Comprehensive Examination

Department of Physics and Astronomy

Stony Brook University

January 2020

General Instructions:

Three problems are given. If you take this exam as a placement exam, you must work on all three problems. If you take the exam as a qualifying exam, you must work on two problems (if you work on all three problems, only the two problems with the highest scores will be counted).

Each problem counts for 20 points, and the solution should typically take approximately one hour.

Use one exam book for each problem, and label it carefully with the problem topic and number and your ID number.

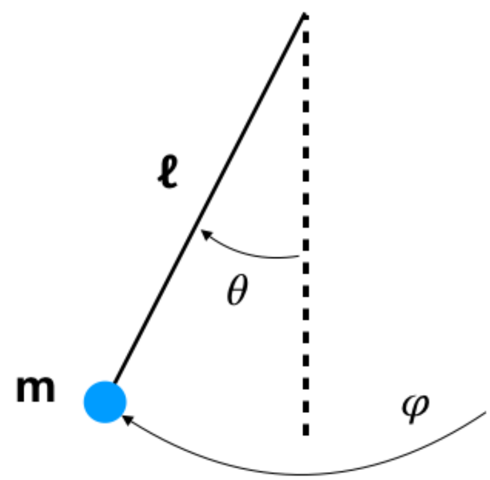
Write your ID number (not your name!) on each exam booklet.

You may use, one sheet (front and back side) of handwritten notes and, with the proctor's approval, a foreign-language dictionary. **No other materials may be used**.

Classical Mechanics 1

Spherical Pendulum

Consider a spherical pendulum: a mass m on a rope of length ℓ attached a frictionless pivot that allows the mass freedom to move in two angular directions. You may assume that the length of the rope is fixed so that the motion of the mass is confined to a sphere with radius ℓ. Do not make small angle approximations in this problem.



(a) (4 points) Determine the Lagrangian of this system in terms of angular variables θ, the angle of the rope w.r.t to the vertical axis, and ϕ the azimuthal motion of the mass.

(b) (4 points) Using the symmetries of the Lagrangian, identify two constants of motion.

(c) (4 points) Using your results from (b), find an implicit solution for the equations of motion $\theta(t)$ and $\phi(\theta)$ (your results should be in terms of integrals that you do not need to do).

(d) (4 points) Using the variational principle, determine the equation(s) of motion for the pendulum and identify the location of stable orbits.

(e) (4 points) Under what conditions are the orbits from (d) stable? Determine the frequency of small oscillations around these orbits (you may leave your answer in terms of θ_c the value of the θ coordinate for a stable circular orbit). Does your answer reduce to the usual frequency of a simple 1-D pendulum in the small θ_c limit? Why or why not?

Classical Mechanics 2

Isotropic oscillator in a magnetic field

Consider a particle of mass m and positive charge q moving in the x, y plane in an isotropic harmonic potential $V(x, y) = \frac{1}{2}m\omega_0^2(x^2 + y^2)$. In addition, the particle is placed in a uniform magnetic field of magnitude B_0 in the z direction.

(a) (2 points) Write down the Lagrangian of the system in cartesian and cylindrical coordinates $x = r\cos\phi$ and $y = r\sin\phi$.

Hint: Use the gauge $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$, and define the magnetic frequency[1] $\omega_B \equiv eB/(2m)$ to simplify the algebra in what follows.

(b) (2 points) Determine the equations of motion in cartesian and cylindrical coordinates.

(c) (4 points) Identify the constant integrals of motion, and write down explicit expressions for these quantities in cartesian and cylindrical coordinates.

(d) (5 points) What are the radii of the stable circular orbits of the particle and the associated angular velocities $\dot{\phi}$. Explicitly interpret the allowed values of $\dot{\phi}$ by drawing a well labeled free body diagram indicating the forces on the particle.

(e) (5 points) Determine the general solution to the equations of motion for $x(t)$ and $y(t)$.

