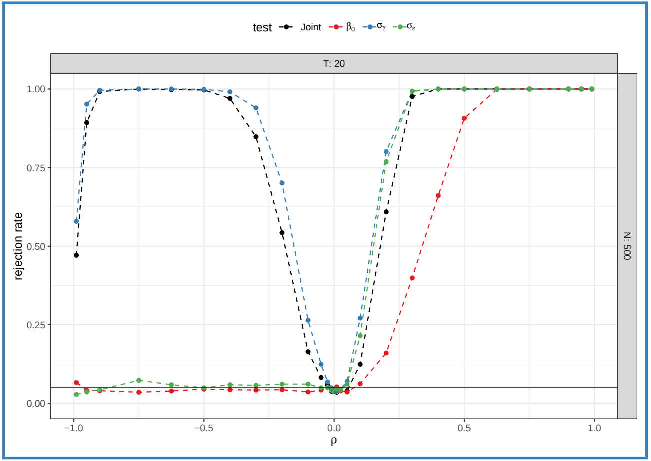
$$\theta = (\beta_0, \sigma_\gamma, \sigma_\varepsilon)$$

Grouping:



5. 4. Rejection Rate Autoregressive Errors

NearFar



DGP:

$$\varepsilon_{n,t} = \rho\varepsilon_{n,t-1} + \tilde{\varepsilon}_{n,t}, \quad \tilde{\varepsilon}_{n,t} \sim \mathcal{N}\left(0, \begin{pmatrix} 1-\rho^2 & 0 \\ 0 & 1-\rho^2 \end{pmatrix}\right)$$

$$\gamma_n = 0$$

$$\text{Cor}(\varepsilon_{n,t_a,1}, \varepsilon_{n,t_b,1}) = \rho^{|t_a - t_b|}$$

$$\text{Cor}(\varepsilon_{n,t_a,2}, \varepsilon_{n,t_b,2}) = \rho^{|t_a - t_b|}$$

Model:

$$\text{Cor}(\varepsilon_{n,t_a,1}, \varepsilon_{n,t_b,1}) = \frac{\sigma_\gamma^2}{1+\sigma_\gamma^2}$$

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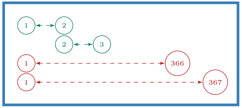
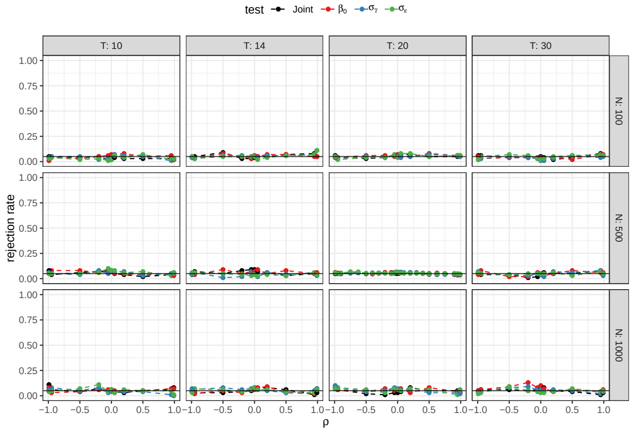


5. 4. Rejection Rate Autoregressive Errors

FirstLast



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Model:

$$\text{Cor}(\varepsilon_{n,t_a,1}, \varepsilon_{n,t_b,1}) = \frac{\sigma_\gamma^2}{1+\sigma_\gamma^2}$$

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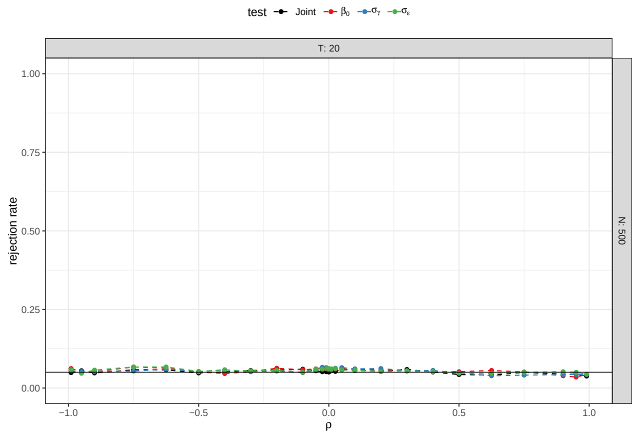

