Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, August 21, 2013

Name:____
(please print)
Identification Number:________

<u>STUDENT</u>: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
<u>PROCTOR</u>: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

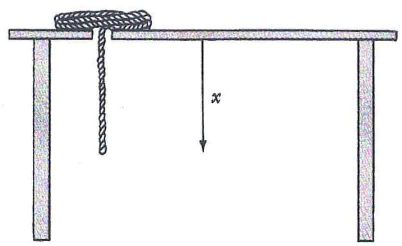
Student's initials
problems handed in:
Proctor's initials

<u>INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS</u>

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **<u>identification number</u>** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **<u>problem number</u>** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**

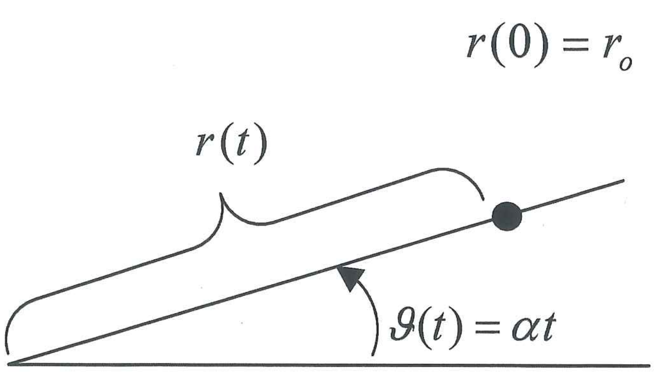
[I-1] **[10]**

A smooth rope is placed above a hole in a table as illustrated in the figure below. One end of the rope falls through the hole at time $t = 0$ s, pulling steadily the remainder of the rope. Find the velocity v and acceleration a of the rope as a function of the distance x to the end of the rope. Ignore all friction. The total length and mass of the rope are L and M, respectively.



[I-2] **[10]**

A particle of mass m *initially rests* on a smooth horizontal plane. The plane is then raised to an inclination angle ϑ at a constant rate, $\vartheta(t) = \alpha t$, causing the particle to slide down the plane. Determine the full motion of the particle, i.e., explicitly solve for $r(t)$. Initially, $r(0) = r_o$.



[I-3] **[10]**

Two harmonic oscillators are placed such that the two masses $m_1 = m_2 = m$ slide against one another, as shown in the figure below. The motion of the two masses is horizontal along the x-direction. The oscillators are identical, having the same spring constant $\kappa_1 = \kappa_2 = \kappa$.

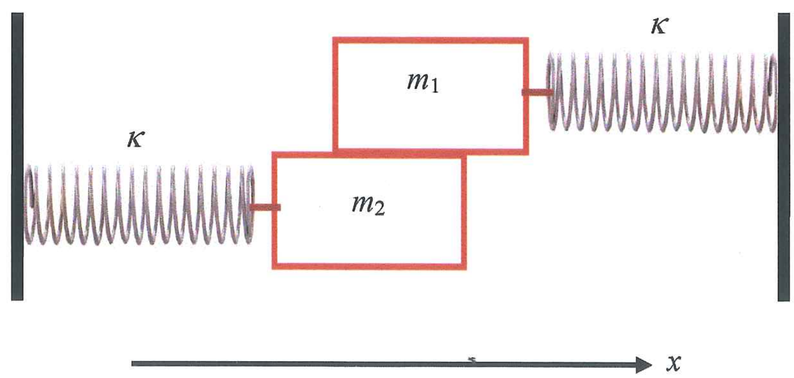
(a) The friction force provides a coupling of the motions proportional to the instantaneous relative velocity. Write down the friction force in terms of the friction constant b.

(b) Find the equation of motion for each mass using Newtonian mechanics.

(c) Solve for all the possible solutions, $x_1(t)$ and $x_2(t)$, depending on the values of κ, m, and b.

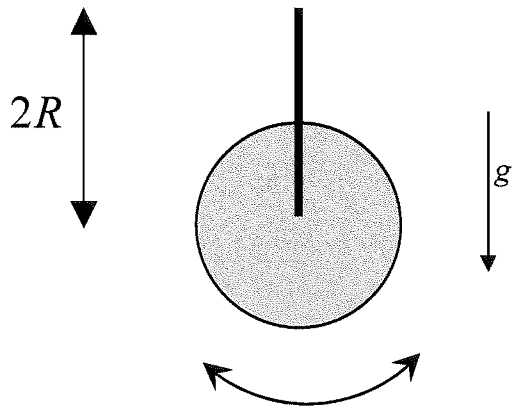
(d) Discuss the physical meaning of the motion of the two masses.

(Note that the natural frequency of the system is $\omega_o^2 = \kappa/m$ and the damping factor is $\beta^2 = b/m$.)



[I-4] **[10]**

A physical pendulum consists of a thin solid disc of mass M and radius R with its center attached to a thin rod of length $2R$ and *negligible mass*. The pendulum can rotate freely in a vertical plane about an axis going through the other end of the rod. The gravitational acceleration is g.



What is the period of this pendulum (for small oscillations)?
(You must express your answer using the parameters and constants given above.)

[I-5] **[10]**

An astronaut takes a round trip voyage to a distant star. The total trip takes 30 years according to the astronaut's watch, 15 years traveling directly away from earth at a constant relative velocity of $0.99c$, and after a very short period of rapid acceleration towards earth, a return trip which takes another 15 years traveling with the relative velocity of $-0.99c$.

(a) How many years does the total trip take as measured from observers on earth?

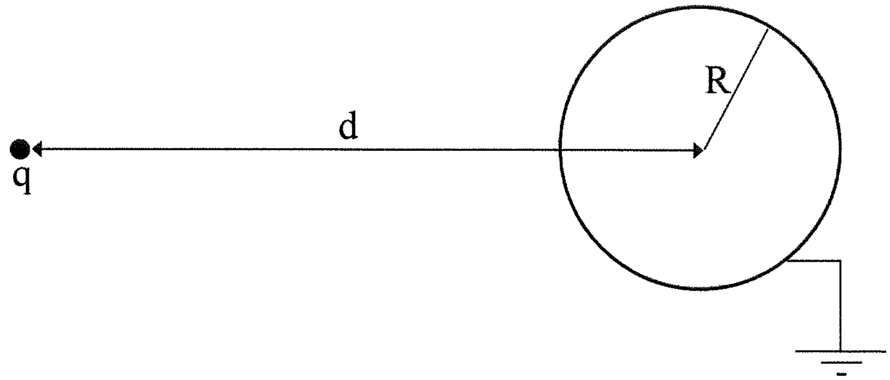
(b) Onboard the spaceship, the astronaut has access to a large telescope. With the telescope, the astronaut can read a very large earth based clock.

(i) The instant *before* the astronaut hits the button to accelerate for the return trip, he/she notes the total time passed by reading the clock through the telescope. How much time has passed according to the telescope clock?

(ii) The instant *after* the acceleration for the return trip ends, he/she again notes the total time passed according to the telescope clock. What is it?

[I-6] [10]

A point charge q is placed at a distance d from the center of a perfectly conducting grounded sphere of radius R.



(a) What are the boundary conditions on the ***E***-field and potential at the surface of the sphere?

(b) Determine the potential *outside* of the sphere.

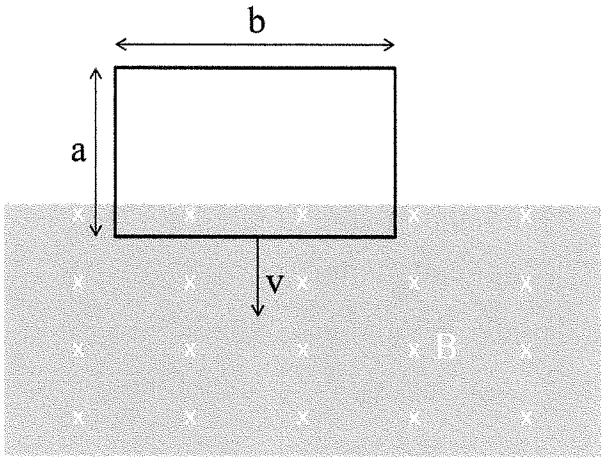
(c) If the sphere were isolated (not grounded) and carried no net charge, what would be the force on the point charge?

[I-7] [10]

A stationary observer detects and observes an incident plane wave with frequency ω striking a moving mirror in vacuum. The constant velocity of the mirror $\mathbf{v}$ is parallel to the wave number of the plane wave $\boldsymbol{k}$. The plane of the mirror is perpendicular to $\boldsymbol{k}$. *Using the boundary conditions that the electromagnetic waves must satisfy at the plane of the mirror*, find the frequency of the reflected wave ω' detected by the stationary observer.

[I-8]   [10]

A rigid rectangular coil of resistance R and dimensions of a and b moves with constant velocity v into a region with a constant magnetic field $\boldsymbol{B}$ (pointing into the page), as shown in the figure below. Derive an expression for the vector force on the coil in terms of the given parameters.



