1-feedback-principles

[AMv2 Ch 2]

goal: introduce fundamental uses and properties of feedback

topics:

1°. mathematical models of systems

  1¹. differential equations (DE)

  1². transfer functions

  1³. block diagrams

2°. effects of feedback

  2¹. disturbance attenuation

  2². unmodeled dynamics

  2³. reference tracking

* read [AMv2 ch 2.2.5] to learn how positive feedback used in digital systems

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1°. mathematical models of systems

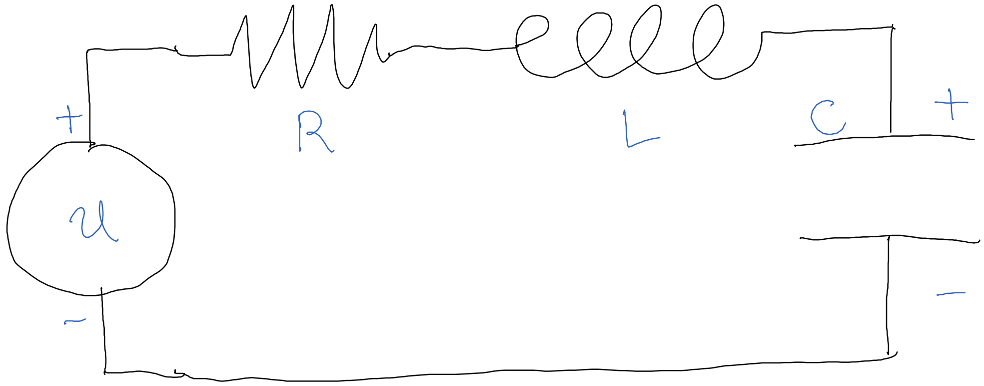
- we will work with several representations of linear control systems:
  - 1¹. differential equations
  - 1². transfer functions
  - 1³. block diagrams

each has advantages & provides insight

1³. block diagrams } insight

* what is a system?

ex: consider an RLC circuit



* how does input voltage u relate to output charge y?

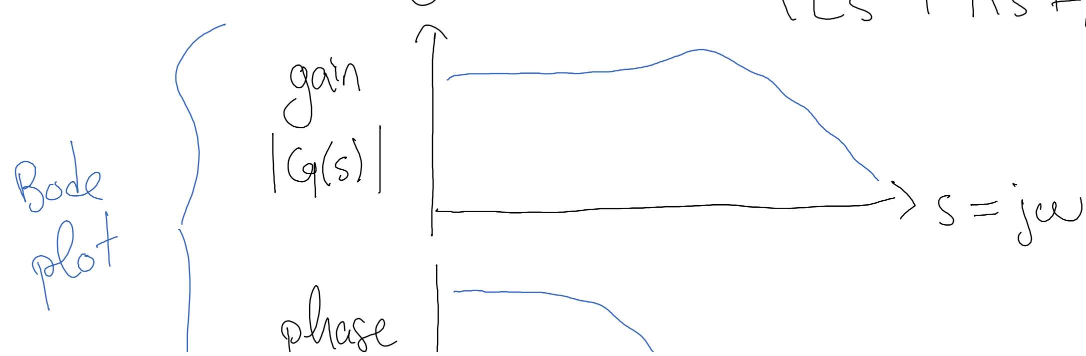
1¹. differential equation

- KVL $\Rightarrow$ $u = Ri + L\frac{d}{dt}i + \frac{1}{c}y$

$$= R\frac{d}{dt}y + L\frac{d^2}{dt}y + \frac{1}{c}y$$

1². transfer function

$$u = e^{st} \Rightarrow y = G(s)\,u = \left(\frac{1}{Ls^2 + Rs + \frac{1}{c}}\right)u$$



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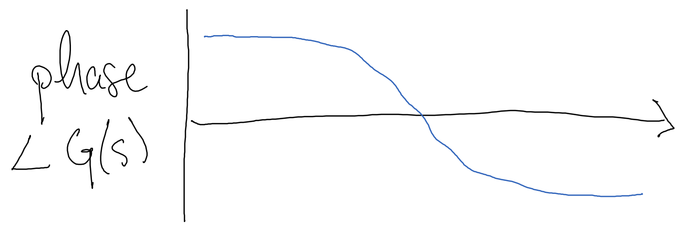
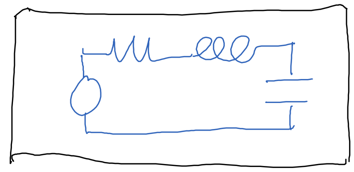


1^3. block diagram $u \longrightarrow$  $\longrightarrow y$

* a <u>system</u> is a mathematical model we can put in a box, i.e. a function that transforms input u to output y

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$1'$. (linear) differential equation (DE): [AMv2 Ch 2] [Nise Ch 3,4]

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y$$

$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u + \cdots + b_n u$$

where u is input

y is output

t is time

and $\{a_k\}, \{b_k\} \subset \mathbb{R}$

(i.e. the a_k's & b_k's are real numbers)

