goal: develop qualitative & quantitative tools to study nonlinear dynamics

topics:

1°. nonlinear dynamics [Nv7 Ch 2] [AMv2 Ch 5]

1¹. trajectories

1². collections of trajectories

1³. equilibrium trajectories

2°. stability

2¹. definition of stability [Nv7 Ch 6]

2². stability of linear DE [AMv2 Ch 6]

2³. parametric stability [Nv7 Ch 8]

* read [AMv2 Ch 5.4] to learn about an advanced test for stability

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1°. nonlinear control systems

- consider a DE with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^p$

$$\dot{x} = f(x,u)$$

- the simplest type of feedback sets u as a function of x, e.g. $u = \alpha(x)$, so

$$\dot{x} = f(x,\alpha(x)) = F(x)$$

→ we'll focus on $\dot{x} = F(x)$ below

1: trajectories

- $x : [0,\infty) \to \mathbb{R}^n$ is a <u>trajectory</u> of DE with <u>initial state</u> $x(0)$

  if $\forall t \geq 0 : \dot{x}(t) = F(x(t))$

  $\hookrightarrow$ so x is continuous <u>and</u> differentiable

* function x satisfies an infinite number of equations

$\hookrightarrow$ we'll assume unique trajectories always exist, but this isn't guaranteed in general; see [AMv2 ex 5.2, 5.3]

<u>ex</u>: damped oscillator (i.e. RLC circuit; spring-mass-damper)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = 0$$

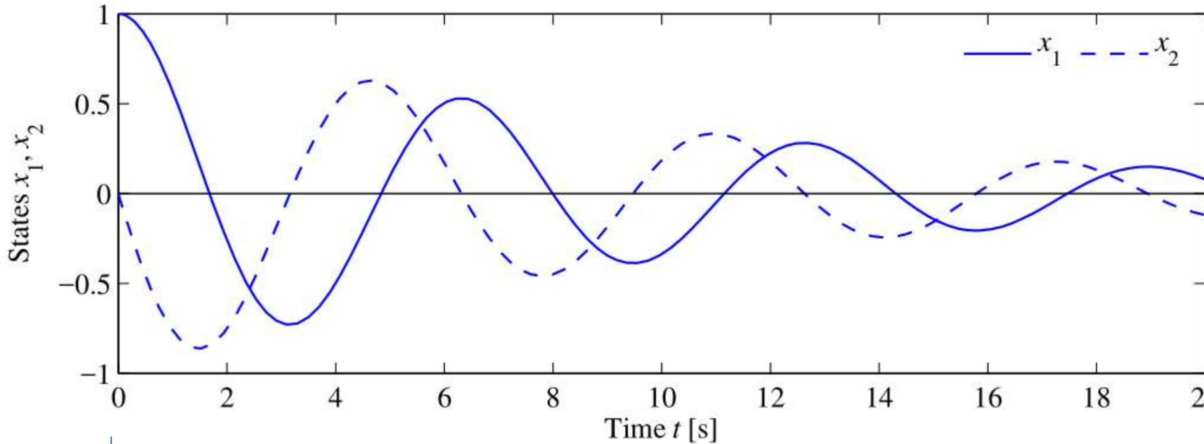
- assume $\zeta < 1$, i.e. lightly damped
- setting $x = (x_1, x_2) = (q, \dot{q}/\omega_0)$,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \omega_0 x_2 \\ -\omega_0 x_1 - 2\zeta\omega_0 x_2 \end{bmatrix} = F(x)$$

$$\Rightarrow x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= e^{-\zeta\omega_0 t}\begin{bmatrix} x_1(0)\cos\omega_d t + \frac{1}{\omega_d}\left(\omega_0\zeta x_1(0) + x_2(0)\right)\sin\omega_d t \\ x_2(0)\cos\omega_d t + \frac{1}{\omega_d}\left(\omega_0^2 x_1(0) + \omega_0\zeta x_2(0)\right)\sin\omega_d t \end{bmatrix}$$