[I-9] [10]

A system of two tiny metal spheres separated by a distance s and connected by a fine wire is shown in **Fig. 1** below. At time t the charge on the upper sphere is $q(t)$, and the charge on the lower sphere is $-q(t)$. Suppose further that we can drive the charge back and forth through the wire, from one end to other, at frequency ω: $q(t)=q_o \cos \omega t$

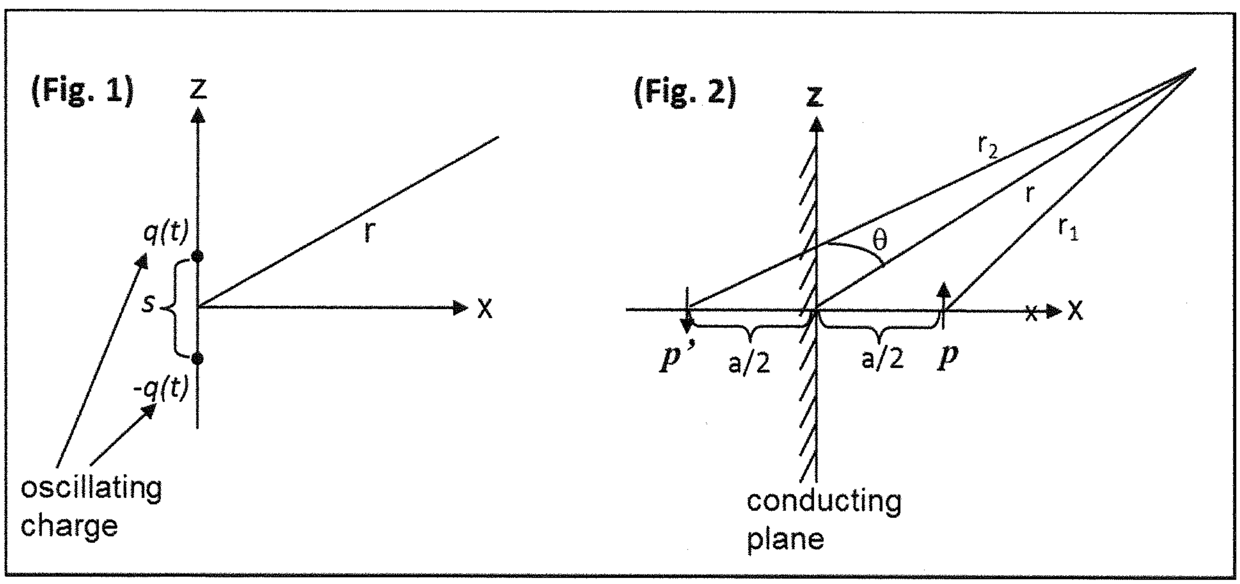
(a) Write down the corresponding, oscillating electric dipole: $\boldsymbol{p}(t)$, *in terms of* the charge q_o and the separating distance s.

(b) In the far field approximation $r >> s$, calculate the vector-potential of this dipole system and express the vector-potential in terms of $\boldsymbol{p}(t)$.

(The next question refers to the system shown in Fig. 2:)

(c) An electric dipole, $\boldsymbol{p}$, oscillates with a frequency ω and amplitude p_o. It is placed at a distant +a/2 from an infinite perfectly conducting plane and the dipole is parallel to the plane as shown in **Fig. 2** below. Find the time-averaged Poynting vector for distance $r >>$ a.

(Hint: there is an imaging dipole, $\boldsymbol{p}'$, formed by the conducting plane.)
(Note: for approximation: $r_1 \approx r - a/2 \sin\theta \cos\varphi$; $r_2 \approx r + a/2 \sin\theta \cos\varphi$; $1/r_1 \approx 1/r_2 \approx 1/r$. You may also neglect the higher order term of 1/r when you calculate the field)



[I-10] [4,3,3]

Consider two Cartesian reference frames S and S'. The reference frames are oriented the same way. The x' axis is parallel to the x axis, y' parallel to y, and z' parallel to z. Reference frame S' is moving relative to S with relativistic velocity V along the x-axis of S. The components of the electric field $\boldsymbol{E}$ and the magnetic flux density $\boldsymbol{B}$ as measured in S are E_x, E_y, E_z and B_x, B_y, B_z, respectively.

(a) Find the Lorentz transformation of the electric field and magnetic flux density. In other words, find all components of the electric field $\boldsymbol{E'}$ and magnetic flux density $\boldsymbol{B'}$ as measured in S' in terms of the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ as measured in S.

(b) Demonstrate that if $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular in S, then they are also perpendicular in S'.

(c) Demonstrate that $E^2 - c^2B^2$ is invariant under Lorentz transformation. Here, c is the speed of light.

Part I

Solution I-1

linear mass density $\sigma = \text{mass/length}$

$$F = \frac{dp}{dt} \text{ becomes}$$

$$m\,g = m\,\dot{v} + \dot{m}\,v$$

where m is the mass of length x of the rope. So

$$m = \sigma x; \dot{m} = \sigma \dot{x}$$

$$\sigma\,x\,g = \sigma\,x\frac{dv}{dt} + \sigma\,\dot{x}\,v$$

$$x\,g = x\frac{dv}{dx}\frac{dx}{dt} + v^2$$

$$x\,g = xv\frac{dv}{dx} + v^2$$

Try a power law solution:

$$v = ax^n; \frac{dv}{dx} = nax^{n-1}$$

Substituting,

$$x\,g = x\left(ax^n\right)\left(nax^{n-1}\right) + a^2\,x^{2n}$$

or

$$x\,g = a^2\,(n+1)\,x^{2n}$$

Since this must be true for all x, the exponent and coefficient of x must be the same on both sides of the equation.

Thus we have: $1 = 2n$ or $n = \frac{1}{2}$

$$g = a^2\,(n+1) \text{ or } a = \sqrt{\frac{2g}{3}}$$

So

$$\boxed{v = \sqrt{\frac{2gx}{3}}}$$

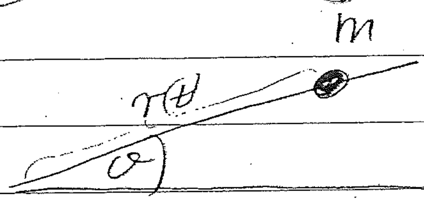
$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \left[\frac{2gx}{3}\right]^{1/2}\frac{g}{3}\left[\frac{2gx}{3}\right]^{-1/2}$$

$$\boxed{a = \frac{g}{3}}$$

[redacted]

Solution

I-2



$$\boxed{r(0) = r_0 \ , \ \dot{r}(0) = 0}$$

$$\theta(t) = \alpha t \qquad (\theta(0) = 0)$$

$$\dot{\theta} = \alpha$$

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta$$

$$= \frac{m}{2}\left(\dot{r}^2 + r^2\alpha^2\right) - mgr\sin(\alpha t)$$

(only a single independent variable, $r(t)$)

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}}$$

$$\boxed{m r \alpha^2 - mg\sin(\alpha t) = m\ddot{r}}$$

$$\ddot{r} = r\alpha^2 - g\sin(\alpha t)$$

2nd order inhomogeneous ordinary diff eq.

I-2 (Con't'd)

homogeneous solutions: $r_H(t) = Ae^{\alpha t} + Be^{-\alpha t}$

particular solution: $r_p = C \sin(\alpha t)$, → **C** to be determined

$$-C\alpha^2 \sin(\alpha t) = C\alpha^2 \sin(\alpha t) - g \sin(\alpha t)$$

$$g = 2C\alpha^2 \Rightarrow \boxed{C = \frac{g}{2\alpha^2}}$$

Thus, the general solution:

$$r(t) = r_H(t) + r_p(t) = \boxed{Ae^{\alpha t} + Be^{-\alpha t} + \frac{g}{2\alpha^2}\sin(\alpha t)}$$

$\boxed{r(0) = r_0 \quad , \dot{r}(0) = 0}$:

$$\begin{cases} A + B = r_0 \\ \alpha A - \alpha B + \dfrac{g}{2\alpha} = 0 \end{cases} \Rightarrow \begin{cases} A + B = r_0 \\ A - B = -\dfrac{g}{2\alpha^2} \end{cases}$$

$$A = \frac{1}{2}\left[r_0 - \frac{g}{2\alpha^2}\right], \quad B = \frac{1}{2}\left[r_0 + \frac{g}{2\alpha^2}\right]$$

$$\boxed{r(t) = r_0 \cosh(\alpha t) - \frac{g}{2\alpha^2}\sinh(\alpha t) + \frac{g}{2\alpha^2}\sin(\alpha t)}$$

I-3 normal mode

(a) friction force:

$$F^{fric} = -b(\dot{\vec{X}}_i - \dot{\vec{X}}_j)$$

(b) Let X_1, X_2 be the respective displacements from their equilibrium positions.

Equation of motion from 2^{nd} law:

$$m\ddot{X}_1 + kX_1 + b(\dot{X}_1 - \dot{X}_2) = 0 \text{ --- } ①$$
$$m\ddot{X}_2 + kX_2 + b(\dot{X}_2 - \dot{X}_1) = 0 \text{ --- } ②$$

① + ②, $m(\ddot{X}_1 + \ddot{X}_2) + k(X_1 + X_2) = 0$

① − ②, $m(\ddot{X}_1 - \ddot{X}_2) + k(X_1 - X_2) + 2b(\dot{X}_1 - \dot{X}_2) = 0$

Redefine coordinates:

Let $\eta_2 \stackrel{def}{=} (X_1 + X_2)$, $\eta_1 \stackrel{def}{=} (X_1 - X_2)$

$$\therefore \begin{cases} m\ddot{\eta}_2 + k\eta_2 = 0 \text{ --- } ③ \\ m\ddot{\eta}_1 + k\eta_1 + 2b\dot{\eta}_1 = 0 \text{ --- } ④ \end{cases}$$

I-3 normal mode

(c) Sol. for η_2 = Simple Harmonic Motion

$$\eta_2 = A\cos(\omega_0 t + \delta) \quad \text{—— ⑤}$$

$$\omega_0^2 \equiv \frac{k}{m}$$

i.e. the resonant freq. #

Sol. for η_1

Let $\eta_1 \overset{\text{def}}{=} e^{\gamma t + \xi}$ —— ⑥

Equ. ④ becomes: $m\gamma^2 + 2b\gamma + k = 0$

$$\therefore \gamma = \frac{-b \pm \sqrt{b^2 - mk}}{m} = -\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{k}{m}}$$

Let $\beta \overset{\text{def}}{=} \frac{b}{m}$ (i.e. the damping factor) ⑦

$$\therefore \gamma = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \quad \text{—— ⑧}$$

If $\beta^2 < \omega_0^2$, $\eta_1(t) = e^{-\beta t}\cos(\sqrt{\beta^2 - \omega_0^2}\, t + \delta)$ ⑨

If $\beta^2 = \omega_0^2$, $\eta_1(t) = e^{-\beta t}$ ⑩

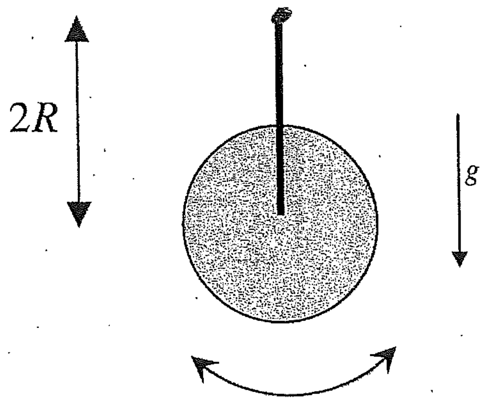
If $\beta^2 > \omega_0^2$, $\eta_1(t) = e^{-\beta t}\, e^{\pm\sqrt{\beta^2 - \omega_0^2}\, t}$ ⑪

#

I-H)   [redacted]

[redacted]

A physical pendulum consists of a thin solid disc of mass M and radius R with its center attached to a thin rod of length $2R$ and *negligible mass*. The pendulum can rotate freely in a vertical plane about an axis going through the other end of the rod. The gravitational acceleration is g. For the following questions you must express all your answers using the parameters above.



