UNIVERSITY OF ILLINOIS AT CHICAGO
DEPARTMENT OF PHYSICS

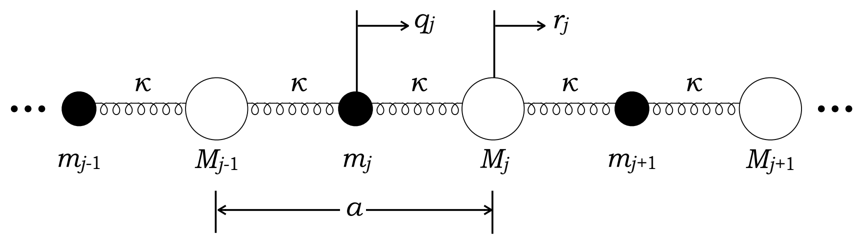
Classical Mechanics
Ph.D. Qualifying Examination

9 January, 2015
9:00 to 12:00

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exams total score.

Problem 1

Consider an infinitely long, linear harmonic chain with particles of two different masses, m and M, and force constant κ, as shown below. Let a be the equilibrium distance between two neighboring particles of the same mass, and let q_j and r_j be the deviations from their equilibrium positions for the j-th particle of mass m, and the j-th paticle of mass M, respectively.



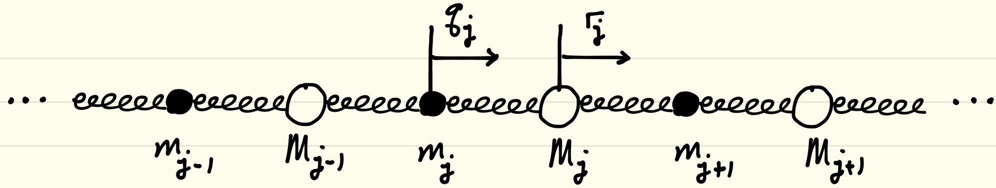
(a) Find the kinetic and potential energies for the system and write down the Lagrangian for the system. Determine the equations of motion for q_j and r_j.

(b) Apply the usual strategy of assuming solutions of the form $q_j = Q_j \exp[i\omega t]$ and $r_j = R_j \exp[i\omega t]$. What are the equations for the amplitudes Q_j and R_j, which result from the equations of motion?

(c) Now, let $Q(k) = \sum_{j=-\infty}^{j=+\infty} Q_j \exp[i\ (jka)]$ and $R(k) = \sum_{j=-\infty}^{j=+\infty} R_j \exp[i\ (jka)]$, where $i = \sqrt{-1}$ and the sum on j is over over all particles of mass m for $Q(k)$ or over all particles of mass M for $R(k)$.

Using the above definitions for $Q(k)$ and $R(k)$, which are identified as the normal modes of the system, perform the sum over all of the amplitudes in part (b) above and obtain the equations for $Q(k)$ and $R(k)$.

(d) Find the normal mode frequencies (i.e. the dispersion relation) $\omega(k)$ for the system. (Note that $k \ \epsilon \ [-\pi/(2a), +\pi/(2a)]$.)

(a) 

The kinetic energy of the system is:

$$\boxed{T = \frac{1}{2}\sum_j \left[m\dot{q}_j + M\dot{r}_j \right]}$$

The potential energy of the system is:

$$\boxed{V = \frac{1}{2}K\left[\cdots + (r_{j-1} - q_j)^2 + (q_j - r_j)^2 + (r_j - q_{j+1})^2 + \cdots \right]}$$

The Lagrangian is then:

$$\boxed{\mathcal{L} = T - V \quad \text{where } T \text{ and } V \text{ are given above}}$$

The EOM for the j-th lattice points are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = m\ddot{q}_j = \frac{\partial \mathcal{L}}{\partial q_j} = -\left[K(r_{j-1} - q_j)(-1) + K(q_j - r_j) \right]$$

$$\Rightarrow \boxed{\ddot{q}_j = \frac{K}{m}\left[r_{j-1} - 2q_j + r_j \right]}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_j} = M\ddot{r}_j = \frac{\partial \mathcal{L}}{\partial r_j} = -\left[K(q_j - r_j)(-1) + K(r_j - q_{j+1}) \right]$$

$$\Rightarrow \boxed{\ddot{r}_j = \frac{K}{M}\left[q_j - 2r_j + q_{j+1} \right]}$$