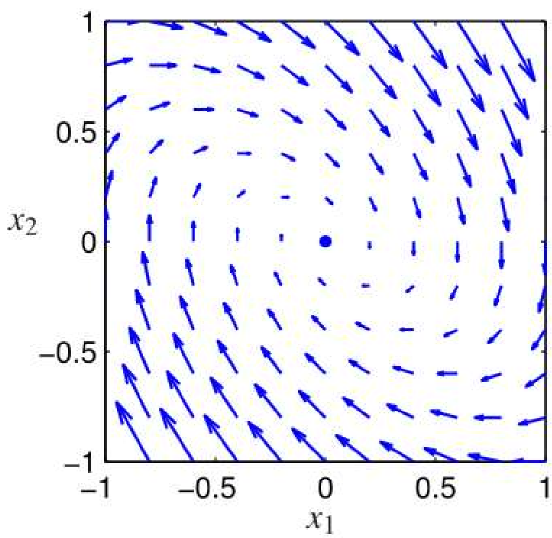
$\to$ verify this function satisfies DE for all $t \geq 0$



↳ initial condition $x(0) = (x_1(0), x_2(0)) = (1, 0)$

1.3. collections of trajectories

- when $x \in \mathbb{R}^2$, $\dot{x} = F(x)$ can be visualized as a <u>vector field</u>



$x = (x_1, x_2)$

$F(x)$

– $F(x)$ specifies the direction (& rate) of change in x:

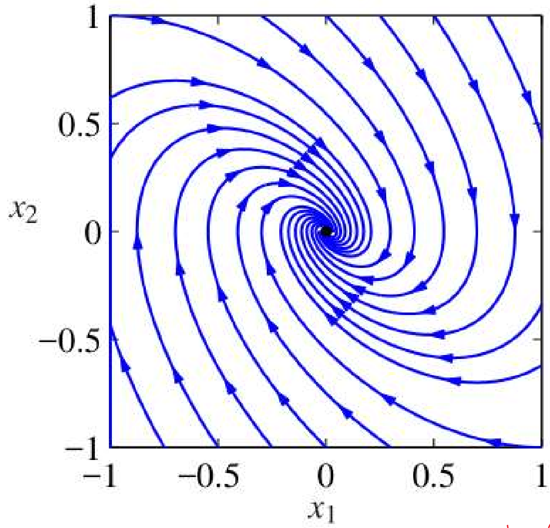
$F(x(t))$

$x(t)$ $x(\tau)$

- overlaying a representative collection of trajectories yields a <u>phase portrait</u>

↳ "phase" terminology is inherited

from physics (classical mechanics)



→ this is the phase portrait for a system we've seen; which system?

1³. equilibrium trajectories

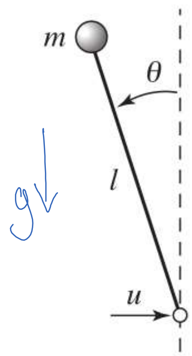
- some trajectories don't move — we say they are at <u>equilibrium</u>:

$$\dot{x}_e = F(x_e) = 0$$

→ importantly, if system is initialized at x_e, it stays there:

$$x(0) = x_e \Rightarrow x(t) = x_e, \text{ all } t \geq 0$$

<u>ex:</u> inverted pendulum



(<u>very</u> simple) model for rocket flight

- state $x = (\theta, \dot\theta)$
- input u – horizontal acceleration of pivot point
- $ml^2 \ddot\theta = mgl \sin\theta - \gamma\dot\theta + l u \cos\theta$

