9-frequency-domain

goal: tools for analysis using transfer functions, root locus plots

1°. frequency domain analysis

1¹. sensitivity functions [AMv2 ch 12.1, 12.2] [Nv7 not covered]

1². root locus [AMv2 Ch 12.5] [Nv7 Ch 9]

* general comment: these techniques were developed before we had cheap computers, so there are many graphing heuristics that are traditionally taught;
→ we'll rely on computers to graph, but still extract intuition

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1°. frequency domain analysis

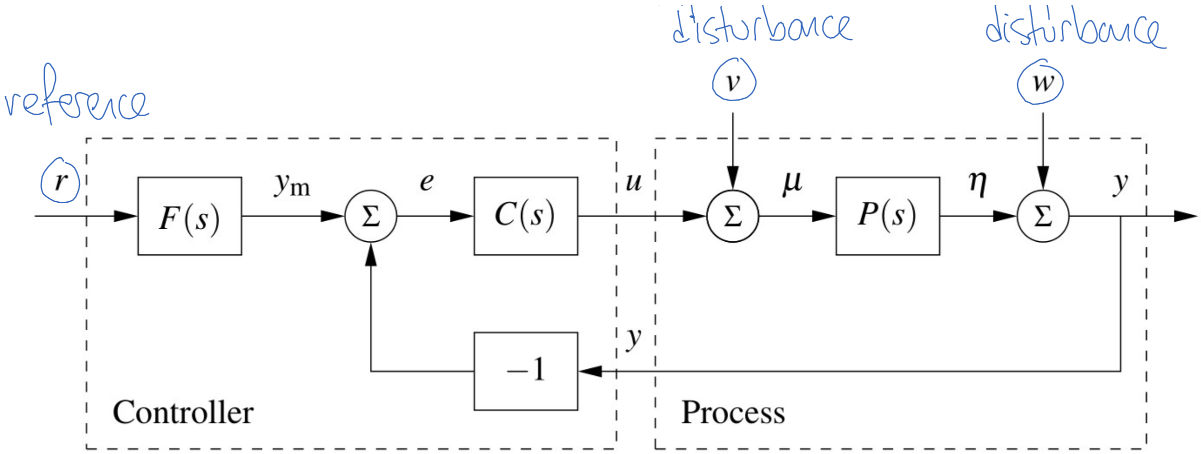
1¹. sensitivity functions

- we now return to the general feedback diagram with 3 external inputs:
  - r : reference
  - v : input disturbance
  - w : output disturbance

input disturbance v

output disturbance w

reference



- note that the process input μ and output η aren't what our controller commands (u) or measures (y)

y	u	e	μ	η	
$\frac{PCF}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{F}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{PCF}{1+PC}$	r
$\frac{P}{1+PC}$	$\frac{-PC}{1+PC}$	$\frac{-P}{1+PC}$	$\frac{1}{1+PC}$	$\frac{P}{1+PC}$	v
$\frac{1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-PC}{1+PC}$	w

* we're particularly concerned with how input & output disturbances v, w map to controller input & output u, y

- neglecting signs, we're focused on:

$S = \frac{1}{1+PC}$ sensitivity $\quad T = \frac{PC}{1+PC}$ complementary sensitivity

↳ note: $S + T = \frac{1+PC}{1+PC} = 1$, thus name makes sense

$PS = \frac{P}{1+PC}$ input sensitivity $\quad CS = \frac{C}{1+PC}$ output sensitivity

→ suppose (open-) loop transfer function

$L(s) = P(s)\,C(s) \to 0$ as $s \to \infty$

(i.e. L is <u>strictly proper</u>),

what can you say about how:

a) high-frequency input disturbance affects output

b) high-frequency output disturbance affects input

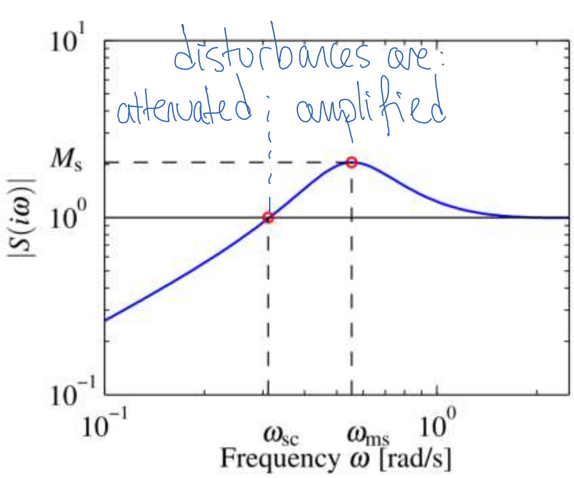
?

