_6-state-and-output-feedback

goal: develop systematic / automated techniques to control physical systems in state-space form

1°. state feedback [AMv2 Ch 7]

1!. stabilization [Nv7 Ch 12.2]

2°. output feedback [AMv2 Ch 8]

2!. observer design [Nv7 Ch 12.5]

1°. state feedback

* start by assuming we (or controller) know whole system state (i.e. "x" in $\dot{x} = Ax + Bu$)

  eg all lumped element voltages / currents in circuit
  all positions and velocities in mechanical system

not: $u \in \mathbb{R} \longrightarrow \boxed{\text{SISO}} \longrightarrow y \in \mathbb{R}$

instead: $u \in \mathbb{R}^p \longrightarrow \boxed{\dot{x} = Ax + Bu} \longrightarrow x \in \mathbb{R}^n$

∘ we've seen that roots of system's characteristic polynomial govern its dynamics & stability

$\longrightarrow$ we'll build up tools that enable us to place these roots given its dynamics & stability

→ we'll build up tools that enable us to place these roots wherever we want to

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1. stabilization

- consider $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

→ we seek to <u>stabilize</u> the system, that is, determine input u as a function of state x, $u(x)$, such that closed-loop system

$$\dot{x} = Ax + Bu(x) \quad \text{is stable}$$

$K \in \mathbb{R}^{p \times n}$

* if we use linear state feedback, $\underbrace{u = -Kx}_{\text{generalized proportional control}}$, then the closed-loop dynamics are:

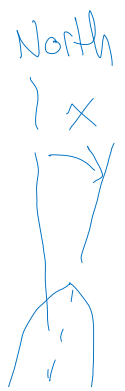
$$\dot{x} = Ax + Bu$$
$$= Ax - BKx = (A - BK)x$$

which is stable if Real part of eigenvalues of $(A-BK)$ are negative: $\mathrm{Re}(\lambda(A-BK)) < 0$

→ given $\dot{x} = ax + bu$, $x \in \mathbb{R}$, $u \in \mathbb{R}$

↳ eg x – vehicle speed
u – engine throttle / force } cruise control
$u = -k(x - r)$, r – reference speed

ex:



x – heading error
u – steering wheel / rudder angle
β – unit conversion between

u — steering wheel / rudder angle

k — unit conversion between heading angle & rudder angle

→ determine the range of values for $k \in \mathbb{R}$ that stabilize the system if $u = -kx$

- given $\dot{x} = ax + bu$, so $u = -kx$ yields $\dot{x} = ax - bkx = (a - bk)x$ which is stable if $\mathrm{Re}(\lambda(a - bk)) < 0$

  i.e. if $(a - bk) < 0$

  $$\iff \frac{a}{b} < k$$

→ more generally, we want to determine entries in $K \in \mathbb{R}^{p \times n}$ to ensure eigenvalues of $A - BK$ are where we want them (i.e. in left-half plane for stability)

- suppose we want eigenvalues to be $\{\lambda_i\}_{i=1}^n$
- then we want the characteristic polynomial of closed-loop system $\dot{x} = (A - BK)x$ to be

$$a(s) = \det(sI - (A - BK))$$

$$= s^n + a_1 s^{n-1} + \cdots + a_{n-1} s^1 + a_n$$

$\{a_i\}_{i=1}^n$ depend on entries of $(A - BK)$ in (potentially) complex way

$$= (s - \lambda_1) \cdot (s - \lambda_2) \cdots (s - \lambda_n)$$

$$= \prod_{i=1}^n (s - \lambda_i)$$

this is just notation that means "multiply stuff together"

$i=1$ } that means multiplying stuff together

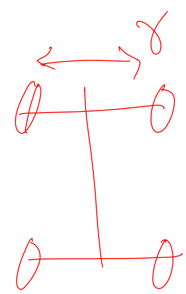
procedure: 1°: compute $\det(sI-(A-BK))$ symbolically

2°: compute $\prod_{i=1}^{n}(s-\lambda_i)$ symbolically

3°: determine what K's entries need to be to make (1°) = (2°)

