ECE 447: Control Systems

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this week: ☑ HW 3 assigned $\longrightarrow$ due Fri Oct 27

☐ week 4 lecture material

☐ tutorial

Q: ☐ post HW 4 at same time as exam?

☐ conv eq ex

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linearization recipe

0°. given $\dot{x} = f(x,u)$ and $\overbrace{x_0 \in \mathbb{R}^n, u_0 \in \mathbb{R}^p}^{\text{equilibrium}}$ s.t. $f(x_0, u_0) = 0$

$(f : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$

1°. compute all partial derivatives $\frac{\partial}{\partial x_j} f_i$ for i,j from 1 to n

$\frac{\partial}{\partial u_k} f_i$ for i from 1 to n, k from 1 to p

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ f_2(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$$

$\delta x = x - x_0$

2°. evaluate & tabulate: $\delta\dot{x} = A \cdot \delta x + B \cdot \delta u$ where $\delta u = u - u_0$

$\frac{\partial f_i}{} \Big|$ $\frac{\partial f_i}{} \Big|$ ...

$$\Bigg\{ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{} & \cdots & & \frac{\partial f_2}{} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & \ddots & \vdots \end{bmatrix}$$

$$\left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0\\u=u_0}} \qquad \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0\\u=u_0}} \qquad \Big\{ A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}$$

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ex: linearize $m\ell^2\ddot\theta = mg\ell\sin\theta - \gamma\dot\theta + \ell\tau\cos\theta.$ at $\theta_0 = 0,\ \tau_0 = 0$, $\dot\theta_0 = 0$

0°: collect data

let $x = \begin{bmatrix}\theta\\ \dot\theta\end{bmatrix} \in \mathbb{R}^2$ $\quad u = [\tau] \in \mathbb{R}^1$ $\qquad x_0 = \begin{bmatrix}0\\0\end{bmatrix} \in \mathbb{R}^2,\ u_0 = [0] \in \mathbb{R}^1$

$$\Rightarrow \dot x = f(x,u) = \begin{bmatrix} \dot\theta \\ (mg\ell\sin\theta - \gamma\dot\theta + \ell\tau\cos\theta)/m\ell^2 \end{bmatrix}$$

1°. compute derivatives

$$\frac{\partial f_1}{\partial x_1} = \frac{\partial\dot\theta}{\partial\theta} = \frac{\partial x_2}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = \frac{\partial\dot\theta}{\partial\dot\theta} = 1$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial}{\partial\theta}(mg\ell\sin\theta + \ell\tau\cos\theta)\frac{1}{m\ell^2} \qquad \frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial\dot\theta}\left(\frac{-\gamma\dot\theta}{m\ell^2}\right) = -\frac{\gamma}{m\ell^2}$$

$$= (mg\ell\cos\theta - \ell\tau\sin\theta)\frac{1}{m\ell^2}$$

$$\frac{\partial f_1}{\partial u_1} = \frac{\partial\dot\theta}{\partial\tau} = 0 \qquad \frac{\partial f_2}{\partial u_1} = \frac{\partial}{\partial\tau}\left(\frac{\ell\tau\cos\theta}{m\ell^2}\right) = \frac{\cancel{\ell}\cos\theta}{m\ell^{\cancel{2}}}$$

2°. evaluate & tabulate @ $x_0 = \begin{bmatrix}\theta_0\\ \dot\theta_0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$, $u_0 = [\tau_0] = [0]$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mg\cancel{\ell}}{m\ell^{\cancel{2}}} & \frac{-\gamma}{m\ell^2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m\ell \end{bmatrix}$$

x_0, u_0 on equilibrium

$$\therefore\ \dot x = f(x,u) \approx A\cdot(x-x_0) + B\cdot(u-u_0) + \underbrace{f(x_0,u_0)}_{0}$$

so $\dot{x} = f(x, u) \simeq A \cdot (x - x_0) + B \cdot (u - u_0) + f(x_0, u_0)$