- [redacted] Find the position of the center of mass of this pendulum, R_{CM}, measured from the axis of rotation.

$$R_{CM} = 2R$$

- [redacted] Determine the moment of inertia of this pendulum, I, about the axis of rotation.

$$I_{CM}^{disc} = \frac{1}{2} MR^2$$

parallel axis theorem: $I = I_{CM}^{disc} + M(2R)^2 = \frac{1}{2}MR^2 + 4MR^2 =$

$$\boxed{= \frac{9}{2}MR^2}$$

- [redacted] the period of this pendulum for small oscillations?.

$$T = 2\pi\sqrt{\frac{I}{M \cdot R_{CM} \cdot g}} = 2\pi\sqrt{\frac{\frac{9}{2}MR^2}{M \cdot 2R \cdot g}} = 2\pi\sqrt{\frac{9}{4}\frac{R}{g}}$$

$$= 2\pi \frac{3}{2}\sqrt{\frac{R}{g}} = \boxed{3\pi\sqrt{\frac{R}{g}}}$$

I-5 Special Relativity (simple, within the context of **Mechanics**):

An astronaut takes a round trip voyage to a distant star. The total trip takes 30 years according to the astronaut's watch, 15 years traveling directly away from earth at a constant relative velocity of 0.99c, and after a very short period of rapid acceleration towards earth, a return trip which takes another 15 years traveling with the relative velocity of -0.99c.

(a) How many years does the total trip take as measured from observers on earth?

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-.99^2}} = 7.088$$

$$t = t_0\gamma = 30years * 7.088 \approx 212.6\, years$$

(b) Onboard the spaceship, the astronaut has access to a large telescope. With the telescope, the astronaut can read a very large earth based clock.

(i) The instant *before* the astronaut hits the button to accelerate for the return trip, he/she notes the total time passed by reading the clock through the telescope. How much time has passed according to the telescope clock?

On the outward leg of the trip, the astronaut sees that the earth based clock appears to be running slow by the stretch factor γ. The clock therefore reads,

$$\frac{15}{\gamma} = 2.11\, yrs$$

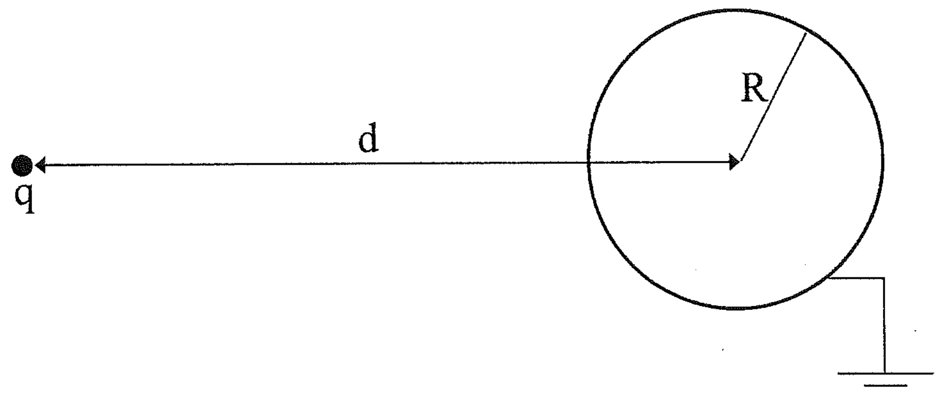
(ii) The instant *after* the acceleration for the return trip ends, he/she again notes the total time passed according to the telescope clock. What is it?

We know that when the rocket arrives back at earth, the astronaut will see 212.6 years through the telescope (the total time the trip takes in the earth frame). We also know that the telescope clock will advance by 2.11 yrs during the return leg of the trip.

$$t - 2.11\, yrs = 210.5\, yrs$$

I-6 Electrostatics or Boundary Value

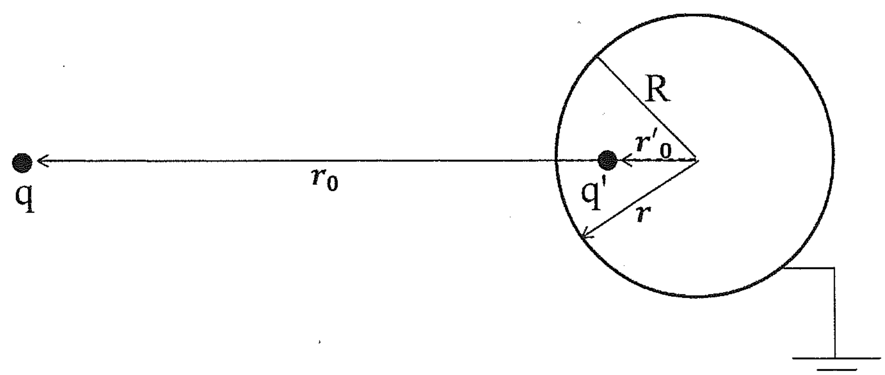
A point charge, q, is placed at a distance d from the center of a perfectly conducting grounded sphere of radius R.



(a) What are the boundary conditions on the E-field and potential at the surface of the sphere?

$$E_T = 0, \qquad E_N = \frac{\sigma}{\epsilon}, \qquad V(R,\theta) = 0$$

(b) Determine the potential outside of the sphere.



The potential of the sphere is 0 (grounded conductor). Using method of images, with an image charge q', located at r_0' yields a potential

$$4\pi\epsilon_0\phi(\boldsymbol{r}) = \left[\frac{q}{|\boldsymbol{r}-\boldsymbol{r}_0|} + \frac{q'}{|\boldsymbol{r}-\boldsymbol{r}_0'|}\right]$$

with,

$$|r - r_0| = \sqrt{r^2 + r_0^2 - 2r \cdot r_0} = \sqrt{R^2 + d^2 - 2Rd\ cos\theta}$$

$$|r - r_0'| = \sqrt{r^2 + r_0'^2 - 2r \cdot r_0'} = \sqrt{R^2 + a^2 - 2Ra\ cos\theta},$$

where $a = |r_0'|$ and $r_0 = d\hat{z}$. For the surface of the sphere,

$$4\pi\epsilon_0\phi(R,\theta) = \left[\frac{q}{\sqrt{R^2 + d^2 - 2Rd\ cos\theta}} + \frac{q'}{\sqrt{R^2 + a^2 - 2Ra\ cos\theta}}\right] = 0$$

$$(R^2 + a^2 - 2Ra\ cos\theta) = (q'/q)^2(R^2 + d^2 - 2Rd\ cos\theta)$$

examining the cosine terms leads to, $-2Ra\cos\theta = -2\left(\frac{q'}{q}\right)^2 Rd\cos\theta,\qquad \frac{q'}{q} = \sqrt{a/d}$

the other terms yield, $(R^2 + a^2) = \frac{a}{d}(R^2 + d^2), \quad \rightarrow \quad a = R^2/d$ (image charge position)

$q' = q\sqrt{a/d} = -qR/d$ (magnitude of image charge)

$$\phi(\boldsymbol{r}) = q/4\pi\epsilon_0\left[\frac{1}{|\boldsymbol{r} - d\hat{\mathbf{z}}|} - \frac{qR/d}{\left|\boldsymbol{r} - \frac{R^2}{d}\hat{\mathbf{z}}\right|}\right]$$

(c) If the sphere were isolated (not grounded) and carried no net charge, what would be the force on the point charge?

Since the sphere is not charged, the image charge needs to be compensated by a secondary image charge, q'' located at the origin of the sphere, where $q'' = -q'$. q'' maintains the boundary conditions (additive constant to the potential over the surface) but kills the monopole moment, long range potential due to the combined images becomes a dipole.

$$F = \frac{q'}{4\pi\epsilon_0}\left(-\frac{1}{|d|^2} + \frac{1}{\left|d - \frac{R^2}{d}\right|^2}\right) = \frac{qR}{4\pi\epsilon_0 d}\left(\frac{1}{|d|^2} - \frac{1}{\left|d - \frac{R^2}{d}\right|^2}\right)\hat{z}$$

Solution

I-7

in the frame where the mirror moves with velocity v
incident electromagnetic plane wave $\parallel \hat{x}$

e.g. $\bar{E} = \bar{E}_0 \cos(\omega t - kx)$ incident wave

$\bar{E}' = \bar{E}_0' \cos(\omega' t + k' x)$ reflected wave

at $x = vt$ the following boundary conditions must be satisfied:

$\bar{E} + \bar{E}' = \bar{0} \quad \forall t$

i.e., $\bar{E}_0 \cos(\omega t - kx)\big|_{x=vt} = -\bar{E}_0' \cos(\omega' t + k' x)\big|_{x=vt} \quad \forall t$

$\bar{E}_0 \cos(\omega t - kvt) = -\bar{E}_0' \cos(\omega' t + k' vt)$

$\omega = kc$ and $\omega' = k' c$

$\Rightarrow \bar{E}_0' = -\bar{E}_0$ and $\omega t - kvt = \omega' t + k' vt \quad \forall t$

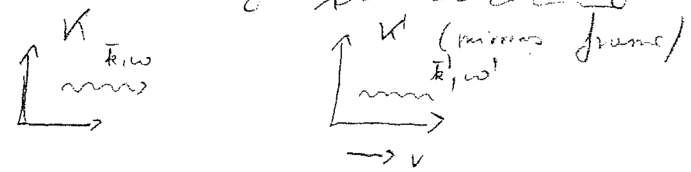
$$\omega\left(1 - \frac{v}{c}\right) = \omega'\left(1 + \frac{v}{c}\right)$$

$$\boxed{\omega' = \omega \frac{1 - v/c}{1 + v/c}}$$

I-7 (cont'd)

