_7-transfer-functions

goal: frequency-domain tools and concepts for analysis & control

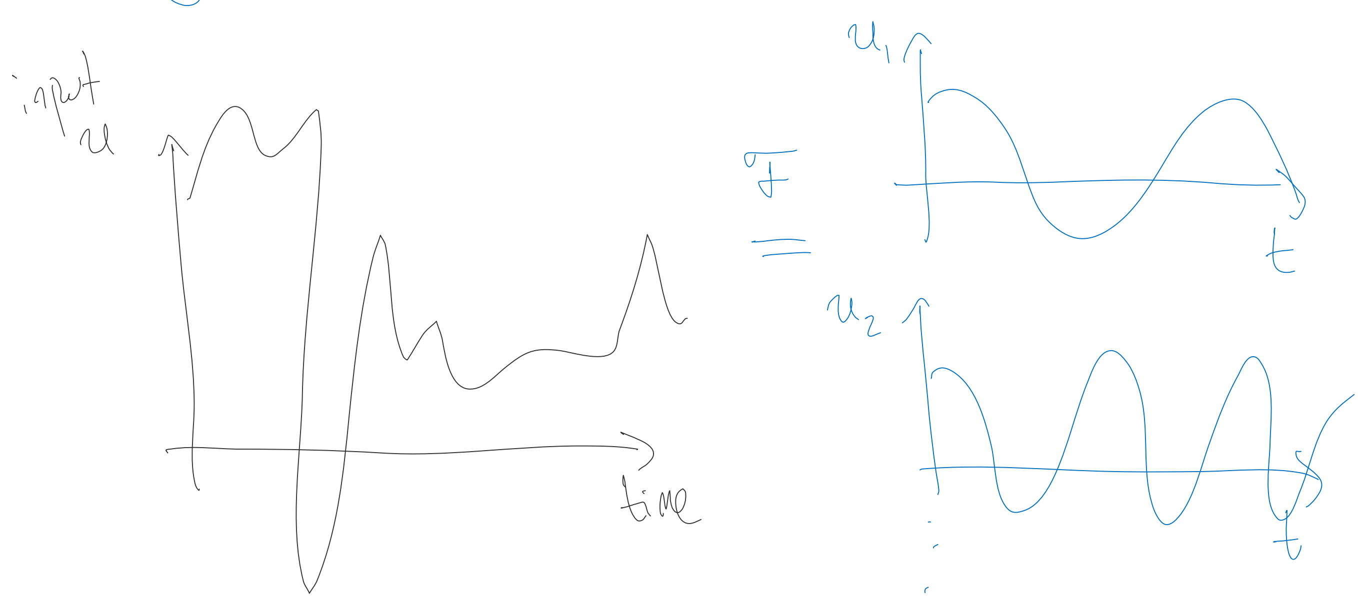
1°. frequency-domain modeling

- 1¹. transfer function of an LTI system [AMv2 Ch 6.3, 9.2] [Nv7 Ch 4.11]
- 1². block diagrams [AMv2 Ch 9.4] [Nv7 Ch 5]
- 1³. poles & zeros [AMv2 Ch 9.5] [Nv7 Ch 4.2, 4.10]
- 1⁴. Bode plot [AMv2 Ch 9.6] [Nv7 Ch 10.1]

1°. frequency domain modeling

- <u>key idea:</u> assuming system is stable, represent/analyze its steady-state response to periodic (sinusoidal) input

$\boxed{*\ \text{LTI}}$ systems' response to complex input is obtained by superimposing response to pure sinusoids



1¹. transfer function of LTI system

1! transfer function of LTI system

- consider the response of LTI system to input signal $u(t)$

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

fact: $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$ } convolution equation

intuition: input $u(\tau)$ applied at time τ adds system's response $e^{A(t-\tau)}Bu(\tau)$ to "initial state" $\tilde{x}(\tau) = Bu(\tau)$

→ verify by differentiating w.r.t. time t

when input $u \equiv 0$, $x(t) = e^{At}x(0)$

if $A = T\Lambda T^{-1}$ where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

then $e^{At} = Te^{\Lambda t}T^{-1}$, $e^{\Lambda t} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_n t})$

note: $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$

thus: $y(t) = Cx(t) + Du(t)$

* linear in $x(0)$ & u

$$= \underbrace{Ce^{At}x(0)}_{\text{homogeneous response to } x(0)} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{particular response to } u}$$

