[AMv2 Ch 2]

goal: introduce fundamental uses and properties of feedback

topics:
1°. mathematical models of systems
  1¹. differential equations (DE)
  1². transfer functions
  1³. block diagrams
2°. effects of feedback
  2¹. disturbance attenuation
  2². unmodeled dynamics
  2³. reference tracking

* read [AMv2 Ch 2.2.5] to learn how positive feedback used in digital systems

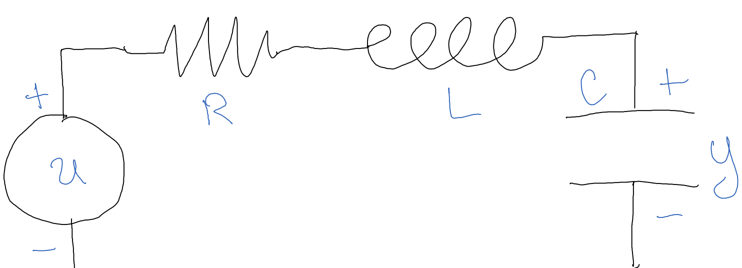
* HW0 assigned - due midnight Fri
* we will answer questions thru noon on Fri
* we will post HW on Fri
→ caveat: won't be solved until Mon
* create video of ipynb → pdf
* " " " " LaTeX

1°. mathematical models of systems
- we will work with multiple representations of linear control systems
  1¹. differential equations
  1². transfer functions
  1³. block diagrams

  each has unique advantages & provides novel insight

* what is a control system?

ex: consider an RLC circuit



→ determine the roots of the characteristic polynomial
$a(s) = Ls^2 + Rs + 1/C$
– are they in left- or right-half plane?

* how does input voltage u relate to output charge y?

* how does input voltage u relate to output charge y? half plane? ↑↑ $\boxed{\text{* WARMUP PROBLEM}}$

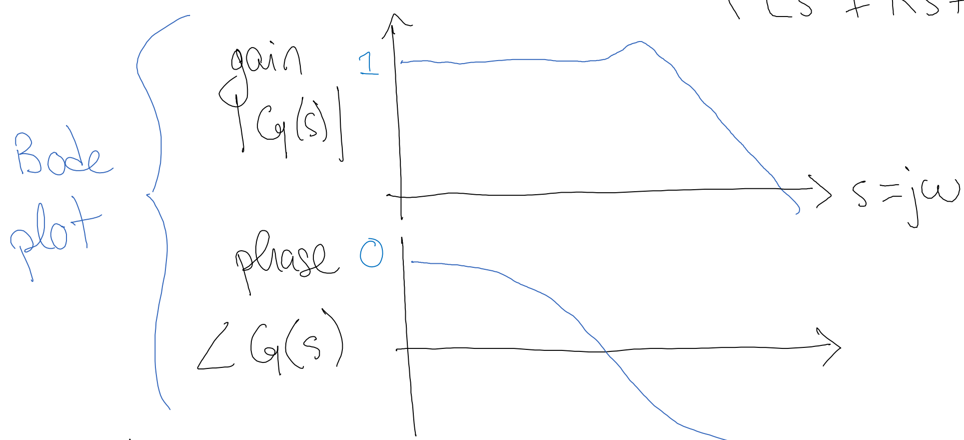
1^1. differential equation

$$\text{KVL} \Rightarrow u = Ri + L\frac{d}{dt}i + \frac{1}{C}y$$
$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{C}y$$

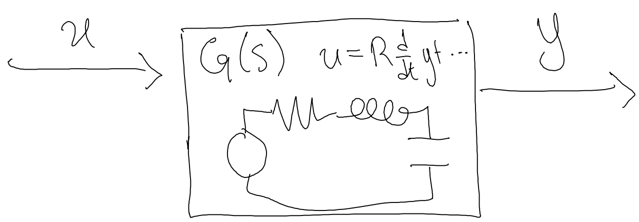
1^2. transfer function

$$u = e^{st} \Rightarrow y = G(s)\,u = \left(\frac{1}{Ls^2 + Rs + \frac{1}{C}}\right)u$$

$\in \mathbb{C}$; called the transfer function



1^3. block diagram



1^1. (linear) differential equation (DE): [AMv2 Ch 2] [Nv7 Ch 3,4]

$$\text{(DE)}\quad \frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = b_1\frac{d^{n-1}}{dt^{n-1}}u + \cdots + b_n u$$

u – input $\quad t$ – time

y – output

where $\{a_k\}_{k=1}^n, \{b_k\}_{k=1}^n \subset \mathbb{R} \iff \forall k : a_k, b_k \in \mathbb{R}$