→ apply this procedure when $A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $B=\begin{bmatrix}\gamma\\ 1\end{bmatrix}$

and (2°) $= s^2-2\zeta\omega s+\omega^2$



$x=\begin{bmatrix}\text{pos}\\ \text{vel}\end{bmatrix}$

$u=$ wheel angle

[AMv2 Ex 7.4]

- $\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad-bc$
- $K\in\mathbb{R}^{p\times n}$, i.e. $K=[k_1\ k_2]$

  so $BK=\begin{bmatrix}\gamma\\ 1\end{bmatrix}[k_1\ k_2]=\begin{bmatrix}\gamma k_1 & \gamma k_2\\ k_1 & k_2\end{bmatrix}$
- $\det(sI-(A-BK)) = s^2+(\gamma k_1+k_2)s+k_1 =$ (1°)

  want: (2°) $= s^2 \quad -2\zeta\omega \quad s+\omega^2$

  want: $\gamma k_1+k_2=-2\zeta\omega$ and $k_1=\omega^2$

  i.e. $\gamma\omega^2+k_2=-2\zeta\omega \iff k_2=-2\zeta\omega-\gamma\omega^2$

2°: output feedback (an <u>output</u> from a system is a measured quantity)

→ for these two systems:

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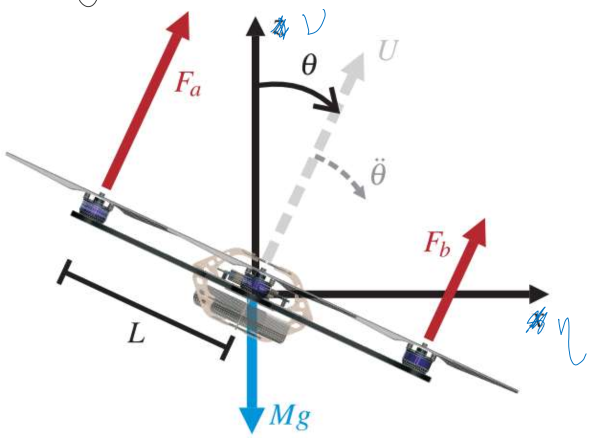
- determine a state vector $x \in \mathbb{R}^n$
- think about how you would measure each state variable (what is the sensor? how reliable/expensive is the sensor?)

ex: quadrotor

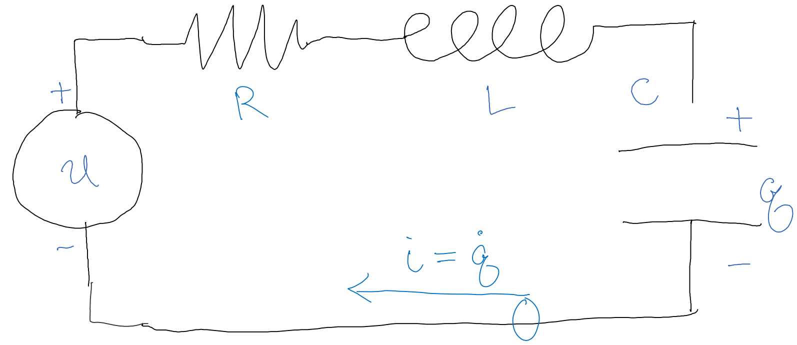


state: (positions, velocities)

measure: - gyro / IMU / accelerometer

- GPS / localization / SLAM

ex: RLC circuit



state: (voltages, currents)

measure: - voltmeter

- ammeter

* last lecture, we derived systematic procedure to design a stabilizing controller that uses all state variables

→ this lecture, we'll derive a systematic procedure to obtain estimates of all state variables using limited number of outputs (i.e. measurement channels / signals)

2'. observers

- to estimate the state of $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

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assume we have access to outputs $y = \underbrace{Cx + Du}$, $y \in \mathbb{R}^q$