Hint: Write down equations of motion for $z \equiv x + iy$ and solve this linear differential equation. Express the final result for $x(t)$ and $y(t)$ in terms of real functions and real constants of integration.

(f) (2 points) Evaluate the angular momentum of the system for the general solution of part (e) and interpret the result.

[1] In Gaussian or Heaviside Lorentz units $\omega_B \equiv eB/(2mc)$.

Classical Mechanics 3

An accelerating frame

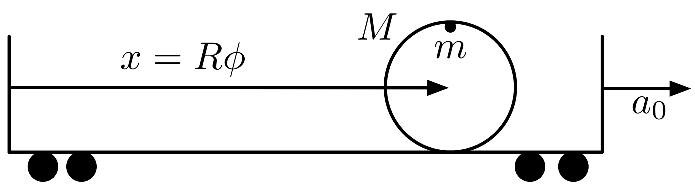
A cylinder with mass M, moment of inertial I, and radius R, starts rolling without slipping from rest in a train that accelerates with constant acceleration a_0. The axis of the cylinder is perpendicular to the motion of the train.

(a) (2 points) Calculate the acceleration of the cylinder using Newton's laws in the lab frame (the ground). Draw a well labelled free body diagram indicating the forces and acceleration in this frame.

(b) (2 points) Calculate the acceleration of the cylinder in the frame of the accelerating train by using Newton's laws in this frame. Again, draw a well labeled free-body diagram indicating the forces and acceleration in this frame.

(c) (5 points) Write down a Lagrangian for the cylinder and calculate its acceleration by solving the Euler Lagrange equations. Is the acceleration consistent with parts (a) and (b)?

Now consider a cylinder-like contraption consisting of a cylindrical ring of mass M and radius R, and a small weight of mass m fixed to the rim of the ring (see below). At time $t = 0$ the cylinder starts to roll without slipping from rest in the accelerating train, and the weight is at the top of its arc as shown in the figure below.



(d) (8 points) Determine the Lagrangian for the angle $\phi(t)$, where $x \equiv R\phi$ is the position of the center of the cylinder relative to the back of the train (see figure). Show that the Lagrangian may (up to total derivatives) be written in a time independent form

$$L = \frac{1}{2} m_{\rm eff}(\phi)\, R^2\, \dot{\phi}^2 - U(\phi)\,, \tag{1}$$

where $m_{\rm eff}(\phi)$ and $U(\phi)$ are specific functions of ϕ.

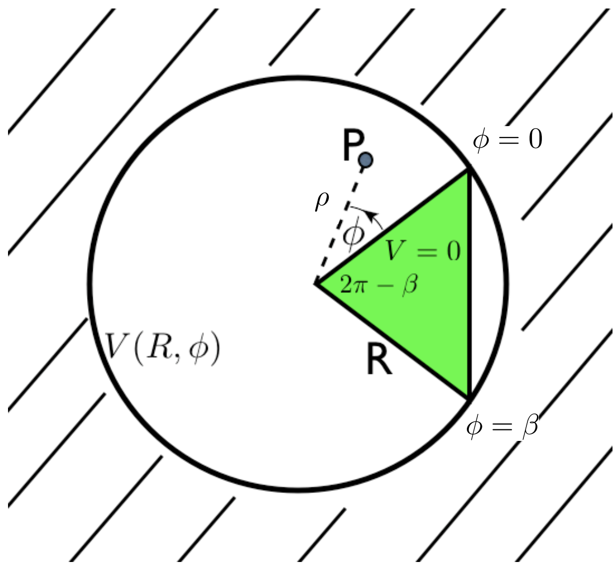
Hint: In the train's frame the acceleration functions like an additional gravitational field of magnitude a_0 pulling the negative x direction.

(e) (3 points) What is the speed of the cylinder after it rolls for two complete turns.