◦ note that (DE) is specified by specified by two polynomial expressions: $a(\square) = \square^n + a_1\square^{n-1} + \cdots + a_n$

"s" is a dummy variable, or "□"

◦ note that (DE) is specified by specified by … driving variable

two polynomial expressions: $a(\square) = \square^n + a_1\square^{n-1} + \dots + a_n$

characteristic poly. $b(\square) = b_1\square^{n-1} + \dots + b_n$

* we'll see that these polynomials govern behavior of (DE)

◦ a "solution" to (DE) is a pair of signals (u, y)

"'" isn't derivative

$u: \mathbb{R} \to \mathbb{R}$, $: t \mapsto u(t)$ $\quad y: \mathbb{R} \to \mathbb{R}$, $: t \mapsto y(t)$

that satisfy (DE) at all times $t \in \mathbb{R}$

→ if u, y solve (DE), show that

$u': \mathbb{R} \to \mathbb{R}$, $: t \mapsto u(t+\tau)$ $\quad y': \mathbb{R} \to \mathbb{R}$, $: t \mapsto y(t+\tau)$

solve (DE) as well

↳ thus (DE) is time-invariant

given: $\forall t \in \mathbb{R}$:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n = b_1 u^{(n-1)}(t) + \dots + b_n$$

want:

$$y'^{(n)}(t) + a_1 y'^{(n-1)}(t) + \dots + a_n = b_1 u'^{(n-1)}(t) + \dots + b_n$$

know: $y'(t) = y(t+\tau)$

so: $\frac{d}{dt} y'(t) = \frac{d}{dt} y(t+\tau)$

→ if $(u_1, y_1), (u_2, y_2)$ solve (DE), show that $(u_1 + \alpha u_2, y_1 + \alpha y_1)$ solve (DE) as well, where $\alpha \in \mathbb{R}$

↳ thus (DE) is linear

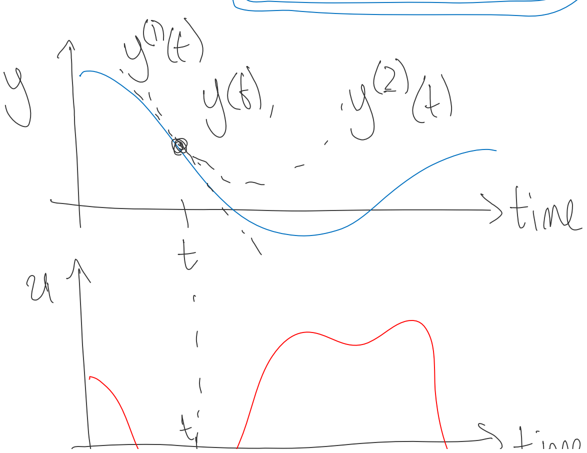
↳ (DE) is linear & time-invariant (LTI)

and it's not hard to show:

$$\frac{d^k}{dt^k} y'(t) = \frac{d^k}{dt^k} y(t+\tau), \quad \frac{d^k}{dt^k} u'(t) = \frac{d^k}{dt^k} u(t+\tau)$$

$$y^{(n)}(t+\tau) + a_1 y^{(n-1)}(t+\tau) + \dots + a_n = b_1 u^{(n-1)}(t+\tau) + \dots + b_n$$

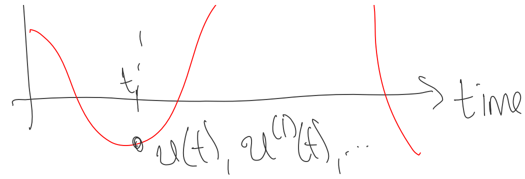
visually:



(DE) says:

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \dots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \dots + b_n u(t)$$

no matter what time t you pick



no matter what time t you pick

<u>fact</u>: every solution to (DE) is a linear combination (i.e. sum) of:

- the <u>homogeneous</u> solution $(u=0)$
- a <u>particular</u> solution $(u \neq 0)$

<u>homogeneous</u>: when $u=0$, i.e. $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$,

y is a linear combination of (complex) exponentials:

$$y(t) = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t} = \sum_{k=1}^{n} c_k e^{s_k t}$$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the <u>roots</u> of the characteristic polynomial $a(s)$, i.e. $a(s_k) = 0$

and $\{c_k\}_{k=1}^n$ are determined by <u>initial condition</u>

$$\left\{ y(0), \dot{y}(0), \ddot{y}(0), \ldots, \frac{d^{n-1}}{dt^{n-1}} y(0) \right\} = \left\{ \frac{d^k}{dt^k} y^{(k)}(0) \right\}_{k=0}^{n-1}$$