(b) Let $q_j = Q_j \exp[i(\omega t)]$ with $Q(k) = \sum_{j=-\infty}^{+\infty} Q_j \exp[ijka]$

and $r_j = R_j \exp[i(\omega t)]$ with $R(k) = \sum_{j=-\infty}^{+\infty} R_j \exp[ijka]$

Then $\ddot{q}_j = -\omega^2 Q_j \exp[i(\omega t)]$ and $\ddot{r}_j = -\omega^2 R_j \exp[i(\omega t)]$

so

$$\boxed{\begin{aligned} &\frac{K}{m} R_{j-1} + \left(\omega^2 - \frac{2K}{m}\right) Q_j + \frac{K}{m} R_j = 0 \\ &\frac{K}{M} Q_j + \left(\omega^2 - \frac{2K}{M}\right) R_j + \frac{K}{M} Q_{j+1} = 0 \end{aligned}}$$

(c) Multiply everything by $\exp[i(jka)]$, then

$$0 = \frac{K}{m} \exp[ika] R_{j-1} \exp[i(j-1)ka] + \left(\omega^2 - \frac{2K}{m}\right) Q_j \exp[i(jka)] + \\ + \frac{K}{m} R_j \exp[i(jka)]$$

$$0 = \frac{K}{M} Q_j \exp[i(jka)] + \left(\omega^2 - \frac{2K}{M}\right) R_j \exp[i(jka)] + \\ + \frac{K}{M} \exp[-ika] Q_{j+1} \exp[i(j+1)ka]$$

Applying the def. for $Q(k)$, $R(k)$ enables us to sum over j:

$$\frac{K}{m} \exp[ika] R(k) + \left(\omega^2 - \frac{2K}{m}\right) Q(k) + \frac{K}{m} R(k) = 0$$

$$\frac{K}{M} Q(k) + \left(\omega^2 - \frac{2K}{M}\right) R(k) + \frac{K}{M} \exp[-ika] Q(k) = 0$$

or

$$\boxed{\begin{aligned} &\left(1 + \exp[ika]\right) \frac{K}{m} R(k) + \left(\omega^2 - \frac{2K}{m}\right) Q(k) = 0 \\ &\left(\omega^2 - \frac{2K}{M}\right) R(k) + \left(1 + \exp[-ika] \frac{K}{M}\right) Q(k) = 0 \end{aligned}}$$

(d) For non-trivial Solutions $Q(k)$ and $R(k)$, it is sufficient to require that:

$$\det \begin{vmatrix} \left(1+\exp[ika]\right)\frac{K}{m} & \left(\omega^2-\frac{2K}{m}\right) \\ \left(\omega^2-\frac{2K}{M}\right) & \left(1+\exp[-ika]\right)\frac{K}{M} \end{vmatrix} = 0$$

Hence,

$$(1+\exp[ika]\frac{K}{m}(1+\exp[-ika]\frac{K}{M} - \left(\omega^2-\frac{2K}{m}\right)\left(\omega^2-\frac{2K}{M}\right) = 0$$

or,

$$\frac{K^2}{mM}\left[1+\exp[ika]+\exp[-ika]\right] - \left[\omega^4 - \omega^2\left(\frac{2K}{M}+\frac{2K}{m}\right) + \frac{4K^2}{mM}\right] = 0$$

Let $\mu = \frac{mM}{m+M}$, then

$$\omega^4 - 2\frac{K}{\mu}\omega^2 + \frac{K^2}{mM}\left\{4 - \left[2+2\cos(ka)\right]\right\} = 0$$

$$\omega^4 - 2\frac{K}{\mu}\omega^2 + \frac{2K^2}{mM}\left\{1-\cos(ka)\right\} = 0 = \omega^4 - 2\frac{K}{\mu}\omega^2 + \frac{2K^2}{mM}\sin^2\left(\frac{ka}{2}\right)$$

Hence,

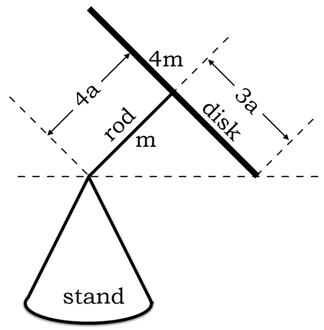
$$\omega^2(k) = \frac{\frac{2K}{\mu} \pm \sqrt{4\frac{K^2}{\mu^2} - 4\frac{K^2}{mM}\left[4\sin^2\left(\frac{ka}{2}\right)\right]}}{2}$$

or,

$$\boxed{\omega^2(k) = \frac{K}{\mu}\left[1 \pm \sqrt{1 - \frac{4\mu^2}{mM}\sin^2\left(\frac{ka}{2}\right)}\right]}$$