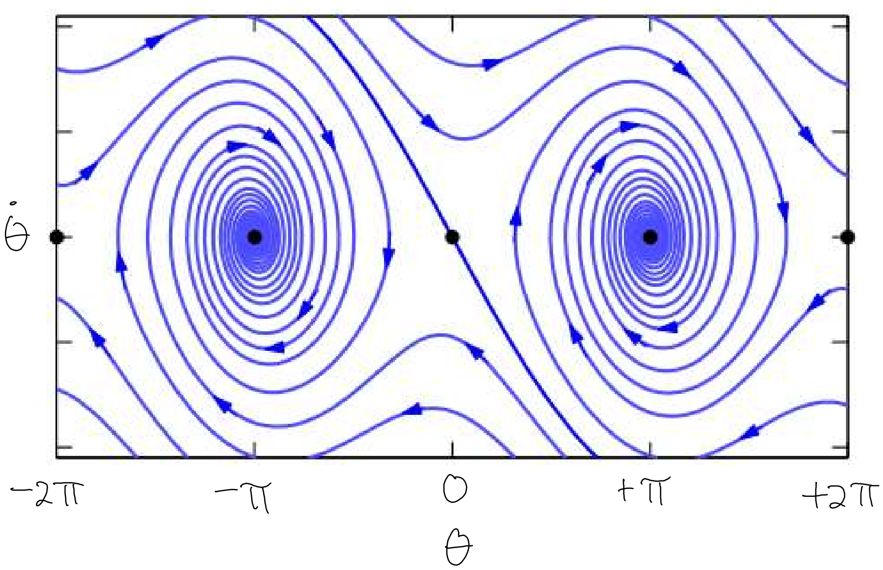
  where γ is coefficient of rotational friction
- with no input $(u = 0)$,

  $\dot x_e = (\dot\theta_e, \ddot\theta_e) = 0 \iff \dot\theta_e = 0,\ \sin\theta_e = 0,$

  so $\theta_e = n\pi,\ n \in \{0, +1, -1, +2, -2, \dots\} = \mathbb{Z}$

→ what is the physical configuration when n even? n odd?

- phase portrait:



2°. stability

- a fundamental goal of feedback is to ensure stability, that is, to ensure trajectories converge to steady-state

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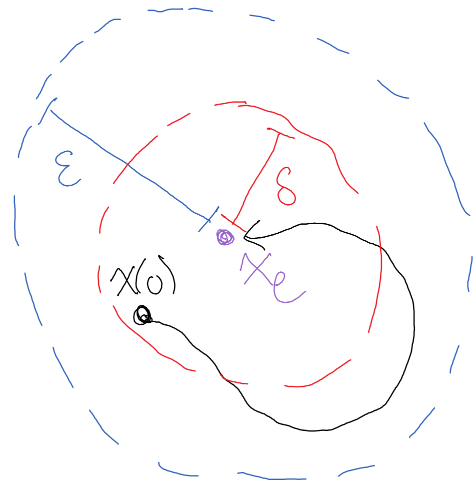
2': definition of stability

- given an equilibrium x_e s.t. $\dot{x}_e = F(x_e) = 0$, we'll say that x_e is <u>stable</u> if trajectories that start close stay close & get closer over time:

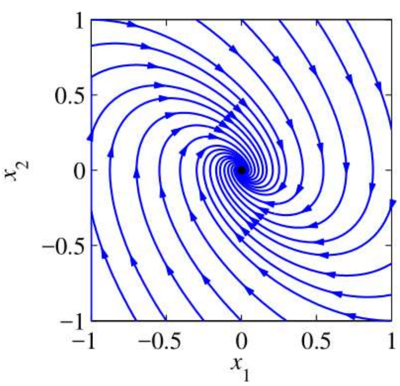


$\forall \varepsilon > 0$: $\longleftarrow$ closeness of $x(t)$ to x_e

$\exists \delta > 0$: $\longleftarrow$ closeness of $x(0)$ to x_e

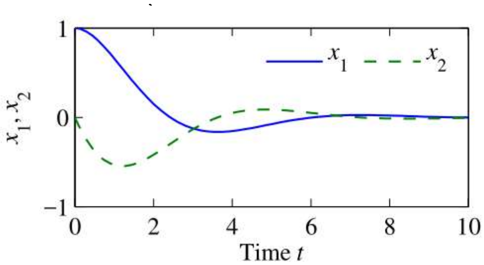
$\|x(0) - x_e\| < \delta \implies \|x(t) - x_e\| < \varepsilon$