$u_0 \in \mathbb{R}^p$ $\quad \in \mathbb{R}^q$

- now we want $G_{yu}(s)$ s.t. $u(t) = u_0 e^{st} \rightsquigarrow y(t) = e^{st} \cdot G_{yu}(s) \cdot u_0$

* recall $\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$, $\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$

- $y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu_0 e^{s\tau}\,d\tau + Du_0 e^{st}$

$$= Ce^{At}x(0) + Ce^{At}\left[\int_0^t \underbrace{e^{-A\tau}e^{s\tau}}_{e^{(sI-A)\tau}}\,d\tau\right]Bu_0 + Du_0 e^{st}$$

$$e^{(sI-A)\tau}$$

→ evaluate the integral — hint: find expression that has $e^{(sI-A)\tau}$ as its τ-derivative

[a simpler problem: consider $A = \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$]

∘ know: $\frac{d}{d\tau} e^{(sI-A)\tau} = (sI-A) e^{(sI-A)\tau}$

so $\frac{d}{d\tau}\left[(sI-A)^{-1} e^{(sI-A)\tau}\right] = e^{(sI-A)\tau}$

$X \cdot e^X = e^X \cdot X \Downarrow e^X X^{-1} = X^{-1} e^X$

$* \; s \notin \{\lambda \in \mathbb{C} : \det(sI-A) = 0\}$

$$\Rightarrow \int_0^t e^{(sI-A)\tau} d\tau = (sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{\tau=t}$$

$$= (sI-A)^{-1}\Big(\underbrace{e^{(sI-A)t}}_{e^{-At}\cdot e^{st}} - I\Big)$$

$$\Rightarrow y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{transient response} \to 0 \text{ if } A \text{ stable}} + \underbrace{\left[C(sI-A)^{-1}B + D\right]}_{\substack{\text{steady-state response} \\ G_{yu}(s) \in \mathbb{R}^{q \times p}}} u_0 e^{st}$$

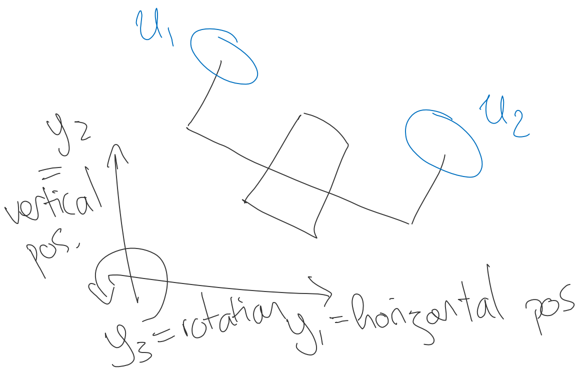
$*$ if A is stable, know $e^{At} \to 0$ as $t \to \infty$

so $y(t) \to G_{yu}(s) u_0 e^{st}$ as $t \to \infty$

note: $G_{yu}(s) \in \mathbb{R}^{q \times p}$; the transfer function from input u_j to output y_i is $[G_{yu}(s)]_{ij}$

$*$ $G_{yu}(s) = C(sI-A)^{-1}B + D$ gives a recipe for determining transfer function matrix from matrices A, B, C, D

ex: quadrotor



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$G_{yu} = \begin{bmatrix} G_{y_1u_1} & G_{y_1u_2} \\ G_{y_2u_1} & G_{y_2u_2} \\ G_{y_3u_1} & G_{y_3u_2} \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

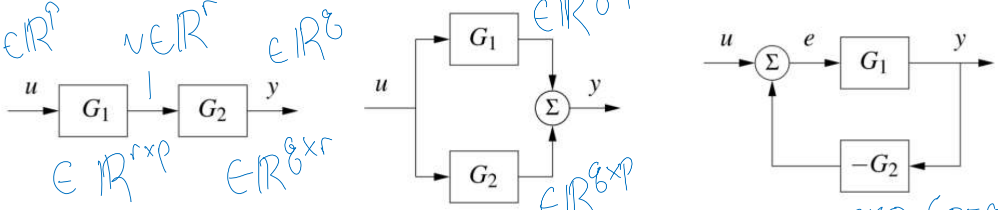
* applying $u_1 = e^{s_1 t}$, $u_2 = e^{s_2 t}$

yields $y_3(t) = \theta(t) = G_{y_3u_1}(s_1)\cdot e^{s_1 t} + G_{y_3u_2}(s_2)\cdot e^{s_2 t}$