Problem 2

A "symmetric top" consists of a thin, uniform, circular disk of mass $4m$ and radius $3a$. A thin, rigid rod, of length $4a$ and mass m is rigidly attached to the center of the disk as shown below. The rod is perpendicular to the disk. The "symmetric top" sits at the apex of a stand as shown below. Choose a coordinate system for the body, which has the $\hat{x}_3$-axis pointing along the direction of the rod.



The Euler angles are defined in the following way: ϕ represents a rotation about the body $\hat{x}_3$-axis, θ represents a rotation about the newly rotated $\hat{x}_1$-axis ($\hat{x}'_1$), and ψ represents a rotation about the newly rotated $\hat{x}_3$-axis ($\hat{x}'_e$). The following relationships for the body's angular velocities ω_1, ω_2, ω_3 then hold in terms of the Euler angles ϕ, θ, ψ:

$$\begin{aligned}\omega_1 &= \dot{\phi}\sin\theta\cos\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}\end{aligned}$$

(a) Obtain the position of the center of mass, and the moments of inertia along the body axes, I_1, I_2, I_3.

(b) The top can rotate freely (i.e. without friction) about the pivot point at the apex of the stand and is subject to a constant gravitational acceleration g. Obtain the Langrangian $\mathfrak{L}$ in terms of the Euler angles, ϕ, θ, and ψ.

(c) Obtain Lagrange's equations of motion for the Euler angles. Identify any conserved quantities.

(d) Determine the minimum spin (rotational velocity) of the disk about the rod, such that the top can precess in a steady motion with the lowest point of the rim of the disk at the same level as the apex of the stand (as shown above).

(a) Choose coords s.t. $|3\rangle$ points along the rod and $|1\rangle, |2\rangle$ are perpendicular. Then,

$$I_3 = \frac{1}{2}(4m)(3a)^2 = \boxed{18ma^2} \quad \text{(uniform disk)}$$

$$I_1 = I_2 = \underbrace{\frac{1}{3}m(4a)^2}_{\text{uniform rod}} + \underbrace{\frac{1}{4}(4m)(3a)^2}_{\text{uniform disk about diameter}} + \underbrace{4m(4a)^2}_{\text{disk about pivot point}}$$

$$= ma^2\left(\frac{16}{3} + 9 + 64\right) = \boxed{\frac{235}{3}ma^2}$$

The CM position along direction of the rod, $|3\rangle$, is $\ell = \frac{(m)(2a) + (4m)(4a)}{m + 4m} = \boxed{\frac{18}{5}a}$

(b) The angular velocities, in terms of the Euler angles are

$$\omega_1 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi$$

$$\omega_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Hence,

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$$

$$= \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

$$V = (5m)g\ell\cos\theta = 18mga\cos\theta$$

So,

$$\boxed{\mathcal{L} = \frac{235}{6}ma^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + 9ma^2(\dot{\phi}\cos\theta + \dot{\psi})^2 - 18mga\cos\theta}$$

(c) EOM: $\frac{\partial \mathcal{L}}{\partial \theta} = I_1 \sin\theta \cos\theta - I_3 \phi (\dot\phi \cos\theta + \psi) \sin\theta + 5mgl \sin\theta$

$$= \sin\theta \left[I_1 \dot\phi^2 \cos\theta - I_3 \dot\phi (\dot\phi \cos\theta + \dot\psi) + 5mgl \right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{d}{dt} I_1 \dot\theta = I_1 \ddot\theta$$

Hence,

★ $$I_1 \ddot\theta = \sin\theta \left[I_1 \dot\phi^2 \cos\theta - I_3 \dot\phi \overbrace{(\dot\phi \cos\theta + \dot\psi)}^{\omega_3} + 5mgl \right]$$

Or, inserting I_1, I_3, and l:

$$\boxed{\frac{235}{3} ma^2 \ddot\theta = \sin\theta \left[\frac{235}{3} ma^2 \dot\phi^2 \cos\theta + 18 ma^2 \dot\phi (\dot\phi \cos\theta + \dot\psi) + 18 mga \right]}$$