Electromagnetism 1

A conducting wedge inside a cylinder

A hollow cylinder of radius R has a potential $V(R, \phi)$ maintained on its surface, as shown in the figure below. A conducting wedge with surfaces at $\phi = 0$ and $\phi = \beta$ and its apex at the symmetry axis of the cylinder (as shown by the shaded region), is placed inside the cylinder and held at a potential of $V = 0$ ($\rho = \sqrt{x^2 + y^2}$ denotes the radial coordinate).



(a) (6 points) Derive the most general form for the potential inside the cylinder for $0 < \phi < \beta$ for a general boundary-condition $V(R, \phi)$. [2]

(b) (2 points) Determine the potential for $0 < \phi < \beta$ resulting from the boundary condition

$$V(R, \phi) = \overline{V}_1 \sin\left(\frac{\pi\phi}{\beta}\right) + \overline{V}_3 \sin\left(\frac{3\pi\phi}{\beta}\right) \tag{1}$$

where $\overline{V}_{1,3}$ are constants.

(c) (6 points) Determine the electric field at each point inside the cylinder for $0 < \phi < \beta$, and determine the surface charge per area on the $\phi = 0$ and $\phi = \beta$ surfaces of the wedge for the boundary conditions of (b).

(d) (3 points) Describe qualitatively the behavior of the electric field (sketch the field lines) and the surface charge per area of part (c) near the tip of the wedge as a function of the wedge apex-angle, $2\pi - \beta$.

---

[2]Possibly useful:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

(e) (3 points) For the situation where $\beta = \pi$ and $\overline{V}_3 = 0$, draw the electric field lines and equi-potential surfaces inside the cylinder.

Electromagnetism 2

Line with rising current

A neutral wire along the z-axis carries a current $I(t)$ that varies with time t as

$$I(t) = \begin{cases} \alpha\, t & t \geq 0 \\ 0 & t < 0 \end{cases}, \tag{1}$$

where α is a positive constant.

a. 9 points Determine the time-dependence of the electric and magnetic fields around the wire at a point $(r, \phi = 0, z = 0)$, in a cylindrical coordinate system where $r = \sqrt{x^2 + y^2}$.

b. 4 points Use your result to determine the fields for long times. Give a physical interpretation of your answer.

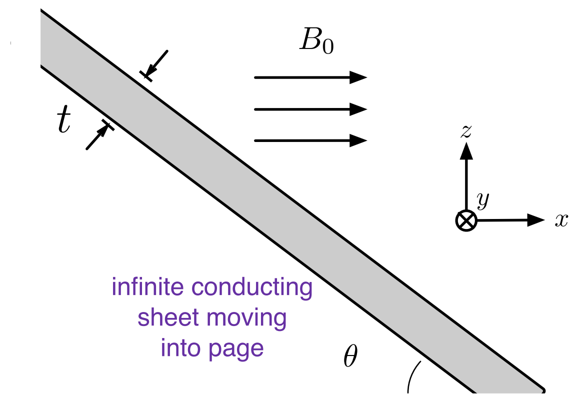
c. 4 points Use your result to determine the fields for short times. Give a physical interpretation of your answer.

d. 3 points Describe briefly the overall physical onset of the fields in time.

Electromagnetism 3

A rotating sphere

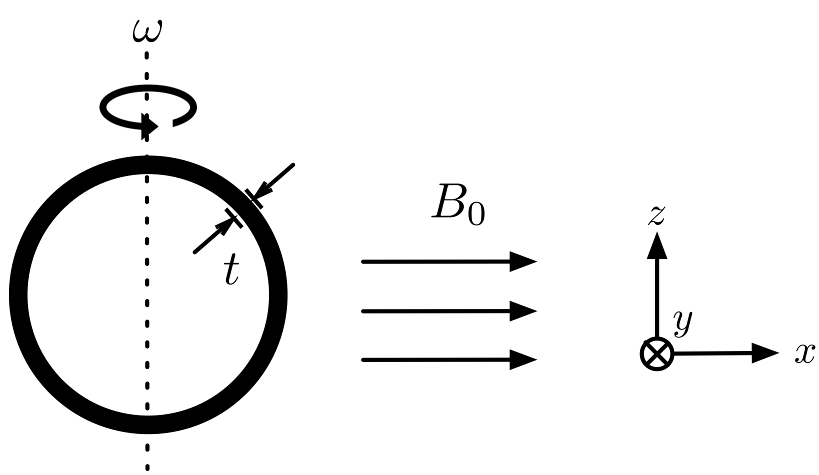
Consider an infinitely large metal sheet of thickness t and conductivity σ, in a homogeneous magnetic field pointing in the x direction, $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$. The sheet has an inclination angle θ relative the x axis (see below), and is moving non-relativistically with velocity v in the y direction (into the page)