and $x(t) \longrightarrow x_e$ as $t \longrightarrow \infty$



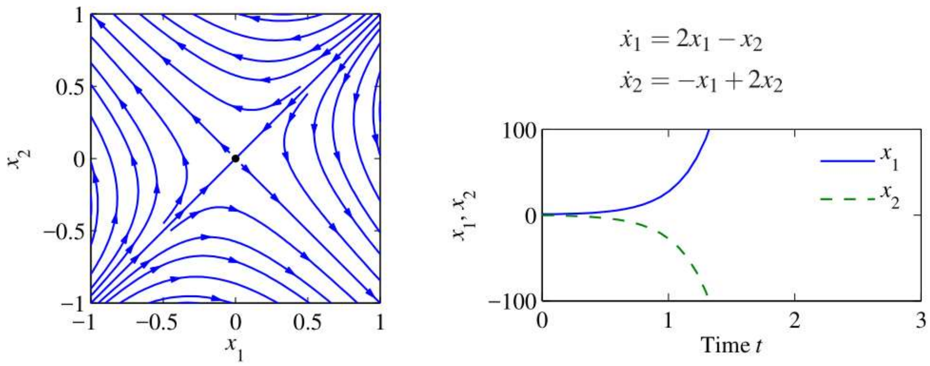
$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - x_2$



- if δ cannot be made arbitrarily large (when ε is large), stability is <u>local</u>; otherwise, stability is <u>global</u>

- if a trajectory is not stable, then we term it <u>unstable</u>:



2². stability of linear DE

- consider $\dot{x} = Ax$

→ what is the shape of A?

- note that 0 (zero; the origin) is an equilibrium for every matrix A, since

$$\dot{0} = A \cdot 0 = 0$$

→ are there other equilibria $x_e \neq 0$?
(what does $Ax_e = 0$ tell you about the matrix A?)

- it turns out that stability of the origin is determined by the set of eigenvalues of A:

$$\lambda(A) = \{ s \in \mathbb{C} \mid \exists v \in \mathbb{C}^n : Av = sv \}$$

↳ v is an eigenvector w/ eigenvalue s

$= \{ s \in \mathbb{C} \mid \det(sI - A) = 0 \}$

↳ $\det : \mathbb{C}^{n \times n} \to \mathbb{R}$ is the determinant function

- recall that $\det(sI - A)$ is a polynomial expression in s,

$$\det(sI - A) = a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n,$$

termed the characteristic polynomial ← this terminology is not accidental — related to characteristic polynomial of transfer functions

- since $\lambda(A)$ is the set of roots of the n-th degree polynomial $\det(sI - A)$, there are (at most) n distinct eigenvalues:

$$\lambda(A) = \{ \lambda_1, \dots, \lambda_j, \dots, \lambda_n \}$$

→ if $s \in \lambda(A)$ and $A \in \mathbb{R}^{n \times n}$, show that $s^* \in \lambda(A)$ (s^* denotes complex-conjugate of $s \in \mathbb{C}$)

- to see how eigenvalues determine stability, consider the special case of diagonal A:

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- clearly $\dot{x}_1 = \lambda_1 x_1$ ← doesn't depend on x_2

$\dot{x}_2 = \lambda_2 x_2$ ← doesn't depend on x_1

so $x_1(t) = e^{\lambda_1 t} x_1(0)$, $x_2(t) = e^{\lambda_2 t} x_2(0)$

→ what condition on λ_1, λ_2 ensure $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is stable?

- more generally for $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$:

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & & \lambda_{n-1} & 0 \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

- clearly $\dot{x}_j = \lambda_j x_j$ (independent of $x_{i \neq j}$)

so $x_j(t) = e^{\lambda_j t} x_j(0)$,

so origin is stable if $\lambda_j < 0$, all $j \in \{1, \cdots, n\}$

- another special case:

- $\dot{x} = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x = Ax$

→ determine eigenvalues of A

(recall $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$)

- we conclude that

$$x_1(t) = e^{\sigma t}(x_1(0) \cos \omega t + x_2(0) \sin \omega t)$$

$$x_2(t) = e^{\sigma t}(-x_1(0) \sin \omega t + x_2(0) \cos \omega t)$$

→ what condition on σ & ω ensure the origin is stable?