Alternative Solution: to $\boxed{I-7}$

Solution using special relativity (relativistic Doppler Effect)



light: $|\bar{k}| = k_0 = \frac{\omega}{c}$

Lorentz tr. for k_0:

I. $k_0' = \gamma\,(k_0 - \bar{\beta}\cdot\bar{k})$ here $\bar{\beta} = \frac{\bar{v}}{c} \uparrow\uparrow \bar{k}$

$$k_0' = \gamma\left(k_0 - \frac{v}{c}\cdot\frac{\omega}{c}\right)$$

$$\omega' = \gamma\left(\omega - \frac{v}{c}\,\omega\right) = \gamma\left(1-\frac{v}{c}\right)\omega = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v}{c}\right)\omega = \sqrt{\frac{1-v/c}{1+v/c}}\;\omega$$

II. after reflection: $\bar{k}' \to -\bar{k}'$ and $\bar{v} \to -\bar{v}$ in

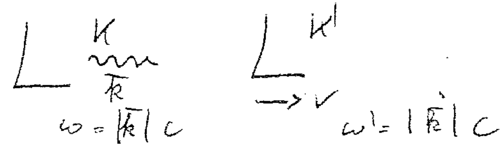
$\omega'_{refl} = \omega'$ inverse transformation (i.e. $\bar{\beta} \uparrow\uparrow \bar{k}'$)

$$\Rightarrow \quad \omega_{refl} = \gamma\left(\omega'_{refl} - \frac{v}{c}\,\omega'_{refl}\right) = \sqrt{\frac{1-v/c}{1+v/c}}\;\omega'_{refl} =$$

$$= \sqrt{\frac{1-v/c}{1+v/c}}\cdot\omega' = \sqrt{\frac{1-v/c}{1+v/c}}\sqrt{\frac{1-v/c}{1+v/c}}\;\omega =$$

$$= \frac{1-v/c}{1+v/c}\,\omega = \frac{c-v}{c+v}\,\omega$$

I-7 cont'd



$$\omega = kc \qquad |\bar{k}| = k_0 = \frac{\omega}{c}$$

$$\boxed{\frac{\omega'}{c} = \gamma\Big(\underbrace{\frac{\omega}{c}}_{k_0} - \frac{\bar{V}}{c}\cdot\bar{k}\Big)}$$

$k_0' = \gamma(k_0 - \bar{\beta}\cdot\bar{k})$
$k_0 \equiv \frac{\omega}{c}$ & $|\bar{k}| = k_0$
both frames

if $\bar{V} \uparrow\uparrow \bar{k}$: $\bar{V}\cdot\bar{k} = V|\bar{k}| = V\cdot\frac{\omega}{c}$

I $\quad \omega' = \gamma\left(\omega - \frac{V}{c}\omega\right)$

$$\omega' = \gamma\left(1 - \frac{V}{c}\right)\omega = \frac{1}{\sqrt{1 - V^2/c^2}}\left(1 - \frac{V}{c}\right)\omega$$

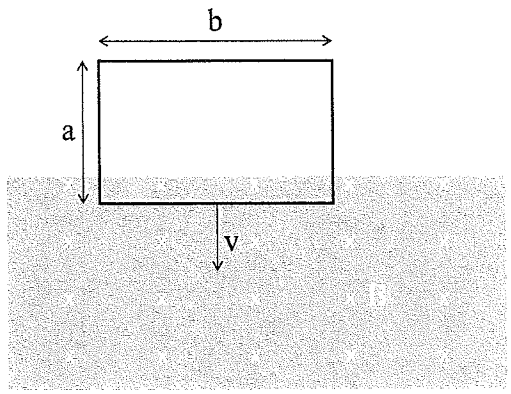
$$= \sqrt{\frac{1 - V/c}{1 + V/c}}\;\omega$$

II after reflection transform back : $V \to -V$
and $\bar{k}' \to -\bar{k}'$, i.e
$\bar{V}$ and $\bar{k}'$ parallel again.

$$\omega_{\text{reflected}} = \sqrt{\frac{1 - V/c}{1 + V/c}}\;\omega' = \frac{1 - V/c}{1 + V/c}\;\omega$$

I-8 Simple Electromagnetism (Biot-Savart, Faraday/Lenz's Law)

A rigid rectangular coil of resistance R and dimensions of a and b moves with constant velocity v into a region with a constant magnetic field B (pointing into the page), as shown in the Figure. Derive an expression for the vector force on the coil in terms of the given parameters.



As it starts to cut across the magnetic field lines, an emf is induced in the coil of magnitude

$$\epsilon = -\int \boldsymbol{B} \times \boldsymbol{v} \cdot d\boldsymbol{l} = -Bvb,$$

producing a current of

$$I = \frac{\epsilon}{R} = -\frac{Bvb}{R}.$$

The minus sign indicates that the current flows counterclockwise. The force on the coil is,

$$F = \left| \int Id\boldsymbol{l} \times \boldsymbol{B} \right| = IbB = -\frac{vb^2B^2}{R}.$$

The direction of the force opposes v.

I-9 EM (dipole)

(a) Osci. electrical Dipole:

$$\vec{p} \equiv q\vec{s},$$

$$\vec{p} = q_0 S \cos(\omega t)\, \hat{e}_z \;\#$$

(b) Vector potential:

Current element: $\vec{I} \equiv \frac{dq}{dt}\hat{z},$

$$\vec{I} = -q_0 \omega \sin\omega t\, \hat{e}_z$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-S/2}^{S/2} \frac{-q_0\omega \sin\omega(t-r/c)}{r}\, dz$$

To 1st order, we replace the integral by its value at the center.

$$\vec{A}(r,t) \cong \frac{\mu_0}{4\pi}\left(\frac{P_0\omega}{r}\right)\sin\omega(t-r/c)\,\hat{e}_z$$

$$\therefore \vec{A}(r,t) \simeq \left(\frac{\dot{\vec{p}}(t)}{r}\right)\frac{\mu_0}{4\pi} \;\#$$

(c) Addition of Vector potential

$$\vec{A} = \left(\frac{\dot{\vec{p}}}{r_1} + \frac{\dot{\vec{p}}'}{r_2}\right)\frac{\mu_0}{4\pi}.$$

$$\vec{A} = -i\frac{\mu_0}{4\pi}(p_0\omega)\left(\frac{e^{ikr_1}}{r_1} - \frac{e^{ikr_2}}{r_2}\right)\hat{e}_z$$

Zinheen

Here, $\vec{p}\,' = -\vec{p} = -p_0\, e^{-i\omega t}\, \hat{e}_z$

Use the following appoximation:

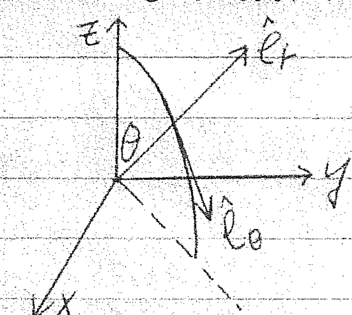
$$\begin{cases} r_1 \cong r - \frac{a}{2}\sin\theta\cos\phi \\ r_2 \cong r + a/2\,\sin\theta\cos\phi \\ \frac{1}{r_1} \cong \frac{1}{r_2} \cong \frac{1}{r} \end{cases}$$

$$\vec{A} \cong \left(\frac{p_0 \omega}{r}\right) i\,\frac{\mu_0}{4\pi}\left(e^{i\frac{ka}{2}\sin\theta\cos\phi} - e^{-i\frac{ka}{2}\sin\theta\cos\phi}\right) e^{i(kr-\omega t)}\,\hat{z}$$

$$\therefore \vec{A} \cong -\frac{\mu_0}{2\pi}\left(\frac{\omega p_0}{r}\right) e^{i(kr-\omega t)} \sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)\hat{z}$$

#

Now, in spherical coordinate $\hat{e}_z = \hat{e}_r\cos\theta - \hat{e}_\theta \sin\theta$



By $\vec{B} = \nabla \times \vec{A}$

$$\therefore \vec{B} \simeq \frac{\hat{e}_\phi}{r}\frac{\partial}{\partial r}(r A_\theta) = \hat{e}_\phi\, \frac{i\omega^2 p_0\, e^{i(kr-\omega t)}}{2\pi \varepsilon_0 c^3 r} \sin\theta \sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)$$

#

By $\vec{E} = c\vec{B} \times \hat{e}_r$

$$\therefore \vec{E} \simeq \hat{e}_\theta \left(\frac{i\omega^2 P_0 e^{i(kr-\omega t)}}{2\pi \varepsilon_0 c^2 r} \right) \sin\theta \, \sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)$$

\#

By $\vec{S} = \frac{\varepsilon_0 c}{2} |E|^2 \hat{e}_r$

$$\therefore \vec{S} \simeq \hat{e}_r \frac{\omega^4 P_0^2}{8\pi^2 \varepsilon_0 c^3} \left(\frac{\sin^2\theta}{r^2} \right) \sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)$$

\#

I-10

(a) generally,

$$F' = \lambda F \tilde{\lambda} \quad \text{with}$$

$$F' = \begin{pmatrix} 0 & B_z & -B_y & -E_x/c \\ -B_z & 0 & B_x & -E_y/c \\ B_y & -B_x & 0 & -E_z/c \\ E_x/c & E_y/c & E_z/c & 0 \end{pmatrix}$$

electromagnetic field tensor

and

$$\lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & & & \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

Lorentz transformation for the configuration of ~~as~~ S and S' as described in the problem statement (standard configuration with standard boost).