1². transfer function: [AMv2 Ch 2] [Nv7 Ch 2]

- from a different perspective, a system transforms input u to output y: $u \longrightarrow \boxed{\text{(LTI) system}} \longrightarrow y$
- when $u(t) = e^{st}$, $s \notin \{s_k\}_{k=1}^n$ ← i.e. s is not a root of $a(s)$

guess $y(t) = G(s) e^{st}$, some $G(s) \in \mathbb{C}$

→ verify: $\frac{d}{dt} u = s e^{st}$, $\frac{d^2}{dt^2} = s^2 e^{st}$, $\ldots$, $\frac{d^{n-1}}{dt^{n-1}} u = s^n e^{st}$

$\frac{d}{dt} y = s G(s) e^{st}$, $\ldots$, $\frac{d^n}{dt^n} y = s^n G(s) e^{st}$

$$\frac{d}{dt} y = s\,G(s)e^{st}, \ \ldots, \ \frac{d^n}{dt^n} y = s^n G(s) e^{st}$$

$\rightarrow$ substituting into (DE): $(s^n + a_1 s^{n-1} + \cdots + a_n)G(s)\cancel{e^{st}}$
$= (b_1 s^{n-1} + \cdots + b_n)\cancel{e^{st}}$

so $$G(s) = \frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}$$ is the <u>transfer function</u>

Thu Oct 3

<u>ex:</u> compute the roots of characteristic polynomial $Ls^2 + Rs + 1/C$

$$s = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

- are these in <u>left-half (complex) plane</u> $\{ z \in \mathbb{C} \mid \underset{\text{"real part"}}{\mathrm{Re}}\, z < 0 \}$?

$\longrightarrow$ yes, assuming: $R, L, C > 0$: $R^2 - 4L/C < R^2$
so $\mathrm{Re}\, s < 0$, i.e. $s \in$ left-half plane

<u>summary</u> & synthesis of 1! & 1? :

- exponential input $\boxed{u(t) = e^{st}}$ to linear time-invariant (LTI) system
yields exp. output $y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{\text{homogeneous response to initial condition}} + G(s)e^{st}$ $= u(t)$

$\{s_k\}_{k=1}^n$ are <u>roots</u> of characteristic polynomial

<u>homogeneous</u> response to initial condition

[recall: $s_k = \sigma_k + j\omega_k$ then $e^{s_k t} = \cancel{e^{\sigma_k t}} e^{j\omega_k t}$]

$\sigma_k = \mathrm{Re}\, s_k < 0$
$\rightarrow 0$ as $t \to \infty$

of characteristic polynomial) (recall: $s_k = \sigma_k + j\omega_k$ then $e^{\ \ } = \cancel{x}\ e^{\ \ }$)

particular response to input signal

→ when is it the case that $\lim_{t\to\infty} y(t) \to G(s)e^{st}$ (when $u(t) = e^{st}$)?

* what must be true of $\{s_k\}_{k=1}^{n}$?

→ $\mathrm{Re}\, s_k < 0$ for all $k \in \{1, \dots, n\}$

∘ terminology:

- static gain: $u(t) = e^{0\cdot t} = 1 \Rightarrow y(t) = G(0) = \frac{b_n}{a_n}$

- given complex numbers $z \in \mathbb{C}$: $|z|$ is magnitude, $\angle z$ is phase

argument = angle = phase

  * $z = re^{j\theta}$ where $r = |z|$, $\theta = \angle z$

- writing $z = \sigma + j\omega$: $\sigma = \mathrm{Re}\, z$ is the real part
  $\omega = \mathrm{Im}\, z$ is the imaginary part

ex: given $u(t) = \sin\omega t = \mathrm{Im}\, e^{j\omega t}$ $\qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$

$y(t) = \mathrm{Im}\left(\boxed{G(j\omega)\,e^{j\omega t}}\right)$ ← just the particular part of output

$\sigma \in \mathbb{R}$, $\mathrm{Im}(\sigma \cdot z) = \sigma \cdot \mathrm{Im}\, z$

$= \mathrm{Im}\left(\boxed{|G(j\omega)|\, e^{j\angle G(j\omega)}}\, e^{j\omega t}\right)$

$z = G(j\omega) = |z|e^{j\angle z}$

$= |G(j\omega)|\, \mathrm{Im}\left(e^{j(\angle G(j\omega) + \omega t)}\right)$

$= \underbrace{|G(j\omega)|}_{\text{gain}} \sin(\omega t + \angle G(j\omega))$

$u(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t \rightsquigarrow y(t) = G(j\omega)e^{j\omega t}$

$\mathrm{Im}\, u(t) = \mathrm{Im}\, e^{j\omega t} = \sin\omega t \rightsquigarrow \mathrm{Im}\, y(t) = \mathrm{Im}\left(G(j\omega)e^{j\omega t}\right)$

→ what happens when $u(t) = e^{s_k t}$, $a(s_k) = 0$?

