General I
Solutions
Fall 2001
16 August 2001

Group I

1. Hamiltonian system (Question and solution from Peter)
   a) n, $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$
   b) $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $H = \sum_{i=1}^{n} p_i \dot{q}_i - L$
   c) By construction, $\frac{dH}{dt} = \frac{\partial H}{\partial t}$
   d) i) Energy
      ii) $L_x \hat{x} + L_y \hat{y}$

2. Energy and potential from wavefunction (Question from Dave, solution from Peter)

Time independent S.E. in 1-D: $-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$,then

$$\psi'' = \frac{d}{dx}\left(-a\frac{\tanh ax}{\cosh ax}\right) = -a\frac{2-\cosh^2 ax}{\cosh^3 ax} = -2a^2\frac{1}{\cosh^2 ax}\psi + a\psi = -\frac{2m}{\hbar^2}(E - V(x))\psi$$

Most logical that x independent part is E: $\begin{aligned} E &= -\frac{\hbar^2 a^2}{2m} \\ V(x) &= -\frac{a^2\hbar^2}{m\cosh^2 ax} \end{aligned}$.

3. Current carrying wire (Question and solution from Peter)

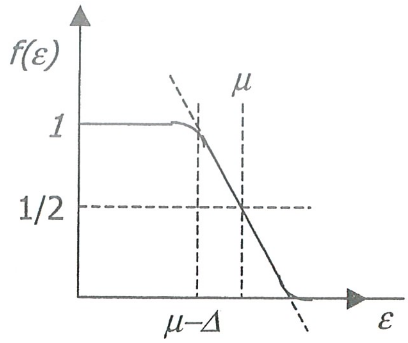
   a) Take loop of radius r around wire and use Ampere's law: $\oint \vec{B}\cdot d\vec{l} = {}^{4\pi}\!/_{c}\, I \Rightarrow \vec{B} = B(r)\hat{\varphi} = \frac{2I}{rc}\hat{\varphi} \Rightarrow \vec{F} = q\frac{\vec{v}}{c}\times\vec{B} = \frac{2Ive}{rc^2}\hat{\rho}$ where $\hat{\rho}$ is radial direction in cylindrical coordinates.
   b) If F' moves along wire with v, then particle is at rest in F' and the Lorentz force is zero. Transform force from F:
   $F_\perp = \frac{dp_\perp}{dt} = \frac{dp'_\perp}{\gamma dt' - \beta\gamma dx'} = \frac{dp'_\perp/dt'}{\gamma + \beta\gamma\, dx'/dt'} = \frac{F'_\perp}{\gamma} \Rightarrow F'_\perp = \gamma\frac{2Ive}{rc^2}$. Can also do problem by transforming currents and charge densities.

4. Fermi energy and chemical potential (Question from Ray, solution from Pete and Ray)

Peters solution:

Energy density of states: $D(x) = a\sqrt{\varepsilon}$ where a is some constant. Then,

$$N = \int_0^\infty f(\varepsilon) D(\varepsilon) d\varepsilon = a \int_0^\infty \frac{\sqrt{\varepsilon}}{e^{\beta(\varepsilon-\mu)}} d\varepsilon,$$ where $\beta = 1/kT$. At $T=0$, all states lie below the Fermi energy, ε_F, which is given by solving $a\int_0^{\varepsilon_F} \sqrt{\varepsilon} d\varepsilon = N = \frac{a}{3}(\varepsilon_F)^{3/2}$. If $T>0$, but lower than the Fermi temperature, then $f(\varepsilon)$ looks like:



Take all states below μ to be filled and linearize the range from $\mu-\Delta$ to $\mu+\Delta$. The slope at $f(\varepsilon)=1/2$ is $-kT/4$ which gives $\Delta=2kT$. Then, to first order in Δ:

$$N = a\left[\frac{1}{3}(\mu-\Delta)^{3/2} + \int_{-\Delta}^{\Delta} \sqrt{x+\mu}\left(-\frac{x}{4kT}+\frac{1}{2}\right)dx\right]$$

$$= \frac{1}{3}a\mu^{3/2}\left(1-\frac{3\Delta}{\mu}\right) + \left(\frac{1}{2}+\frac{x}{4\mu}-\frac{x}{8kT}\right)\Bigg|_{-\Delta}^{\Delta}$$

Equate this to $\frac{1}{3}a(\varepsilon_F)^{3/2} = \frac{a}{3}\mu^{3/2}\left(1+\frac{3}{2}\frac{y}{\mu}\right)$ where $\varepsilon_F = \mu + y$ which shows $y>0$ and the chemical potential goes below the Fermi energy as the temperature rises above zero.