(a) (4 points) Determine the charge per area on the surfaces of the sheet. Draw a sketch.

(b) (4 points) Determine the current density in the sheet and the energy dissipated per surface area.

Now consider a thin metal spherical shell of thickness t, conductivity σ, and radius R. The shell is placed in the same magnetic field directed along the x axis, $\boldsymbol{B} = B_0\hat{\boldsymbol{x}}$. The sphere is rotated with angular velocity ω around the z axis.



(c) (4 points) Use part (b) to *estimate* the total energy dissipated in the sphere per time in terms of σ, ω, R, t and B_0 up to an order one numerical factor.

(d) (3 points) Determine charge per area on on the inner and outer surfaces of the sphere and the radial component of the electric field in the metal.

(e) (3 points) Determine the electrostatic potential in the metal consistent with (d) by solving the Laplace equation. Find all components of the electric field in the metal.

Some formulas on the Laplace equation in spherical coordinates are compiled below.

(f) (2 points) Determine the current density $\boldsymbol{J}$ in the sphere.

The Laplace equation in spherical coordinates:

- Gradient in spherical coordinates:

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\,\hat{\boldsymbol{\phi}}$$

- Laplacian in spherical coordinates:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

- Separated solutions to the Laplace equation take the form:

$$\psi(r,\theta,\phi) = \sum_{\ell m}\left(\frac{A_{\ell m}}{r^{\ell+1}} + B_{\ell m}r^{\ell}\right)Y_{\ell m}(\theta,\phi)\,.$$

  Here $Y_{\ell m}(\theta,\phi)$ are spherical harmonics and $A_{\ell m}$ and $B_{\ell m}$ are constants.

- The lowest spherical harmonics are:

$$\begin{aligned}
Y_{0,0} =& \frac{1}{\sqrt{4\pi}} \\
Y_{10} =& \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{1,\pm1} =& \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{+i\phi} \\
Y_{2,0} =& \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \\
Y_{2,\pm1} =& \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi} \\
Y_{2,\pm2} =& \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{\pm i2\phi}
\end{aligned}$$

Quantum Mechanics 1

Angular momentum and Wigner functions

(a) *(5 pts)* Use the properties of the angular momentum operators $\vec{J} = \{J_x,\ J_y\ J_z\}$, and the properties of the standard angular momentum eigenstates $|j, m\rangle$ to find the average and the magnitude of the fluctuations of the x component of the angular momentum in the eigenstates $|j, m\rangle$:

$$\langle j, m|J_x|j, m\rangle, \quad \langle j, m|J_x^2|j, m\rangle \, .$$

For a given j, find the value of m which minimizes the standard deviation σ of J_x. Provide a brief (no more than two sentences) qualitative interpretation of the result.

(b) *(5 pts)* Using again the properties of $\vec{J}$, calculate the rotated operator

$$e^{i\beta J_y/\hbar} J_z e^{-i\beta J_y/\hbar} \, .$$

Compare the obtained expression to the classical vector rotation.

(c) *(7 pts)* From the results of parts (a) and (b), calculate the following characteristics of the Wigner functions $d_{m'm}(\beta)$:

$$\sum_{m'} m' |d_{m'm}(\beta)|^2, \quad \sum_{m'} m'^2 |d_{m'm}(\beta)|^2.$$

As a reminder, $d_{m'm}(\beta) = \langle j, m'|e^{-i\beta J_y/\hbar}|j, m\rangle$.

