[AMv2 Ch 2]

<u>goal</u>: introduce fundamental uses and properties of <u>feedback</u>

<u>topics</u>:
- 1°. mathematical models of systems
  - 1¹. differential equations (DE)
  - 1². transfer functions
  - 1³. block diagrams
- 2°. effects of feedback
  - 2¹. disturbance attenuation
  - 2². unmodeled dynamics
  - 2³. reference tracking

* read [AMv2 Ch 2.2.5] to learn how <u>positive</u> feedback used in digital systems

* HW0 assigned - due midnight Fri

* we will answer questions thru noon on Fri

* we will post HW on Fri

→ caveat: won't be solved until Mon

* create video of ipynb → pdf

* " " " " LaTeX

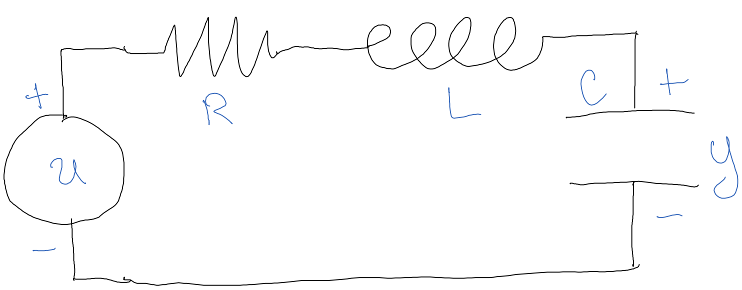
1°. mathematical models of systems

- we will work with multiple representations of linear control systems
  - 1¹. differential equations
  - 1². transfer functions
  - 1³. block diagrams

  each has unique advantages & provides novel insight

* what is a control system?

<u>ex</u>: consider an RLC circuit



→ determine the <u>roots</u> of the characteristic polynomial

$a(s) = Ls^2 + Rs + 1/C$

– are they in left- or right-half plane?

* how does input voltage u relate to output charge y?

* how does input voltage u relate to output charge y?

half plane :

* WARMUP PROBLEM

1¹. differential equation

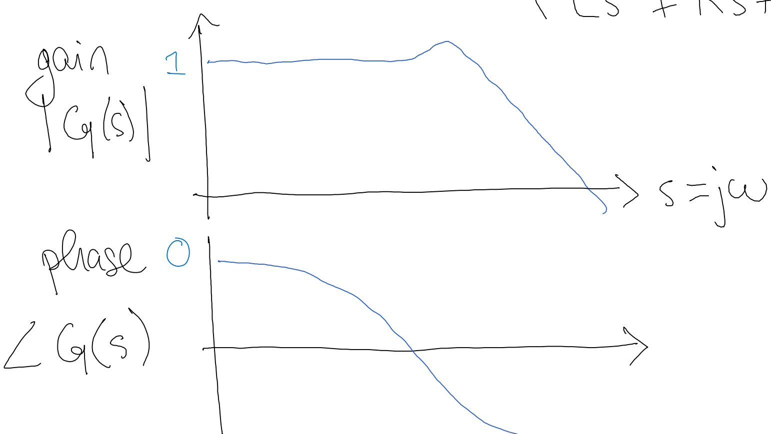
$$\text{KVL} \Rightarrow u = Ri + L\frac{d}{dt}i + \frac{1}{c}y$$
$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{c}y$$

1². transfer function

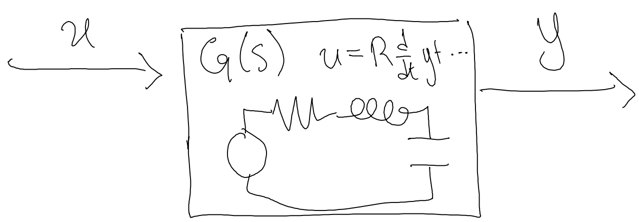
$$u = e^{st} \Rightarrow y = G(s)u = \left(\frac{1}{Ls^2 + Rs + \frac{1}{c}}\right)u$$

