goal: design stabilizing controllers and estimating observers

1°. state feedback [AMv2 Ch 7]
  1¹. stabilization [Nv7 Ch 12.2]
  1². integral feedback

2°. output feedback [AMv2 Ch 8]
  2¹. observer design [Nv7 Ch 12.5]
  2². closing the loop

1°. state feedback

- as we've seen, the roots of an LTI system's characteristic polynomial govern its behavior, e.g. stability

→ we'll build tools that enable us to place these roots where we want them (& determine when/if it's possible to do so)

1'. stabilization

∘ consider the LTI system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^p$$

- we seek to <u>stabilize</u> the system, that is, determine input u as a function of state x s.t. closed-loop system is asymptotically stable
- if we use linear (i.e. proportional) state feedback, $u = -Kx$, then the closed-loop dynamics are

$$\dot{x} = Ax + Bu$$
$$= Ax - BKx$$
$$= (A - BK)x,$$

which is asymptotically stable if $\mathrm{Re}(\lambda(A-BK)) < 0$, i.e. all eigenvalues of $A-BK$ have negative real part

→ given $\dot{x} = ax + bu$, $x \in \mathbb{R}$, $u \in \mathbb{R}$, determine the range of values for $K \in \mathbb{R}$ that stabilize the system if $u = -Kx$

- more generally, if we want the closed-loop system to have $\{\lambda_i\}_{i=1}^m$ as its set of eigenvalues (i.e. $\mathrm{Re}\,\lambda_i < 0\}$) then we're trying to determine entries of <u>feedback</u> / <u>gain</u> <u>matrix</u> $K \in \mathbb{R}^{p \times n}$ s.t.

$$\lambda(A - BK) = \{\lambda_i\}_{i=1}^m$$

i.e. the <u>characteristic polynomial</u> of closed-loop system $\dot{x} = (A - BK)x$,

$$a(s) = \det(sI - (A - BK))$$
$$= s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^1 + a_n$$
$$= (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$$
$$= \prod_{i=1}^{n} (s - \lambda_i)$$

<u>ex:</u> 7.4 vehicle steering

- $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$

(γ is ratio of wheelbase and distance from CoM to rear wheel)

- applying linear state feedback

$$u = -Kx = -k_1 x_1 + -k_2 x_2$$

yields closed-loop dynamics

$$\dot{x} = (A-BK)x = \begin{bmatrix} -\gamma k_1 & 1-\gamma k_2 \\ -k_1 & -k_2 \end{bmatrix}$$

with characteristic polynomial

$$\det(sI-(A-BK)) = s^2 + (\gamma k_1 + k_2)s + k_1$$

→ choose k_1, k_2 so that closed-loop system has characteristic polynomial

$$a(s) = s^2 + 2\zeta_c \omega_c s + \omega_c^2$$

(i.e. behaves like stable 2nd-order sys)

* we can choose k_i's to place eigenvalues of $A-BK$ where we want by solving a system of equations!

↳ this technique is termed <u>pole placement</u>, because eigenvalues are poles of the system's transfer function

- the <u>place</u> command in the Control

Systems Toolbox does this for you (even for MIMO systems)

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12. integral feedback

- the preceding <u>pole placement</u> technique <u>stabilizes</u> the LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

i.e. drives the state (hence, output) to zero asymptotically

→ if we're given a non-zero reference output r, we'll augment this approach with <u>integral feedback</u>, i.e. apply $u(x, z) = -Kx - K_I z$ where $\dot{z} = y - r$

$$\iff z(t) = \int_0^t y(\tau) - r(\tau)\, d\tau$$

- this dynamic compensator creates a new state variable: $z \in \mathbb{R}$

– the augmented system is still LTI:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

- integral will equilibrate to value z_e for which $\dot{z}_e = r - y_e = r - Cx_e = 0$

  i.e. $r = Cx_e$

→ express the equilibrium state x_e in terms of equilibrium integral z_e

* since integrator has its own state, must perform control design on augmented system!