(d) *(3 pts)* Characteristics of the Wigner functions calculated in part (c) can be viewed as the average and the fluctuations of an operator. Identify the operator and calculate its standard deviation. Make a brief comparison to part (a).

Quantum Mechanics 2

Two particles in a box – perturbative, adiabatic & diabatic changes

Two spin-1/2 fermions of mass μ interact only though a "ferromagnetic" spin-spin interaction:

$$V = -u\vec{\sigma}_1 \cdot \vec{\sigma}_2 \,, \quad u > 0,$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) *(3 pts)* What is the spin contribution $E_{s,t}$ to the particle energy in the singlet and the triplet states, respectively.

(b) *(5 pts)* The particles are confined to the one-dimensional interval $0 < x < a$ (box with width a). What are the (total) eigenenergies $E_{n,m}$ and the wavefunctions $\psi_{n,m}(x_1, x_2)$ of the two particles? Indicate when the ground state is singlet or triplet and explain why.

(c) *(4 pts)* A perturbation potential

$$U(x) = \begin{cases} \delta & x > a/2 \\ -\delta & x < a/2 \end{cases}.$$

is applied to the particles. Find the first-order correction to the energy of a singlet ground state.

(d) *(4 pts)* Consider making changes to the box potential by adjusting the width to a different value b. If this width is changed smoothly and slowly, how does the wave function and energy of the particles change with time if they start in the ground state in the original box?

(e) *(4 pts)* Consider starting with the system in the lowest energy singlet state and adjusting the size of the box suddenly to $b = 2a$. How does the wave function and energy of the particles change with time? Calculate the probability p of still finding the particles in the (new) ground state right after the sudden expansion.

Quantum Mechanics 3

Isotropic 3D harmonic oscillator

The Hamiltonian of an isotropic 3D harmonic oscillator of mass μ and frequency ω is:

$$H = \frac{p^2}{2\mu} + \frac{\mu\omega^2 r^2}{2} .$$

(a) *(5 pts)* Show that the time-independent Schrödinger equation separates in Cartesian coordinate system. Determine the eigenenergies E_n and their degeneracies g_n of the stationary states of the oscillator.

(b) *(5 pts)* Express the Cartesian components of the angular momentum operator $\vec{L} = \{L_x, L_y, L_z\}$ in terms of the raising and lowering operator $a_j^\dagger$, a_j, where $j = x, y, z$. Find the time dependence of $\vec{L}$ in Heisenberg representation.

(c) *(5 pts)* Calculate the operator L^2 of the angular momentum and express it through the operator n of the total number of the excitations, and operators Q and $Q^\dagger$:

$$n = \sum_j a_j^\dagger a_j \, , \quad Q = \sum_j a_j^2 \, .$$

(d) *(5 pts)* Consider the subspace of the degenerate energy eigenstates $|n_x, n_y, n_z\rangle$ of the oscillator with energy $E = (7/2)\hbar\omega$. Using the previous results in this problem, construct the state with vanishing angular momentum, $|L^2 = 0\rangle$, in this subspace.

Statistical Mechanics 1

Specific heat, magnetic susceptibility, and the Stoner instability

Consider electrons in a band with a density of states (for a single spin) of the form

$$g(\epsilon) = g_0 \left(1 - \frac{\epsilon^2}{\epsilon_0^2}\right) , \tag{1}$$

for $-\epsilon_0 < \epsilon < \epsilon_0$ and zero otherwise. The band is half-filled[5] at $T = \mu = 0$. Assuming first that the electrons do not interact, the grand potential is

$$\Omega(T, V, \mu, H) = -VT \sum_{\sigma} \int_{-\epsilon_0}^{\epsilon_0} \mathrm{d}\epsilon \, g(\epsilon) \ln[1 + e^{-\beta(\epsilon + 2\mu_B \sigma H - \mu)}] \, . \tag{2}$$

Here $\beta = 1/T$, V is the volume, μ_B is the Bohr magneton, H is the magnetic field and $\sigma = \pm 1/2$ is the electron spin.