→ note: we're restricting attention to <u>linear</u> observations of state/input

$C \in \mathbb{R}^{q \times n}$
$D \in \mathbb{R}^{q \times p}$

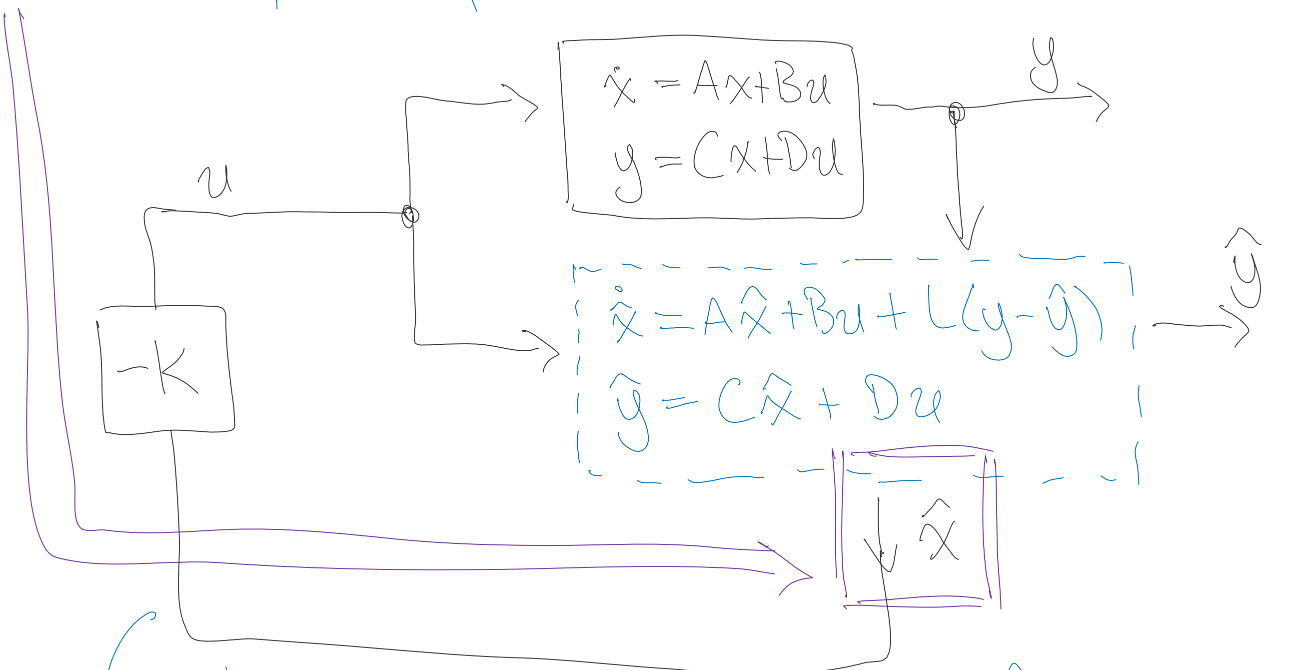
* to solve this problem, we'll construct another system termed an <u>observer</u>:

compare w/ $K \in \mathbb{R}^{p \times n}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\underbrace{y - \hat{y}}_{\text{output error}})$$
$$\hat{y} = C\hat{x} + Du$$

$L \in \mathbb{R}^{n \times q}$

* unlike x, all of observer state $\hat{x}$ is known to us



→ idea: if $\hat{x} \simeq x$ then $u = -K\hat{x} \simeq -Kx$, so this input will control original system

→ to see why this works, determine the dynamics of $\underbrace{e = x - \hat{x}}$ ($\dot{e}$ shouldn't depend on $x, \hat{x}, y, \hat{y}$, or u)

$e \in \mathbb{R}^n$

- $\dot{e} = \dot{x} - \dot{\hat{x}}$

$e \in \mathbb{R}^n$

- $\dot{e} = \dot{x} - \dot{\hat{x}}$
$$= (Ax + \cancel{Bu}) - (A\hat{x} + \cancel{Bu} + L(y - \hat{y}))$$
$$= Ax - A\hat{x} - L(Cx - C\hat{x})$$
$$= A(x - \hat{x}) - LC(x - \hat{x}) = \overbrace{(A - LC)}^{\in \mathbb{R}^{n \times n}} e$$

* so if $L \in \mathbb{R}^{n \times q}$ is s.t. $\mathrm{Re}(\lambda(A - LC)) < 0$
  we know $e(t) = e^{(A-LC)t} e(0) \longrightarrow 0$ as $t \to \infty$
  $\Longrightarrow \hat{x}(t) \longrightarrow x(t)$ as $t \to \infty$!

