ECE 447: Control Systems

goal: frequency-domain controller synthesis

(a) Nyquist stability criterion if $L = PC$ has no poles in right-half $\mathbb{C}$:
then $\frac{L}{1+L} = \frac{PC}{1+PC}$ is stable $\iff$ Ω does not encircle $-1 \in \mathbb{C}$

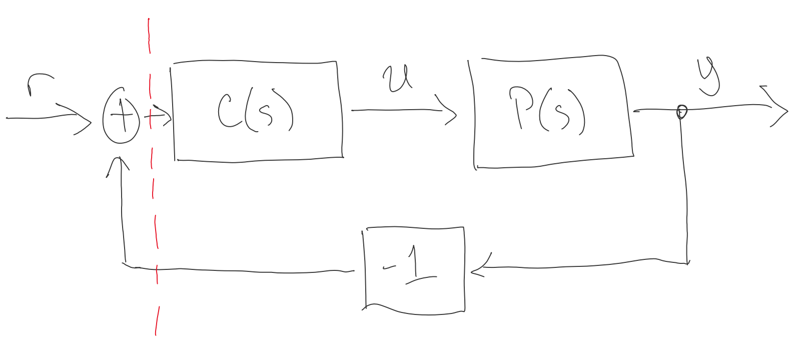
(b) stability margins gain margin g_m: distance from Ω to -1 in $|L|$
phase margin φ_m: distance from Ω to -1 in $\angle L$

(c) root locus can predict effect of large and small proportional feedback gain using poles, zeros, and #poles − #zeros of process P

(d) proportional-integral-derivative (PID)

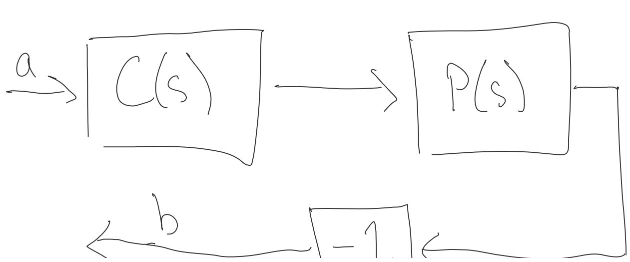
(a) Nyquist stability criterion [AMv2 Ch 10.1, 10.2] [Nv7 Ch10.3]

- key idea: assess stability, robustness, & sensitivity of closed-loop systems by studying open-loop systems



closed-loop system has transfer function

$$G_{yr} = \frac{PC}{1+PC} = \frac{L}{1+L}$$



open-loop system has transfer function

$G_{ab} = -PC = -L$

$\xleftarrow{\;b\;} \boxed{-1} \longleftarrow$ $\Big\}$ $G_{ba} = -PC = -L$

- we'll consider 2 ways the open-loop transfer function tells us about stability of the closed-loop system:
  1°. algebraic observation 2°. thought experiment

1°. algebraic observation: what does $L(s) = P(s)C(s)$ say about

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{L(s)}{1+L(s)}\,?$$

→ what happens if $\exists\, s^* \in \mathbb{C}$ s.t. $L(s) = P(s)C(s) = -1$?

– then as $s \to s^*$: $|G_{yr}(s)| = \left|\dfrac{P(s)C(s)}{1+P(s)C(s)}\right| \xrightarrow{s \to s^*} \left|\dfrac{-1}{1-1}\right| \to \infty$

* practically speaking: system response is <u>unbounded</u> (unstable) for inputs $\simeq e^{s^* t}$

- but practically speaking, we're only concerned with $s = j\omega$, so we're only worried if $\exists\, \omega^* \in \mathbb{R}$ s.t. $L(j\omega^*) = P(j\omega^*)C(j\omega^*) = -1$

2°. thought experiment $e^{st} \longrightarrow \boxed{L} \longrightarrow \boxed{-1} \longrightarrow -L(s)e^{st}$

what happens when we close feedback loop?

