don't be scared!

in class { • midterm exam Thu Oct 31
→ review Tue Oct 29 }

goal: qualitative & quantitative analysis of linear system behavior & relation to nonlinear system behavior

watch lec 3 video! → • HW 3 assigned

• mid-quarter assessment Thu Oct 24

topics:

1°. linear systems

1¹. linearization [Nv7 Ch 2.11] [AMv2 Ch 6.4]

1². linearity [Nv7 Ch 2.10, 3] [AMv2 Ch 6.1]

1³. matrix exponential [AMv2 Ch 6.2]

1⁴. input/output response [AMv2 Ch 6.3]

1⁵. frequency response [Nv7 Ch 10]

1°. linear systems

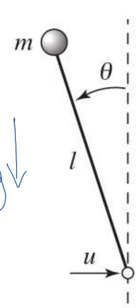
Few physical elements display truly linear characteristics. For example the relation between force on a spring and displacement of the spring is always nonlinear to some degree. The relation between current through a resistor and voltage drop across it also deviates from a straight-line relation. However, if in each case the relation is reasonably *linear, then it will be found that the system behavior will be very close to that obtained by assuming an ideal, linear physical element, and the analytical simplification is so enormous that we make linear assumptions wherever we can possibly do so in good conscience.*

Robert H. Cannon, *Dynamics of Physical Systems*, 1967 [Can03].

* the important question is not "is my system linear?" but rather "is linearity a useful approximation?"
→ if yes, then leverage linearity in analysis & synthesis

1¹. linearization

ex: inverted pendulum



g↓

(very simple) model for rocket flight

$$f(\theta,\dot\theta,u) = \begin{bmatrix}\dot\theta\\ \ddot\theta\end{bmatrix} = \frac{d}{dt}\begin{bmatrix}\theta\\ \dot\theta\end{bmatrix}$$

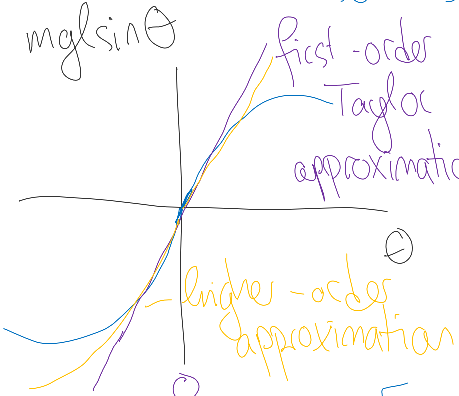
- previously determined dynamics $ml^2\ddot\theta = mgl\sin\theta - \gamma\dot\theta + lu\cos\theta$

have equilibria at: $\dot\theta_e = 0$, $\theta_e = k\cdot\pi$, $k\in\mathbb{Z}$ $\quad k = 0, 1, -1, \dots$

→ compute first-order Taylor series about equilibrium $(\theta_e,\dot\theta_e) = (\boxed{k\pi},0)$, $u_e = 0$

$$\begin{aligned}
ml^2\ddot\theta = mgl\sin\theta - \gamma\dot\theta + lu\cos\theta \;\simeq\;& \overbrace{\ddot\theta(\theta_e,\dot\theta_e,u_e)}^{0\text{th-order}} &&= 0\\
&+ \frac{\partial}{\partial\theta}\ddot\theta(\theta_e,\dot\theta_e,u_e)\cdot(\theta-\theta_e) && + mgl\cdot(\theta-\cancel{\theta_e}^{\,k\pi})\\
&+ \frac{\partial}{\partial\dot\theta}\ddot\theta(\theta_e,\dot\theta_e,u_e)\cdot(\dot\theta-\dot\theta_e) && - \gamma\cdot(\dot\theta-\cancel{\dot\theta_e}^{\,0})\\
&+ \frac{\partial}{\partial u}\ddot\theta(\theta_e,\dot\theta_e,u_e)\cdot(u-u_e) && + l\cancel{\cos\theta_e}^{\,\pm1}\cdot(u-\cancel{u_e}^{\,0})\\
& && - l\cancel{u_e\sin\theta_e}^{\,0}\cdot(\theta-\theta_e)
\end{aligned}$$

first-order Taylor approximation



$$ml^2\ddot\theta = mgl\sin\theta - \gamma\dot\theta + lu\cos\theta$$

$$\left[\Rightarrow ml^2\frac{\partial}{\partial\theta}\ddot\theta = mgl\cos\theta - lu\sin\theta\right]_{\theta=\theta_e=k\pi} = \pm mgl$$