<u>Lorentz transformation:</u>

$[\beta = \frac{v}{c}, \ \gamma = (1-\beta^2)^{-1}]$

$$E_x' = E_x$$

$$B_x' = B_x$$

$$E_y' = \gamma(E_y - \beta c B_z)$$

$$B_y' = \gamma(B_y + \beta E_z/c)$$

$$E_z' = \gamma(E_z + \beta c B_y)$$

$$B_z' = \gamma(B_z - \beta E_y/c)$$

$\boxed{I-10}$ (Cont'd)

(b) $\vec{E} \perp$ to $\vec{B} \Rightarrow \vec{E}\cdot\vec{B} = 0$

If $\vec{E}\cdot\vec{B}=0$ in S, $\vec{E}' \perp \vec{B}'$ if $\vec{E}\cdot\vec{B}$ is invariant under Lorentz transformation.

$$\vec{E}'\cdot\vec{B}' = E_x' B_x' + B_y' E_y' + E_z' B_z'$$

$$= E_x B_x + \gamma^2 (E_y - \beta c B_z)(B_y + \beta E_z/c)$$

$$+ \gamma^2 (E_z + \beta c B_y)(B_z - \beta E_y/c)$$

$$= E_x B_x + E_y B_y + E_z B_z = \vec{E}\cdot\vec{B}$$

(c) $$E'^2 = E_x'^2 + E_y'^2 + E_z'^2$$

$$= E_x^2 + \gamma^2 (E_y^2 + E_z^2) + \gamma^2 \beta^2 c^2 (B_y^2 + B_z^2)$$
$$- 2\gamma^2 \beta c (E_y B_z - E_z B_y)$$

and $$c^2 B'^2 = \gamma^2 \beta^2 (E_y^2 + E_z^2) + c^2 (B_x^2 + \gamma^2 (B_y^2 + B_z^2))$$
$$- 2\gamma^2 \beta c (E_x B_y - E_z B_y)$$

$\Downarrow$ $E'^2 - c^2 B'^2 = E^2 - c^2 B^2$ with $\gamma^2(1-\beta^2) = 1$.

Solutions

Physics PhD Qualifying Examination
Part II – Friday, August 23, 2013

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] [10]

Consider a particle in a one-dimensional potential well of width L with infinitely high walls:

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases}.$$

At $t = 0$ the wave function of the particle is

$$\psi(x,0) = \begin{cases} \sqrt{\dfrac{2}{L}} & \text{for } \quad L/4 < x < 3L/4 \\ 0 & \text{otherwise} \end{cases}.$$

At time $t > 0$ we measure the energy of the particle. What is the probability that the energy of the particle is ***greater than*** the ground-state energy of the particle in the potential-well?

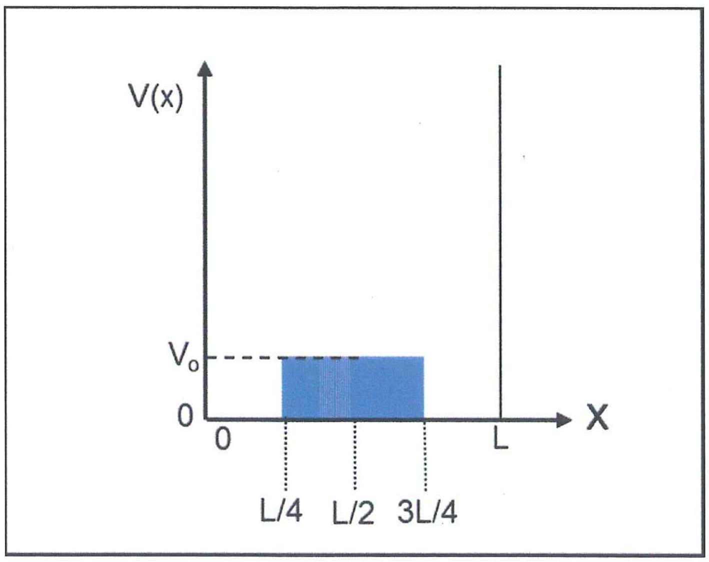
[II-2] **[10]**

Consider a particle of mass m in an infinitely high potential well, as shown in the figure below. Suppose that the step (marked by the dark-shaded area) at the bottom of the well can be considered as a small perturbation.

(a) Use first-order perturbation theory to calculate the eigenenergies E_n of the particle in the potential well.

(b) What are the first-order corrected wavefunctions Ψ_n?

(c) If the particle is an electron, how do the frequencies emitted by the perturbed system compare with those of the unperturbed system?



[II-3] **[10]**

Consider an electron spin and an arbitrary unit vector $\vec{e} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$ in the three-dimensional space, specified by the polar (ϑ) and the azimuth (φ) angles. In the usual S_z-representation the electron spin operator can be expressed in terms of the Pauli matrices

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}, \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Determine the properly normalized eigenvectors and eigenvalues (in terms of ϑ and φ) of the following operator:

$$\sigma_e \equiv \vec{e}\cdot\vec{\sigma}.$$

($\vec{e}\cdot\vec{\sigma}$ is the scalar product of the vectors $\vec{e}$ and $\vec{\sigma}$. In units of $\hbar/2$, this is the operator for the spin projected along the direction $\vec{e}$.)

Now we measure S_z, and we find that it is $\hbar/2$:

(b) What is the probability that the component of the spin along the direction $\vec{e}$ is $+\hbar/2$?

(c) What is the probability that the component of the spin along the direction $\vec{e}$ is $-\hbar/2$?

(d) What is the expectation value of the spin along the direction $\vec{e}$?

[II-4] **[10]**

In the quantum theory of scattering from a fixed potential, we get the following expression for the asymptotic form of the wave function,

$$\psi(r) \xrightarrow{r\to\infty} e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r} .$$

(a) If the entire Hamiltonian is spherically symmetric, give an argument that the scattering amplitude f should be independent of the angle ϕ.

(b) Why cannot this argument be extended (considering rotation about any axis) to conclude that f should be independent of θ as well?

(c) Reconsider part (b) in the case where the incident energy approaches zero.

(d) What is the formula for the scattering cross section in terms of f?

(e) What is the formula for the first Born approximation for f? (Be sure to define all quantities introduced. You need not worry about simple dimensionless factors like 2 or π).

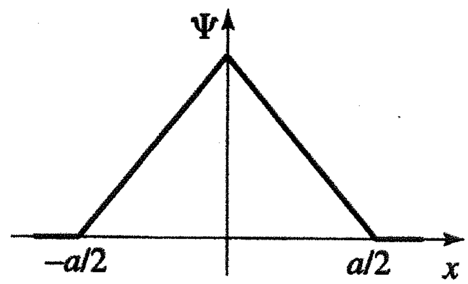
(f) Under what conditions is the Born approximation valid?

[II-5] **[10]**

Suppose the wave function for a particle is given by the symmetric wave function illustrated in the figure below:

$$\psi(x) = \sqrt{\frac{12}{a^3}}\left(\frac{a}{2} - |x|\right) \quad \mathit{for}\ |x| \le a/2$$

$$\psi(x) = 0 \quad \mathit{for}\ |x| \le a/2$$



(a) Calculate the uncertainty Δx in the position of the particle.

(b) Calculate the uncertainty Δp in the momentum p of the particle.

[II-6] **[10]**

The Hamiltonian of a classical radiation field (as opposed to quantized) is given by

$$H = \frac{\vec{p}^2}{2m_e} + e\phi(x) - \frac{e}{m_e c}\vec{A}\cdot\vec{p} \ ,$$

where the vector potential is

$$\vec{A} = 2A_0\hat{\varepsilon}\cos(\frac{\omega}{c}\hat{n}\cdot\hat{x} - \omega t)$$

for a monochromatic field.

(a) Write down Fermi's golden rule.

(b) For the absorption case, estimate the transition rate from energy state $|i\rangle$ to state $|n\rangle$ of an atom excited by this electromagnetic wave. State $|n\rangle$ is a higher energy state relative to state $|i\rangle$.

(c) Given the energy density of classical electromagnetic wave below:

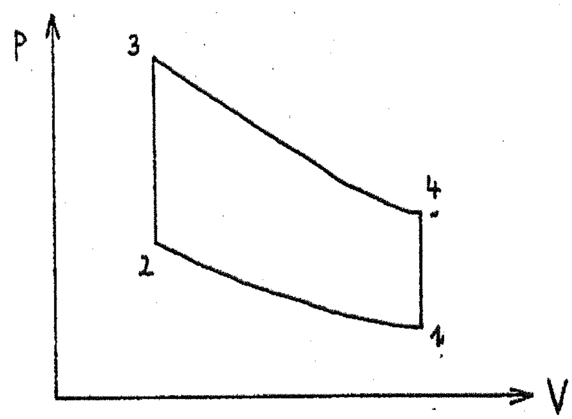
$$u = \frac{1}{2}(\frac{E_{\max}^2}{8\pi} + \frac{B_{\max}^2}{8\pi}) \ ,$$

calculate the absorption cross section which is defined as:

$$\frac{\text{(energy/unit) absorbed by the atom(i} \rightarrow \text{n)}}{\text{energy flux of the radiation field}} \ .$$

[II-7] **[10]**

Consider the pressure-volume (*P-V*) diagram of an Otto engine illustrated in the figure below. Calculate the efficiency η of the Otto engine as a function of the compression ratio $\varepsilon = V_1/V_2$. Treat the gasoline / air mixture in the combustion chamber of the engine as a *monatomic ideal gas*. The lines connecting points 1 and 2 as well as points 3 and 4 represent adiabatic processes; the lines connecting points 2 and 3 as well as points 4 and 1 represent isochoric processes.



[II-8] **[10]**

The Helmholtz free energy of a gas is given by $F(T,V) = -\frac{a}{3}T^4 V$, where a is a positive constant. The gas is initially at temperature T and has volume V. Then the gas undergoes (Gay-Lussac — Joule) "free expansion" from V to $16V$. (In this process the gas suddenly and adiabatically expands into vacuum, i.e., no work is done.)

Your answers must be expressed in terms of the initial temperature T, the initial volume V, and the constant a, but may not necessarily involve all of them.

(a) Obtain the final temperature T_2 of the gas.

(b) Calculate the total entropy change ΔS of this gas during the above free expansion.

[II-9] **[10]**

A system consists of N independent localized (hence, distinguishable) particles. The single-particle energy spectrum has infinitely many energy levels, but precise information is only available on the lowest two levels. The energy of the single-particle ground state and the first excited state are 0 and 2ε, with degeneracies $g_1 = 4$ and $g_2 = 1$, respectively.

Obtain the low-temperature behavior of the entropy of the system $S(N,T)$.

[Note that you need to obtain the asymptotic T-dependence at low temperature, $S(N,T)$ when $\varepsilon / kT >> 1$, not just the trivial limiting value $\lim_{T \to 0} S(N,T)$.]