→ what happens to e^{st} if (i) $|L(s)| < 1$ – attenuated, i.e. → 0

(ii) $|L(s)| > 1$ – amplified, i.e. → ∞

(iii) $|L(s)| = 1$ – sustained

when we close the loop?

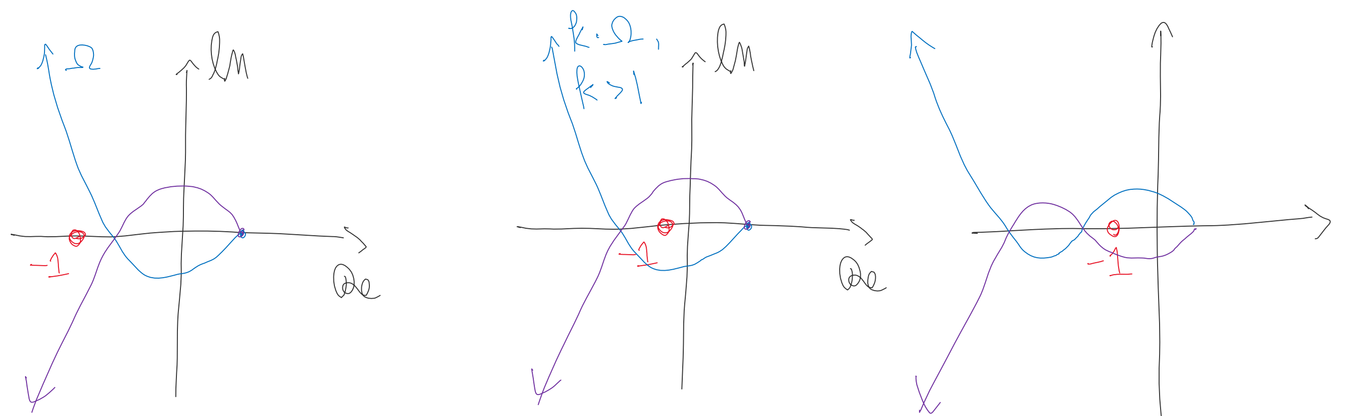
- conclude again that $L(s) = -1$, i.e. $|L(s)| = 1$, $\angle L(s) = \pi = 180°$ is a critical point for L along imaginary axis

* it turns out that the $\boxed{\text{graph of } L(j\omega) \text{ – Nyquist plot} \quad \Omega = \{ L(j\omega) \in \mathbb{C} : -\infty < \omega < +\infty \}}$

thm: (Nyquist stability criterion) ← application of argument principle

if L has no poles in the right-half plane

then $\frac{L}{1+L} = \frac{PC}{1+PC}$ is stable $\iff$ Ω does not encircle $-1 \in \mathbb{C}$