13. block diagrams

- we've seen examples of block diagram representation & algebra

$\in \mathbb{R}^p$ $\quad v \in \mathbb{R}^r$ $\quad \in \mathbb{R}^q$ $\quad \in \mathbb{R}^{q \times p}$



$\in \mathbb{R}^{r\times p}$ $\quad \in \mathbb{R}^{q \times r}$ $\quad \in \mathbb{R}^{q\times p}$ $\quad$ SISO ($p=q=1$)

(a) $G_{yu}(s) = G_2(s)G_1(s)$ $\quad$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ $\quad$ (c) $G_{yu}(s) = \dfrac{G_1(s)}{1+G_1(s)G_2(s)} = (I + \cancel{G_2 G_1})^{-1} G_1$

→ rederive / verify these formulas for $p, q \neq 1$

(c) $y = G_1 e = G_1(u - G_2 y) = G_1 u - G_1 G_2 y$

$\iff y + G_1 G_2 y = G_1 u$ $\qquad$ * note: $G_1 \in \mathbb{R}^{q\times p}$, $G_2 \in \mathbb{R}^{p \times q}$

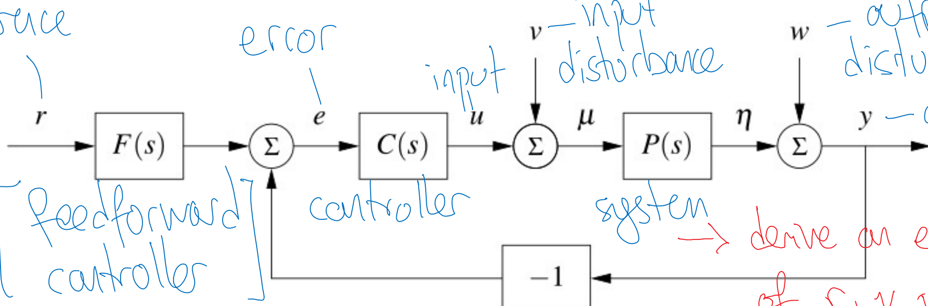
$= (I + G_1 G_2) y = G_1 u \iff$

$\iff y = (I + G_1 G_2)^{-1} G_1 u$ so $G_{yu} = (I + G_1 G_2)^{-1} G_1$

$y \in \mathbb{R}^{1\times q}$ $\quad y + yH_1H_2 = y(I + H_1 H_2)$

- consider a more detailed feedback diagram

reference $\quad$ error $\quad$ input $\quad$ v – input disturbance $\quad$ w – output disturbance $\quad$ y – output



[feedforward controller] $\quad$ controller $\quad$ system

→ derive an expression for e in terms of r, v, w, i.e. find G_{er}, G_{ev}, G_{ew} s.t. $e = G_{er} r + G_{ev} v + G_{ew} w$

ex: cruise control

r: desired speed $\qquad$ C: cruise control

u: control input, ie torque / gas pedal

u : control input, ie torque / gas pedal

v : input disturbance, eg headwind, slope

w : output disturbance, eg sensor noise, ADC quantization

y : output (speed)

P : car dynamics

torque $\rightarrow$ speed

eg $P = \frac{b}{s+a}$

$$\rightarrow e = Fr - y = Fr - (w + \eta) = Fr - w - P\mu$$

$$= Fr - w - P(v + u) = Fr - w - Pv - PCe$$

$$\Longleftrightarrow e + PCe = Fr - w - Pv$$

$$\Longleftrightarrow (I + PC)e = Fr - w - Pv$$

$$\Longleftrightarrow e = (I+PC)^{-1}(Fr - w - Pv)$$

$$p, q \neq 1 \quad = \underbrace{(I+PC)^{-1}F}_{G_{er}} \cdot r \underbrace{- (I+PC)^{-1}}_{G_{ew}} \cdot w \underbrace{- (I+PC)^{-1} P}_{G_{ev}} \cdot v$$

$$= \frac{F}{1+PC} r - \frac{1}{1+PC} w - \frac{P}{1+PC} v$$

$$y = \underbrace{\frac{(w + Pv + PCFr)}{1+PC}}_{\text{not a transfer function b/c it mixes signals } (w,v,r) \text{ with transforms}} = \underbrace{\frac{1}{1+PC}}_{G_{yw}} w + \underbrace{\frac{P}{1+PC}}_{G_{yv}} v + \underbrace{\frac{PCF}{1+PC}}_{G_{yr}} r$$

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13. poles & zeros

- consider the SISO transfer function $G(s) = \frac{b(s)}{a(s)}$
- G is a rational function that is ratio of polynomials