$\in \mathbb{C}$; called the transfer function

Bode plot



1³. block diagram



1¹. (linear) differential equation (DE): [AMv2 Ch 2] [Nv7 Ch 3,4]

$$\text{(DE)} \quad \frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = b_1\frac{d^{n-1}}{dt^{n-1}}u + \cdots + b_n u$$

u – input t – time

y – output

where $\{a_k\}_{k=1}^n, \{b_k\}_{k=1}^n \subset \mathbb{R} \iff \forall k : a_k, b_k \in \mathbb{R}$

- note that (DE) is specified by specified by two polynomial expressions: $a(\square) = \square^n + a_1\square^{n-1} + \cdots + a_n$

"s" is a dummy variable or "□"

∘ note that (DE) is specified by specifying two polynomial expressions:

$$a(\square) = \square^n + a_1 \square^{n-1} + \cdots + a_n$$
$$b(\square) = b_1 \square^{n-1} + \cdots + b_n$$

characteristic poly.

dummy variable

* we'll see that these polynomials govern behavior of (DE)

∘ a "solution" to (DE) is a pair of signals (u, y)

$$u: \mathbb{R} \to \mathbb{R} \qquad y: \mathbb{R} \to \mathbb{R}$$
$$: t \mapsto u(t) \qquad : t \mapsto y(t)$$

that satisfy (DE) at all times $t \in \mathbb{R}$

" ' " isn't derivative

→ if u, y solve (DE), show that

$$u': \mathbb{R} \to \mathbb{R} \qquad y': \mathbb{R} \to \mathbb{R}$$
$$: t \mapsto u(t+\tau) \qquad : t \mapsto y(t+\tau)$$

solve (DE) as well

↳ thus (DE) is time-invariant

given: $\forall t \in \mathbb{R}$:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n = b_1 u^{(n-1)}(t) + \cdots + b_n$$

want:

$$y'^{(n)}(t) + a_1 y'^{(n-1)}(t) + \cdots + a_n = b_1 u'^{(n-1)}(t) + \cdots + b_n$$

know: $y'(t) = y(t+\tau)$

so: $\frac{d}{dt} y'(t) = \frac{d}{dt} y(t+\tau)$

→ if $(u_1, y_1), (u_2, y_2)$ solve (DE), show that $(u_1 + \alpha u_2, y_1 + \alpha y_1)$ solve (DE) as well, where $\alpha \in \mathbb{R}$

↳ thus (DE) is linear

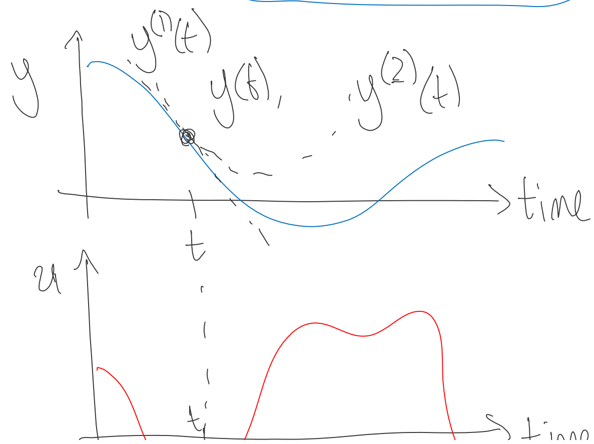
↳ (DE) is linear & time-invariant (LTI)

and it's not hard to show:

$$\frac{d^k}{dt^k} y'(t) = \frac{d^k}{dt^k} y(t+\tau), \quad \frac{d^k}{dt^k} u'(t) = \frac{d^k}{dt^k} u(t+\tau)$$

$$y^{(n)}(t+\tau) + a_1 y^{(n-1)}(t+\tau) + \cdots + a_n = b_1 u^{(n-1)}(t+\tau) + \cdots + b_n$$

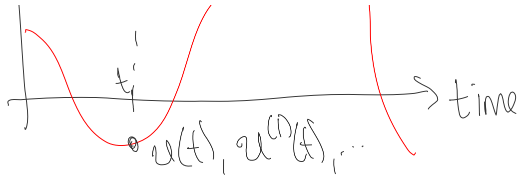
visually:



(DE) says:

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t) = b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)$$

no matter what time t you pick



no matter what time t you pick

<u>fact</u>: every solution to (DE) is a linear combination (i.e. sum) of:

- the <u>homogeneous</u> solution $(u=0)$
- a <u>particular</u> solution $(u \neq 0)$