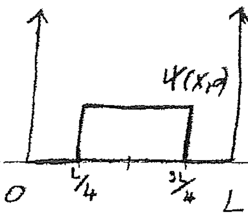
[II-10] **[10]**

Consider a *two-dimensional extreme-relativistic* ($\varepsilon = cp$) free electron gas confined to an area $A = L^2$. The number of electrons is N.

Obtain P_o, the pressure of the system at $T = 0$. You must express your answer in terms of the electron (number) density N / A (and other relevant fundamental constants).

Part - II

$\boxed{\text{II-1}}$



$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \qquad E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2$$

$$|\psi(0)\rangle = \sum_n a_n|\psi_n\rangle$$

$$\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} & \frac{L}{4} < x < \frac{3L}{4} \\ 0 & x < \frac{L}{4},\ x > \frac{3L}{4} \end{cases}$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\sum_n a_n|\psi_n\rangle = \sum_n a_n e^{-\frac{i}{\hbar}\hat{H}t}|\psi_n\rangle =$$

$$= \sum_n a_n e^{-\frac{i}{\hbar}E_n t}|\psi_n\rangle$$, where

$$a_n = \langle\psi_n|\psi(0)\rangle = \int_{L/4}^{3L/4}\sqrt{\frac{2}{L}}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{L/4}^{3L/4}\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2}{L}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/4}^{3L/4}$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right)\right) = \begin{cases} \frac{2}{n\pi}\frac{2}{\sqrt{2}}(\pm 1) & n = \text{odd, and } \begin{matrix} +1: n = 1, 5, 9, \ldots \\ -1: n = 3, 7, 11, \ldots \end{matrix} \\ \emptyset & n = \text{even} \end{cases}$$

$$\boxed{\psi(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)}$$

$$p_n(t) = \left|a_n e^{-\frac{i}{\hbar}E_n t}\right|^2 = |a_n|^2 = \begin{cases} \frac{8}{n^2\pi^2} & \text{if } n = \text{odd} \\ \emptyset & \text{if } n = \text{even} \end{cases}$$

$$\sum_{n=1}^{\infty} p_n = 1 \qquad (\text{can check})$$

$$P(E_n) = p_n$$

$$P(E > E_1) = p_2 + p_3 + \ldots = 1 - p_1 = \boxed{1 - \frac{8}{\pi^2}} \simeq 0.189$$

II-2 Perturbation (t-indep.)

① Solution - particle in a 1D infinite well

$$E_n = n^2 E_1, \quad \text{where } E_1 = \frac{\hbar^2 \pi^2}{2m L^2}, \; n=1,2,\cdots$$

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \; \#$$

② 1^{st} order perturbation:

Eigen Energy: $E_n = E_n^0 + H'_{nn}$

Eigen function: $\psi_n = \psi_n^0 + \sum_{i \neq n} \frac{H'_{in}}{E_n^0 - E_i^0} \psi_i^0 \quad \#$

Here, $H'_{nn} = \langle \psi_n | H' | \psi_n \rangle$

$H'_{in} = \langle \psi_i^0 | H' | \psi_n^0 \rangle$

③ perturbation potential

$$H' = V_0, \quad \frac{1}{4}L \leq x \leq \frac{3}{4}L$$

$$H'_{nn} = \frac{2V_0}{L} \int \sin^2\left(\frac{n\pi x}{L}\right) dx$$

Math: $\int \sin^2 y \, dy = \frac{1}{2}(y - \cos y \sin y)$

$$\therefore \begin{cases} H'_{nn} = \frac{1}{2} V_0, & n = \text{even} \\ H'_{nn} = \frac{1}{2} V_0 + \frac{V_0}{n\pi}, & n = 1, 5, 9, \cdots \\ H'_{nn} = \frac{1}{2} V_0 - \frac{V_0}{n\pi}, & n = 3, 7, 11, \cdots \end{cases}$$

\#

II -2 perturbation (t-indep.)

④ $$H'_{in} = \frac{2V_0}{L}\int_{L/4}^{3L/4} \sin\frac{i\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

from Math $\int \sin ax \sin bx\, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$

$$a = i\pi/L$$
$$b = n\pi/L$$

$$H'_{in} = \frac{2V_0}{L}\left[\frac{\sin(i-n)\pi x/L}{2(i-n)\pi/L} - \frac{\sin(i+n)\pi x/L}{2(i+n)\pi/L}\right]_{L/4}^{3L/4}$$

$$\therefore H'_{in} = \frac{V_0}{\pi}\left[\sin(i-n)\tfrac{3}{4}\pi - \sin(i-n)\tfrac{\pi}{4}\right]$$
$$-\frac{V_0}{\pi}\left[\sin(i+n)\tfrac{3}{4}\pi - \sin(i+n)\tfrac{\pi}{4}\right]$$ #

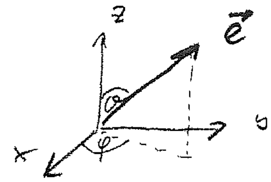
⑤ Transition frequencies : $\Delta E = \hbar\omega$

$$E_n^0 = n^2 E_1$$

$$E'_n = E_n^0 + H'_{nn}$$

$$\hbar\omega = (E'_n - E'_l) = (E_n^0 - E_l^0) + (H'_{nn} - H'_{ll})$$

$$\therefore \hbar\omega = (n^2 - l^2)E_1^0 + (H'_{nn} - H'_{ll})$$ #



$\boxed{\text{II - 3}}$

a) $\sigma_e = \vec{e}\cdot\vec{\sigma} = \sin\theta\cos\varphi\,\sigma_x + \sin\theta\sin\varphi\,\sigma_y + \cos\theta\,\sigma_z$

$$= \sin\theta\cos\varphi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \sin\theta\sin\varphi\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix}0 & \sin\theta\cos\varphi\\ \sin\theta\cos\varphi & 0\end{pmatrix} + \begin{pmatrix}0 & -i\sin\theta\sin\varphi\\ i\sin\theta\sin\varphi & 0\end{pmatrix} + \begin{pmatrix}\cos\theta & 0\\ 0 & -\cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}$$

eigenvalues & eigenvectors: $\qquad \sigma_e|s_i\rangle = \lambda_i|s_i\rangle \quad i=1,2$

$$\begin{vmatrix}\cos\theta-\lambda & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta-\lambda\end{vmatrix} = (\lambda-\cos\theta)(\lambda+\cos\theta) - \sin^2\theta = 0$$

$\lambda^2 - \cos^2\theta - \sin^2\theta = 0$

$\lambda^2 = 1 \qquad\qquad \boxed{\lambda = \pm 1}$ (regardless of the direction $\vec{e}$

$\lambda = 1:$ $(\cos\theta - 1)\,a_1 + \sin\theta\, e^{-i\varphi}\, a_2 = 0$

$$a_2 = \frac{1-\cos\theta}{\sin\theta}\, e^{i\varphi} a_1 = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\, e^{i\varphi} a_1 = \frac{\sin(\theta/2)}{\cos(\theta/2)}\, e^{i\varphi}\, a$$

normalized eigenvector:

$$|s_1\rangle = \begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\,e^{i\varphi}\end{pmatrix}$$

II-3 (cont'd)

$\lambda = -1$: $(\cos\theta + 1)\, a_1 + \sin\theta\, e^{-i\varphi} a_2 = 0$

$$a_2 = -\frac{1+\cos\theta}{\sin\theta} e^{i\varphi} a_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} e^{i\varphi} a_1 = -\frac{\cos(\theta/2)}{\sin(\theta/2)} e^{i\varphi} a_1$$

normalized eigenvector:

$$|s_2\rangle = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)\, e^{i\varphi} \end{pmatrix}$$

b) $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad |s\rangle = c_1 |s_1\rangle + c_2 |s_2\rangle$

$$c_1 = \langle s_1 | s\rangle = \left(\cos(\theta/2),\ \sin(\theta/2)\, e^{-i\varphi}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\cos(\theta/2)}$$

$$P_1 = |c_1|^2 = \cos^2(\theta/2)$$

c) $$c_2 = \langle s_2 | s\rangle = \left(-\sin(\theta/2),\ \cos(\theta/2)\, e^{-i\varphi}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{-\sin(\theta/2)}$$

$$P_2 = |c_2|^2 = \sin^2(\theta/2)$$

d) $$\langle s | \sigma_e | s\rangle = \left(\langle s_1| c_1^* + \langle s_2| c_2^*\right) \sigma_e \left(c_1|s_1\rangle + c_2|s_2\rangle\right)$$

$$= \left(\langle s_1| c_1^* + \langle s_2| c_2^*\right)\left(c_1|s_1\rangle - c_2|s_2\rangle\right) =$$

$$= |c_1|^2 - |c_2|^2 = \cos^2(\theta/2) - \sin^2(\theta/2) = \cos(\theta)$$

$$\langle s | S_e | s\rangle = \langle s | \tfrac{\hbar}{2}\sigma_e | s\rangle = \boxed{\tfrac{\hbar}{2}\cos(\theta)}$$

(also simply follows from $\langle \sigma_e\rangle = P_1(+1) + P_2(-1)$)

II-4 Scattering Theory (Born Approx.)

In the quantum theory of scattering from a fixed potential, we get the following expression for the asymptotic form of the wave function,

$$\psi(r) \xrightarrow{r\to\infty} e^{ikz} + f(\theta,\phi)e^{ikr}/r\,.$$

(a) If the entire Hamiltonian is spherically symmetric, give an argument that the scattering amplitude f should be independent of the angle ϕ.

The incident wave $e^{ikz} = e^{ikrcos\theta}$ is an eigenstate of $\widehat{L_z}$, with eigenvalue $m = 0$. If the Hamiltonian is spherically symmetric, the angular momentum is conserved and the outgoing wave is still an eigenstate, with same eigenvalue.

$$\widehat{L_z} f(\theta,\phi) = \frac{\hbar}{i}\frac{\partial}{\partial\phi} f(\theta,\phi) = mf(\theta,\phi) = 0$$

$$\frac{\partial f(\theta,\phi)}{\partial\phi} = 0 \longrightarrow f(\theta,\phi) = f(\theta)$$

(b) Why cannot this argument be extended (considering rotation about any axis) to conclude that f should be independent of θ as well?

The asymptotic form of the wavefunction $\psi(r)$ is not an eigenfunction of L^2, hence the argument cannot be extended to θ.