◦ consider the SISO transfer function $G = \frac{b(s)}{a(s)}$

* G is a rational function, that is, ratio of polynomials
  - roots of a are termed poles
  - roots of b are termed zeros
  - difference between degree of a & b termed relative degree

    G is proper if ≥ 0, strictly proper if > 0

◦ let's relate these concepts to $\dot{x} = Ax + Bu$, $y = Cx + Du$

- poles $= \lambda(A)$, i.e. eigenvalues of A
- zeros are complex numbers $s \in \mathbb{C}$ s.t.

  $$u(t) = e^{st} \text{ yields } y(t) = 0$$

* if state is fully actuated (B square) or fully measured (C square) then there are no zeros

◦ since poles & zeros govern system's behavior, often visualize them with a pole-zero diagram:

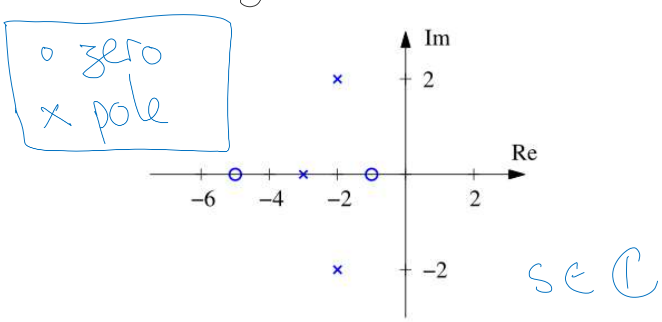

Figure 9.9: A pole zero diagram for a transfer function with zeros at −5 and −1 and poles at −3 and $-2 \pm 2j$. The circles represent the locations of the zeros, and the crosses the locations of the poles. A complete characterization requires we also specify the gain of the system.

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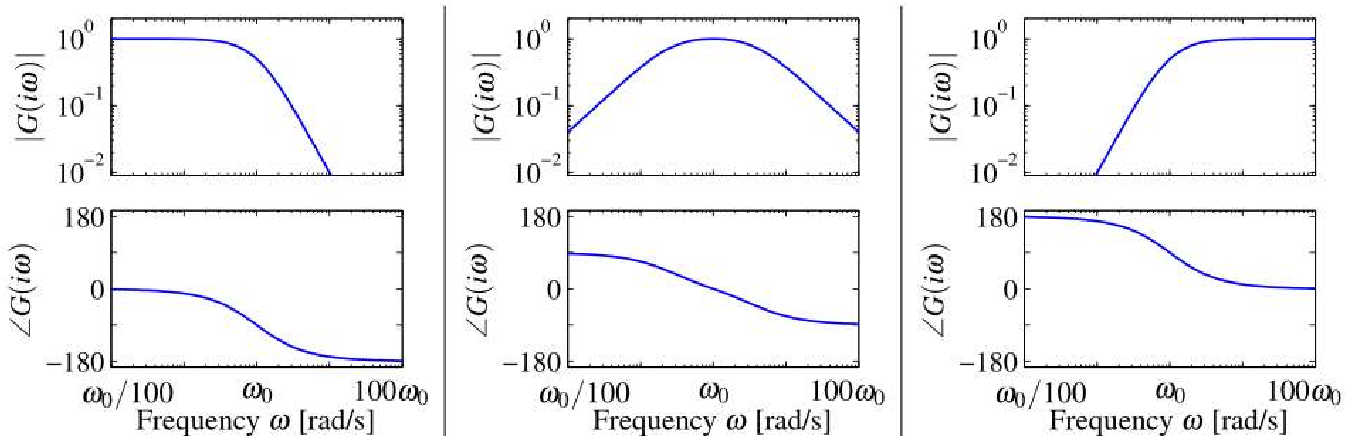
14. Bode plot

◦ since a complex exponential input $e^{j\omega t}$ yields

1. Bode plot

- since a complex exponential input $e^{j\omega t}$ yields output $G(j\omega)e^{j\omega t}$ $= |G(j\omega)|e^{j\omega(t+\angle G(j\omega))}$

it's useful to visualize <u>gain</u> $|G(j\omega)|$ and <u>phase</u> $\angle G(j\omega)$ as frequency ω varies



$G(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(a) Low-pass filter

$G(s) = \dfrac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(b) Band-pass filter

$G(s) = \dfrac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

(c) High-pass filter

Figure 9.17: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

→ which of these would you want for G_{yr}, G_{yn}, G_{yw}? (in cruise control example)