Also, $\frac{\partial \mathcal{L}}{\partial \phi} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi}$, so

★★ $$\frac{\partial \mathcal{L}}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + I_3 \overbrace{(\dot\phi \cos\theta + \dot\psi)}^{\omega_3} \cos\theta = \text{Const.}$$

$$= \boxed{\frac{235}{3} ma^2 \dot\phi \sin^2\theta + 18 ma^2 (\dot\phi \cos\theta + \dot\psi) \cos\theta = \text{Const.}}$$

Finally, $\frac{\partial \mathcal{L}}{\partial \psi} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\psi}$, so

★★★ $$\frac{\partial \mathcal{L}}{\partial \dot\psi} = I_3 (\dot\phi \cos\theta + \dot\psi) = I_3 \omega_3 = \text{Const.} \Rightarrow \omega_3 = \text{const}$$

$$\boxed{18 ma^2 (\dot\phi \cos\theta + \dot\psi) = 18 ma^2 \omega_3 = \text{Const.}}$$

(b) Lowest point of rim = apex of stand: $\cos\theta = \frac{3}{5}$

Steady precession motion: $\dot{\theta} = 0$

$$\dot{\phi} \equiv \Omega = \text{Const}$$

$$\ddot{\phi} = \ddot{\theta} = \ddot{\psi} = 0$$

Combining ★★★ with ★: $I_1\ddot{\theta} = \sin\theta\left(I_1\dot{\phi}^2\cos\theta - I_3\dot{\phi}\omega_3 + 5mgl\right)$

Combining ★★★ with ★★: $\frac{d}{dt}\left[I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta\right] = 0$

Now $\ddot{\theta} = 0 \Rightarrow I_1\dot{\phi}^2\cos\theta - I_3\omega_3 + 5mgl = 0$

Or, $\Omega^2 - \Omega\frac{I_3\omega_3}{I_1\cos\theta} + \frac{5mgl}{I_1\cos\theta} = 0$

Inserting I_1, I_2, I_3, l, and $\cos\theta$, we get

$$\Omega^2 - \frac{18ma^2\omega^3}{\frac{235}{3}ma^2\left(\frac{3}{5}\right)}\Omega + \frac{5mg\left(\frac{18}{5}a\right)}{\frac{235}{3}ma^2\left(\frac{3}{5}\right)} = 0$$

Or, $\Omega^2 - \frac{18}{47}\omega_3\Omega + \frac{18}{47}\frac{g}{a} = 0$

$$\Rightarrow \Omega = \frac{9\omega_3}{47} \pm \sqrt{\left(\frac{9}{47}\right)^2\omega_3^2 - \frac{18}{47}\frac{g}{a}}$$

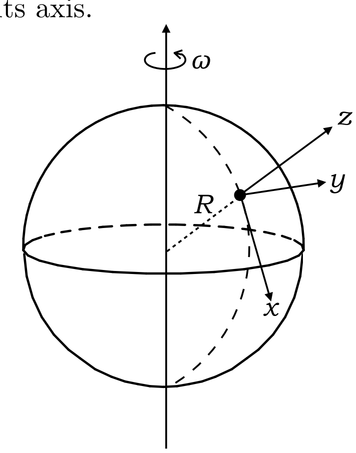
The physical solutions must be real, so

$$\omega_3^2 \geq \left(\frac{47}{9}\right)^2\frac{18}{47}\frac{g}{a} = \frac{47}{9}2\frac{g}{a} = \frac{94}{9}\frac{g}{a}$$

Thus, $\boxed{\omega_3 \geq \sqrt{\frac{94}{9}\frac{g}{a}}}$

Problem 3

A point particle is constrained to move on the surface of the Earth without friction. The origin of a local coordinate system is oriented such that the positive x-axis points south, the positive y-axis points east, and the positive z-axis points perpendicular to the Earth's surface, as shown below. Let the radius of the Earth be R and ω be the angular velocity of the Earth's rotation about its axis.



(a) Assume the particle moves with a velocity of $\vec{v} = v_x\hat{x} + v_y\hat{y}$. What is the horizontal component of the Coriolis acceleration? Write down the resulting set of coupled, nonlinear, second-order differential equations of motion for the positions x and y.