- more generally, consider nonlinear system

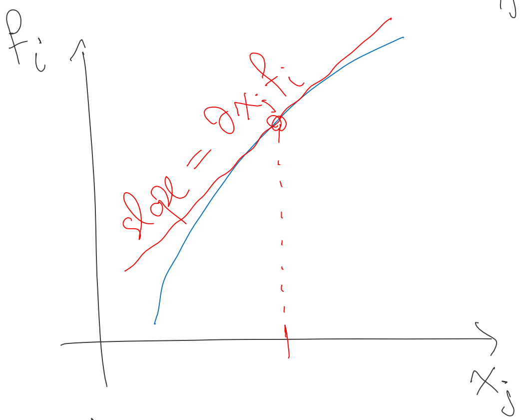
$$(NL) \quad \dot x = f(x,u), \quad x\in\mathbb{R}^n,\ u\in\mathbb{R}^p$$

- assume f is continuously differentiable (smooth)
- assume there is an equilibrium $x_e\in\mathbb{R}^n$, $u_e\in\mathbb{R}^p$: $f(x_e,u_e) = 0$

- consider the (Jacobian) derivative

$$\partial f = [\partial f]^n = \overbrace{[\partial_{x_1}f_1 \;\; \partial_{x_2}f_1 \;\; \cdots \;\; \partial_{x_n}f_1]}^{n\text{ columns}}\;)$$

$$\frac{\partial}{\partial x} f = \left[\frac{\partial}{\partial x_j} f_i \right]_{i,j=1}^{n} = \left.\begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 & \cdots & \partial_{x_n} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} f_n & \partial_{x_2} f_n & \cdots & \partial_{x_n} f_n \end{bmatrix}\right\} n \text{ rows}$$



let $A = \frac{\partial}{\partial x} f(x_e, u_e) \in \mathbb{R}^{n \times n}$

and $$\frac{\partial}{\partial u} f = \left[\frac{\partial}{\partial u_k} f_i \right]_{\substack{i=1 \\ k=1}}^{\substack{i=n \\ k=p}} = \left.\overbrace{\begin{bmatrix} \partial_{u_1} f_1 & \partial_{u_2} f_1 & \cdots & \partial_{u_p} f_1 \\ \partial_{u_1} f_2 & \vdots & & \vdots \\ \vdots & & & \\ \partial_{u_1} f_n & \cdots & & \partial_{u_p} f_n \end{bmatrix}}^{p \text{ columns}}\right\} n \text{ rows}$$

- let $A = \frac{\partial}{\partial x} f(x_e, u_e) \in \mathbb{R}^{n \times n}$, $B = \frac{\partial}{\partial u} f(x_e, u_e) \in \mathbb{R}^{n \times p}$

– then trajectories $\dot{\delta x} = A \cdot \delta x + B \cdot \delta u$ (L), $\delta x \in \mathbb{R}^n$, $\delta u \in \mathbb{R}^p$
approximate trj's of (NL) near (x_e, u_e):
* if $\|x(t) - x_e\|$, $\|u(t) - u_e\|$ small then

$$x(t) \simeq x_e + \delta x(t)$$

<u>fact:</u> (L) is stable $\Longleftrightarrow$ (NL) is stable near (x_e, u_e)

1°. Linearity • a linear time-invariant system in state-space form:

(L) $\dot{x} = Ax + Bu$ $x \in \mathbb{R}^n \Rightarrow A \in \mathbb{R}^{n \times n}$
$u \in \mathbb{R}^p \Rightarrow B \in \mathbb{R}^{n \times p}$

(L) $\dot{x} = Ax + Bu$ $\quad x \in \mathbb{R}^n \Rightarrow A \in \mathbb{R}^{n\times n}$