→ what happens when $u(t) = e^{s_k t}$, $a(s_k) = 0$?
- what does the transfer function tell us?
- " " (DE) tell us?

→ same Q's for $u(t) = e^{s_\ell t}$, $b(s_\ell) = 0$

→ s_k termed a <u>pole</u>, s_ℓ termed a <u>zero</u>

→ when is it the case that $\lim_{t\to\infty} y(t) \to G(s) e^{st}$ (when $u(t) = e^{st}$)?

* what must be true of $\{s_k\}_{k=1}^n$?

→ $\operatorname{Re} s_k < 0$ for all $k \in \{1, \dots, n\}$

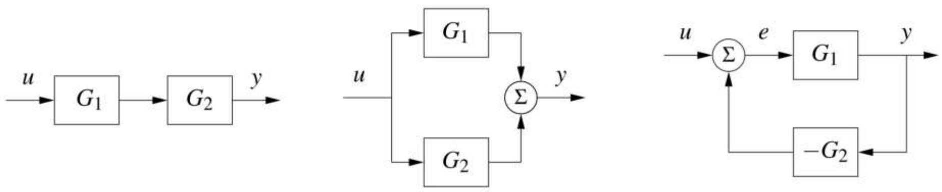
<u>def</u>: say LTI system is <u>stable</u> if all roots of char. poly. are in the left-half plane, i.e.

* Routh (1831–1907) & Hurwitz (1859–1919) found criteria for stability using only coefficients (i.e. <u>not</u> the roots) of characteristic polynomial $a(s)$

(roots of $a(s)$ have negative real part) (if and only if) (algebraic conditions on $\{a_k\}_{k=1}^n$ hold)

roots of $a(s)$ have negative real part	$\iff$ algebraic conditions
$a_0 s^2 + a_1 s + a_2$ ($a_0 \neq 0$)	$\frac{a_1}{a_0}, \frac{a_2}{a_0} > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0$, $a_1 a_2 > a_3$, $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$

1.3. block diagrams [AMv2 Ch 2] [Nv7 Ch 5]



(a) $G_{yu}(s) = G_2(s)G_1(s)$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ (c) $G_{yu}(s) = \frac{G_1(s)}{1+G_1(s)G_2(s)}$

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2. effects of feedback

- there are many uses & types of feedback; we'll focus on these important cases:

  2.1. disturbance attenuation

  2.2. unmodeled dynamics

  2.3. reference tracking

* read [AMv2 ch 2] to learn about other uses & types of feedback

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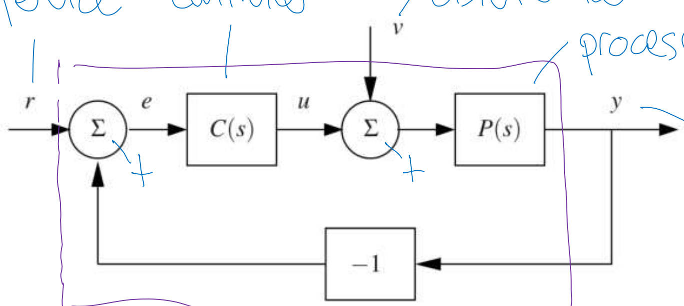
2.1. disturbance attenuation [AMv2 Ch 2.3]

- consider the block diagram: (standard "negative feedback" form)

reference controller disturbance process, i.e. physical system or plant



output

* this diagram is a precise mathematical statement about how signals (→'s) are transformed (□'s)

ex: how does output y relate to external inputs r, v?

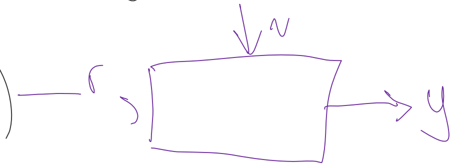
i.e. find an equation of the form $y = G_{yr} r + G_{yv} v$

$y = P(u+v) = P(v + C e)$

i.e. find an equation of the form $y = G_{yr} r + G_{yv} v$

$$y = P(u+v) = P(v + Ce)$$

$$= P(v + C(r-y))$$



$[\, Y(s) = P(s)(U(s)+V(s))$

or $y(t) = (h_p * (u+v))(t) \,]$

impulse response of P

* solve for y i.t.o. r, v