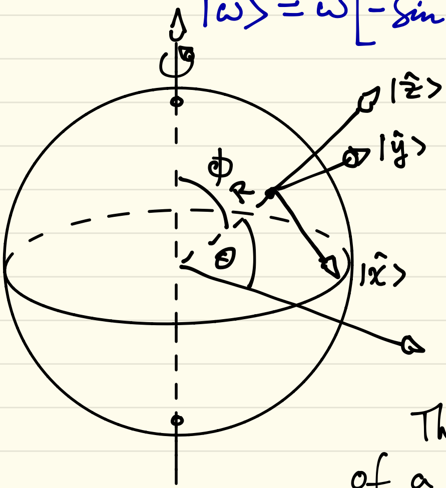
(b) Now let the origin of the coordinate system be located on the Earth's equator. Assume the following:

- the radius of the Earth is very large compared with x, the distance of the particle from the equator: $x \ll R$
- the velocity of the particle perpendicular to the equator is much less than the velocity of the particle along the equator: $|v_x| \ll |v_y|$

If, at time $t = 0$, the particle is located at the origin with initial velocity $\vec{v}(0) = v_x\hat{x} + v_y\hat{y}$, determine the positions $x(t)$ and $y(t)$ on the surface of the Earth as a function of time.

(c) Draw a rough sketch of the trajectory from part (b).

$$|\omega\rangle = \omega\left[-\sin\phi\,|\hat{x}\rangle + \cos\phi\,|\hat{z}\rangle\right]$$



$$|v_{rot}\rangle = v_x|\hat{x}\rangle + v_y|\hat{y}\rangle$$

ⓐ The velocity and acceleration of a particle in a fixed frame, relative to a rotating frame is:

$$|v_{fix}\rangle = |v_{rot}\rangle + |\omega\rangle \times |r_{rot}\rangle$$

$$|a_{fix}\rangle = |a_{rot}\rangle + |\omega\rangle \times |v_{rot}\rangle + |\omega\rangle \times |v_{rot}\rangle + |\omega\rangle \times \left(|\omega\rangle \times |r_{rot}\rangle\right)$$

Hence the Coriolis acceleration is

$$|a_c\rangle = 2|\omega\rangle \times |v_{rot}\rangle = 2\omega\left(-\sin\phi|\hat{x}\rangle + \cos\phi|\hat{z}\rangle\right) \times \left(v_x|\hat{x}\rangle + v_y|\hat{y}\rangle\right)$$

$$= 2\omega\left(-v_x\sin\phi|\hat{z}\rangle + v_x\cos\phi|\hat{y}\rangle - v_y\cos\phi|\hat{x}\rangle\right)$$

The (x,y) horizontal components are:

$$|a_{cH}\rangle = 2\omega\cos\phi\left(v_x|\hat{y}\rangle - v_y|\hat{x}\rangle\right) = -2\omega\cos\phi\left(v_y|\hat{x}\rangle - v_x|\hat{y}\rangle\right)$$

$$\therefore \boxed{= -2\omega\sin\theta\left(v_y|\hat{x}\rangle - v_x|\hat{y}\rangle\right)}$$

The ODE's for x and y are

$$\boxed{\begin{aligned} \ddot{x} &= -2\omega \sin\theta \, \dot{y} \\ \ddot{y} &= 2\omega \sin\theta \, \dot{x} \end{aligned}}$$

(b) Assume $|x| \ll R$, then $\sin\theta \approx \theta = \frac{x}{R}$ and hence,

$$\ddot{x} = -2\omega \frac{x}{R} \dot{y}$$

Now further assume that $|v_x| \ll |v_y|$, then

$$\ddot{y} = 2\omega \frac{x}{R} \dot{x} \approx 0$$

$$\Rightarrow \dot{y} = v_y \Rightarrow \ddot{x} = -2\omega \frac{x}{R} v_y \Rightarrow \ddot{x} + 2 v_y \frac{\omega}{R} x = 0$$

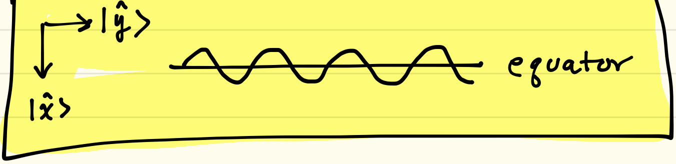
$$\Rightarrow x = A \sin(\Omega t) + B \cos(\Omega t) \quad \text{with } \Omega = \sqrt{\frac{2 v_y \omega}{R}}$$

$$y = C + v_y t \qquad x(0) = y(0) = 0 \Rightarrow B = C = 0$$

$$\dot{x}(0) = v_x \Rightarrow A = \frac{v_x}{\Omega}$$

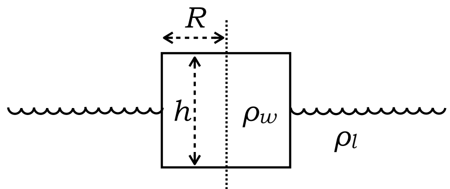
$$\Rightarrow \boxed{x(t) = \frac{v_x}{\Omega} \sin(\Omega t) \; ; \quad y(t) = v_y t}$$