ex: 7.8 cruise control

- linearizing around an equilibrium speed v_e, throttle u_e yields

$$\dot{\nu} = a\nu + bu + (\text{disturbance}),$$

$$y = v \simeq v_e + \nu$$

$$\dot{z} = y - r$$

- in state-space form:

$$\dot{x} = \underbrace{A}\,\underbrace{x} + \underbrace{B}\,u + w$$

$$\frac{d}{dt}\begin{bmatrix} \nu \\ \end{bmatrix} = \begin{bmatrix} a & 0 \\ & \end{bmatrix}\begin{bmatrix} \nu \\ \end{bmatrix} + \begin{bmatrix} b \\ \end{bmatrix} u + \begin{bmatrix} 0 \\ \end{bmatrix}$$

$$\frac{d}{dt}\underbrace{\begin{bmatrix} v \\ z \end{bmatrix}} = \underbrace{\begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix}}\underbrace{\begin{bmatrix} v \\ z \end{bmatrix}} + \underbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}\mu + \begin{bmatrix} 0 \\ v_e - r \end{bmatrix}$$

$$+ \begin{bmatrix} \text{(disturbance)} \\ 0 \end{bmatrix}$$

- suppose we want the closed-loop characteristic polynomial

$$s^2 + a_1 s + a_2$$

by applying $\mu = -Kx$

$$-k_p v - k_I z$$

$\rightarrow$ compute closed-loop characteristic polynomial & use it to find k_p, k_I

$\det(sI - (A - BK))$

$$= s^2 + (bk_p - a)s + bk_I$$

so setting $k_p = \frac{a_1 + a}{b}$, $k_I = \frac{a_2}{b}$

gives desired closed-loop

2°. output feedback

- sensing is expensive — it's rarely practical or affordable to directly measure every state variable

→ we'll derive tools that enable us to estimate & control the system state using a small number of <u>outputs</u>

2! observers

∘ to estimate the state of an LTI system

$$\dot{x} = Ax + Bu$$

using only its output

$$y = Cx + Du$$

we'll construct another LTI system termed an <u>observer</u>:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x} + Du$$

→ the state $\hat{x}$ of this system is known to us – we implement a simulation of it

– intuitively, the output error $y - \hat{y}$ is fed back to control observer system state until $\hat{x} \simeq x$

– to see why this works, consider

the dynamics of the error $e = x - \hat{x}$

→ determine a DE for $\dot{e}$ in terms of e
(i.e. $\dot{e}$ shouldn't depend on $x, \hat{x}, y, \hat{y}$)

- $\dot{e} = \dot{x} - \dot{\hat{x}}$

$$= (Ax + Bu) - (A\hat{x} + Bu + L(y - \hat{y}))$$
$$= Ax - A\hat{x} - L(Cx - C\hat{x})$$
$$= (A - LC)e$$

* if $\operatorname{Re} \lambda(A - LC) < 0$, then error dynamics are asymptotically stable, $\dot{e} \to 0$, which means observer state converges to real state: $\hat{x} \to x$!

- remarkably, observer design reduces to pole placement !

ex: 8.3 vehicle steering

- $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$
$y = Cx$, $C = [1 \ 0]$

→ compute observer error dynamics $(A - LC)$ and characteristic polynomial
(what is L's shape?) $L \in \mathbb{R}^{2 \times 1}$

$$-(A-LC) = \begin{bmatrix} \ell_1 & 1 \\ \ell_2 & 0 \end{bmatrix}$$

$$\Rightarrow \det(sI-(A-LC)) = s^2+\ell_1 s+\ell_2$$

* all we need to do is choose $\ell_1, \ell_2 > 0$ to ensure $\mathrm{Re}\,\lambda(A-LC) < 0$, i.e. observer error goes to zero asymptotically

→ the <u>place</u> command in Control Sys Toolbox can do this for you (just make sure to transpose A & C!)

2°. closing the loop

- now consider what happens when we control the real system with the estimated state:

$$\dot{x} = Ax + Bu \quad \Big| \quad \dot{\hat{x}} = A\hat{x} + Bu + L(y-\hat{y})$$

$$y = Cx + Du \quad \Big| \quad \hat{y} = C\hat{x} + Du$$

- we already saw that $e = x - \hat{x}$

$\Rightarrow \dot{e} = (A-LC)e$

→ determine the dynamics of x when we apply feedback $u = -K\hat{x}$:

in terms of x & e
(i.e. substitute out for $\hat{x}$)

$$\dot{x} = Ax + Bu$$
$$= Ax - BK\hat{x}$$
$$= Ax - BK(x - e)$$
$$= (A - BK)x + BKe$$

- combining these DE gives a single closed-loop LTI system:

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix}$$

* remarkably, the system is block diagonal, so its characteristic polynomial factors into the product:

$$a(s) = \det(sI - (A - BK)) \cdot \det(sI - (A - LC))$$

→ so if we (separately) design full-state feedback K and observer error feedback L, then real system can be controlled with estimated state!