We will start with zero magnetic field, $H = 0$.

(a) (3 points) Derive an expression from Ω for the number of electrons N and the energy of the electrons E in terms of $g(\epsilon)$, T, V and μ (you may leave any integrals unevaluated).

(b) (1 point) Compute N at half filling, i.e. the value of N at T=μ=0.

(c) (2 points) Show that the chemical potential is independent of temperature (and hence zero) if the number of electrons is kept at its half filling value of part (b).

(d) (5 points) For low temperatures and N at half filling, compute the specific heat, $C = \frac{1}{V}\frac{\partial E}{\partial T}$, keeping the lowest term in the expansion $C = c_0 + c_1 T + c_2 T^2 + \ldots$.

Formulate your answer in terms of ϵ_0, g_0 and numerical coefficients. Some helpful integrals are given at the end.

Now we look at non-zero magnetic field, assuming $\mu_B H \ll T$.

(e) (5 points) For low temperature and N at half filling, compute the magnetization M (where M is proportional to H) and the susceptibility, $\chi = \frac{1}{V}\frac{\partial M}{\partial H} = \frac{1}{V}\frac{M}{h}$, keeping the lowest term in the expansion $\chi = b_0 + b_1 T + b_2 T^2 + \ldots$.

Formulate your answer in terms of ϵ_0, g_0 and numerical coefficients. Some helpful integrals are given at the end.

(f) (1 point) What is the value of the ratio $\frac{C/T}{\chi/\mu_B^2}$? (Note: If you do this correctly, this quantity is independent of the density of states.)

[5]This description is relevant in transition metals, where the Fermi energy lies in a narrow band originating from the d electrons.

Now we turn to interacting electrons. We represent the interaction by a "mean field" contribution to the energy:

$$E_{int} = \gamma \frac{N_+ N_-}{V} \tag{3}$$

where γ is the so-called Stoner coupling constant and N_+ and N_- are the number of electron in the spin up and spin down state, respectively. This interaction (the exchange interaction) describes the fact that Coulomb repulsion between the electrons is reduced for electrons with parallel spins, when the spacial component of the wave function has to be anti-symmetric.

(g) (3 points) At zero temperature the system develops spontaneous magnetization if $\gamma > \gamma_{crit}$. What is the value of γ_{crit} ?

Integrals you may need:

We define

$$\alpha_n \equiv \int_{-\infty}^{\infty} \frac{x^n e^x}{(e^x+1)^2} \,. \tag{4}$$

The first few coefficients are

$$\alpha_0 = 1\,, \tag{5}$$
$$\alpha_1 = 0\,, \tag{6}$$
$$\alpha_2 = \frac{\pi^2}{3}\,, \tag{7}$$
$$\alpha_3 = 0\,, \tag{8}$$
$$\alpha_4 = \frac{7\pi^4}{15}\,. \tag{9}$$

Statistical Mechanics 2

Thermodynamics of weak solutions

This problem explores thermodynamics of a weak solution consisting N_s molecules of solvent and N molecules of solute (minor component) with small relative concentration $c = N/N_s \ll 1$. The Gibbs potential of such *dilute* solution can be approximated as

$$\Phi(p, T, N_s, N) \approx N_s \mu_{s0} + N\phi(p, T) + NT \log \frac{N}{N_s} \tag{1}$$

where μ_{s0} is the chemical potential of pure solvent ($N = 0$), $\phi(p, T)$ is the contribution of a single solute molecule to the Gibbs potential, and the third term describes the entropy correction due to indistinguishable nature of solute molecules and is closely related to the "Boltzmann's factor" in the ideal gas. Note that this Gibbs potential is extensive, $\Phi(p, T, N_s, N) = N_s f(p, T, c)$, $c = N/N_s$.

