[AMv2 Ch 3 & 4]

goal: further develop modeling tools
& apply them to physical phenomena

topics:

1°. modeling — [AMv2 Ch 3] [Nv7 Ch 3,4,5]

- 1¹. concepts
- 1². state space models
- 1³. numerical simulation

2°. examples

- 2¹. RLC circuit
- 2². quadrotor
- 2*. cruise control
- 2*. population dynamics

} [AMv2 Ch 4] from prior year; not required Fa19

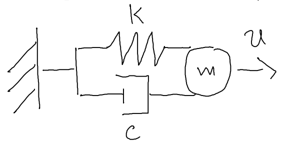
1°. modeling

1¹. concepts

- a model is a mathematical representation of a physical phenomenon, eg
  - mechanical – electrical – biological
  - computational system

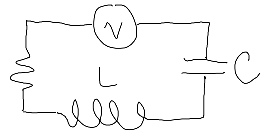
  ↳ we've already seen 3 types of model:
  - i) differential equation
  - ii) transfer function
  - iii) block diagram
- some models are simple, instantaneous relationships, e.g. given a closed circuit, the voltages & currents in each lumped element are related mathematically via Kirchoff's laws
- we'll focus instead on dynamical models

wherein quantities of interest
(positions & velocities ; currents & voltages)
change over time – we've already seen an example:

ex: spring-mass-damper: 

position q and velocity $\dot{q}$ interact over time
via (DE) $\underbrace{m}_{\text{mass}}\ddot{q} + \underbrace{c}_{\text{damping}}\dot{q} + \underbrace{k}_{\text{stiffness}} q = u$ – input force

note: given initial $(q(0), \dot{q}(0))$ and input
force $u:[0,\infty)\to\mathbb{R}$, the (DE)
$: t \mapsto u(t)$
determines $(q(t), \dot{q}(t))$ for all $t \geq 0$

ex: RLC circuit 

capacitor charge q & current $\dot{q}$
interact over time
via (DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = V$

note: given initial $(q(0), \dot{q}(0))$ and input
voltage $V:[0,\infty)\to\mathbb{R}$, the (DE)
$: t \mapsto u(t)$
determines $(q(t), \dot{q}(t))$ for all $t \geq 0$

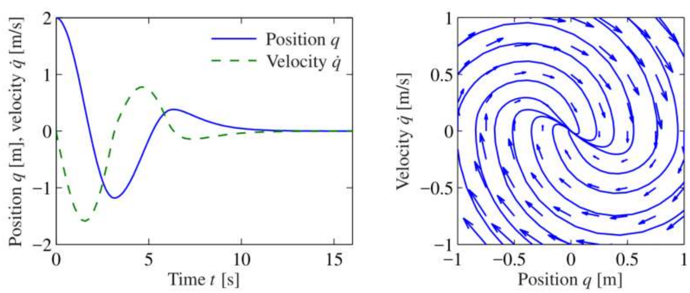


Figure 3.2: Illustration of a state model. A state model gives the rate of change of the state as a function of the state. The plot on the left shows the evolution of the state as a function of time. The plot on the right, called a *phase portrait*, shows the evolution of the states relative to each other, with the velocity of the state denoted by arrows.

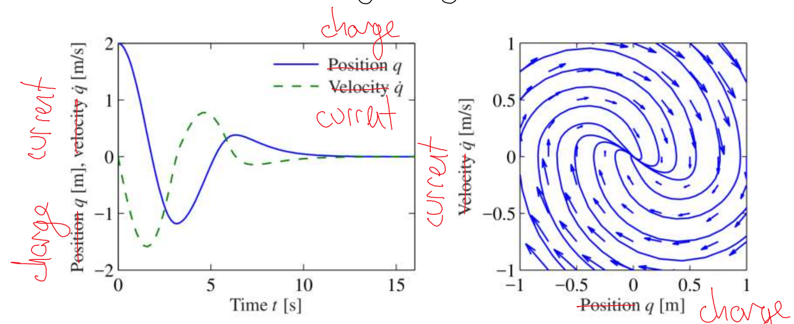


Figure 3.2: Illustration of a state model. A state model gives the rate of change of the state as a function of the state. The plot on the left shows the evolution of the state as a function of time. The plot on the right, called a *phase portrait*, shows the evolution of the states relative to each other, with the velocity of the state denoted by arrows.