ex: vehicle steering $\dot{x} = Ax + Bu$, $y = Cx$ $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

→ compute observer error dynamics $(A - LC)$ and characteristic polynomial

- $L \in \mathbb{R}^{2 \times 1}$ - $A - LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$

- $\det(sI - (A - LC)) = s^2 + l_1 s + l_2$

* all we need to do is choose l_1, l_2 to ensure
  $\mathrm{Re}(\lambda(A - LC)) < 0$
  i.e. observer error goes to zero

* numerically, can use ctrl. place:
  - know place$(A, B) \rightsquigarrow K$ s.t. $\mathrm{Re}(\lambda(A - BK)) < 0$
  - so $(\text{place}(A^T, C^T))^T \rightsquigarrow L$ s.t. $\mathrm{Re}(\lambda(A - LC)) < 0$

* since $(A - LC)^T = A^T - C^T L^T$

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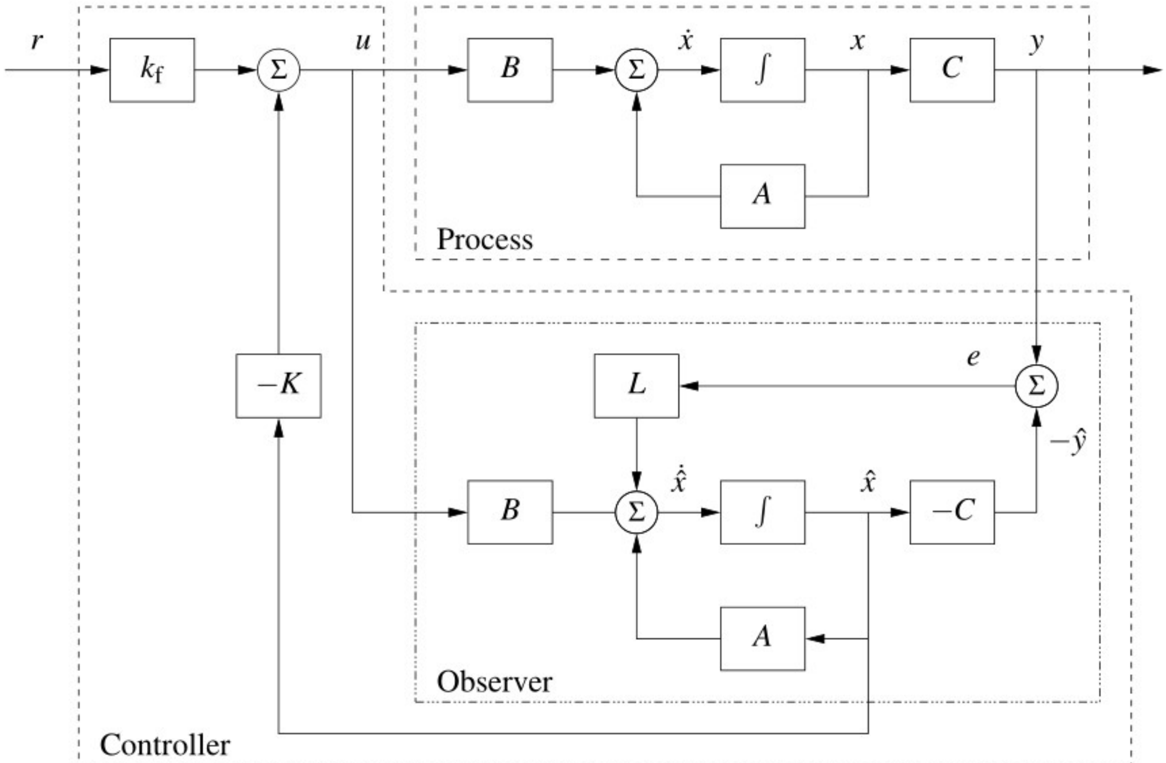

Figure 8.7: Block diagram of an observer-based control system. The observer uses the measured output y and the input u to construct an estimate of the state. This estimate is used by a state feedback controller to generate the corrective input. The controller consists of the observer and the state feedback; the observer is identical to that in Figure 8.5.