3-nonlinear-dynamics-stability

goal: develop qualitative & quantitative tools to study nonlinear dynamics

<u>topics</u>:

1°. nonlinear dynamics [Nv7 Ch 2] [AMv2 Ch 5]

1¹. trajectories & visualization

1². equilibrium trajectories

2°. stability

2¹. definition of stability [Nv7 Ch 6]

2². stability of linear DE [AMv2 Ch 6]

2³. parametric stability [Nv7 Ch 8]

* read [AMv2 Ch 5.4] to learn about an advanced test for stability

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1°. nonlinear control systems

- consider a DE with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$

$$\dot{x} = f(x, u)$$

- the simplest type of feedback sets u as a function of x, e.g. $u = \alpha(x)$, so

$$\dot{x} = f(x, \alpha(x)) = F(x)$$

→ we'll focus on $\dot{x} = F(x)$ below

1¹. trajectories & visualization

– $x : [0, \infty) \to \mathbb{R}^n$ is a trajectory of DE with initial state $x(0)$

1. trajectories & visualization

- $x : [0,\infty) \to \mathbb{R}^n$ is a <u>trajectory</u> of DE with <u>initial state</u> $x(0)$

  if $\forall t \geq 0 : \dot{x}(t) = F(x(t))$

  $\hookrightarrow$ so x is continuous <u>and</u> differentiable

* function x satisfies an infinite number of equations

  $\hookrightarrow$ we'll assume unique trajectories always exist, but this isn't guaranteed in general; see [AMv2 ex 5.2, 5.3]

<u>ex</u>: damped oscillator (i.e. RLC circuit; spring-mass-damper)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0$$

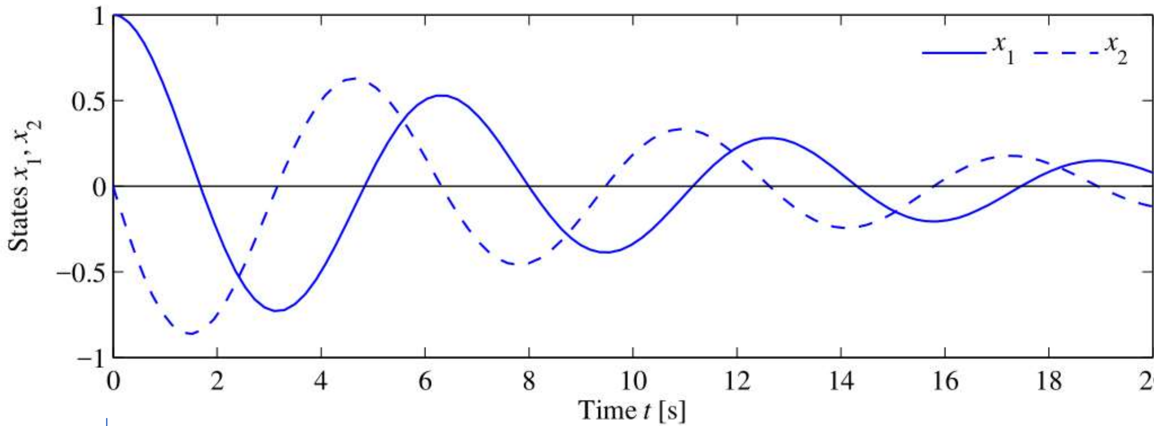
- assume $\zeta < 1$, i.e. lightly damped
- setting $x = (x_1, x_2) = (q, \dot{q}/\omega_0)$,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \omega_0 x_2 \\ -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{bmatrix} = F(x)$$