<u>homogeneous</u>: when $u=0$, i.e. $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$,

y is a linear combination of (complex) exponentials:

$$y(t) = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t} = \sum_{k=1}^{n} c_k e^{s_k t}$$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the <u>roots</u> of the characteristic polynomial $a(s)$, i.e. $a(s_k) = 0$

and $\{c_k\}_{k=1}^n$ are determined by <u>initial condition</u>

$$\left\{ y(0), \dot{y}(0), \ddot{y}(0), \ldots, \frac{d^{n-1}}{dt^{n-1}} y(0) \right\} = \left\{ \frac{d^k}{dt^k} y^{(k)}(0) \right\}_{k=0}^{n-1}$$

1°. transfer function: [AMv2 Ch 2] [Nv7 Ch 2]

- from a different perspective, a system transforms input u to output y: $\xrightarrow{u}$ $\boxed{\text{(LTI) system}}$ $\xrightarrow{y}$
- when $u(t) = e^{st}$, $s \notin \{s_k\}_{k=1}^n$ ← i.e. s is not a root of $a(s)$

guess $y(t) = G(s) e^{st}$, some $G(s) \in \mathbb{C}$

→ verify: $\frac{d}{dt} u = s e^{st}$, $\frac{d^2}{dt^2} = s^2 e^{st}$, $\ldots$, $\frac{d^{n-1}}{dt^{n-1}} u = s^n e^{st}$

$\frac{d}{dt} y = s G(s) e^{st}$, $\ldots$, $\frac{d^n}{dt^n} y = s^n G(s) e^{st}$

$\frac{d}{dt} y = s\, G(s) e^{st}, \ \ldots, \ \frac{d^n}{dt^n} y = s^n G(s) e^{st}$

→ substituting into (DE): $(s^n + a_1 s^{n-1} + \cdots + a_n) G(s) \cancel{e^{st}} = (b_1 s^{n-1} + \cdots + b_n) \cancel{e^{st}}$

so $G(s) = \dfrac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \dfrac{b(s)}{a(s)}$ is the <u>transfer function</u>

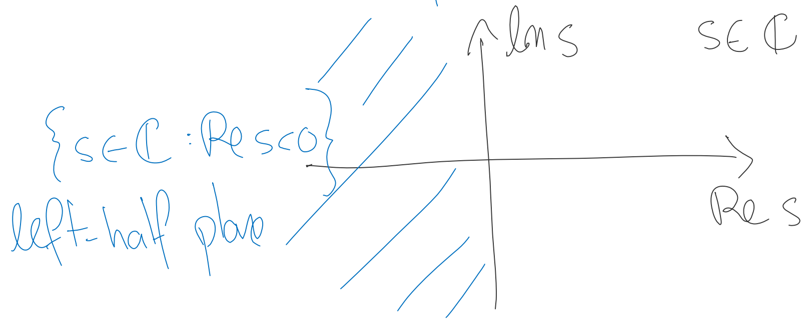
Thu Oct 3

<u>ex:</u> compute the roots of characteristic polynomial $Ls^2 + Rs + 1/C$

$$s = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

- are these in <u>left-half (complex) plane</u> $\{z \in \mathbb{C} \mid \underbrace{\mathrm{Re}}_{\text{"real part"}}\, z < 0\}$?

→ yes, assuming: $R, L, C > 0$: $R^2 - 4L/C < R^2$
so $\mathrm{Re}\, s < 0$, i.e. $s \in$ left-half plane



<u>summary</u> & synthesis of 1! & 1?:

- exponential input $\boxed{u(t) = e^{st}}$ to linear time-invariant (LTI) system yields exp. output $y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{\text{homogeneous response to initial condition}} + G(s) e^{st}$, $e^{st} = u(t)$

$\{s_k\}_{k=1}^{n}$ are <u>roots</u> of characteristic polynomial

<u>homogeneous</u> response to initial condition

[recall: $s_k = \sigma_k + j\omega_k$ then $e^{s_k t} = \cancel{e^{\sigma_k t}} e^{j\omega_k t}$]

$\sigma_k = \mathrm{Re}\, s_k < 0$
$\to 0$ as $t \to \infty$

of characteristic polynomial) [recall: $s_k = \sigma_k + j\omega_k$ then $e \quad = \cancel{x}\, e \quad]$

particular response to input signal

$\longrightarrow$ when is it the case that $\lim_{t\to\infty} y(t) \to G(s)e^{st}$ (when $u(t) = e^{st}$)?