Ω does not encircle -1 Ω does encircle -1 Ω does not encircle -1

↳ * this condition is not necessary for stability, but relaxing

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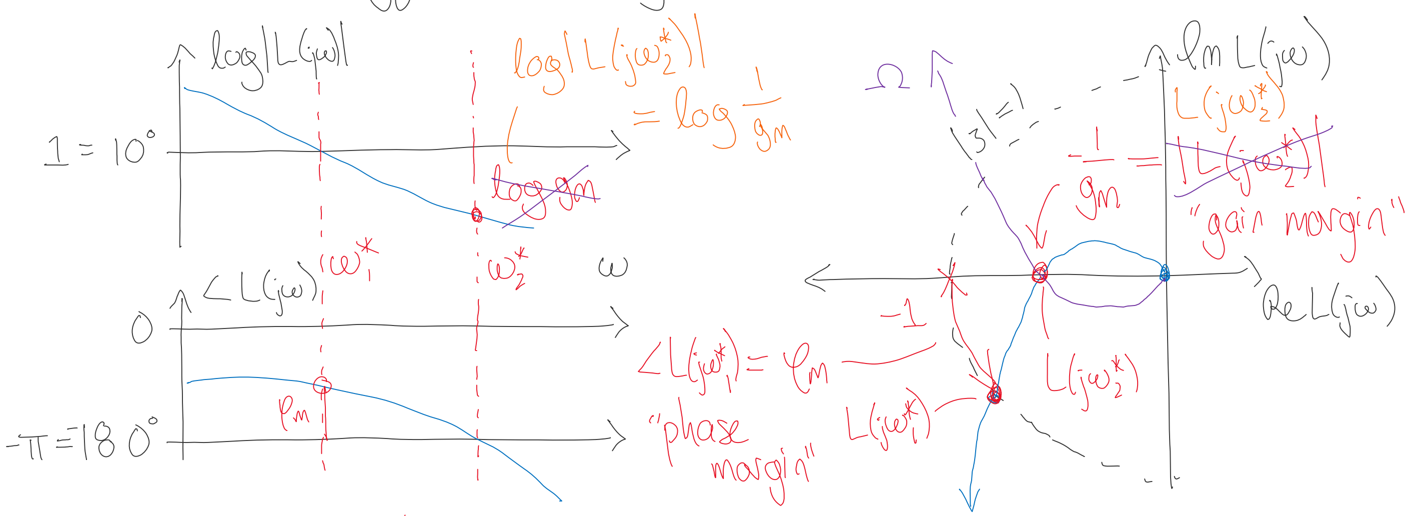
(b) stability margins [AMv2 Ch 10.3] [Nv7 Ch 10.7]

- given that a closed-loop system $\frac{PC}{1+PC}$ is stable, $L = PC$

  we can use Nyquist stability criterion to assess robustness:



⟶ use Bode plot of L to sketch Nyquist plot

* what if we know $L = PC$ only approximately, i.e. $\tilde{L} = \tilde{P}\tilde{C} \simeq L$?

  eg. if we have model uncertainty/inaccuracy in process $\tilde{P} \simeq P$

  eg. if we have implementation error in controller $\tilde{C} \simeq C$

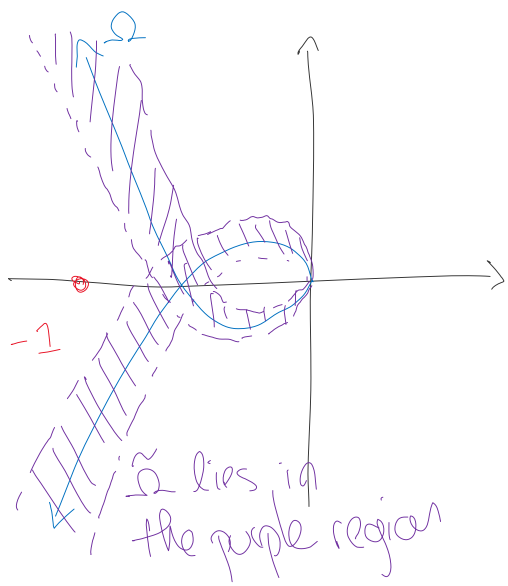
  from components, amplifiers, A2D having errors/tolerances

⟶ Nyquist stability criterion gives a robustness measurement:

  how far is Ω from $-1 \in \mathbb{C}$?

* if $\tilde{C} \simeq C$ and $\tilde{P} \simeq P$ then $\tilde{L} = PC \simeq \tilde{P}\tilde{C}$ so $\tilde{\Omega} \simeq \Omega$:

↑ ⟶ so measuring distance from



$\to$ so measuring distance from $\Omega \subset \mathbb{C}$ to $-1 \in \mathbb{C}$ gives a margin of stability:

g_m: distance from Ω to -1 if we only change $|L|$

φ_m: distance from Ω to -1 if we only change $\angle L$

(c) root locus [AMv2 Ch 12.5] [Nv7 Ch 9]