$$\Rightarrow x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

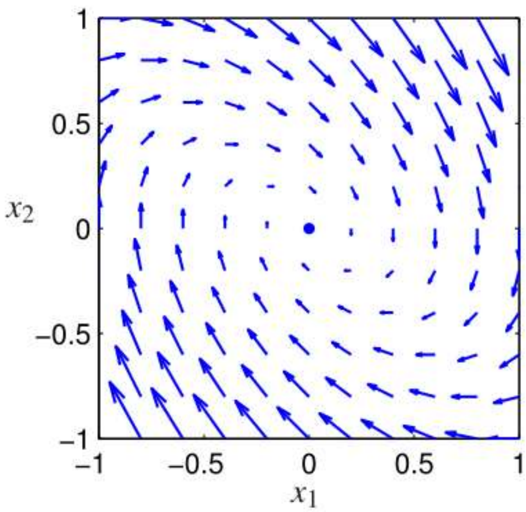
$$= e^{-\zeta\omega_0 t}\begin{bmatrix} x_1(0)\cos\omega_d t + \frac{1}{\omega_d}(\omega_0\zeta x_1(0) + x_2(0))\sin\omega_d t \\ x_2(0)\cos\omega_d t + \frac{1}{\omega_d}(\omega_0^2 x_1(0) + \omega_0\zeta x_2(0))\sin\omega_d t \end{bmatrix}$$

$\to$ verify this function satisfies DE for all $t \geq 0$



↳ initial state $x(0) = (x_1(0), x_2(0)) = (1, 0)$

- since $x \in \mathbb{R}^2$, $\dot{x} = F(x)$ can be visualized as a <u>vector field</u>



$x = (x_1, x_2)$

$F(x)$

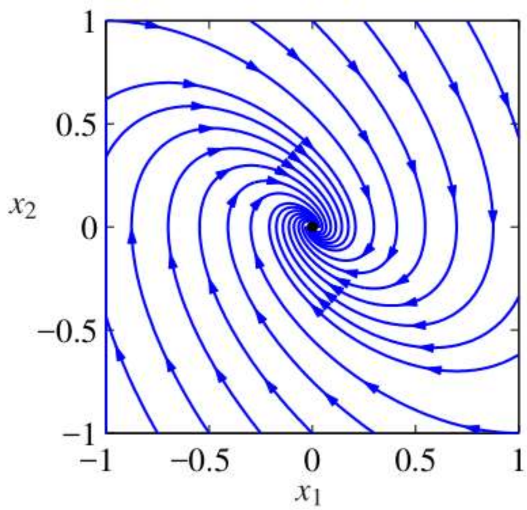
– $F(x)$ specifies the direction (& rate) of change in x:

$F(x(t))$

$x(t)$ $x(\tau)$

- overlaying a representative collection of trajectories yields a <u>phase portrait</u>

↳ "phase" terminology is inherited from physics (classical mechanics)



1.2: equilibrium trajectories

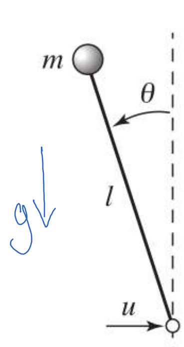
- some trajectories don't move — we say they are at <u>equilibrium</u>:

$$\dot{x}_e = F(x_e) = 0$$

→ importantly, if system is initialized at x_e, it stays there:

$$x(0) = x_e \Rightarrow x(t) = x_e, \text{ all } t \geq 0$$

<u>ex:</u> inverted pendulum



(<u>very</u> simple) model for rocket flight

–

– state $x = (\theta, \dot{\theta})$

– input u – horizontal acceleration

of pivot point

- $ml^2 \ddot{\theta} = mgl \sin\theta - \gamma \dot{\theta} + l u \cos\theta$

  where γ is coefficient of rotational friction

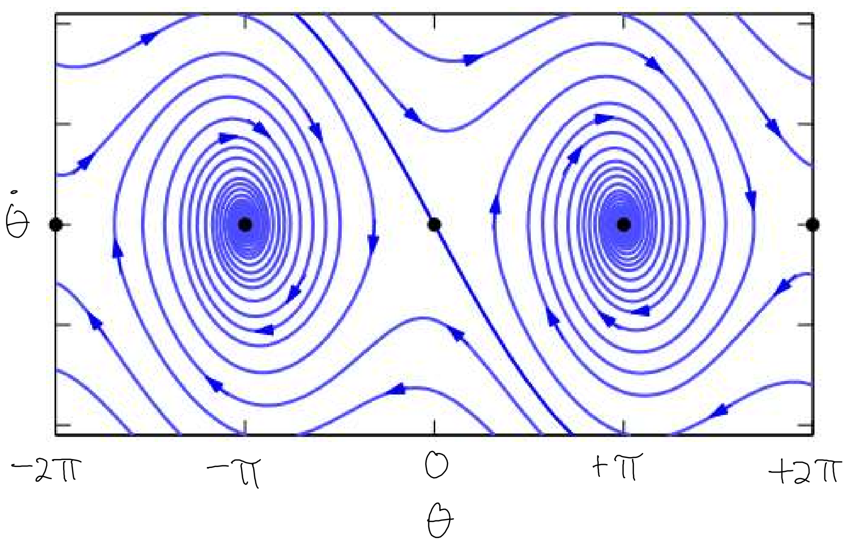
- with no input $(u = 0)$,

  $\dot{x}_e = (\dot{\theta}_e, \ddot{\theta}_e) = 0 \iff \dot{\theta}_e = 0, \ \sin\theta_e = 0,$

  so $\theta_e = n\pi, \ n \in \{0, +1, -1, +2, -2, \ldots\} = \mathbb{Z}$

$\longrightarrow$ what is the physical configuration when n even? n odd?

- phase portrait:



2°. stability

- a fundamental goal of feedback is to ensure <u>stability</u>, that is, to ensure trajectories converge to steady-state

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2'. definition of stability

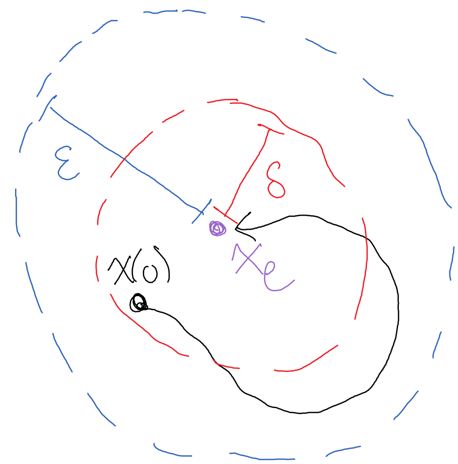
- given an equilibrium x_e s.t. $\dot{x}_e = F(x_e) = 0$, we'll say that x_e is <u>stable</u> if trajectories that start close stay close & get closer over time:

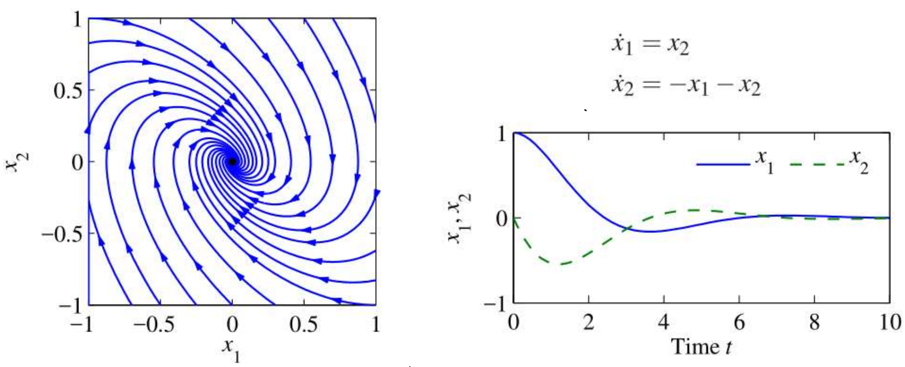
$\forall \varepsilon > 0$: $\longleftarrow$ closeness of $x(t)$ to x_e