* what must be true of $\{s_k\}_{k=1}^{n}$?

$\longrightarrow$ $\operatorname{Re} s_k < 0$ for all $k \in \{1, \dots, n\}$

∘ terminology:

- static gain: $u(t) = e^{0\cdot t} = 1 \Rightarrow y(t) = G(0) = \frac{b_n}{a_n}$

- given complex numbers $z \in \mathbb{C}$: $|z|$ is magnitude, $\angle z$ is phase = angle = argument
  - * $z = re^{j\theta}$ where $r = |z|$, $\theta = \angle z$

- writing $z = \sigma + j\omega$: $\sigma = \operatorname{Re} z$ is the real part
  - $\omega = \operatorname{Im} z$ is the imaginary part

ex: given $u(t) = \sin\omega t = \operatorname{Im} e^{j\omega t}$ $\qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$

$$y(t) = \operatorname{Im}\left(\boxed{G(j\omega)e^{j\omega t}}\right) \leftarrow \text{just the particular part of output}$$

$$= \operatorname{Im}\left(\boxed{|G(j\omega)|e^{j\angle G(j\omega)}}\, e^{j\omega t}\right)$$

$$= |G(j\omega)| \operatorname{Im}\left(e^{j(\angle G(j\omega) + \omega t)}\right)$$

$$= \underbrace{|G(j\omega)|}_{\text{gain}} \sin(\omega t + \angle G(j\omega))$$

$\sigma \in \mathbb{R}$
$\operatorname{Im}(\sigma\cdot z) = \sigma\cdot\operatorname{Im} z$

$z = G(j\omega)$
$= |z|e^{j\angle z}$

$$u(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t \rightsquigarrow y(t) = G(j\omega)e^{j\omega t}$$

$$\operatorname{Im} u(t) = \operatorname{Im} e^{j\omega t} = \sin\omega t \rightsquigarrow \operatorname{Im} y(t) = \operatorname{Im}\left(G(j\omega)e^{j\omega t}\right)$$

$\longrightarrow$ what happens when $u(t) = e^{s_k t}$, $a(s_k) = 0$?

→ what happens when $u(t) = e^{s_k t}$, $a(s_k) = 0$?
- what does the transfer function tell us?
- " " (DE) tell us?

→ same Q's for $u(t) = e^{s_\ell t}$, $b(s_\ell) = 0$

→ s_k termed a <u>pole</u>, s_ℓ termed a <u>zero</u>

→ when is it the case that $\lim_{t \to \infty} y(t) \to G(s) e^{st}$ (when $u(t) = e^{st}$)?

* what must be true of $\{s_k\}_{k=1}^{n}$?

→ $\operatorname{Re} s_k < 0$ for all $k \in \{1, \dots, n\}$

def: say LTI system is <u>stable</u> if all roots of char. poly. are in the left-half plane, i.e.

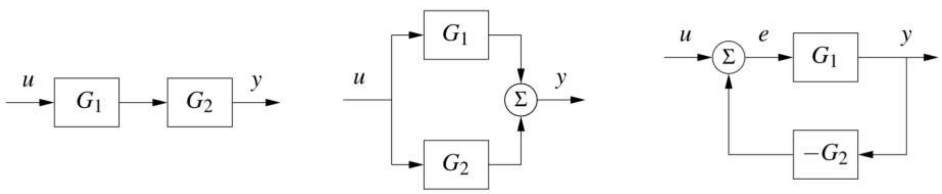
* Routh (1831–1907) & Hurwitz (1859–1919) found criteria for stability using only coefficients (i.e. <u>not</u> the roots) of characteristic polynomial $a(s)$

(roots of $a(s)$ have negative real part) (if and only if) (algebraic conditions on $\{a_k\}_{k=1}^{n}$ hold)

$\overset{1}{a_0} s^2 + a_1 s + a_2$	$\Longleftrightarrow$ $\frac{a_1}{a_0}, \frac{a_2}{a_0} > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0$, $a_1 a_2 > a_3$, $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$