Ray's solution:

The density of states is proportional to $\sqrt{\varepsilon}$, therefore the tail of the Fermi-Dirac distribution above μ adds more than is lost from below μ. Therefore, μ must decrease in order for the integral to remain constant.

Either solution is fine, but if the second course is chosen, the argument must be precisely stated.

Group II

1. Balloon on a train (Question and solution from Peter)

According to the Equivalence Principle, we can replace the constant acceleration by a gravitational acceleration. Then, choosing *x* as the direction of the train's motion and *y* as up, $\vec{F} = -\frac{g}{10}\hat{x} - g\hat{y} \Rightarrow tan\theta = 0.1$. Balloon moves *forward* (in direction of train's motion) from vertical.

2. 1D relativisitic gas (Question and solution from Peter Fisher)

The entropy is given by $S = -\left.\frac{\partial F}{\partial T}\right|_L$ where the free energy is $F = -kT\ln Z$. For a 1D gas, the partition function is $Z = 2\int_o^\infty e^{-E/kT}\frac{L}{hc}dE = \frac{2LkT}{hc}$ where leading factor of two takes into account the particle motion in both the positive and negative directions. Then $F = -kT\ln\left(\frac{2LkT}{hc}\right) \Rightarrow S = k\left(\ln\frac{2kT}{hc}+1\right)$. The result is not consistent with the third law; the ultrarelativistic approximation does not hold as when *kT~m*, where *m* is the mass of the particles.

3. Expectation of *x* and *p* (Question and solution from Dave)

For <*x*>, $\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\left\langle\left[x, -\frac{\hbar^2}{2m}p^2 + U(x)\right]\right\rangle$. *x* and *U(x)* commute and $[x, p^2] = i\hbar p \Rightarrow \frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}$ which is just the classical velocity. *p* commutes with the kinetic part of *H* so

$\frac{d\langle p\rangle}{dt} = \frac{1}{i\hbar}\langle[p, U(x)]\rangle = -\left\langle\frac{\partial U}{\partial x}\right\rangle = \langle F(x)\rangle \neq F(\langle x\rangle)$ where last inequality shows the QM result is not the same as the classical.

4. Two blocks and two pulleys (Question from Dave, solution from Peter)

Most straightforward to use Lagrangian with constraint for fixed length of rope:

$L = \frac{1}{2}\left(m_1\dot{x}^2 + m_2\dot{y}^2\right) + m_1 g x$ where (referring to the diagram with the problem) x

$g(x,y) = 0 = x + 2y + d - l$

increases downward and y increases to the right. The rope has length l and d accounts for all the rope not taken up by x and y. Also from the constraint, we have $\ddot{x} = -2\ddot{y}$. Using the Euler-Lagrange equations with undetermined multiplier gives

$$\begin{array}{c} m_1\ddot{x} - m_1 g + \lambda = 0 \\ m_2\ddot{y} + 2\lambda = 0 \end{array} \Rightarrow \begin{array}{c} -2m_1\ddot{x} + 2m_1 g - 2\lambda = 0 \\ -\frac{m_2\ddot{x}}{2} + 2\lambda = 0 \end{array} \Rightarrow \ddot{x} = \frac{4m_1 g}{4m_1 + m_2}$$

Check limits: $m_1 >> m_2$, acceleration is g, : $m_1 << m_2$, acceleration goes to zero.

Group III

1. Yo-yo (Question from Dave, solution from Peter)
The torque on the yo-yo is $\tau = rF$ which gives an angular acceleration of $\tau = I\dot{\omega} \Rightarrow \alpha = \dot{\omega} = \tau/I$. The net force is $F_{net} = F - mg = ma_y \Rightarrow a_y = F/m - g$ is for no acceleration $a_y = 0 \Rightarrow g = F/m \Rightarrow F = mg$ which gives an angular acceleration of $\omega = rmg/I$ where the moment of inertia is $I = MR^2/2$..