1°: state-space models

- generalizing the preceding example,
  let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ denote <u>state vector</u>

  and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in \mathbb{R}^p$ denote <u>input vector</u>

  – then the state could change in time
  according to a

differential equation $\frac{d}{dt}x = \dot{x} = f(x,u)$

or

difference equation $x^+ = f(x,u)$

– we'll refer to both as a (DE);
to distinguish them notationally,
write $x(t)$ for state of differential eqn. at continuous time $t \in \mathbb{R}$

and $x[k]$ for state of difference eqn. at discrete time $k \in \mathbb{N}$

– in either case, $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$
$: (x,u) \mapsto \dot{x}$ or x^+

is assumed to be a smooth function so that derivatives $\frac{d}{dx}f$, $\frac{d}{du}f$ exist

→ in spring-mass-damper:

$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, so $\frac{d}{dt}x = \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = f(x,u)$

where $\ddot{q} = \frac{1}{m}(u - c\dot{q} - kq)$

→ in RLC circuit:

$\ddot{q} = \frac{1}{L}(v - R\dot{q} - \frac{1}{C}q)$

– the (DE) is <u>linear</u> if f is linear:

$f(x,u) = Ax + Bu$, A & B are matrices

→ determine shapes of matrices A, B

<u>ex</u>: we previously saw linear (DE)

$$\frac{d^n}{dt^n}y + a_1 \frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = u$$

– letting $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{d^{n-1}}{dt^{n-1}}y \\ \frac{d^{n-2}}{dt^{n-2}}y \\ \vdots \\ \frac{d}{dt}y \end{bmatrix}$

$$\begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \qquad \begin{bmatrix} \frac{d}{dt} y \\ y \end{bmatrix}$$

yields

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 x_1 - \cdots - a_n x_n \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A \in \mathbb{R}^{n \times n}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B \in \mathbb{R}^{n \times 1}} u$$

$$= Ax + Bu$$

↳ so the two notions of linear (DE) coincide

<u>ex:</u> embedded system

- consider the PI controller

$$u(t) = k_P e(t) + k_I \int_0^t e(\tau)\, d\tau$$

$$= k_P e(t) + k_I x(t)$$

so that $\dot{x}(t) = e(t)$ (DE)

↳ i.e. $x(t) = \int_0^t e(\tau)\, d\tau$ is the <u>controller's</u> state

- to implement on an embedded system, microprocessor will measure error at sampling intervals $t = \Delta, 2\Delta, 3\Delta, \ldots$
- approximating derivative in (DE) yields

$$\frac{x((k+1)\Delta) - x(k\Delta)}{\Delta} \simeq \dot{x}(k\Delta) = e(k\Delta)$$

where the digital controller satisfies

$$\tilde{x}[k+1] = \tilde{x}[k] + \Delta\, \tilde{e}[k]$$

where $\tilde{x}[k] \simeq x(k\Delta)$, $\tilde{e}[k] \simeq e(k\Delta)$

* digital controller easy to implement on microprocessor since it requires only addition and multiplication operations

→ how does performance depend on Δ?
 – think about the limits as $\Delta \to 0$ or ∞

1³. numerical simulation

- the discrete approximation employed in the digital controller suggests a computational approach to study dynamical systems in state-space form:

$$(DE)\quad \dot{x} = f(x,u),\quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

 – approximating derivative in (DE) yields

$$\frac{x((k+1)\Delta) - x(k\Delta)}{\Delta} \simeq \dot{x}(k\Delta) = f(x(k\Delta), u(k\Delta))$$

 whence the difference equation

$$(\widetilde{DE})\quad \tilde{x}[k+1] = \tilde{x}[k] + \Delta f(\tilde{x}[k], u(k\Delta))$$

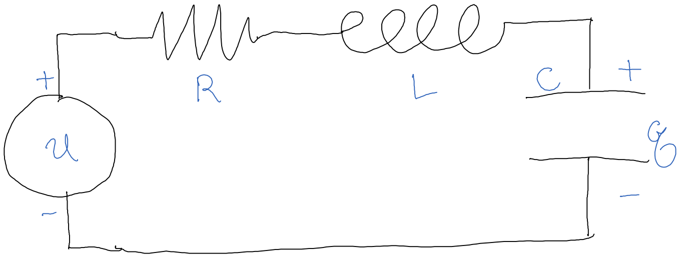
 yields the approximation $\tilde{x}[k] \simeq x(k\Delta)$

* importantly, $(\widetilde{DE})$ can be inductively applied on a computer to compute $\tilde{x}$, whereas given a nonlinear function f, you generally can't determine solution x to (DE)

→ how does approximation depend on Δ?
 – think about the limits as $\Delta \to 0$ or ∞

2°. examples

2¹. RLC circuit



- treating voltage u as input and capacitor charge q as output,

$$L\ddot{q} + R\dot{q} + 1/C\, q = u$$

- letting $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^2$ denote circuit state,

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ (-R\dot{q} - 1/C\, q + u)/L \end{bmatrix}$$

$$= f((q,\dot{q}),u) = f(x,u)$$

→ show that f is <u>linear</u>:

$$f(x,u) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/L \end{bmatrix}} u$$