- consider a process $P(s) = \dfrac{b(s)}{a(s)} = \dfrac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$

  that we seek to control using proportional feedback: $C(s) = k > 0$

- then we know the closed-loop transfer function is

$$\frac{PC}{1+PC} = \frac{k\frac{b}{a}}{1 + k\frac{b}{a}} \cdot \frac{a}{a} = \frac{k\,b(s)}{a(s) + k \cdot b(s)}$$

$\to$ so the closed-loop characteristic polynomial is

$$\tilde{a}(s) = a(s) + k \cdot b(s)$$

* we'll analyze roots of $\tilde{a}$ in two regimes: large & small k

1°. small $k > 0$: as $k \to 0$, $\tilde{a} \to a$, so roots of $\tilde{a} \to$ roots of a

2°: large $k>0$ and $s\in\mathbb{C}$: as $k, |s|\to\infty$,

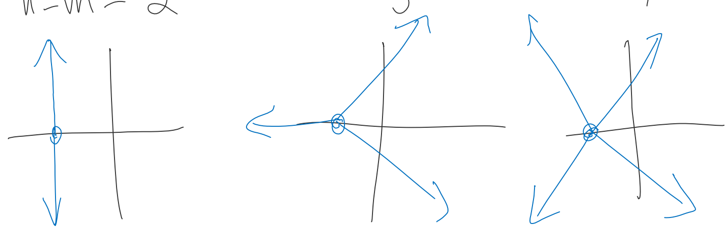
$$\tilde{a}(s) = b(s)\cdot\left(\frac{a(s)}{b(s)} + k\right) \simeq b(s)\cdot\left(\frac{s^{n-m}}{b_0} + k\right)$$

* assuming $n>m$, so P is <u>strictly proper</u>, ie causal,

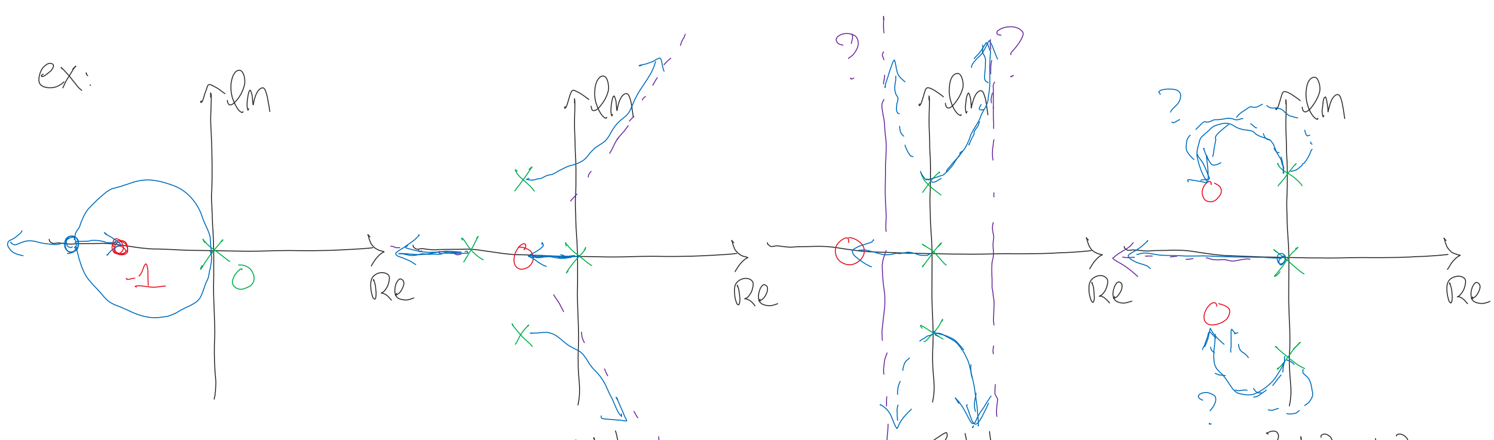
the roots of $\tilde{a}(s) \longrightarrow \begin{cases} \text{roots of } b(s) \\ \underline{\text{and}}\ \sqrt[n-m]{-b_0 k} \end{cases}$