$\exists \delta > 0$: $\longleftarrow$ closeness of $x(0)$ to x_e

$$\| x(0) - x_e \| < \delta \implies \| x(t) - x_e \| < \varepsilon$$

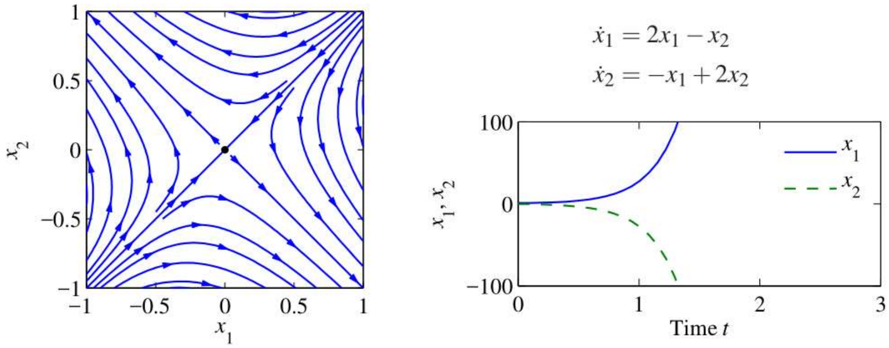
and $x(t) \longrightarrow x_e$ as $t \longrightarrow \infty$





- if δ cannot be made arbitrarily large (when ε is large), stability is <u>local</u>; otherwise, stability is <u>global</u>

∘ if a trajectory is not stable, then we term it <u>unstable</u>:



2^{2}. stability of linear DE

- consider $\dot{x} = Ax$

→ what is the shape of A?

- note that 0 (zero; the origin) is an equilibrium for every matrix A, since

$$\dot{0} = A \cdot 0 = 0$$

→ are there other equilibria $x_e \neq 0$?
(what does $Ax_e = 0$ tell you about the matrix A?)

- it turns out that stability of the origin is determined by the set of <u>eigenvalues</u> of A:

$$\lambda(A) = \{ s \in \mathbb{C} \mid \exists v \in \mathbb{C}^n : Av = sv \}$$

↳ v is an <u>eigenvector</u> w/ <u>eigenvalue</u> s

$$= \{ s \in \mathbb{C} \mid \det(sI - A) = 0 \}$$

↳ $\det : \mathbb{C}^{n \times n} \to \mathbb{R}$ is the <u>determinant function</u>

- recall that $\det(sI - A)$ is a polynomial expression in s,

$$\det(sI - A) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n,$$

termed the characteristic polynomial ← this terminology is not

$\det(sI - A) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$,

termed the characteristic polynomial ← this terminology is not accidental — related to characteristic polynomial of transfer functions

- since $\lambda(A)$ is the set of roots of the n-th degree polynomial $\det(sI - A)$, there are (at most) n distinct eigenvalues:

$$\lambda(A) = \{\lambda_1, \ldots, \lambda_j, \ldots, \lambda_n\}$$

→ if $s \in \lambda(A)$ and $A \in \mathbb{R}^{n \times n}$, show that $s^* \in \lambda(A)$ (s^* denotes complex-conjugate of $s \in \mathbb{C}$)

- to see how eigenvalues determine stability, consider the special case of diagonal A:

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- clearly $\dot{x}_1 = \lambda_1 x_1$ ← doesn't depend on x_2

  $\dot{x}_2 = \lambda_2 x_2$ ← doesn't depend on x_1

  so $x_1(t) = e^{\lambda_1 t} x_1(0)$, $x_2(t) = e^{\lambda_2} x_2(0)$

→ what condition on λ_1, λ_2 ensure $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is stable?

- more generally for $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$:

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & \lambda_{n-1} & 0 \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} \vdots & & & & \\ 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

- clearly $\dot{x}_j = \lambda_j x_j$ (independent of $x_{i \neq j}$)

  so $x_j(t) = e^{\lambda_j t} x_j(0)$,

  so origin is stable if $\lambda_j < 0$, all $j \in \{1, \cdots, n\}$

- another special case:
  - $\dot{x} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x = Ax$

  $\rightarrow$ determine eigenvalues of A

  (recall $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$)

  - we conclude that

$$x_1(t) = e^{\sigma t}(x_1(0)\cos\omega t + x_2(0)\sin\omega t)$$

$$x_2(t) = e^{\sigma t}(-x_1(0)\sin\omega t + x_2(0)\cos\omega t)$$

  $\rightarrow$ what condition on σ & ω ensure the origin is stable?