- note that (DE) is specified by two polynomial expressions,

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$$
$$b(s) = b_1 s^{n-1} + \cdots + b_n$$

↳ called the <u>characteristic</u> polynomial

* these polynomials, and their algebraic properties, tell us a lot about (DE)

- a "solution" to (DE) is a pair of <u>signals</u> $u : \mathbb{R} \to \mathbb{R} : t \mapsto u(t)$ $\quad y : \mathbb{R} \to \mathbb{R} : t \mapsto y(t)$

↳ i.e. smooth functions of time

that satisfy (DE) at all times $t \in \mathbb{R}$

→ if u, y solve (DE), show that

$u' : \mathbb{R} \to \mathbb{R} : t \mapsto u(t+\tau)$ $\quad y' : \mathbb{R} \to \mathbb{R} : t \mapsto y(t+\tau)$

solve (DE) as well

↳ thus (DE) is <u>time-invariant</u>

→ if $(u_1, y_1), (u_2, y_2)$ solve (DE), show that $(u_1 + \alpha u_2, y_1 + \alpha y_1)$ solve (DE) as well, where $\alpha \in \mathbb{R}$

↳ thus (DE) is <u>linear</u>

↳ (DE) is linear & time-invariant (LTI)

fact: every solution to (DE)
is a linear combination of:
the homogeneous solution (i.e. $u=0$)
& a particular solution ($u \neq 0$)

fact: when $u=0$, the solution to

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$$

is a linear combination of complex exponentials

$$y(t) = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$$
$$= \sum_{k=1}^{n} c_k e^{s_k t}$$

assuming they are distinct

where $\{s_k\}_{k=1}^{n}$ are the roots of the characteristic polynomial $a(s)$,
i.e. $a(s_k) = 0$ for $k = 1, \cdots, n$
and the coefficients $\{c_k\}$ are determined by the initial condition

$$y(0), \dot{y}(0), \ddot{y}(0), \cdots, \frac{d^n}{dt^n} y(0)$$
$$= \left\{ \frac{d^k}{dt^k} y(0) \right\}_{k=1}^{n}$$

fact: since a_k's are real, the roots are:
real -or- complex conjugate pairs
– real root s_k ⇝ real exponential $e^{s_k t}$

- real root $s_k \rightsquigarrow$ real exponential $e^{s_k t}$
  - ↳ decays to 0 (zero) if $s_k < 0$,
    constant if $s_k = 0$,
    increases if $s_k > 0$
- complex conjugate pair $s_k = \sigma \pm j\omega \in \mathbb{C}$
  $\rightsquigarrow e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t)$
  - ↳ oscillating within exponential envelope determined by $\sigma \in \mathbb{R}$

1². transfer function: [AMv2 Ch 2] [Nise Ch 2]

- from a different perspective, a system transforms input u to output y:

$$\text{input signal} \xrightarrow{u} \boxed{\text{system}} \xrightarrow{y} \text{output signal}$$

- when $u(t) = e^{st}$, $s \notin \{s_k\}$,
  and system is linear & time-invariant,
  guess $y(t) = G(s)\, e^{st}$
  - ↳ some frequency-dependent $G : \mathbb{C} \to \mathbb{C}$

→ verify the following:

$\frac{d}{dt} u = s e^{st}$, $\frac{d^2}{dt^2} u = s^2 e^{st}$, ..., $\frac{d^n}{dt^n} u = s^n e^{st}$

$\frac{d}{dt} y = s\, G(s)\, e^{st}$, ..., $\frac{d^n}{dt^n} y = s^n G(s)\, e^{st}$

$\frac{d}{dt} y = s\,G(s)\,e^{st}, \ \ldots, \ \frac{d^n}{dt^n} y = s^n G(s) e^{st}$

- substituting into (DE) yields

$$(s^n + a_1 s^{n-1} + \cdots + a_n)\, G(s)\, e^{st}$$
$$= (b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n)$$

so $G(s) = \dfrac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \dfrac{b(s)}{a(s)}$

is the (frequency) transfer function that tells us how (complex) exponential inputs transform into exponential outputs

→ recall from signals class that any input can be expressed as a (possibly infinite) sum of complex exponentials via the Fourier transform,

* so $G(s)$ contains all the information

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summary & synthesis of 1! & 1? :

- exponential input $u(t) = e^{st}$ to linear system yields exp. out. $y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{\text{homogeneous response to initial state}} + G(s) e^{st}$

homogeneous response to initial state

particular response to input signal

- terminology:

  - static gain $G(0) = \frac{b_n}{a_n} = y$ for $u = e^{0 \cdot t} = 1$
  - magnitude $|z|$, argument / phase / angle $\angle z$ of a complex number $z = r e^{j\theta}$ are defined by $|z| = r$, $\angle z = \theta$
  - real part $\mathrm{Re}\, z$, imaginary part $\mathrm{Im}\, z$ of $z = \sigma + j\omega$ defined $\mathrm{Re}\, z = \sigma$, $\mathrm{Im}\, z = \omega$