$u \in \mathbb{R}^p \Rightarrow B \in \mathbb{R}^{n\times p}$

- given initial state $x_0 \in \mathbb{R}^n$ & control input $u : [0,\infty) \to \mathbb{R}^p$
  $: t \mapsto u(t)$
  - $x_h : [0,\infty) \to \mathbb{R}^n$ is the <u>homogeneous</u> solution
    if $x_h(0) = x_0$ and $\frac{d}{dt} x_h(t) = A\, x_h(t)$
    i.e. x_h satisfies (L) with zero input $(u=0)$
  - $x_p : [0,\infty) \to \mathbb{R}^n$ is the <u>particular</u> solution (associated with u)
    if $x_p(0) = 0$ and $\frac{d}{dt} x_p(t) = A\, x_p(t) + Bu(t)$

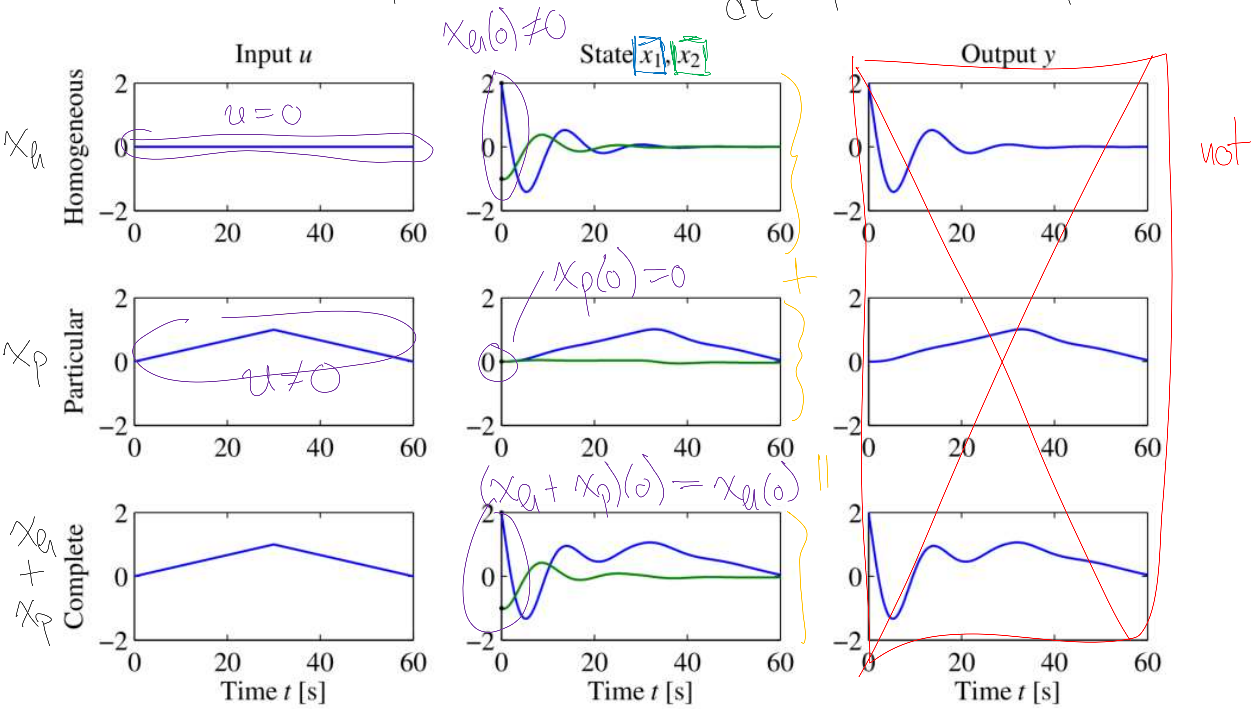


Figure 6.1: Superposition of homogeneous and particular solutions. The first row shows the input, state, and output corresponding to the initial condition response. The second row shows the same variables corresponding to zero initial condition but nonzero input. The third row is the complete solution, which is the sum of the two individual solutions.