  - more generally, A could be block-diagonal:

$$\dot{x} = Ax = \begin{bmatrix} \sigma_1 & \omega_1 & & & \\ -\omega_1 & \sigma_1 & & \text{\huge 0} & \\ & & \ddots & & \\ & \text{\huge 0} & & \sigma_m & \omega_m \\ & & & -\omega_m & \sigma_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2m-1} \\ x_{2m} \end{bmatrix}$$

  - now eigenvalues $\lambda(A) = \{\sigma_k \pm j\omega_k\}_{k=1}^{m}$ and

$x_{2k-1}(t) = e^{\sigma_k t}\left(x_{2k-1}(0)\cos\omega_k t + x_{2k}(0)\sin\omega_k t\right)$

$x_{2k}(t) = e^{\sigma_k t}\left(-x_{2k-1}(0)\sin\omega_k t + x_{2k}(0)\cos\omega_k t\right)$

so origin is stable if $\sigma_k = \mathrm{Re}\,\lambda_k$

- most systems aren't (block) diagonal, but many can be transformed to be:

– if $\lambda(A)$ consists of n distinct eigvals, there exists invertible $T \in \mathbb{R}^{n\times n}$
  i.e. $\det T \neq 0$
  such that $TAT^{-1} \in \mathbb{R}^{n\times n}$ is (block) diagonal

– applying the change-of-coordinates $z = Tx$
  yields $\dot{z} = T\dot{x} = TAx = TAT^{-1}z$

→ show that $\lambda(A) = \lambda(TAT^{-1})$

– now given a trajectory $z : [0,\infty) \to \mathbb{R}^n$ for $\dot{z} = TAT^{-1}z$,
  $x : [0,\infty) \to \mathbb{R}^n$ defined by $x(t) = T^{-1}z(t)$
  is a trajectory for $\dot{x} = Ax$,
  so stability of z determines stability of x

2^3. parametric stability

- when designing a feedback controller, we've seen that model parameters can limit stabilizing controller parameters
  (e.g. PI control of 2nd-order system)

  semicolon implies μ doesn't vary with time

– consider $\dot{x} = F(x;\mu)$ where $x \in \mathbb{R}^n$ – states

$\mu \in \mathbb{R}^k$ – parameters

* vector μ can have model parameters (e.g. R, L, C; M, C, K; ...) and/or controller parameters (e.g. $k_P, k_I, k_D, \ldots$)

- since equilibrium x_e satisfies $\dot{x}_e = F(x_e; \mu) = 0$, the equilibrium generally varies with parameters: $x_e(\mu)$

∘ in linear systems, the equilibrium won't move: $\dot{0} = A(\mu) \cdot 0 = 0$ (we're always interested in $x_e = 0$)

- plotting eigenvalues $\lambda(A(\mu)) : \mathbb{R} \rightrightarrows \mathbb{C}$ special notation for multi-valued function
as parameter μ varies $: \mu \mapsto \{\lambda_j\}_{j=1}^n$
is termed a <u>root locus diagram</u> — <u>root</u> = eigenvalues
<u>locus</u> = image/graph of eigenvalues in $\mathbb{C}$

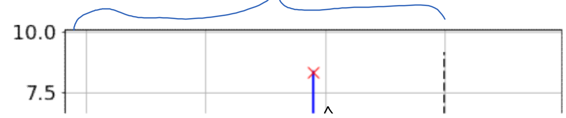
<u>ex:</u> proportional-integral control of first-order system

∘ $P(s) = \dfrac{b}{s+a}$ ∘ $C(s) = k_P + k_I \dfrac{1}{s}$

∘ $G_{yv} = \dfrac{P}{1+PC} = \dfrac{bs}{s^2 + (a + bk_P)s + bk_I}$

∘ choosing parameters $a = 1$, $b = 1$, $k_P = 10$, roots of $s^2 + (a + bk_P)s + bk_I$