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Parameter:

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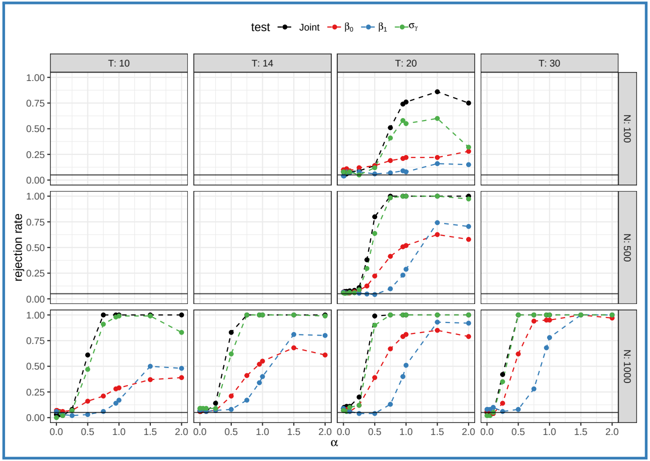
Grouping:

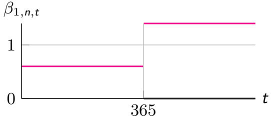
5. 5. Rejection Rate Shift

NearFar



DGP:

$$\beta_{1,n,t} = \begin{cases} 1 - \alpha, & t < 365 \\ 1 + \alpha, & t \geq 365 \end{cases}$$



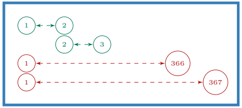
Model:

$$\beta_{1,n,t} = \beta_1$$

Parameter:

$$\theta = (\beta_0, \beta_1, \sigma_\gamma)$$

Grouping:



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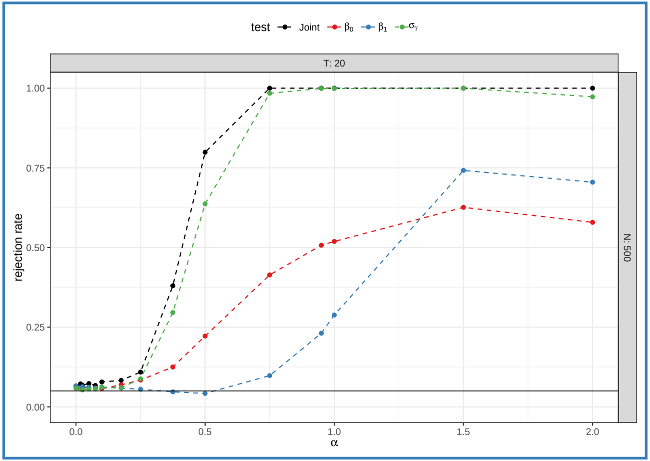
NearFar



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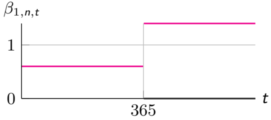

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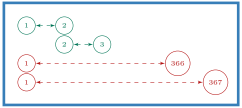

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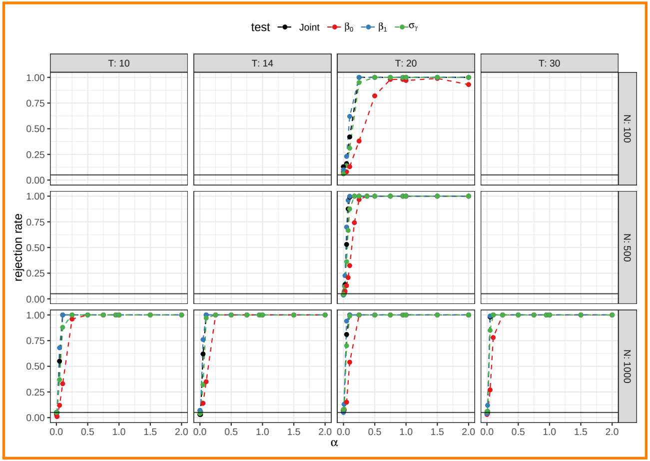