13 block diagrams [AMv2 Ch 2] [Nv7 Ch 5]

1³. block diagrams [AMv2 Ch 2] [Nv7 Ch 5]



(a) $G_{yu}(s) = G_2(s)G_1(s)$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ (c) $G_{yu}(s) = \dfrac{G_1(s)}{1 + G_1(s)G_2(s)}$

2: effects of feedback

- there are many uses & types of feedback; we'll focus on these important cases:

  2¹. disturbance attenuation

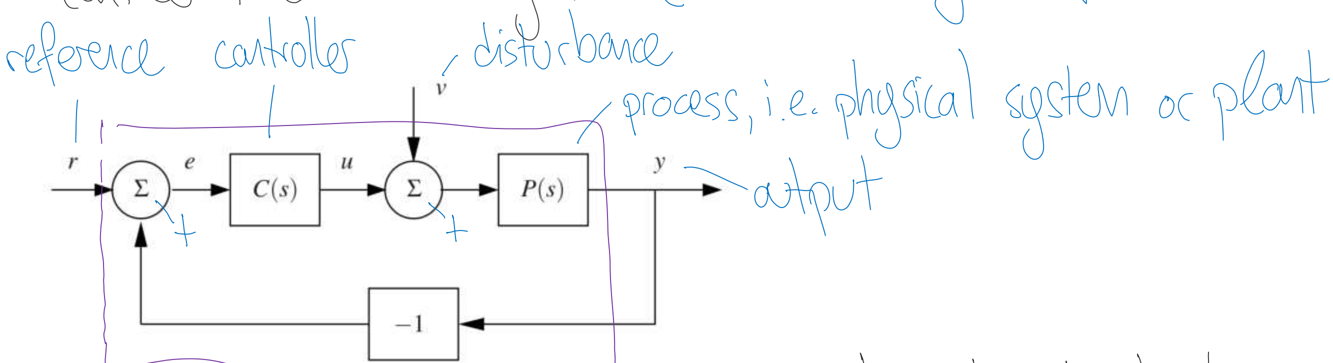
  2². unmodeled dynamics

  2³. reference tracking

* read [AMv2 ch 2] to learn about other uses & types of feedback

2¹. disturbance attenuation [AMv2 Ch 2.3]

- consider the block diagram: (standard "negative feedback" form)



* this diagram is a precise mathematical statement about how signals (→'s) are transformed (□'s)

ex: how does output y relate to external inputs r, v?

i.e. find an equation of the form $y = G_{yr} r + G_{yv} v$

i.e. find an equation of the form $y = G_{yr} r + G_{yv} v$

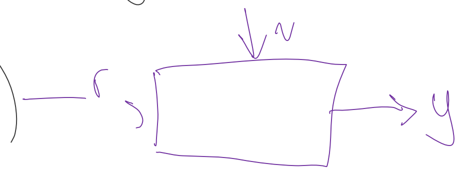
$y = P(u+v) = P(v + Ce)$

$= P(v + C(r-y))$

[$Y(s) = P(s)(U(s) + V(s))$

or $y(t) = (h_p * (u+v))(t)$]

impulse response of P



* solve for y i.t.o. r, v

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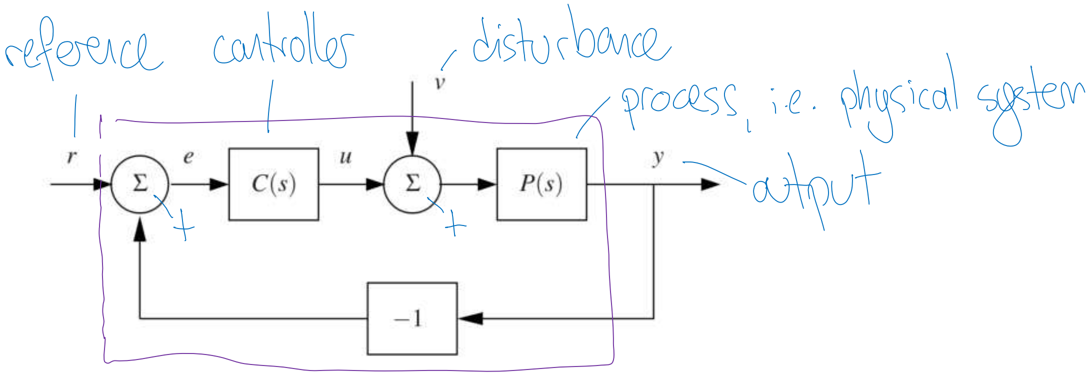
Tue Oct 8