→ i.e. find $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$

2^2. quadrotor

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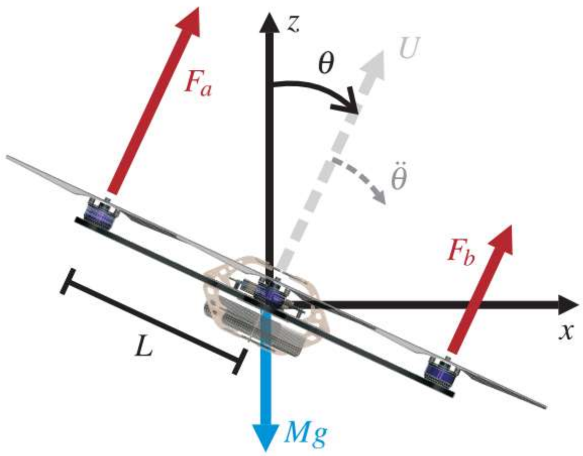
May 3-8, 2010, Anchorage, Alaska, USA

[ICRA]

A Simple Learning Strategy for High-Speed Quadrocopter Multi-Flips

Sergei Lupashin, Angela Schöllig, Michael Sherback, Raffaello D'Andrea



$$
\begin{aligned}
M\ddot{z} &= (F_a + F_b + F_c + F_d)\cos\theta - Mg && (1)\\
M\ddot{x} &= (F_a + F_b + F_c + F_d)\sin\theta && (2)\\
I_{yy}\ddot{\theta} &= L(F_a - F_b)\,, && (3)
\end{aligned}
$$

$\eta = x$ (horizontal)

$\nu = z$ (vertical)

$$M\ddot{\eta} = F\sin\theta$$

$$M\ddot{\nu} = -Mg + F\cos\theta$$

$$I\ddot{\theta} = \tau$$

- where $F = F_a + F_b + F_c + F_d$ is the sum of thrusts from all 4 rotors
  $\tau = L(F_a - F_b)$ is the net torque around the roll axis

- with $q = (\eta, \nu, \theta) \in \mathbb{R}^3$ denoting positions and
  $\dot{q} = \frac{d}{dt} q = (\dot{\eta}, \dot{\nu}, \dot{\theta}) \in \mathbb{R}^3$ denoting velocities,
  the state is $x = (q, \dot{q}) \in \mathbb{R}^6$, input is $u = (F, \tau) \in \mathbb{R}^2$,
  so dynamics are $\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q}(x,u) \end{bmatrix} = f(x,u)$

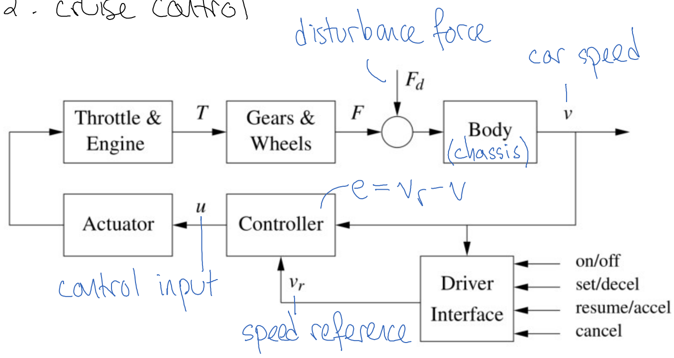
  where $\ddot{q}(x,u) = \begin{bmatrix} F/M \sin\theta \\ -g + F/M \cos\theta \\ \tau/I \end{bmatrix}$, $f : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^6 : (x,u) \mapsto \dot{x}$

- if we measure positions (η, ν), eg with GPS or motion capture,
  the output function is $y = (\eta, \nu) = h(x)$ where $h : \mathbb{R}^6 \to \mathbb{R}^2 : x \mapsto (\eta, \nu)$

→ determine an equilibrium (x^*, u^*) such that $f(x^*, u^*) = 0$

* determine all equilibria

2*: cruise control



- force balance on car chassis yields

$$m\dot{v} = F - F_d$$

  - m is mass of car, passengers, load
  - v is velocity / speed of car
  - F is engine force, proportional to control signal $0 \leq u \leq 1$ that specifies throttle position (which in turn specifies fuel injection rate), and varying with engine (angular) speed ω:

$$F = \frac{Ru}{r} T(\omega), \quad \omega = \frac{R}{r} v$$

$$T(\omega) = T_m \cdot \left(1 - \beta\left(\frac{\omega}{\omega_m} - 1\right)^2\right)$$

where: T_m - max torque @ speed ω_m
R – gear ratio r – wheel radius

- $F_d = F_g + F_r + F_a$ is disturbance force:
  F_g – gravitational force from road slope
  F_r – rolling/road friction
  F_a – aerodynamic drag