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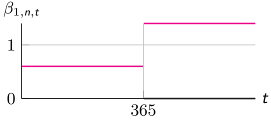

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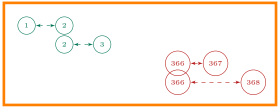

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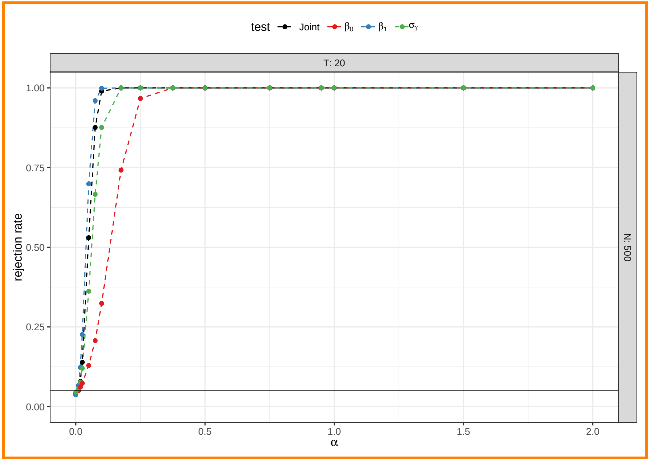

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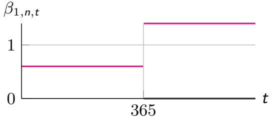

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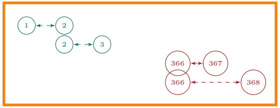
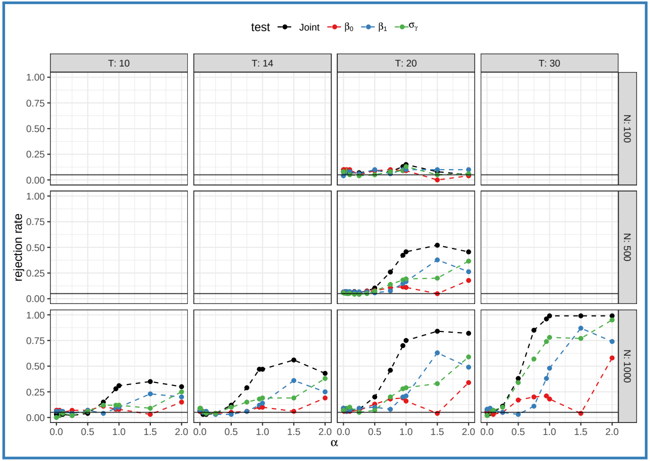


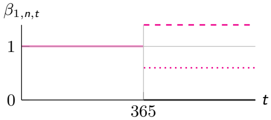
5. 6. Rejection Rate Split

NearFar



DGP:

$$\beta_{1,n,t} = \begin{cases} 1, & t < 365 \\ 1+\alpha, & t \geq 365 \text{ and } n \in P_1 \\ 1-\alpha, & t \geq 365 \text{ and } n \in P_2 \end{cases}$$



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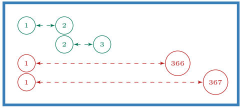

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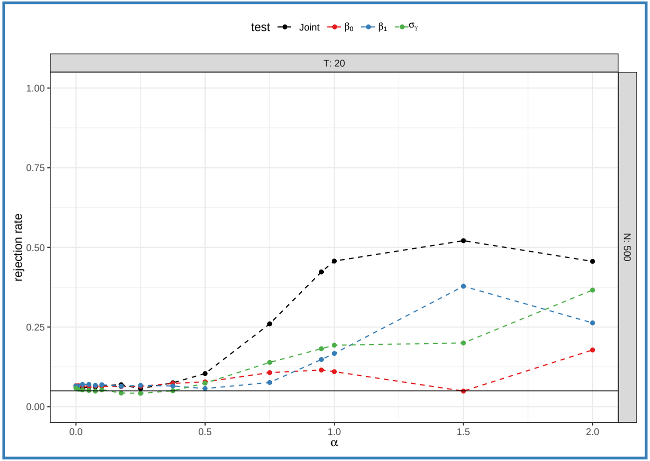
NearFar