- more generally, A could be block-diagonal:

$$\dot{x} = Ax = \begin{bmatrix} \sigma_1 & \omega_1 & & & \\ -\omega_1 & \sigma_1 & & 0 & \\ & & \ddots & & \\ & 0 & & \sigma_m & \omega_m \\ & & & -\omega_m & \sigma_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2m-1} \\ x_{2m} \end{bmatrix}$$

- now eigenvalues $\lambda(A) = \{\sigma_k \pm j\omega_k\}_{k=1}^m$ and

$$x_{2k-1}(t) = e^{\sigma_k t}\left(x_{2k-1}(0)\cos\omega_k t + x_{2k}(0)\sin\omega_k t\right)$$

$$x_{2k}(t) = e^{\sigma_k t}\left(-x_{2k-1}(0)\sin\omega_k t + x_{2k}(0)\cos\omega_k t\right)$$

so origin is stable if $\sigma_k = \mathrm{Re}\,\lambda_k$

- most systems aren't (block) diagonal, but many can be transformed to be:

- if $\lambda(A)$ consists of n distinct eigvals, there exists invertible $T \in \mathbb{R}^{n\times n}$ i.e. $\det T \neq 0$

  such that $TAT^{-1} \in \mathbb{R}^{n\times n}$ is (block) diagonal

- applying the change-of-coordinates $z = Tx$

  yields $\dot{z} = T\dot{x} = TAx = TAT^{-1}z$

→ show that $\lambda(A) = \lambda(TAT^{-1})$

- now given a trajectory $z : [0,\infty) \to \mathbb{R}^n$ for $\dot{z} = TAT^{-1}z$,

  $x : [0,\infty) \to \mathbb{R}^n$ defined by $x(t) = T^{-1}z(t)$

  is a trajectory for $\dot{x} = Ax$,

so stability of z determines stability of x

2^3. parametric stability

- when designing a feedback controller, we've seen that model parameters can limit stabilizing controller parameters (e.g. PI control of 2nd-order system)

semicolon implies μ doesn't vary with time

- consider $\dot{x} = F(x;\mu)$ where $x \in \mathbb{R}^n$ – states
  $\mu \in \mathbb{R}^k$ – parameters

  * vector μ can have model parameters (e.g. R, L, C; M, C, K; ...) and/or controller parameters (e.g. $k_P, k_I, k_D, \ldots$)

- since equilibrium x_e satisfies $\dot{x}_e = F(x_e;\mu) = 0$, the equilibrium generally varies with parameters: $x_e(\mu)$

- in linear systems, the equilibrium won't move: $\dot{0} = A(\mu) \cdot 0 = 0$ (we're always interested in $x_e = 0$)

special notation for multi-valued function

- plotting eigenvalues $\lambda(A(\mu)) : \mathbb{R} \rightrightarrows \mathbb{C}$
  as parameter μ varies $: \mu \mapsto \{\lambda_j\}_{j=1}^n$
  is termed a <u>root locus diagram</u> — <u>root</u> = eigenvalues
  <u>locus</u> = image/graph of eigenvalues in $\mathbb{C}$

