[AMv2 Ch 3 & 4]

goal: further develop modeling tools & apply them to physical phenomena

topics:

1°. modeling [AMv2 Ch 3] [Nv7 Ch 3,4,5]

1¹. concepts

1². state space models

1³. numerical simulation

2°. examples

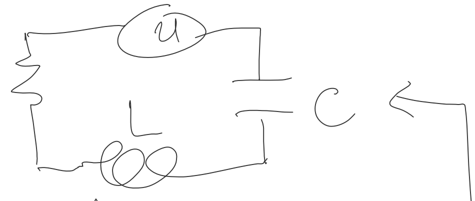
2¹. RLC circuit

2². quadrotor

1°. modeling

1¹. concepts

ex: spring-mass-damper: R



input u - voltage output q - charge on C

KVL $\Rightarrow$ charge q and current $\dot{q}$ interact over time according to (DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$

*note that initial condition $(q(0), \dot{q}(0))$ and input $u : [0,\infty) \to \mathbb{R}$
$: t \mapsto u(t)$
determine $(q(t), \dot{q}(t))$ for all $t \geq 0$

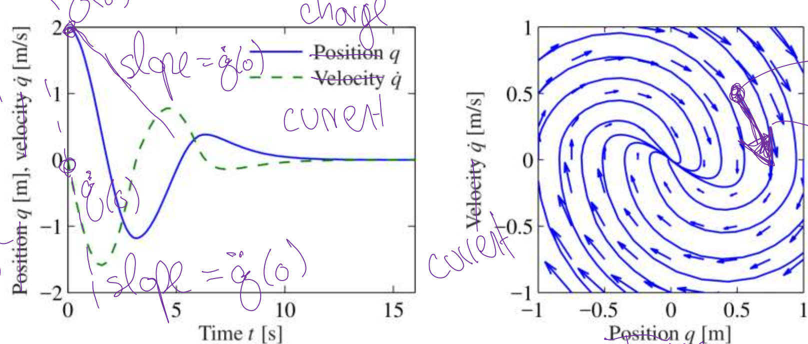


Figure 3.2: Illustration of a state model. A state model gives the rate of change of the state as a function of the state. The plot on the left shows the evolution of the state as a function of time. The plot on the right, called a *phase portrait*, shows the evolution of the states relative to each other, with the velocity of the state denoted by arrows.

$u = 0$, i.e. homogeneous solution

$\ddot{q} = \frac{1}{L}\left(v - R\dot{q} - \frac{1}{C} q\right)$

$(q, \dot{q})$ plane is state space (or phase space)

↳ overlay trajectories ⇝ phase portrait

↳ overlay vectors ⇝ vector field

1.2 state-space models

∘ generalizing the preceding example,

ex: RLC circuit

1. State space models

- generalizing the preceding example,

  let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ denote <u>state vector</u>

  and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in \mathbb{R}^p$ denote <u>input vector</u>

- then the state changes in time according to

  $$\text{(DE)} \quad \frac{d}{dt} x = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = f(x,u)$$

  where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$
  $: (x,u) \mapsto \dot{x}$

<u>ex:</u> RLC circuit

$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, so $x_1 = q$, $x_2 = \dot{q}$

$u = [v]$, so $u_1 = v$ (voltage)

$$L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$$

$$\dot{x} = f(x,u) = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{1}{L}\left(v - R\dot{q} - \frac{1}{C} q\right) \end{bmatrix}$$

- focus on <u>linear</u> f: $\exists A \in \mathbb{R}^{n \times n}, B^{n \times p}$ s.t. $f(x,u) = Ax + Bu = \dot{x}$

  $*$ $f(x_1 + \alpha x_2, u) = A(x_1 + \alpha x_2) + Bu$
  $= Ax_1 + \alpha A x_2 + Bu = f(x_1, 0) + \alpha f(x_2, 0) + f(0,u)$

- we previously saw linear (DE) $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = u$