(c) the motion is oscillatory and stable about the equator



Problem 4

A cylindrical block of wood of mass density ρ_w, radius R, and height h is partially immersed in a liquid of mass density ρ_l and then released, as shown in the figure.



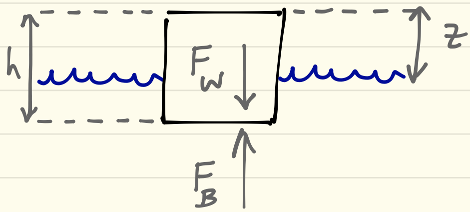
(a) What is the equilibrium height of the block above the water level z_{eq}?

(b) If the block was initially slightly raised, so that $z_0 \equiv z(t = 0) > z_{\text{eq}}$, and then released, calculate $z(t)$ assuming no viscosity.

(c) Now assume that the liquid is viscous, and that the viscous force is proportional to the velocity, as given by $F_v = -bv$. How is the motion of the block modified? Write down the equation of motion.

(d) What is the condition on the viscous parameter b for the motion to be critically damped?

ⓐ Free body diagram:



Area of Block

$$A = \pi R^2$$

The volume of the block:

$$V = Ah$$

The displaced volume is:

$$\Delta V = A(h-z)$$

The mass of displaced liquid is: $m_\ell = \rho_\ell \Delta V$

Hence, the buoyant force on the block of wood is:

$$F_B = m_\ell g = \rho_\ell \Delta V g = \rho_\ell (h-z) A g$$

The buoyant force must equal the weight of the block so:

$$F_B = m_\ell g = F_w = m_w g \Rightarrow m_w g = \rho_\ell (h - z_{eq}) A g$$

Now $m_w = \rho_w A h$ Therefore: $\rho_w A h g = \rho_\ell (h - z_{eq}) A g$

Hence:

$$h - z_{eq} = \frac{\rho_w}{\rho_\ell} h \Rightarrow \boxed{z_{eq} = h\left(1 - \frac{\rho_w}{\rho_\ell}\right)}$$

(b) When the block is raised to $z > z_{eq}$, there will be a restoring force on the block of:

$$F_B = \rho_\ell (h - [z_{eq} + z]) A g = \rho_\ell (h - z_{eq}) A g - \underbrace{\rho_\ell z A g}_{F_R}$$

The restoring force is then

$$m_\ell \ddot{z} = F_R = -(\rho_\ell A g) z \equiv -k z$$

Or $\quad m_\ell \ddot{z} + k z = 0 \quad \Rightarrow \quad \ddot{z} + \frac{k}{m_\ell} z = 0$

This will be SHO motion with

$$\omega_0^2 = \frac{k}{m} = \frac{\rho_\ell A g}{\rho_w A h} = \frac{\rho_\ell}{\rho_w} \frac{g}{h}$$

Hence:

$$z(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \underbrace{+ z_{eq}}_{\text{Motion takes place about equilibrium}}$$

$$z(0) = z_0 \quad \Rightarrow \quad A = z_0 - z_{eq}$$

$$\dot{z}(0) = 0 \quad \Rightarrow \quad B = 0$$

$$z(t) = A\cos(\omega_0 t) + z_{eq} = \boxed{(z_0 - z_{eq})\cos\left(\sqrt{\frac{\rho_\ell}{\rho_w}\frac{g}{h}}\, t\right) + z_{eq}}$$

ⓒ In the presence of a viscous damping force, the EOM are:

$$m_\ell \ddot{z} = -b\dot{z} - kz \qquad \text{or} \qquad \boxed{\ddot{z} + \frac{b}{m_\ell}\dot{z} + \omega_0^2 z = 0}$$