2. HD in equilibrium (Question and hand written solution from Dave)

Treat each diatomic molecule as harmonically bound system. Then the oscillation frequency is $\omega \propto \mu^{-1/2}$ where *m* is the reduced mass of the system. The zero point energy of H_2 is E_o=*6,000k*. Then

$$H_2 + D_2 \leftrightarrow 2HD + \Delta E$$

$$E_o + \sqrt{\frac{1}{2}}E_o = 2E_o\sqrt{\frac{3}{4}} + \Delta E \quad \Rightarrow \Delta E = 0.025E_o$$ which favors the H_2+D_2 state.

$$E_o + 0.707E_o = 1.732E_o + \Delta E$$

Finally, $\frac{[H_2][D_2]}{[2HD]} = e^{\Delta E/kT}$ which gives $N_{HD} = 2\exp(\Delta E/kT)N_{H^2} = 2\exp(-0.5)N_{H^2}$.

3. Incoherent transmission through a dielectric slab

For transmitted power from dielectric 1 to *n*, $t = \frac{4n}{(1+n)^2}$ and $r = \left(\frac{n-1}{n+1}\right)^2$ (Both are easy to work out if you forget.) So, beam hits the slab and fraction t^2 gets through both interfaces and a fraction *rt* is reflected from second interface. This then reflects back from the first interface and contributes r^2t^2 to the outgoing power. Each successive reflection gives another factor of r^2 and

$$t^2 + r^2t^2 + r^4t^2 + \cdots = t^2\sum_{i=0}^{\infty} r^2 = \frac{t^2}{1-r^2} = \frac{2n}{n^2+1}$$ for the transmitted power. The field is then $E_o\sqrt{\frac{2n}{n^2+1}}$.

4. Relativistic electron (Question and solution from Peter)

For a relativistic electron, $p = \beta\gamma m_e c = \frac{\partial L}{\partial \dot{x}} \Rightarrow \alpha = m_e c^2$

HD IN EQUILIBRIUM

CONSIDER THIS AS A TWO-STATE SYSTEM

$$\begin{array}{ll} & \text{——— } 2HD \\ \Delta E \uparrow & \\ & \text{——— } H_2 + D_2 \end{array}$$

GROUND STATE (ZERO POINT VIBRATIONAL) ENERGY OF A DIATOMIC MOLECULE $\epsilon_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{K}{\mu}}$.

K's ARE THE SAME, μs ARE DIFFERENT

$$\mu_{H_2} = \frac{m_p m_p}{m_p + m_p} = \frac{1}{2} m_p \qquad \epsilon_{0,H_2} \text{ GIVEN AS } 6000\,k$$

$$\mu_{D_2} = \frac{1}{2} m_D = 2\mu_{H_2} \quad \Rightarrow \epsilon_{0,D_2} = \frac{1}{\sqrt{2}}\, 6000\,k$$

$$\mu_{HD} = \frac{2 m_p^2}{3 m_p} = \frac{2}{3} m_p = \frac{4}{3}\mu_{H_2} \Rightarrow \epsilon_{0,HD} = \frac{\sqrt{3}}{2}\, 6000\,k$$

$$2HD: \quad E = 2 \cdot \frac{\sqrt{3}}{2} 6000 = \sqrt{3}\, 6000 = (1.732) 6000$$

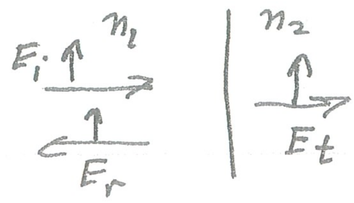
$$H_2 + D_2: \quad E = 6000 + \frac{1}{\sqrt{2}} 6000 = \underline{(1.707)\, 6000}$$

$$\Delta E = (0.025)\, 6000 = 150\,k$$

$$\frac{N_{HD}}{N_{H_2}} = 2\, e^{-\Delta E/kT} = \underline{\underline{2\, e^{-0.5}}}$$