  $F_g = m\,g\,\sin\theta$, θ – road slope, g – gravitational constant

  $F_r = m\,g\,C_r \frac{v}{|v|}$, C_r – coefficient of friction

  $F_a = \frac{1}{2}\rho\,C_d\,A\,|v|\,v$, ρ – air density
  C_d – aerodynamic drag (shape-dependent)
  A – frontal area of car

* see [AMv2, Ch 4.1] for parameter values

- taken together,

$$m\dot{v} = \underbrace{\frac{R}{r}\,u\,T\left(\frac{R}{r}v\right)}_{F} \underbrace{\underbrace{-m\,g\sin\theta}_{F_g} \underbrace{- m\,g\,C_r\frac{v}{|v|}}_{F_r} \underbrace{-\frac{1}{2}\rho\,C_d\,A\,v^2}_{F_a}}_{F_d}$$

→ what is the system state? control input?

- let's apply PI control:

$$u(t) = k_P e(t) + k_I\int_0^t e(\tau)\,d\tau,\quad e = v_r - v$$

with $z = \int_0^t e(\tau)\,d\tau$ denoting control state,

$$\dot{z} = v_r - v,\quad u = k_P(v_r - v) + k_I z$$

- applying analysis of PI control from previous lecture yields nice conclusions:
  - when disturbance force F_d is constant, the steady-state error will be zero,
  $$e = v_r - v = 0 \Rightarrow v = v_r\,!$$
  - this conclusion is valid in the presence of unmodeled dynamics, so long as
    i) they're stable
    ii) the PI gains aren't too large

relative to their characteristic time constant

→ (why) does the prior analysis of PI control apply to this nonlinear system?

2*: population dynamics

- let x denote population of an organism
  - assuming birth & death rates are proportional to the current population,

$$\dot{x} = bx - dx = (b-d)\,x,$$

where b is birth rate, d is death rate;

→ how does this model behave over time?

this is a linear model that diverges to $+\infty$ if $b > d$, converges to 0 if $b < d$

→ (when) is this a reasonable model?
(when) isn't it " "?

- more realistically, the environment permits a <u>carrying capacity</u> $k > 0$, and the birth rate decreases as the population approaches k:

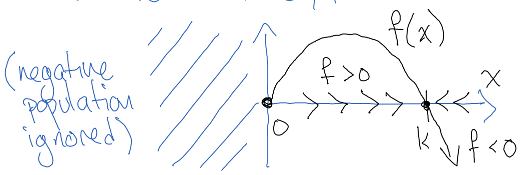
$\dot{x} = r\,x\left(1 - \frac{x}{k}\right)$ where $r > 0$ is <u>growth rate</u>

↳ termed the <u>logistic</u> growth model

→ how does this model behave over time?

- the graph of f in $\dot{x} = f(x)$ tells us a lot about model's behavior:

- since $r, k > 0$, f is a concave parabola with roots at $0, k$:



* $x(t) \to k$ from any $x(0) > 0$ as $t \to \infty$?

→ $x = 0$ or k termed <u>equilibria</u>; why?

- now we'll consider two interacting populations:

let $H \geq 0$ denote number of hares (prey),

$L \geq 0$ " " lynxes (predator)

and consider the state-space DE

$$\dot{H} = r H \left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}$$

$$\dot{L} = b \frac{aHL}{c+H} - dL$$

↳ note: first term is logistic growth

where r – growth rate of hares, k – carrying capacity of hares

a – predation rate, c – limits predation at low H

b – growth rate of lynxes, d – death rate of lynxes

- with $x = (H, L)$, $\dot{x} = f(x)$, it's not as easy to visualize graph of f

→ how many dimensions are needed to graph $\{(x, f(x))\}$?

- however, it's still helpful to visualize salient features of f, eg equilibria, that is, x_e s.t. $\dot{x}_e = f(x_e) = 0$

– second equation is simpler, so start there:

$\dot{L} = 0 \iff L_e = 0$ or $H_e = \frac{cd}{ab-d}$

– substituting into first eqn,

$$(H_e, L_e) = (0,0),\ (0,k),\ \left(\frac{cd}{ab-d},\ \frac{bcr(abk - cd - dk)}{(ab-d^2)k}\right)$$

are all equilibria

→ which of these 3 are ecologically feasible?

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* important practical note: all models are inaccurate; the robustness inherent in feedback enables us to use simple models to control complex phenomena (see previous lecture on unmodeled dynamics)

"all models are wrong but some are useful"
– George Box (statistician), 1978