- the sensitivity transfer functions

$S,\ T,\ PS,\ CS$

can be used to assess (or specify) performance of a closed-loop system

– for example, the maximum gain M_s of the sensitivity function S is related to the stability margin s_m via $M_s = \dfrac{1}{s_m}$



(a) Gain curves

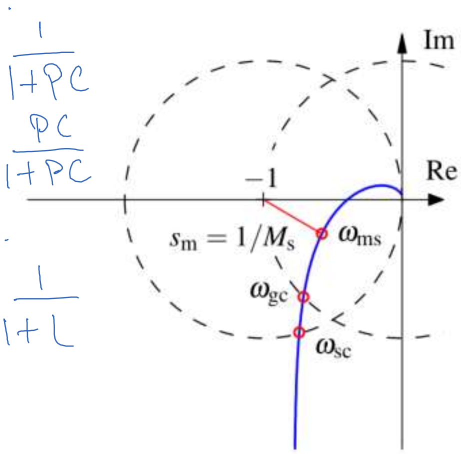
note:

$S = \dfrac{1}{1+PC}$

$L = \dfrac{PC}{1+PC}$

so...

$S = \dfrac{1}{1+L}$



(b) Nyquist plot

Frequency ω [rad/s]

(a) Gain curves

(b) Nyquist plot

– specifications may be based on peak gain or corresponding frequency ω_{ms}, crossover frequency ω_{sc} (smallest freq for which gain equals one (1)), bandwidth, ...

→ how would you measure these performance specifications empirically? (suppose you have access to signals r, u, y in the block diagram, i.e. you can measure and/or alter additively)

1^2. root locus

- the main control design technique we've used so far is eigenvalue assignment via full-state feedback
  – the resulting controller is complex, since it requires constructing an observer with the same complexity as the original system
- we'll now investigate how much can be accomplished with the simplest possible controller: a single gain (i.e. proportional control)

- next week we'll elaborate to a gain, an integrator, and a differentiator (i.e. proportional-integral-derivative control, PID)

∘ consider the system transfer function

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$= b_0 \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

- assume P is proper, i.e. $n \geq m$
- consider negative feedback with pure gain controller $C(s) = k$

∘ the closed-loop transfer function is $\frac{k \cdot P}{1 + k \cdot P}$, which has characteristic polynomial

$$a_k(s) = a(s) + k\, b(s)$$

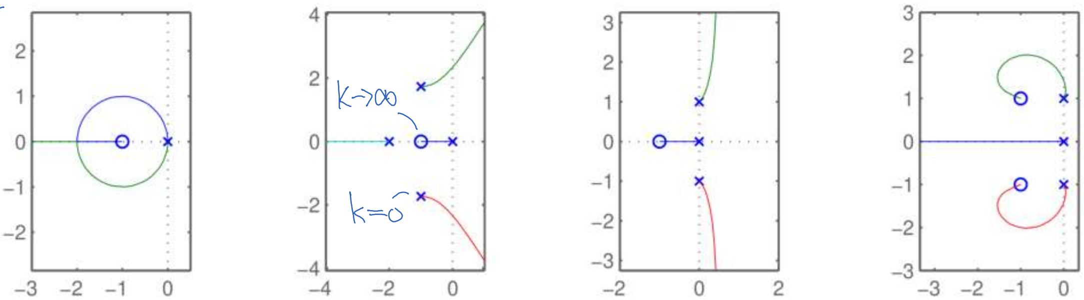
- closed-loop stability is determined by the roots of a_k, which vary with k
- the graph of the roots of a_k in the complex plane $\mathbb{C}$ as k varies is termed the <u>root locus</u>
- since $a_0 = a$, the roots of a determine the starting point

- for k large, it turns out that the roots of a_k will approach $n-m$ equally-spaced asymptotes

ex:

$$P_a(s) = k\frac{s+1}{s^2}, \qquad P_b(s) = k\frac{s+1}{s(s+2)(s^2+2s+4)},$$
$$P_c(s) = k\frac{s+1}{s(s^2+1)}, \qquad P_d(s) = k\frac{s^2+2s+2}{s(s^2+1)}. \tag{12.18}$$

start at "x", tend to "o"



(a) $P_a(s)$ (b) $P_b(s)$ (c) $P_c(s)$ (d) $P_d(s)$

Figure 12.18: Examples of root loci for processes with the transfer functions $P_a(s)$, $P_b(s)$, $P_c(s)$, and $P_d(s)$ given by equation (12.18).

→ which of these systems can be stabilized by proportional feedback?
(can the gain be arbitrarily large?)

(a) yes – any $k>0$ works
(b) yes – k can't be too large, otherwise two poles have $\mathrm{Re} \geq 0$
(c) no – two poles have $\mathrm{Re} \geq 0$ for all $k \geq 0$
(d) yes – not clear from diagram whether small k works, but all k sufficiently large do

→ how would you use the root locus to determine whether a system can be stabilized with proportional feedback?

→ how would you use the root locus to determine whether a stable system can track a non-zero reference with integral feedback?

- before computation was cheap, many heuristics were developed to enable control engineers to draw root locus diagrams by hand
- now, we can rely on the computer to draw the diagram, and focus our effort on interpreting the diagram