system is stable when all roots have negative real part

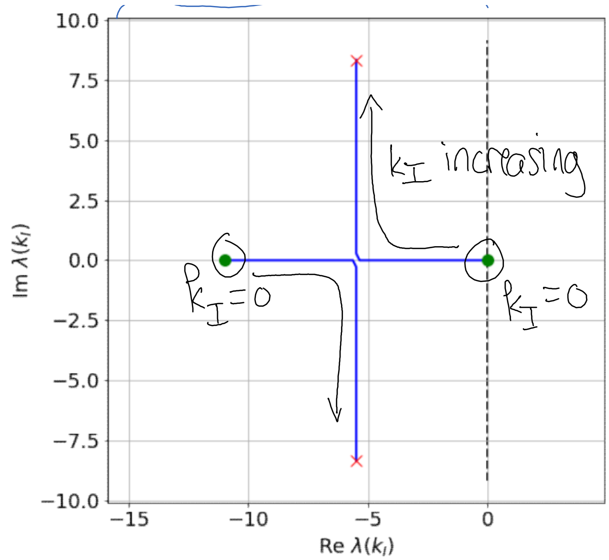


roots of $s^2 + (a + bk_P)s + bk_I$

are $-\frac{11}{2} \pm \frac{1}{2}\sqrt{121 - 4k_I}$;

plotting these in complex plane as a function of k_I yields:

"root locus" diagram →

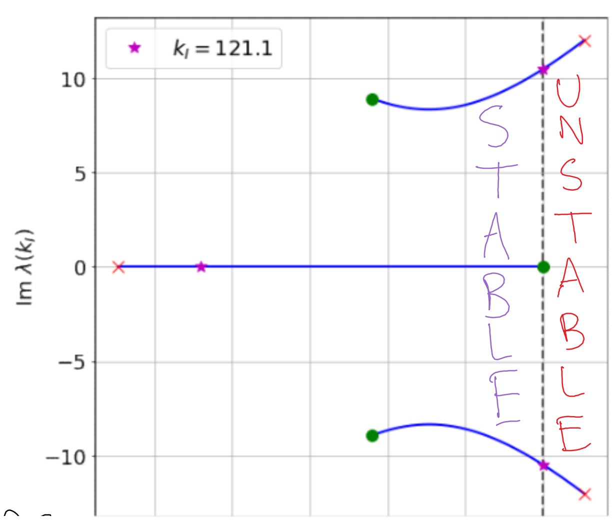


* indicates that k_I can be arbitrarily large, but that isn't physically realistic — large k_I will yield large inputs, which can excite unmodeled dynamics (of sensors, actuators, E&M coupling, vibrations, ...)

- including unmodeled dynamics yields $P(s) = \frac{b}{(s+a)(1+sT)}$

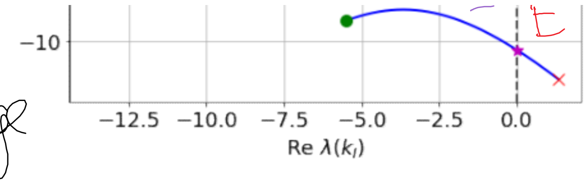
so $G_{yv} = \frac{P}{1+PC} = \frac{bs}{Ts^3 + (1+aT)s^2 + (a+bk_P)s + bk_I}$

- using parameters $a=1$, $b=1$, $T=\frac{1}{10}$, $k_P=10$ and plotting roots of G_{yv} in complex plane yields:



* importantly, two roots leave the left half-plane for $k_I > 121.1$, so the system