$(k = \rho_\ell A g)$

ⓓ Hence $\beta = \frac{b}{2m_\ell}$ which

for $\beta^2 < \omega_0^2$ will have oscillatory solutions of the form

$$z(t) = \exp[-\beta t]\{A\cos(\omega t) + B\sin(\omega t)\}\ ; \quad \omega^2 = \omega_0^2 - \beta^2$$

$$\Rightarrow \frac{b}{2\pi\rho_w R^2 h} < \sqrt{\frac{\rho_\ell}{\rho_w}\frac{g}{h}} \quad \Rightarrow \quad b < 2\pi R^2\sqrt{\rho_\ell \rho_w g h}$$

for $\beta^2 = \omega_0^2$ will have critically damped solutions

$$z(t) = (A + Bt)\exp[-\beta t]$$

$$\Rightarrow \frac{b}{2\pi\rho_w R^2 h} = \sqrt{\frac{\rho_\ell}{\rho_w}\frac{g}{h}} \quad \Rightarrow \quad \boxed{b = 2\pi R^2\sqrt{\rho_\ell \rho_w g h}}$$

for $\beta^2 > \omega_0^2$ will have overdamped solutions

$$z(t) = \exp[-\beta t]\{A\exp[-\omega_2 t] + B\exp[+\omega_2 t]\}\ ; \quad \omega_2^2 = \beta^2 - \omega_0^2$$

$$\Rightarrow \frac{b}{2\pi\rho_w R^2 h} > \sqrt{\frac{\rho_\ell}{\rho_w}\frac{g}{h}} \quad \Rightarrow \quad b > 2\pi R^2\sqrt{\rho_\ell \rho_w g h}$$

Problem 5

A particle of mass m moves in an attractive central potential $V(r) = -\frac{1}{\beta}\frac{k}{r^\beta}$, where k and β are constants. Assume that the angular momentum L of the particle is not zero.

(a) Write down the Lagrangian. Show that the angular momentum L of the particle is conserved.

(b) Determine the total energy of the system in terms of m, r, $\dot{r}$, k, β, and L. What is the kinetic energy term, which is only a function of the radial velocity of the particle? What is the effective potential energy term $V_{\text{eff}}(r)$, which is only a function of radial position of the particle?

(c) Sketch the effective potential $V_{\text{eff}}(r)$ above as a function of r for the following three cases

- $\beta < 0$
- $2 > \beta > 0$
- $\beta > 2$

For what values of β does a stable circular orbit exist? For what values of β are all orbits bounded?

(d) For those values of β which support a stable circular orbit, calculate the radius, r_0, of the stable circular orbit in terms of m, k, β, and L.

(e) Let $r = r_0 + \delta r$. Derive the equation of motion for radial deviations, $\delta r(t)$, assuming δr is small. Under what conditions will the perturbed orbit be closed?

ⓐ $$\boxed{\mathcal{L} = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 + \frac{1}{\beta}\frac{k}{r^\beta}}$$

The EOM are:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 = \frac{\partial \mathcal{L}}{\partial \theta} = mr^2\ddot{\theta} \Rightarrow mr^2\dot{\theta} = L = \text{const}$$

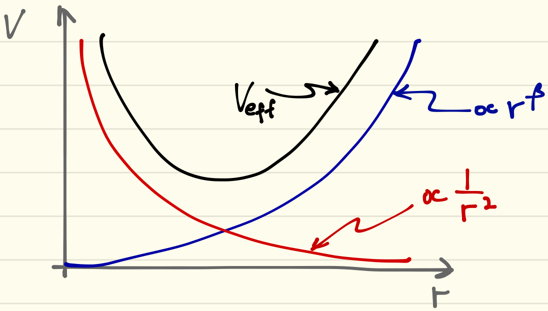
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - \frac{k}{r^{\beta+1}} = \frac{L^2}{mr^3} - \frac{k}{r^{\beta+1}}$$

ⓑ The total Energy is:

$$\boxed{E = \underbrace{\frac{1}{2}m\dot{r}^2}_{K(\dot{r})} + \underbrace{\frac{L^2}{2mr^2} - \frac{1}{\beta}\frac{k}{r^\beta}}_{V_{eff}(r)}}$$