INCOHERENT TRANSMISSION

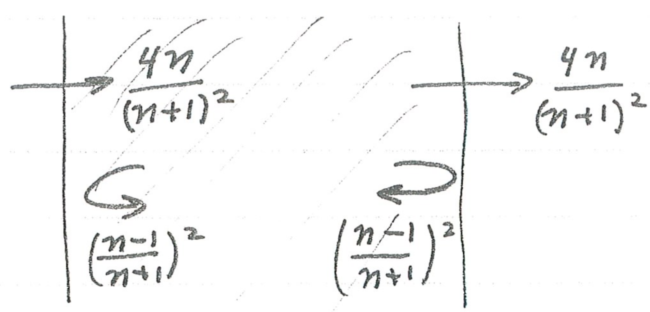
E



$$E_r = \frac{n_1 - n_2}{n_1 + n_2} E_i$$

$$E_t = \frac{2 n_1}{n_1 + n_2} E_i$$

POWER $\quad E \cdot B = n E^2$



LET $\zeta = \left(\frac{n-1}{n+1}\right)^4$

$$P_{OUT}/P_{IN} = \left(\frac{4n}{(n+1)^2}\right)^2 \left\{ 1 + \zeta + \zeta^2 + \zeta^3 \cdots \right.$$

$$= \frac{16 n^2}{(n+1)^4} \frac{1}{1-\zeta} = \frac{16 n^2}{(n+1)^4 - (n-1)^4}$$

$$(n+1)^4 - (n-1)^4 = \left[(n+1)^2 + (n-1)^2\right]\left[(n+1)^2 - (n-1)^2\right] = \left[2n^2 + 2\right]\left[4n\right]$$

$$P_{OUT}/P_{IN} = \frac{2n}{n^2+1} \qquad E_{OUT}/E_{IN} = \sqrt{\frac{2n}{(n^2+1)}}$$

Group IV

1. Relativistic collision (Question and solution from Peter)

In lab frame, proton (mass m_p) has energy and momentum *(E,p)*, electron *(m*

In lab frame, proton (mass m_p) has energy and momentum (E,p), electron $(m_e,0)$. Then velocity of CM frame is $\beta^* = p/_{E+m_e} \Rightarrow \gamma^* = \frac{E+m_e}{\sqrt{2m_e(E+m_e)+m_p^2}}$.

In CM frame, electron energy-momentum before collision is $(\gamma^* m, -\beta^*\gamma^* m)$, after $(\gamma^* m, \beta^*\gamma^* m)$ for zero degree scattering, which is where maximum energy transfer occurs. Then, electron energy in lab frame after the collision is

$$E_{e,after} = \gamma^* m + (\gamma^*\beta^*)^2 m = T_{e,after} + m \Rightarrow T_{e,after} = \frac{2T(T+2m_p)}{2m_e(T+m_p+m_e)+m_p^2} m_e .$$

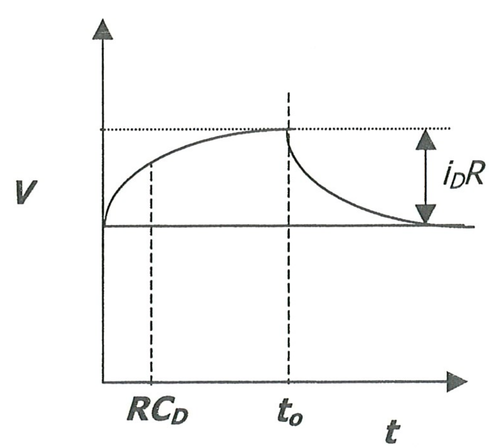
2. Detector bias (Question and solution from Peter)

At $t=0$ $V_D(0)=V$. Then, $V_D(t)=\frac{Q}{C_D} \Rightarrow \dot{V}_D(t)=\frac{\dot{Q}}{C_D}=\frac{i_D(t)-i_R(t)}{C_D}$ where i_D is as defined in the problem and i_R is the current flowing *into* the voltage supply through R and Q is the charge on the upper plate of the capacitor. Also, $I_R=(V_D-V)/R$. Then

$$\dot{V}_D = \frac{dV_D}{dt} = \frac{Ri_d - V_D + V}{RC_D} \Rightarrow \int_{V_D(0)}^{V_D(t)} \frac{dV_D}{Ri_d - V_D + V} = \int_0^t \frac{dt'}{RC_D} \Rightarrow -\ln\frac{Ri_d - V_D(t)+V}{Ri_d - V_D(0)+V} = \frac{t}{RC_D}$$