<u>ex:</u> scalar system

- consider $\dot{x} = ax + bu$ with $x_0 \in \mathbb{R}$, $u = \alpha \sin \omega t$

$\to$ homogeneous solution $x_h(t) = e^{at} x_0$

→ homogeneous solution $x_h(t) = e^{at} x_0$

→ particular solution $x_p(t) = -\frac{\alpha}{a^2+\omega^2}\left(-\omega e^{at} + \omega \cos \omega t + a \sin \omega t\right)$

→ verify $x_h(0) = x_0$, $\frac{d}{dt} x_h(t) = a\, x_h(t)$

$x_p(0) = 0$, $\frac{d}{dt} x_p(t) = a\, x_p(t) + b\, u(t)$

1.3. matrix exponential (i.e. the homogeneous solution)

→ read [AMv2 "Eigenvalues and Modes"]

↳ interesting discussion of eigenvalues, eigenvectors, and coordinate choice

14. input/output response

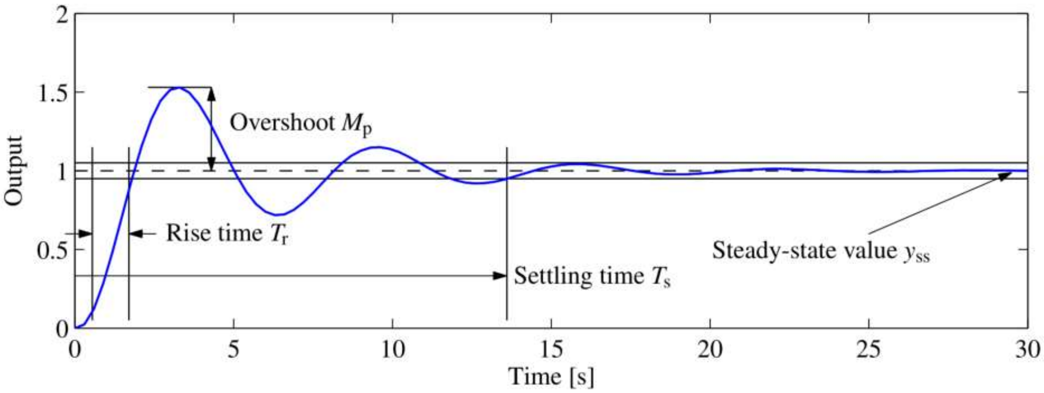
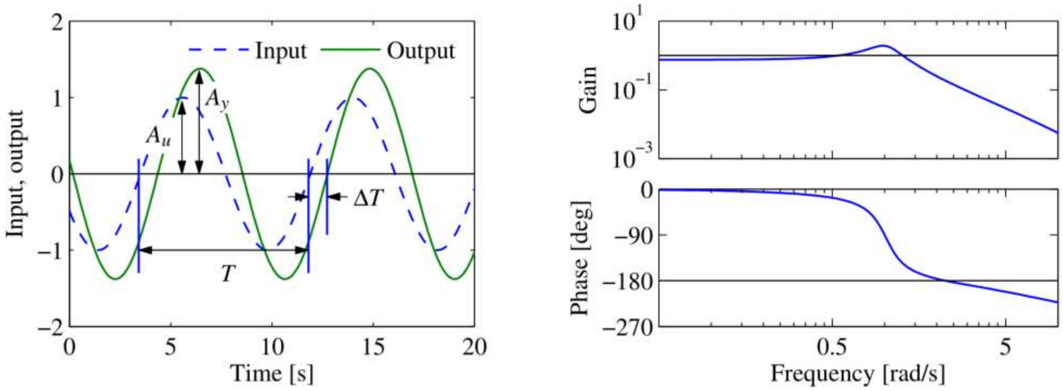


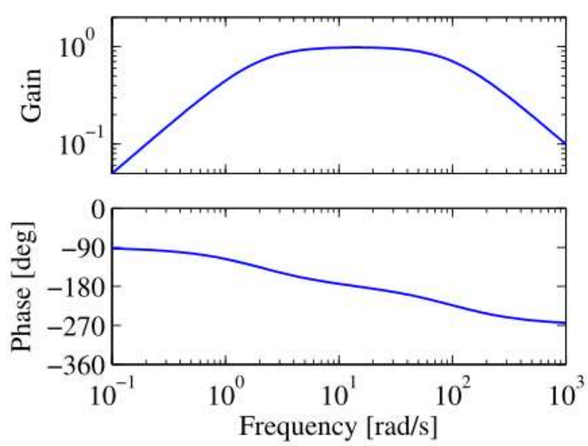
Figure 6.9: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

15. frequency response



(a) Input/output response

(b) Frequency response



(b) Frequency response