→ so as $k, |s|\to\infty$ the closed-loop poles converge to:

zeros of P (i.e. roots of $b(s)$) or $(n-m)$-th "roots of unity"



ex:



$P = \dfrac{s+1}{s^2}$

poles: 2 @ $0\in\mathbb{C}$

zeros: 1 @ $-1\in\mathbb{C}$

$n-m: 2-1=1$

$P = \dfrac{s+1}{s(s+2)(s^2+2s+4)}$

poles: 1 @ 0
1 @ −2
2 @ $-1\pm j$

zeros: 1 @ −1

$n-m = 4-1 = 3$

$P = \dfrac{s+1}{s(s^2+1)}$

poles: 1 @ 0
2 @ $\pm j$

zeros: 1 @ −1

$n-m=3-1=2$

$P = \dfrac{s^2+2s+2}{s(s^2+1)}$

poles: 1 @ 0
2 @ $\pm j$

zeros: $-1\pm j\omega$

$n-m=3-2=1$

$n-m = 4-1=3$ $n-m=3-1=2$ $n-m=3-2=1$

* know system stable for all $k>0$ large

* know system is unstable for $k>0$ too large

* system is unstable for all $k>0$

* know $k>0$ large will stabilize system

(d) proportional-integral-derivative (PID) [AMv2 Ch 11] [Nv7 Ch 9.4]

- if we take a step back, what have we really learned to do?

* take prior knowledge of process model and design feedback controllers to shape stability & performance

ex: $$\boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y-\hat{y}) \\ \hat{y} &= C\hat{x} + Du \\ u &= -K\hat{x} \qquad x-\hat{x}\to 0 \end{aligned}}$$

$\uparrow y$ $\downarrow u$

- uses a TON of prior knowledge i.e. A, B, C, D
- uses n-dimensional state i.e. $x \in \mathbb{R}^n \Rightarrow \hat{x} \in \mathbb{R}^n$

* can we get away with less?

ex: what if we think of 3 types of error $e = r - y$

1°. present error $k_P e$ – proportional

2°. past error $k_I \int e$ – integral

3°. future error $k_D \dot{e}$ – derivative

- uses minimal* prior knowledge
- uses 1-dim state

3°. future error $\quad k_D \dot{e}$ – derivative $\quad$ state

$$e \to \boxed{k_P e + k_I \int e + k_D \dot{e}} \to u$$

$$e \to \boxed{\frac{k_P s + k_I + k_D s^2}{s}} \to u$$

PID transfer function $G_{ue}^{PID}(s)$

$$u = k_P e + k_I \int e + k_D \dot{e} \overset{\mathcal{L}}{\rightsquigarrow} u = k_P e + k_I \frac{1}{s} e + k_D s e$$

$$\Leftrightarrow \frac{u}{e} = k_P + k_I \frac{1}{s} + k_D s$$

$$\Leftrightarrow \frac{u}{e} = \frac{k_P s}{s} + \frac{k_I}{s} + \frac{k_D s^2}{s}$$

* how to choose k_P, k_I, k_D?

↳ widely-used rules developed by Zeigler & Nichols in the 1940s

↳ guaranteed to work for process $\frac{e^{-sT_c}}{a+s} = P(s)$

1°. set $k_I, k_D = 0$

2°. increase k_P until system oscillates
@ gain k_c w/ period T_c

3°. Nyquist stability criterion implies loop transfer function
$L = k_c P$ passing through critical point $-1 \in \mathbb{C}$ @ $\omega_c = \frac{2\pi}{T_c}$

type	k_c	T_c	T_d
P	$k_c/2$		
PI	$\frac{2}{5} k_c$	$\frac{4}{5} T_c$	
PID	$\frac{3}{5} k_c$	$\frac{1}{2} T_c$	$0.125\, T_c$