2'. disturbance attenuation [AMv2 Ch 2.3]

∘ consider the block diagram:

* ipynb on github → Colab



→ determine how output y depends on reference r, disturbance d by 1°: translating the block diagram to equations and 2°: manipulating these equations to solve for y in terms of r, v

∘ $y = P\cdot(v+u)$ ← * this is true in freq- or time-domain

$= P\cdot v + P\cdot u = Pv + PCe = Pv + PC(r-y)$

$= Pv + PCr - PCy$

$\iff y + PCy = Pv + PCr$

$\iff (1+PC)y = Pv + PCr$

$\iff y = \dfrac{P}{1+PC} v + \dfrac{PC}{1+PC} r$

* assume $1 + PC \neq 0$; we'll see this is related to stability

∘ so in particular, $G_{yv}(s) = \dfrac{P(s)}{1+P(s)C(s)}$

"transfer function from v to y"

∘ so in particular, $G_{yv}(s) = \dfrac{\quad}{1+P(s)C(s)}$

notation means "transfer function from v to y"

ex: consider first-order process $\underbrace{P(s) = \dfrac{b}{s+a} \iff \dot{y} + ay = bu}$

assume $a, b > 0$

$\dot{y} = \frac{d}{dt}y$ $\qquad \dot{y} = -ay + bu$

∘ interpret
- y as velocity of car
- u throttle/gas pedal
- r desired velocity
- a air resistance, wheel friction
- b conversion from force to acceleration
- v road slope, head/tailwind

- mixing fluids
- velocity of a mass
- low-pass filters
- lumped circuit elements (inductor $v = L\frac{d}{dt}i$, capacitor $i = C\frac{d}{dt}v$)

→ determine transfer function G_{yv} when $C(s) = k_p$

$$G_{yv} = \frac{P}{1+PC} = \frac{b/s+a}{1 + bk_p/s+a} = \boxed{\frac{b}{s+(a+bk_p)}}$$

* I wanted G_{yv} as a rational function (i.e. ratio of polynomials)

* Routh-Hurwitz stability criterion (R-H)

this system is stable if all roots of characteristic polynomial

$a(s) = s + (a + bk_p)$ are negative

i.e. if $-(a + bk_p) < 0$

* R-H constrains values for k_p: $\quad k_p > \frac{-a}{b}$

∘ assuming $k_p > \frac{-a}{b}$, constant slope v_0 yields $\boxed{v = v_0 e^{0 \cdot t}}$

$$y(t) = y(0)\underbrace{e^{-(a+bk_p)t}}_{\to 0} + G_{yv}(0)\,v$$

$$\to y_0 = G_{yv}(0)\,v_0 = \frac{b}{a+bk_p}v_0$$

∘ in comparison, without feedback ($k_p = 0$), $y_0 = \frac{b}{a}v_0$

→ including proportional control $C(s) = k_p$ $\underbrace{\text{attenuates disturbance}}$

$$\left(\frac{b}{a+bk_p} < \frac{b}{a}\right)$$

$$\left(\frac{b}{a+bk_p} < \frac{b}{a}\right)$$

- try proportional-integral control:

$$u(t) = k_p e(t) + k_I \int_0^t e(\tau)\,d\tau$$

$$\Leftrightarrow \dot{u} = k_p \dot{e} + k_I e \qquad * \text{assume } e(0)=0$$

freq domain

$$\Leftrightarrow s u = k_p s e + k_I e$$

$$\Leftrightarrow s \frac{u}{e} = k_p s + k_I \Leftrightarrow \frac{u}{e} = \frac{k_p s + k_I}{s} = k_p + \frac{k_I}{s}$$

$$= C(s)$$

$$\rightarrow \text{verify } G_{yv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a+bk_p)s + bk_I}$$

$*$ note: $v = v_0$ constant $\Rightarrow y_0 = G_{yv}(0)v_0 = 0$!

$\rightarrow$ constant disturbance yields <u>zero</u> steady-state error