(c) Reconsider part (b) in the case where the incident energy approaches zero.

As $E \to 0$ $(k \to 0)$, the incident wave consist mainly of the $l = 0$ partial wave; other partial waves have very small amplitudes and can be neglected. In this case, the spherical symmetry of H approximately conserves L^2, therefore,

$$\widehat{L^2} f(\theta) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right] f(\theta) = 0\,.$$

hence,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) f(\theta) = 0.$$

The only solution of which, where $f(\theta)$ has appropriate wavefunction properties is for

$$\frac{\partial f(\theta)}{\partial\theta} = 0.$$

(d) What is the formula for the scattering cross section in terms of f?

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2.$$

(e) What is the formula for the first Born approximation for f? (Be sure to define all quantities introduced. You need not worry about simple dimensionless factors like 2 or π).

In the first Born approximation, for scattering from a central field $V(r')$, f is given by

$$\begin{aligned} f(\theta, \phi) &= -\frac{m}{2\pi\hbar^2} \int V(r') e^{-iq\cdot r'} d^3 r' \\ &= -\frac{2m}{\hbar^2 q} \int_0^\infty r' V(r') \sin(qr') dr', \end{aligned}$$

where, $q = k - k_0$, with k and k_0 being the momenta of the particle before and after the scattering, respectively.

(f) Under what conditions is the Born approximation valid?

The validity of the Born approximation requires that the interaction potential is small compared with the energy of the incident particle.

Solution II-5:

From the even symmetry of Ψ, $\langle x \rangle = 0$.

$$\begin{aligned}
\langle x^2 \rangle &= 2\int_0^{a/2} \frac{12}{a^3} x^2 \left(\frac{a}{2} - x\right)^2 dx \\
&= \frac{24}{a^3} \int_0^{a/2} \left[\left(\frac{a}{2}\right)^2 x^2 - ax^3 + x^4\right] dx \\
&= \frac{24}{a^3} \left[\frac{1}{3}\left(\frac{a}{2}\right)^5 - \frac{a}{4}\left(\frac{a}{2}\right)^4 + \frac{1}{5}\left(\frac{a}{2}\right)^5\right] \\
&= \frac{3a^2}{4}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{a^2}{40}
\end{aligned}$$

Thus

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{2\sqrt{10}}$$

II-5
(Cont'd)

$$\langle P_x \rangle = \int_{-a/2}^{+a/2} \sqrt{\frac{12}{a^3}} \left(\frac{a}{2} - |x|\right) - i\hbar \frac{d}{dx} \sqrt{\frac{12}{a^3}} \left(\frac{a}{2} - |x|\right) dx$$

$$= \frac{12}{a^3} \int_{-\frac{a}{2}}^{0} \left(\frac{a}{2} - x\right)\left(-i\hbar \frac{d}{dx}\right)\left(\frac{a}{2} - x\right) dx$$

$$+ \frac{12}{a^3} \int_{0}^{a/2} \left(\frac{a}{2} - x\right)\left(-i\hbar \frac{d}{dx}\right)\left(\frac{a}{2} - x\right) dx$$

$$= \frac{12}{a^3} \int_{-\frac{a}{2}}^{0} \left(\frac{a}{2} - x\right)(-i\hbar)(-dx)$$

$$+ \frac{12}{a^3} \int_{0}^{a/2} \left(\frac{a}{2} - x\right)(-i\hbar)(-dx)$$

$$= i\hbar \frac{12}{a^3} \left[\int_{-a/2}^{0} \left(\frac{a}{2} - x\right) dx + \int_{0}^{a/2} \left(\frac{a}{2} - x\right) dx \right]$$

$$= i\hbar \frac{12}{a^3} \left[\left(\frac{a}{2} x - \frac{1}{2} x^2\right)_{-a/2}^{0} + \left(\frac{a}{2} x - \frac{1}{2} x^2\right)_{0}^{a/2} \right]$$

$$= i\hbar \frac{12}{a^3} \left[-\left(\frac{a}{2}\left(-\frac{a}{2}\right) - \frac{1}{2}\frac{a^2}{4}\right) + \left(\frac{a}{2}\cdot\frac{a}{2} - \frac{1}{2}\frac{a^2}{4}\right) \right]$$

$$= i\hbar \frac{12}{a^3} \left[\frac{a^2}{4} + \frac{a^2}{8} + \frac{a^2}{4} - \frac{a^2}{8} \right] = i\hbar \frac{12}{a^3} \cdot 2 \frac{a^2}{4}$$

$$= i\hbar \frac{6}{a}$$

II-5
(cont'd)

$$\langle p_x^2 \rangle = \int_{-a/2}^{+a/2} \frac{\sqrt{12}}{a^3}\left(\frac{a}{2} - |x|\right)\left(-i\hbar \frac{d}{dx}\right)^2 \sqrt{\frac{12}{a^3}}\left(\frac{a}{2} - |x|\right) dx$$

$$= \frac{12}{a^3}\left[\int_{-a/2}^{0}\left(\frac{a}{2} - x\right)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\left(\frac{a}{2} - x\right) dx\right.$$

$$\left. + \int^{a/2}\left(\frac{a}{2} - x\right)\left(-\hbar^2 \frac{d^2}{dx^2}\right)\left(\frac{a}{2} - x\right) dx\right]$$

$$= \frac{12}{a^3}\left[\int_{-a/2}^{0}(-\hbar^2)\left(\frac{a}{2} - x\right)\underbrace{\frac{d^2}{dx^2}\left(\frac{a}{2} - x\right)}_{=0} dx + \right.$$

$$\left.\int_0^{a/2}(-\hbar^2)\left(\frac{a}{2} - x\right)\underbrace{\frac{d^2}{dx^2}\left(\frac{a}{2} - x\right)}_{0} dx\right]$$

$$\langle p_x^2 \rangle = 0$$

$$\Delta p = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2} = \sqrt{-\left(i\hbar \frac{6}{a}\right)^2}$$

$$= \sqrt{-\left(i^2 \hbar^2 \frac{36}{a^2}\right)}$$

$$= \sqrt{\hbar^2 \frac{36}{a^2}} = \hbar \frac{6}{a}$$

$$\Delta x \, \Delta p = \frac{a}{2\sqrt{10}} \cdot \hbar \frac{6}{a} = \frac{\hbar}{2} \frac{6}{\sqrt{10}} \approx \frac{\hbar}{2} \cdot 1.9 \geq \frac{\hbar}{2}$$

II – 6 perturbation (t-depend.)

(a) Fermi's Golden Rule

$$W_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2 \delta(E_n - E_i) \quad \text{①}$$

#

$$V_{ni} = \langle n|V|i\rangle$$

Where V is the perturbation term

(b) $\cos(\frac{\omega}{c}\hat{n}\cdot\vec{x} - \omega t) = \frac{1}{2}\left[e^{i(\frac{\omega}{c})\hat{n}\cdot\vec{x}-\omega t} + e^{-i(\frac{\omega}{c})\hat{n}\cdot\vec{x}+\omega t}\right]$

$$\vec{A} = A_0\hat{\varepsilon}\Big[\underbrace{e^{i\frac{\omega}{c}\hat{n}\cdot\vec{x}-\omega t}}_{\text{responsible for absorption}} + \underbrace{e^{-i\frac{\omega}{c}\hat{n}\cdot\vec{x}+\omega t}}_{\text{responsible for stimu. emission}}\Big]$$

For Absorption

$$V_{ni} = -\frac{eA_0}{m_0 c}\left(e^{i\frac{\omega}{c}\hat{n}\cdot\vec{x}}\ \hat{\varepsilon}\cdot\vec{p}\right)_{ni}$$

$$W_{ni} = \frac{2\pi}{\hbar}\frac{e^2}{m_0^2 c^2}|A_0|^2\left|\langle n|e^{i\frac{\omega}{c}\hat{n}\cdot\vec{x}}\ \hat{\varepsilon}\cdot\vec{p}|i\rangle\right|^2 \delta(E_n - E_i - \hbar\omega)$$

② #

(c) $U = \left(\frac{1}{2}\ \frac{E_{max}^2}{8\pi} + \frac{B_{max}^2}{8\pi}\right)$

$\therefore$ Energy flux $= cU = \frac{1}{2\pi}\frac{\omega^2}{c}|A_0|^2$ # ③

(Note: $E = \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$, $\vec{B} = \nabla\times\vec{A}$)

II — 6 perturbation (t-depend.)

No. ②/2

from equ-② and equ-③

$$\sigma_{abs.} = \frac{(\hbar w)\left[\frac{2\pi}{\hbar}\frac{e^2}{m_e^2 c^2}|A_0|^2 \left|\langle n| e^{i\frac{w}{c}\hat{n}\cdot\vec{x}}\,\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2 \delta(E_i - E_n - \hbar w)\right.}{\frac{1}{2\pi}\frac{w^2}{c}|A_0|^2}$$

$$\sigma_{abs} = \frac{4\pi\hbar^2}{2w}\left(\frac{e^2}{\hbar c}\right)\left|\langle n| e^{i\frac{w}{c}\hat{n}\cdot\vec{x}}\,\hat{\varepsilon}\cdot\vec{p}\,|i\rangle\right|^2 \delta(E_i - E_n - \hbar w)$$

#

Solution II-7

engine efficiency = work done / heat absorbed

$$\oint dU = 0 = \oint(\delta Q + \delta A) = 0 = Q + A \quad -A = Q = Q_1 + Q_2$$

heat absorbed Q_1 $\qquad dU = \underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{C_V} dT + \underbrace{\left(\frac{\partial U}{\partial V}\right)_V}_{0} dV$

$$\eta = \frac{Q_1 + Q_2}{Q_1} = 1 + \frac{Q_2}{Q_1}$$

Isochoren $\quad V = \text{const.} \quad dV = 0 \quad dA = 0$

$$\delta Q = dU = c_V\, dT$$

$2 \to 3 \quad Q_1 = c_V \int_{T_2}^{T_3} dT = c_V (T_3 - T_2) \qquad Q_1 > 0$

$4 \to 1 \quad Q_2 = c_V \int_{T_4}^{T_1} dT = c_V (T_1 - T_4) \qquad Q_2 < 0$

Adiabaten $\quad dS = 0 \qquad S = \int_{T_0}^{T} c_V(T) \frac{dT}{T} + R \ln \frac{V}{V_0} + \text{const.}$