- define $x_k = \frac{d^{n-k}}{dt^{n-k}} y$ $\iff$ $\frac{d^n}{dt^n} y = -a_1 \frac{d^{n-1}}{dt^{n-1}} y - \cdots - a_n y + u$

so $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d^{n-1}}{dt^{n-1}} y \\ \frac{d^{n-2}}{dt^{n-2}} y \\ \vdots \\ \frac{d}{dt} y \\ y \end{bmatrix} \Rightarrow \dot{x} = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d^n}{dt^n} y \\ \frac{d^{n-1}}{dt^{n-1}} y \\ \vdots \\ \frac{d^2}{dt^2} y \\ \frac{d}{dt} y \end{bmatrix} = \begin{bmatrix} \frac{d^n}{dt^n} y \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix}$

$$\dot{x} = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d^n}{dt^n} y \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} -a_1 \frac{d^{n-1}}{dt^{n-1}} y - \cdots - a_n y + u \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} -a_1 x_1 - a_2 x_2 - \cdots - a_n x_n + u \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} = Ax + Bu$$

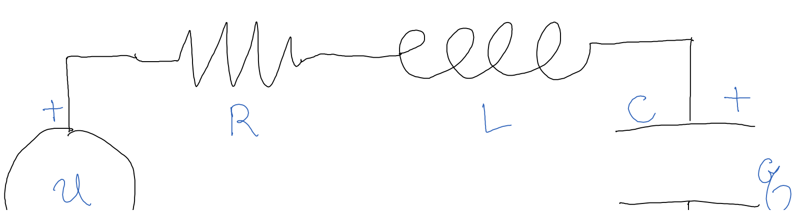
for some $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$

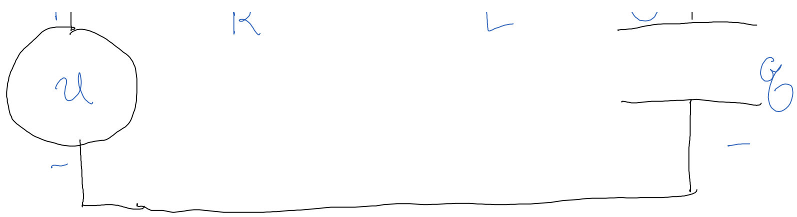
$$\Rightarrow \dot{x} = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{A \in \mathbb{R}^{n \times n}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B \in \mathbb{R}^{n \times 1}} u$$

1°. numerical simulation

2°. examples

2¹. RLC circuit





2². quadrotor

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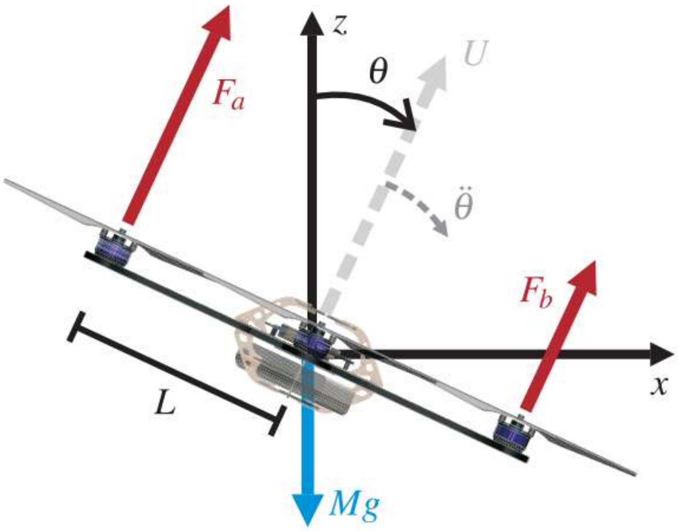
A Simple Learning Strategy for High-Speed Quadrocopter Multi-Flips

Sergei Lupashin, Angela Schöllig, Michael Sherback, Raffaello D'Andrea



$$M\ddot{z} = (F_a + F_b + F_c + F_d)\cos\theta - Mg \quad (1)$$
$$M\ddot{x} = (F_a + F_b + F_c + F_d)\sin\theta \quad (2)$$
$$I_{yy}\ddot{\theta} = L(F_a - F_b)\,, \quad (3)$$