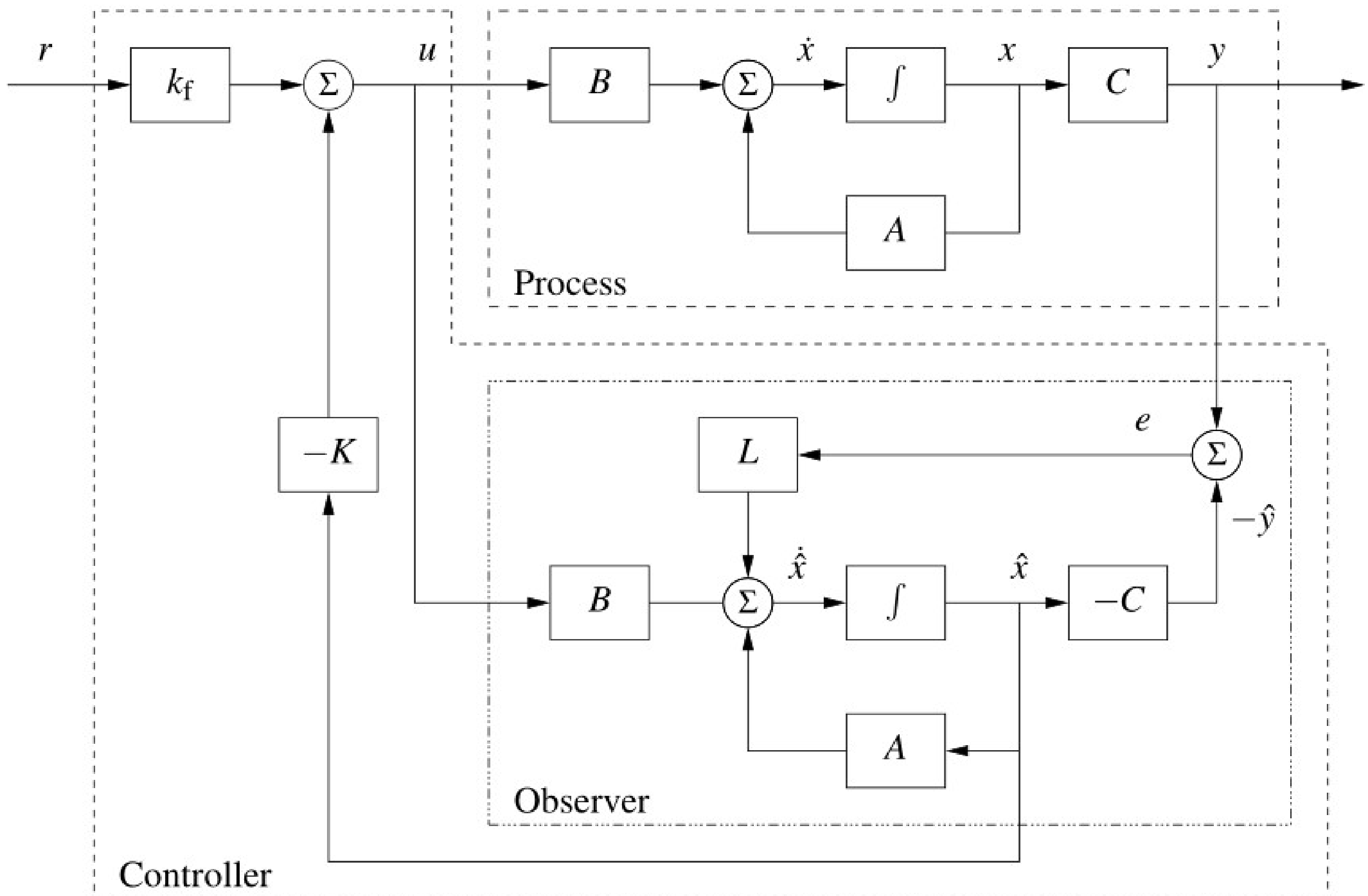

Figure 8.7: Block diagram of an observer-based control system. The observer uses the measured output y and the input u to construct an estimate of the state. This estimate is used by a state feedback controller to generate the corrective input. The controller consists of the observer and the state feedback; the observer is identical to that in Figure 8.5.

ex: 8.4 vehicle steering

- $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

→ determine the closed-loop dynamics of the observer $\hat{x}$ using state feedback gain K, observer gain L

(what are the shapes of K & L ?)