$S_1 = S_2 \qquad c_V \ln T_1 + R \ln V_1 = c_V \ln T_2 + R \ln V_2$

(1) $S_3 = S_4 \qquad c_V \ln T_3 + R \ln V_2 = c_V \ln T_4 + R \ln V_1$

$$\text{I} \quad c_V \ln \frac{T_1}{T_2} = R \ln \frac{V_2}{V_1}$$

$$\text{II} \quad c_V \ln \frac{T_3}{T_4} = R \ln \frac{V_1}{V_2} = -R \ln \frac{V_2}{V_1}$$

$\text{I} + \text{II} \quad \ln \frac{T_1}{T_2} = -\ln \frac{T_3}{T_4}$

$$\frac{T_1}{T_2} = \frac{T_4}{T_3} \qquad \frac{T_1}{T_4} = \frac{T_2}{T_3}$$

$$\eta = 1 + \frac{T_1 - T_4}{T_3 - T_2} = 1 - \left(\frac{T_4 - T_1}{T_3 - T_2}\right) = 1 - \frac{T_4}{T_3} \frac{\left(1 - \frac{T_1}{T_4}\right)}{\left(1 - \frac{T_2}{T_3}\right)} = 1 - \frac{T_4}{T_3}$$

aus (1)

$$\frac{T_4}{T_3} = \left(\frac{V_1}{V_2}\right)^{\left(1 - \frac{c_p}{c_V}\right)} \qquad \eta = 1 - \varepsilon^{\left(1 - \frac{c_p}{c_V}\right)} \qquad c_p = \frac{5}{2} R$$

$$\eta = 1 - \varepsilon^{-2/3} \qquad c_V = \frac{3}{2} R$$

II-8

~~Problem~~

The Helmholtz free energy of a gas is given by $F(T,V) = -\frac{a}{3}T^4V$.

(a) ~~...~~ The gas initially is at temperature T and has volume V. This gas then undergoes (Gay-Lussac — Joule) "free expansion" from V to $16V$.
Obtain the final temperature T_2 of the gas.
(You must express your answer in terms of the variables and constants given, i.e. initial temperature T, initial volume V, and the constant a, but your final answer may not necessarily include all of them.)

$$F = U - TS \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{3}aT^3V$$

$$U = F + TS = -\frac{a}{3}T^4V + \frac{4}{3}aT^4V = aT^4V$$

Joule free expansion: $U =$ const: $\quad U(T_1, V_1) = U(T_2, V_2) \qquad V_2 = 16V$

$$aT^4V = aT_2^4(16V)$$

$$T_2^4 = \frac{T^4}{16} \implies \boxed{T_2 = \frac{T}{2}}$$

(b) ~~...~~ Calculate the total entropy change of this gas during the above free expansion. (As in part (a), you must express your answer in terms of the initial temperature and volume, T and V, and possibly the constant a.)

$$\Delta S = S_2 - S_1 = \frac{4}{3}aT_2^3V_2 - \frac{4}{3}aT_1^3V_1^3 = \frac{4}{3}a\left\{\left(\frac{T}{2}\right)^3(16V) - T\cdot V\right\}$$

$$= \frac{4}{3}aT^3V\{2-1\} = \boxed{\frac{4}{3}aT^3V}$$

II - [redacted]9

[redacted]

A system consists of N independent localized (hence, distinguishable) particles. The single-particle energy spectrum has infinitely many energy levels, but precise information is only available on the lowest two levels. The energy of the single-particle ground state and the first excited state are 0 and 2ε, with degeneracies $g_1 = 4$ and $g_2 = 1$, respectively.

[redacted] Obtain the low-temperature behavior of the heat capacity of the system, $C(N,T)$ $(\varepsilon / kT >> 1)$.

$$Z = g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT} + \dots \simeq g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT}$$

$$= 4 \cdot 1 + 1 \cdot e^{-2\varepsilon/kT} = 4\left\{1 + \frac{1}{4} e^{-2\varepsilon/kT}\right\}$$

$$\ln Z \cong \ln(4) + \ln\left\{1 + \frac{1}{4} e^{-2\varepsilon/kT}\right\} \overset{\varepsilon/kT >> 1}{\simeq} \ln(4) + \frac{1}{4} e^{-\frac{2\varepsilon}{kT}}$$

$x << 1$

$\ln(1+x) \simeq x$

$$= \ln(4) + \frac{1}{4} e^{-\beta 2\varepsilon}$$

$$U = -N \frac{\partial}{\partial \beta} \ln Z = N \frac{\varepsilon}{2} e^{-2\beta\varepsilon} = N \frac{\varepsilon}{2} e^{-\frac{2\varepsilon}{kT}}$$

II #9) Cont'd.

Obtain the low-temperature behavior of the entropy of the system $S(N,T)$ $(\varepsilon / kT >> 1)$.

$$F = -NkT \ln 2 \qquad \text{(Boltzmann statistics)}$$

$$F = U - TS \quad \Rightarrow \quad S = \frac{1}{T}(U - F)$$

$$S(N,T) \cong \frac{1}{T}\left\{ N\frac{\varepsilon}{2} e^{-2\varepsilon/kT} + NkT\left(\ln(4) + \frac{1}{4} e^{-2\varepsilon/kT}\right)\right\}$$

$$= Nk\ln(4) + \frac{N\varepsilon}{2T} e^{-2\varepsilon/kT} + \frac{Nk}{4} e^{-2\varepsilon/kT}$$

$$= Nk\ln(4) + \left(\frac{1}{4}Nk\left(\frac{2\varepsilon}{kT}\right) + \frac{Nk}{4}\right) e^{-2\varepsilon/kT}$$

$$= Nk\ln(4) + \frac{Nk}{4}\left(1 + \frac{2\varepsilon}{kT}\right) e^{-2\varepsilon/kT}$$

$$\boxed{\simeq Nk\ln(4) + \frac{Nk}{4}\left(\frac{2\varepsilon}{kT}\right) e^{-2\varepsilon/kT}}$$

~~(e) (2 pts) What is the entropy of the system at zero temperature, i.e. lim S(N,T)~~

$\boxed{\text{II} - 10}$

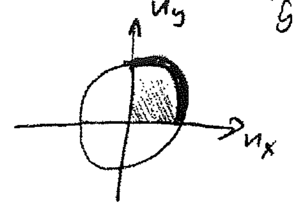
[redacted]

Consider a *two-dimensional* extreme-relativistic ($\varepsilon = cp$) free electron gas confined to an area $A = L^2$. The number of electrons is N.

[redacted] [redacted] Obtain the density of states $g(\varepsilon)$.

$$\varepsilon = cp = c\hbar k = c\hbar \frac{\pi}{L}\sqrt{(n_x^2 + n_y^2)} =$$

$$= \frac{ch}{2L}\underbrace{(n_x^2 + n_y^2)}_{n^2}{}^{1/2} = \frac{ch}{2L} n$$



$n_y = 1, 2, \ldots$

$$d\mathcal{N}(\varepsilon) = \frac{\partial \mathcal{N}}{\partial \varepsilon} d\varepsilon$$

$$n = \frac{2L}{ch}\varepsilon \qquad \mathcal{N}(\varepsilon) = \frac{1}{4}\pi n^2 = \frac{1}{4}\pi \frac{4L^2}{c^2h^2}\varepsilon^2 = \frac{\pi A}{c^2h^2}\varepsilon^2$$

$$g(\varepsilon) = (2s+1)\frac{\partial \mathcal{N}}{\partial \varepsilon} = 2 \cdot \frac{\pi A}{c^2h^2} 2\varepsilon$$

$$= \boxed{\frac{4\pi A}{c^2h^2}\varepsilon}$$

[redacted] [redacted] Find the Fermi energy and express it in terms of the density N/A. $T = 0$

$$N = \int_0^{\varepsilon_F} g(\varepsilon)d\varepsilon = \frac{4\pi A}{c^2h^2}\int_0^{\varepsilon_F}\varepsilon\, d\varepsilon = \frac{2\pi A}{c^2h^2}\varepsilon_F^2$$

$$\boxed{\varepsilon_F = \frac{ch}{\sqrt{2\pi}}\left(\frac{N}{A}\right)^{1/2}}$$

II-10 (cont'd)

[redacted] Find U_o, the internal energy of the system at $T=0$. You must express your answer explicitly in terms of N and A.

$$U_o = \int_0^{\varepsilon_F} g(\varepsilon)\,\varepsilon\,d\varepsilon = \frac{4\pi A}{h^2c^2}\int_0^{\varepsilon_F} \varepsilon^2\,d\varepsilon = \frac{4\pi A}{3c^2h^2}\,\varepsilon_F^3 =$$

$$= \frac{2}{3} N \varepsilon_F$$

$$\boxed{= \frac{2}{3} N \frac{ch}{\sqrt{2\pi}}\left(\frac{N}{A}\right)^{1/2}}$$

$$\boxed{= \frac{2}{3} A\cdot \frac{ch}{\sqrt{2\pi}}\left(\frac{N}{A}\right)^{3/2}}$$

$$= \frac{4\pi A}{3c^2h^2}\frac{c^3h^3}{(2\pi)^{3/2}}\left(\frac{N}{A}\right)^{3/2} =$$

$$= \frac{2}{3}\frac{ch}{\sqrt{2\pi}} A\cdot\left(\frac{N}{A}\right)^{3/2}$$

[redacted] Obtain P_o, the pressure of the system at $T=0$. You must express your answer in terms of the density N/A.

$\varepsilon \sim p^s$ d-dim

use $PA = \frac{s}{d} U$, where $s=1$, $d=2$

$$P_o = \frac{1}{2}\frac{U_o}{A} = \boxed{\frac{1}{3}\frac{ch}{\sqrt{2\pi}}\left(\frac{N}{A}\right)^{3/2}}$$

---

or alternatively, $F = U - TS$ } for all temperatures

$$P = -\left(\frac{\partial F}{\partial A}\right)_{T,N}$$

specifically, for $T=0$:

$$P_o = -\left(\frac{\partial U_o}{\partial A}\right) = \frac{1}{3}\frac{ch}{\sqrt{2\pi}}\left(\frac{N}{A}\right)^{3/2}$$