(A) [3pt] How much mechanical work can be extracted in a process where such solution is further diluted by a factor of 2 (assuming constant temperature and pressure)?

(B) [3pt] Compute the chemical potential of **solvent** molecules for solute concentration $c > 0$.

(C) [5pt] Consider a volume of solution ($c_1 = c$) and a volume of pure solvent ($c_2 = 0$) separated by a heat-conducting membrane impermeable to the solute molecules. What is the condition for their equilibrium? Find the osmotic pressure acting on the membrane $\Delta p = p_1 - p_2$.

(D) [5pt] Now consider equilibrium of salty water and ice (which cannot contain any significant amount of salt). How does the melting temperature change for fixed $c = \text{const}$?

Hint: examine how the derivation of Clausius-Clapeyron equation $\frac{dP}{dT} = \frac{s_1 - s_2}{v_1 - v_2}$ is modified if one of the phases can contain solute.

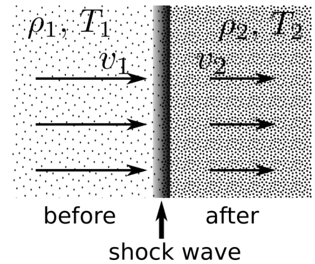
(E) [4pt] Estimate the shift in the ice melting temperature in equilibrium with 1g NaCl/liter salt water.

Some useful physical quantities (at 1 atm) are given on the next page:

Water specific heat	$\approx 4.2\ \mathrm{kJ/(kg \cdot K)}$
Ice specific heat	$\approx 2.1\mathrm{kJ/(kg \cdot K)}$
Ice melting latent heat (1atm)	$L = 334\ \mathrm{kJ/kg}$
Standard atomic weight, Na	$\mu_{Na} \approx 22.99\ \mathrm{g/mol}$
Standard atomic weight, Cl	$\mu_{Cl} \approx 35.45\ \mathrm{g/mol}$

Statistical Mechanics 3

Shock wave in ideal gas



Consider a shock wave steadily propagating through an ideal gas from *right to left*: the gas in front of the shock is initially in the state 1, and behind the shock wave it is in the state 2. The density and temperature change from (ρ_1, T_1) to (ρ_2, T_2) only within the thin wave front. The gas before and after the shock wave is in equilibrium.

It is convenient to consider the reference frame (see figure) in which the shock wave is stationary while the density, temperature, and velocity of the gas change as the gas passes from left (state 1) to right (state 2) through the front. Assume also that the constant-volume heat capacity per molecule $C_V/N = c$ of the gas is independent of density and temperature.

(A) [3pt] What is the entropy of such gas in equilibrium at temperature T and pressure P?

Even though the density, temperature, and velocity change discontinuously at the front, the following quantities are conserved (neglect any vertical motion of the gas as a whole):

$$\rho_1 v_1 = \rho_2 v_2 \,, \tag{1}$$
$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \,, \tag{2}$$
$$\frac{\varepsilon_1 + p_1}{\rho_1} + \frac{1}{2} v_1^2 = \frac{\varepsilon_2 + p_2}{\rho_2} + \frac{1}{2} v_2^2 \,, \tag{3}$$

where $\varepsilon_{1,2} = E_{1,2}/V_{1,2}$ are the *volume* densities of the gas internal energy.

(B) [3pt] Explain the physical origin and derive the conservation laws (1,2,3).

(C) [4pt] Using these conservation laws, find the change of the temperature $y = T_2/T_1$ if the density is increased by factor $x = \rho_2/\rho_1 > 1$ behind the shock wave.

(D) [5pt] How does the entropy of the gas change? Compute the change of entropy per particle $\Delta s = s_2 - s_1$ and provide qualitative explanation for its sign (you may expand in $(x - 1) \ll 1$).

(E) [5pt] Compute the speed of the shock wave $u = v_1$ in the reference frame of the initial gas and compare it to the adiabatic speed of sound u_1. What about v_2 and u_2 (the same behind the shock wave)?