ex: (linearized) bicycle

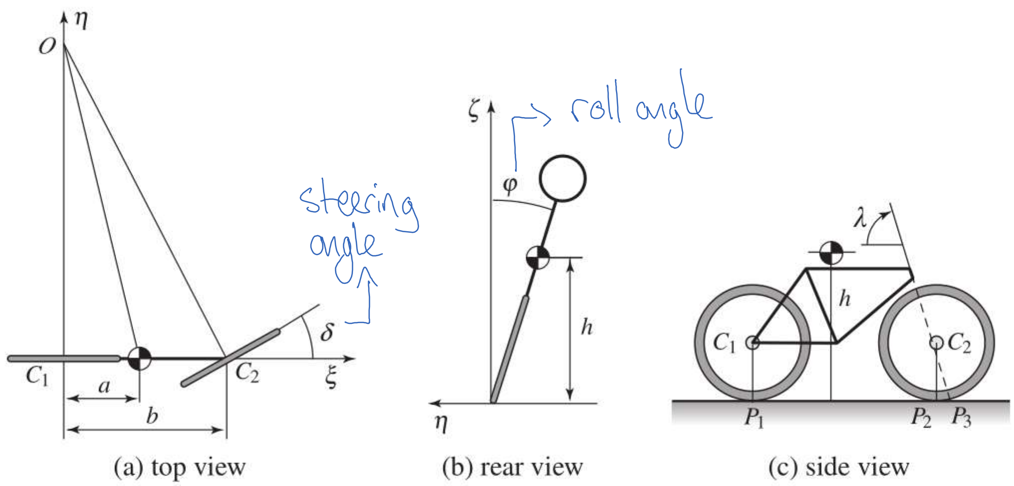


Figure 4.5: Schematic views of a bicycle. The steering angle is δ, and the roll angle is φ. The center of mass has height h and distance a from a vertical through the contact point P_1 of the rear wheel. The wheel base b is the distance between P_1 and P_2, and the trail c is the distance between P_2 and P_3.

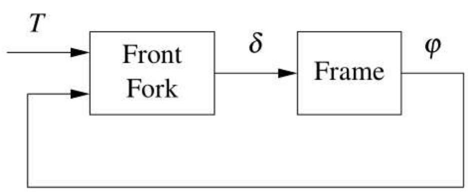


Figure 4.6: Block diagram of a bicycle with a front fork. The steering torque applied to the handlebars is T, the roll angle is φ and the steering angle is δ. Notice that the front fork creates a feedback from the roll angle φ to the steering angle δ that under certain conditions can stabilize the system.

- whipple (linearized) model

velocity of rear wheel

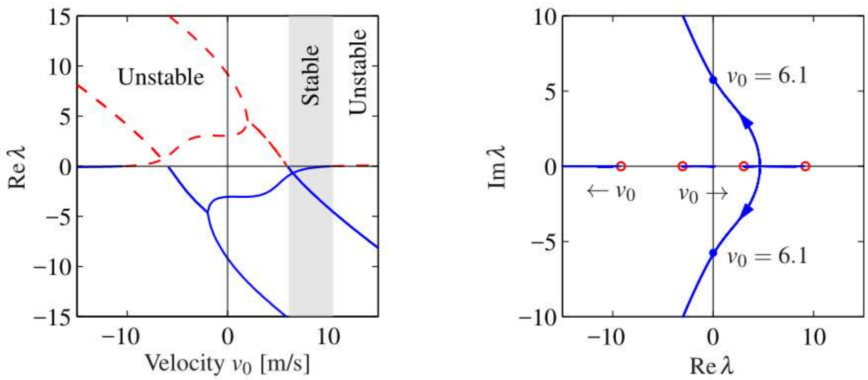
$$\underbrace{M}_{\in \mathbb{R}^{2\times 2}} \begin{bmatrix} \ddot{\varphi} \\ \ddot{\delta} \end{bmatrix} + \underbrace{C}_{\in \mathbb{R}^{2\times 2}} v_0 \begin{bmatrix} \dot{\varphi} \\ \dot{\delta} \end{bmatrix} + \underbrace{(K_0 + K_2 v_0^2)}_{\in \mathbb{R}^{2\times 2}} \begin{bmatrix} \varphi \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix}$$

- $x = (x_1, x_2, x_3, x_4) = (\varphi, \delta, \dot{\varphi}, \dot{\delta})$

2x2 identity matrix

$$\Rightarrow \dot{x} = Ax = \begin{bmatrix} 0 & I \\ -M^{-1}(K_0 + K_2 v_0^2) & -M^{-1} C v_0 \end{bmatrix} x$$

$$\Rightarrow \dot{x} = Ax = \begin{bmatrix} 0 & I \\ -M^{-1}(K_0 + K_2 v_0^2) & -M^{-1} C v_0 \end{bmatrix} x$$



(a) Stability diagram (b) Root locus diagram

Figure 5.18: Stability plots for a bicycle moving at constant velocity. The plot in (a) shows the real part of the system eigenvalues as a function of the bicycle velocity v_0. The system is stable when all eigenvalues have negative real part (shaded region). The plot in (b) shows the locus of eigenvalues on the complex plane as the velocity v is varied and gives a different view of the stability of the system. This type of plot is called a *root locus diagram*.