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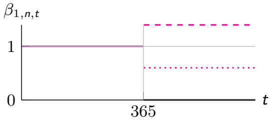

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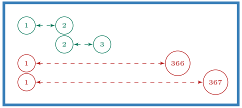
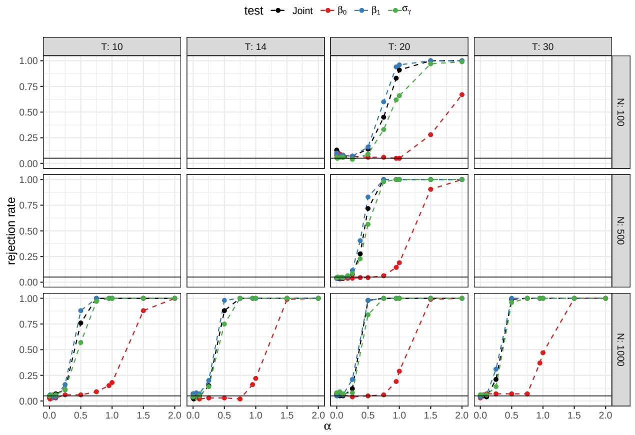


5. 6. Rejection Rate Split

FirstLast



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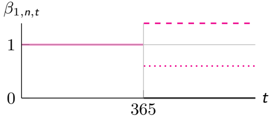

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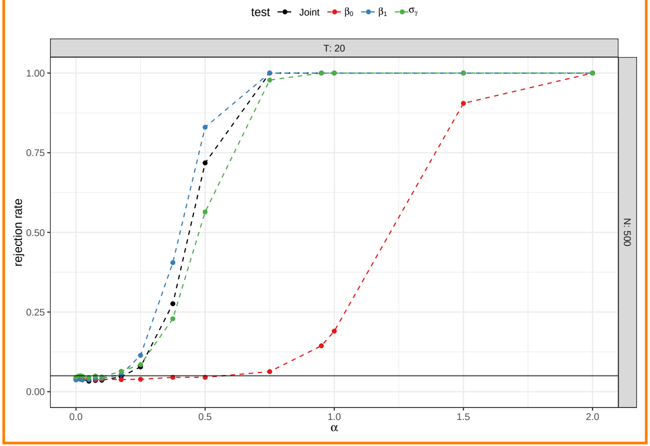

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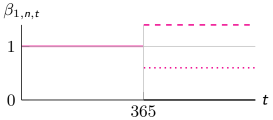

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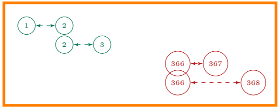

Conclusions

6. Conclusions

- Detect structural changes over time.
- Detect changes in dependencies between observations.
- No need to explicitly model dynamic effects.
- Theoretical results demonstrated in simulation study.

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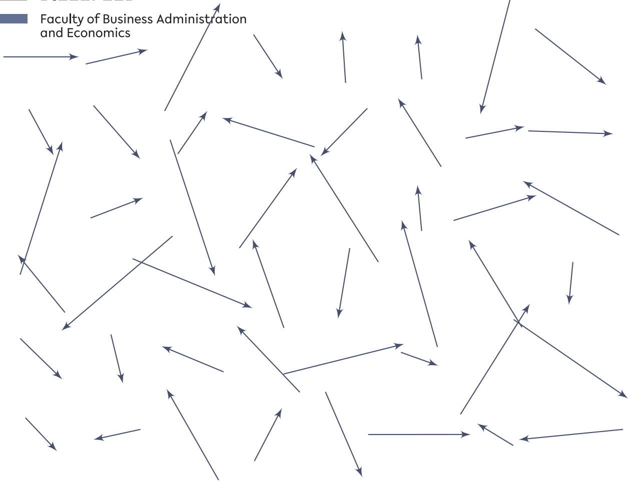
Faculty of Business Administration and Economics



Thank you!

Questions & Remarks

Contact: sebastian.buescher@uni-bielefeld.de