for $K_I > 121.1$, so the system is <u>unstable</u> when k_I is too large



<u>ex:</u> (linearized) bicycle

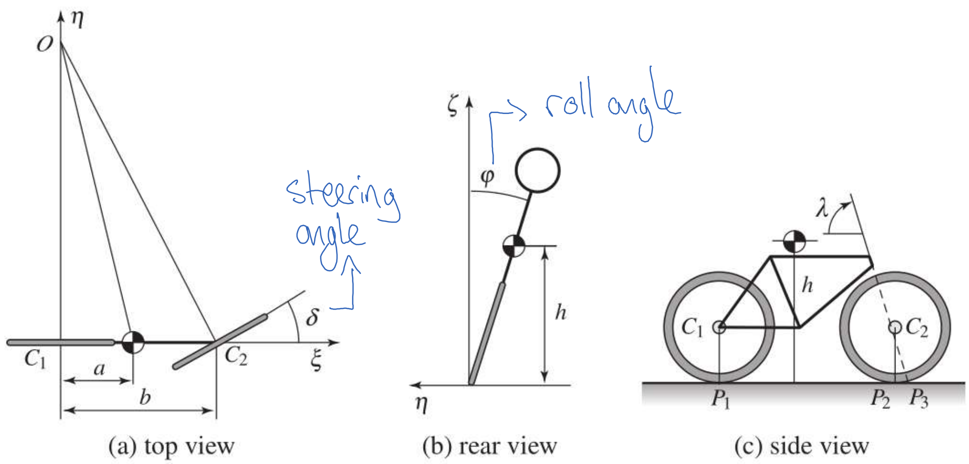


Figure 4.5: Schematic views of a bicycle. The steering angle is δ, and the roll angle is φ. The center of mass has height h and distance a from a vertical through the contact point P_1 of the rear wheel. The wheel base b is the distance between P_1 and P_2, and the trail c is the distance between P_2 and P_3.

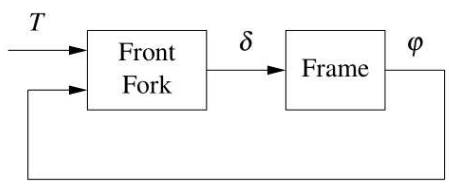


Figure 4.6: Block diagram of a bicycle with a front fork. The steering torque applied to the handlebars is T, the roll angle is φ and the steering angle is δ. Notice that the front fork creates a feedback from the roll angle φ to the steering angle δ that under certain conditions can stabilize the system.

- Whipple (linearized) model

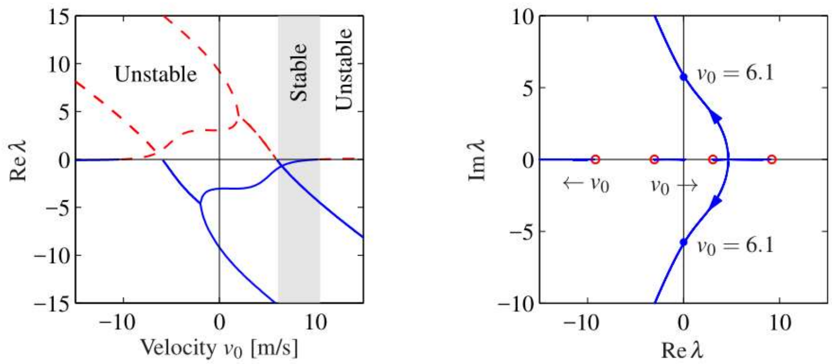
velocity of rear wheel

$$\underbrace{M}_{\in\mathbb{R}^{2\times 2}}\begin{bmatrix}\ddot{\varphi}\\ \ddot{\delta}\end{bmatrix} + \underbrace{C}_{\in\mathbb{R}^{2\times 2}} v_0 \begin{bmatrix}\dot{\varphi}\\ \dot{\delta}\end{bmatrix} + \underbrace{(K_0 + K_2 v_0^2)}_{\in\mathbb{R}^{2\times 2}}\begin{bmatrix}\varphi\\ \delta\end{bmatrix} = \begin{bmatrix}0\\ T\end{bmatrix}$$

- $x = (x_1, x_2, x_3, x_4) = (\varphi, \delta, \dot{\varphi}, \dot{\delta})$

2x2 identity matrix

$$\Rightarrow \dot{x} = Ax = \begin{bmatrix} 0 & I \\ -M^{-1}(K_0 + K_2 v_0^2) & -M^{-1} C v_0 \end{bmatrix} x$$



(a) Stability diagram (b) Root locus diagram

Figure 5.18: Stability plots for a bicycle moving at constant velocity. The plot in (a) shows the real part of the system eigenvalues as a function of the bicycle velocity v_0. The system is stable when all eigenvalues have negative real part (shaded region). The plot in (b) shows the locus of eigenvalues on the complex plane as the velocity v is varied and gives a different view of the stability of the system. This type of plot is called a *root locus diagram*.