Then,

$1-\frac{V_D(t)}{Ri_D}+\frac{V}{Ri_D} = e^{-t/RC_D} \Rightarrow V_D(t) = Ri_D\left(1-e^{-t/RC_D}\right)+V$. After several time constants, V_D reaches a maximum of Ri_D+V. When the current shuts off, the capacitor discharges with the same time constant.



3. Bohr-Sommerfeld quantization (Question from Dave, solution from Dave)

Bohr-Sommerfeld quantization says $\oint p dx = nh$ for a periodic system. For $U=mgz$, $p = \sqrt{2mE_n}\left(1 - \frac{z}{z_n}\right)^{1/2}$ where $z_n = \frac{E_n}{mg}$ is the max. height above the surface. Then $2\int_0^{z_n} p(z)dz = 2\int_0^{z_n}\sqrt{2m^2 g z_n\left(1 - z/z_n\right)}dz = nh$. Take $u=z/z_n$ and

$2\sqrt{2m^2 g z_n^3}\int_0^1 (1-u)^{1/2} du = nh \Rightarrow E_n = \left(\frac{1}{8}\frac{n^2h^2}{I^2}mg^2\right)^{1/3}$ where $I = \int_0^1 (1-u)^{1/2} du = \frac{2}{3}$.

The integral is actually quite easy: Taylor expand the integrand, integrate the infinite series and the resum the series.

4. A non-ideal gas (Question from Peter)

First law: $dE = TdS - PdV = \left.\frac{\partial E}{\partial T}\right|_V dT + \left.\frac{\partial E}{\partial V}\right|_T dV$ so $\left.\frac{\partial S}{\partial T}\right|_V = \left.\frac{1}{T}\frac{\partial E}{\partial T}\right|_V = 2bV^{2/3}$

Maxwell relation from Free energy: $\left.\frac{\partial S}{\partial V}\right|_T = \left.\frac{\partial P}{\partial T}\right|_V = \frac{4}{3}bV^{-1/3}T$. $dS(V,t)$ is a perfect differential, so integrate first expression w.r.t T: $S = 2bV^{2/3}T + f(V)$ then differentiate to show $f'(V)=0$ which gives $S = 2bV^{2/3}T + S_o$.

Group V

1. Charged sheets (Question and solution from Peter)

Choose x to lie along sheet 2 (positive to the right) and y to lie along sheet 1 (positive to left). Then in region I the fields are

$$\vec{E}_1 = 2\pi\sigma_o\hat{x}$$

$$\vec{E}_2 = 2\pi\sigma_2(-\cos\theta\hat{x} + \sin\theta\hat{y})$$ σ_3 and σ_2 are unknown. For there to be no field in

$$\vec{E}_3 = -2\pi\sigma_3\hat{y}$$

region I, x and y components must cancel. This gives a system of two linear equations:

$$\begin{aligned} 2\pi(\sigma_2\cos\theta) &= 2\pi\sigma_o \\ 2\pi(\sigma_2\sin\theta - \sigma_3) &= 0 \end{aligned} \Rightarrow \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & -1 \end{pmatrix}\begin{pmatrix} \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} \sigma_o \\ 0 \end{pmatrix}$$

The determinant of the matrix is just -cosθ. Solving gives

$$\begin{aligned} \sigma_2 &= \sigma_o/\cos\theta \\ \sigma_3 &= \sigma_o\tan\theta \end{aligned}$$

From the uniqueness theorem, the solution is unique.