ⓒ

for $\beta < 0$: $V_{eff}(r) = \frac{1}{\beta}kr^\beta + \frac{L^2}{2mr^2}$

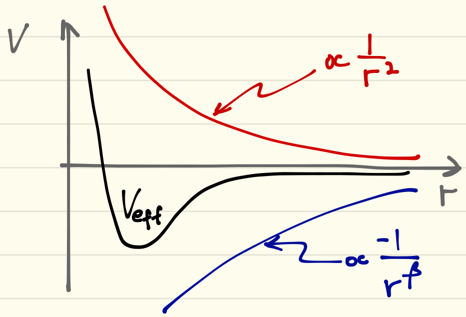


Supports stable circular orbit

All orbits are bounded

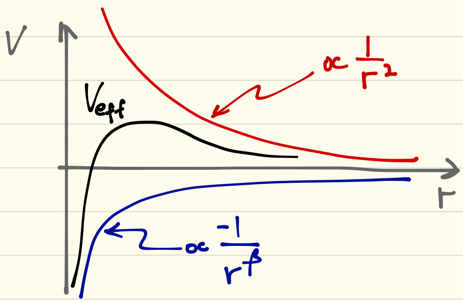
(c) continued

for $0 < \beta < 2$ $\quad V_{eff}(r) = -\frac{1}{\beta}\frac{k}{r^{\beta}} + \frac{L^2}{2mr^2}$



$\boxed{\text{Supports stable circular orbit}}$

for $\beta > 2$ $\quad V_{eff}(r) = -\frac{1}{\beta}\frac{k}{r^{\beta}} + \frac{L^2}{2mr^2}$



$\boxed{\text{Supports circular orbit, but not stable}}$

(d) Stable circular orbits exist for $\beta < 0$ and $0 < \beta < 2$
The condition for a circular orbit is

$$\frac{d}{dr}V_{eff}(r) = 0 \Rightarrow \frac{k}{r_0^{\beta+1}} - \frac{L^2}{mr_0^3} = 0$$

$$\Rightarrow r_0^{2-\beta} = \frac{L^2}{mk} \Rightarrow \boxed{r_0 = \left(\frac{L^2}{mk}\right)^{\frac{1}{2-\beta}}}$$

(e) Assume $r = r_0 + \delta r$

Now, $m\ddot{r} = F_{eff}(r) = -\frac{\partial}{\partial r} V_{eff}(r) = -\frac{k}{r^{\beta+1}} + \frac{L^2}{2mr^3}$

So, $$m\delta\ddot{r} = -\frac{k}{(r_0+\delta r)^{\beta+1}} + \frac{L^2}{m(r_0+\delta r)^3}$$

$$= -\frac{k}{r_0^{\beta+1}(1+\delta r/r_0)^{\beta+1}} + \frac{L^2}{mr_0^3(1+\delta r/r_0)^3}$$

$$\approx -\frac{k}{r_0^{\beta+1}}\left[1-(\beta+1)\frac{\delta r}{r_0}\right] + \frac{L^2}{mr_0^3}\left[1-3\frac{\delta r}{r_0}\right] \qquad (1+\varepsilon)^n \approx (1+n\varepsilon)$$

$$= \underbrace{\left[-\frac{k}{r_0^{\beta+1}} + \frac{L^2}{mr_0^3}\right]}_{=0 \text{ from part (d)}} - \left[-\frac{k(\beta+1)}{r_0^{\beta+1}}\frac{1}{r_0} + \frac{3L^2}{mr_0^4}\right]\delta r$$

also $r_0^{2-\beta} = \frac{L^2}{mk} \rightarrow k = \frac{L^2}{mr_0^{2-\beta}}$

$$= -\left[-\frac{k(\beta+1)}{r_0^{\beta+2}} + \frac{3L^2}{mr_0^4}\right]\delta r = -\left[-\left(\frac{L^2}{mr_0^{2-\beta}}\right)\left(\frac{\beta+1}{r_0^{\beta+2}}\right) + \frac{3L^2}{mr_0^4}\right]\delta r$$

$$= -\frac{L^2}{mr_0^4}(2-\beta)\,\delta r \Rightarrow \delta\ddot{r} + \underbrace{\frac{L^2}{m^2r_0^4}(2-\beta)}_{\omega_0^2}\delta r = 0$$

So $\omega_0 = \frac{L}{mr_0^2}\sqrt{2-\beta}$ also, we have $\omega_\theta = \dot{\theta} = \frac{L}{mr_0^2}$

The orbit is closed when the frequency of small osill satisify: $n\omega_0 = j\omega_\theta$ n, j integers

Hence, when $\boxed{\sqrt{2-\beta} = \frac{j}{n}}$