2. Phased array radar

a) Take continuous dipole along z. From segment dz in direction θ, the phase shift is $\phi(z) = \frac{2\pi z}{\lambda}\sin\theta$ and the field strength is proportional to

$$\int_{-Nl/2}^{Nl/2}\cos(\phi(z))dz = -\frac{\lambda}{2\pi\sin\theta}\sin\phi(z)\Big|_{-Nl/2}^{Nl/2} = -\frac{\lambda}{\pi\sin\theta}\sin\left(\frac{\pi Nl}{\lambda}\sin\theta\right).$$ Taking θ small gives $P = \frac{\lambda^2}{\pi^2\theta^2}\sin^2\left(\frac{\pi Nl\theta}{\lambda}\right)$. Beam width is then $\frac{\lambda}{Nl}$.

b) Next max. in power corresponds to minimum in field:

$$\theta\frac{\pi Nl}{\lambda}\cos\left(\frac{\pi Nl\theta}{\lambda}\right) - \sin\left(\frac{\pi Nl\theta}{\lambda}\right) = 0 \Rightarrow \frac{\pi Nl\theta}{\lambda} = \tan\left(\frac{\pi Nl\theta}{\lambda}\right)$$

c) Next min. is $\theta = \frac{2\lambda}{Nl}$, so estimate that the first max. is $\theta = \frac{3\lambda}{2Nl} \Rightarrow$ ratio is

$$\left(\frac{\frac{2Nl}{3\pi}}{Nl}\right)^2 = \left(\frac{2}{3\pi}\right)^2$$

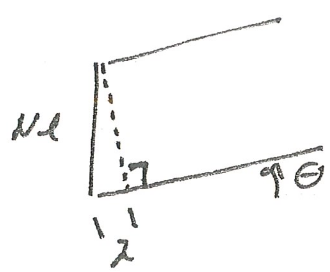
PHASED ARRAY RADAR

USE A VECTOR REPRESENTATION OF THE COMPLEX PHASE. WHEN ALL DIPOLES RADIATE IN PHASE AND ALL PATH LENGTHS ARE EQUAL ($\theta = 0$) ALL CONTRIBUTIONS ADD IN PHASE.

$$\xrightarrow{\quad E_1 \quad} \qquad I_1 \equiv E_1^{\,2}$$

a) A ZERO OCCURS AS A FUNCTION OF θ WHEN THE PHASE DIFFERENCE ACROSS THE ARRAY IS A MULTIPLE OF 2π.

FIRST ZERO:

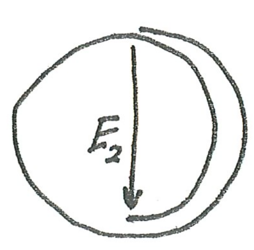


$$\theta\Big|_{\text{FIRST ZERO}} \approx \frac{\lambda}{N\ell} \Rightarrow \text{BEAM WIDTH} = \frac{2\lambda}{NL} \cdot \frac{1}{2} = \underline{\underline{\frac{\lambda}{NL}}}$$

b) $\theta_{\text{SECOND DIFF. MAX}}$ IS ABOUT HALF WAY BETWEEN THE 1^{ST} & 2^{ND} ZERO

$$\Rightarrow \theta_2 \approx \underline{\underline{\frac{3}{2}\frac{\lambda}{N\lambda}}}$$

c)



TOTAL LENGTH IS STILL E_1

$$\frac{3}{2}(\pi E_2) = E_1 \quad \Rightarrow E_2 = \frac{2}{3\pi} E_1$$

$$\underline{\underline{I_2/I_1 = \left(\frac{2}{3\pi}\right)^2}} \approx \frac{1}{22}$$

3. Ground state energy

$\langle\psi|H|\psi\rangle$ for any normalized ψ gives an upper bound to the ground state energy.

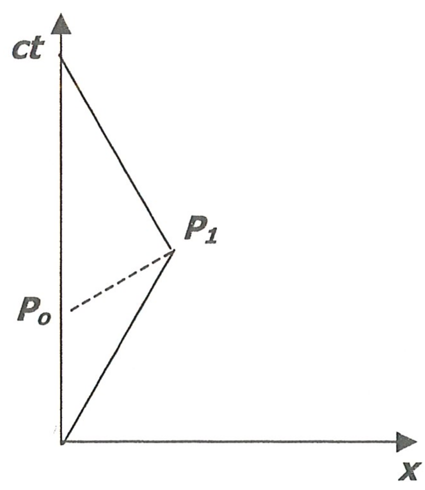
Then $\psi' = \left(\frac{1}{2\pi\sigma^2}\right)^{-1/4}\left(-\frac{x}{2\sigma^2}\right)e^{-x^2/4\sigma^2}$ and $\psi'' = \left(\frac{1}{2\pi\sigma^2}\right)^{-1/4}\left[\left(-\frac{x}{2\sigma^2}\right)^2 - \frac{1}{2\sigma^2}\right]e^{-x^2/4\sigma^2}$

which gives $\left\langle\frac{p^2}{2m}\right\rangle = \frac{1}{8}\frac{\hbar^2}{m\sigma^2}$. For the potential, $\langle V\rangle = a\langle x^4\rangle = 3a\sigma^4$. Then

$\frac{d\langle H\rangle}{d\sigma} = -\frac{1}{4}\frac{\hbar^2}{m\sigma^3} + 12a\sigma^3 = 0 \Rightarrow \sigma = \left(\frac{1}{48}\frac{\hbar^2}{am}\right)^{1/6}$. Putting back in for H gives

$\langle H\rangle = \frac{3}{16}\frac{1}{\sigma^2}\frac{\hbar^2}{m} = \left(\frac{3}{4}\right)^{4/3} a^{1/3}\left(\frac{\hbar^2}{m}\right)^{2/3} > E_o$.

4. Relativistic astronauts (Question from Peter, solution from Peter)



a) $\beta=3/5$ gives $\gamma=5/4$ and $t'_o=\gamma t_o=5/4$ *y*
b) *A* follows $x=vt$ and the light signal follows $x=c(t-t_o)$. Equating and solving gives $x_1=3/2ly$
c) *A* will have aged 4 y, *E* will have aged 5 y.

GROUND STATE ENERGY

$\langle\psi|\mathcal{H}|\psi\rangle$ FOR ANY NORMALIZED ψ GIVES AN UPPER BOUND TO THE GROUND STATE ENERGY.

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + a x^4 \qquad \psi = (2\pi\sigma^2)^{-1/4} e^{-\frac{x^2}{4\sigma^2}}$$

$$\psi' = (\ )^{-1/4}\left(-\frac{x}{2\sigma^2}\right) e^{-x^2/4\sigma^2}$$

$$\psi'' = (\ )^{-1/4}\left(\frac{-x}{2\sigma^2}\right)^2 e^{-\frac{x^2}{4\sigma^2}} - (\ )^{-1/4}\frac{1}{2\sigma^2} e^{-x^2/4\sigma^2}$$

$$\left\langle \frac{p^2}{2m}\right\rangle = -\frac{\hbar^2}{2m}\Big[\underbrace{\left(\frac{1}{2\sigma^2}\right)^2 \langle x^2\rangle_{\sigma^2} - \frac{1}{2\sigma^2}}_{-\frac{1}{4}\frac{1}{\sigma^2}}\Big] = \frac{1}{8}\frac{\hbar^2}{m\sigma^2}$$

$$\langle V(x)\rangle = a\langle x^4\rangle = 3a\sigma^4, \quad \langle\mathcal{H}\rangle = \frac{1}{8m}\frac{\hbar^2}{\sigma^2} + 3a\sigma^4$$

$$\frac{d\langle\mathcal{H}\rangle}{d\sigma} = -\frac{1}{4}\frac{\hbar^2}{m\sigma^3} + 12a\sigma^3 = 0 \quad\Rightarrow\quad \sigma = \left(\frac{1}{48}\frac{\hbar^2}{am}\right)^{1/6}$$

$$\langle\mathcal{H}\rangle = \frac{1}{\sigma^2}\Big(\frac{1}{8}\frac{\hbar^2}{m} + \underbrace{3a\sigma^6}_{\frac{1}{16}\frac{\hbar^2}{m}}\Big) = \frac{3}{16}\frac{1}{\sigma^2}\frac{\hbar^2}{m}$$

$$= \left(\frac{3}{4}\right)^{4/3} a^{1/3}\left(\frac{\hbar^2}{m}\right)^{2/3} > E_0$$

Tom Greytak