- note: given $u(t) = \sin \omega t = \mathrm{Im}\, e^{j\omega t}$

$$
\begin{aligned}
y(t) &= \mathrm{Im}\left(G(j\omega) e^{j\omega t} \right) \\
&= \mathrm{Im}\left(|G(j\omega)| e^{j \angle G(j\omega)} e^{j\omega t} \right) \\
&= |G(j\omega)| \, \mathrm{Im}\, e^{j(\angle G(j\omega) + j\omega t)} \\
&= \underbrace{|G(j\omega)|}_{\text{gain}} \sin(\omega t + \underbrace{\angle G(j\omega)}_{\text{phase}})
\end{aligned}
$$

→ what happens when $u(t) = e^{s_k t}$, $a(s_k) = 0$?

- what does the transfer function tell us?
- " " (DE) tell us?

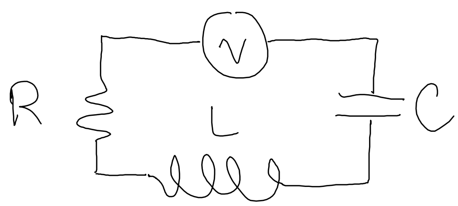
– " " (DE) tell us?

→ same Q's for $u(t) = e^{s_\ell t}$, $b(s_\ell) = 0$

↳ s_k termed a <u>pole</u>, s_ℓ termed a <u>zero</u>

<u>ex:</u> consider an RLC circuit

$$\text{(DE)} \quad L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$$



} q is charge on capacitor

– can determine transfer function by inspection:

$$G(s) = \frac{1}{Ls^2 + Rs + 1/C}$$

→ show that $G(s) \simeq \begin{cases} C, & s \text{ small} \\ 1/Ls^2, & s \text{ large} \end{cases}$

– what is the interpretation of the circuit's behavior in these two input regimes (small, large)?

∘ given an LTI system $u \rightarrow \boxed{\text{LTI sys}} \rightarrow y$,

since $u(t) = e^{st} \rightsquigarrow y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}} + G(s) e^{st}$

we want to know whether (homogeneous) encoded we want to know whether $\begin{pmatrix}\text{homogeneous} \\ \text{response}\end{pmatrix}$ encoded in c_k's

is <u>stable</u>, i.e. decays to zero for any initial state

– in other words, we want to know if roots of characteristic polynomial $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n = 0$ have negative real part

↳ you know the quadratic formula; analogous formulae exist for cubic & quartic (i.e. 3rd- & 4th-order) polynomials, but generally don't for higher-orders!

– Routh (1831–1907) & Hurwitz (1859–1919) found necessary & sufficient criteria for stability using only coefficients $\{a_k\}$ (i.e. <u>not</u> roots $\{s_k\}$):

roots of $a(s)$ have negative real part if and only if ($\iff$) $\begin{pmatrix}\text{algebraic condition} \\ \text{on coeff's } \{a_k\}\end{pmatrix}$

$s^2 + a_1 s + a_2$	$a_1, a_2 > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0$, $a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0$, $a_1 a_2 > a_3$, $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$

<u>ex:</u> RLC circuit $L\ddot{q} + R\dot{q} + \frac{1}{C} q$

stable $\iff$ R/L, $1/LC > 0$

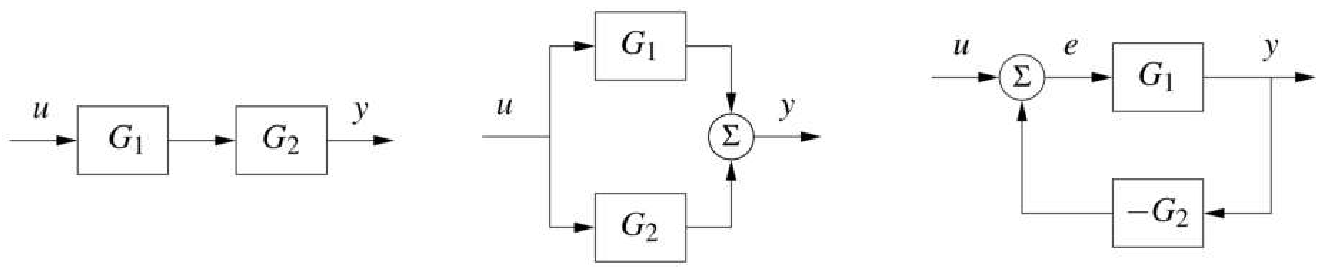
stable $\iff$ $R/L,\ 1/CL > 0$

↳ matches our intuition

(→ is there a physical interpretation of $R, L,$ or $C \leq 0$?)

1.3: <u>block diagrams</u> provide a third kind of math model, particularly useful for specifying system interconnections:

[AMv2 Ch 2]
[Nise Ch 5]



(a) $G_{yu}(s) = G_2(s)G_1(s)$ (b) $G_{yu}(s) = G_1(s) + G_2(s)$ (c) $G_{yu}(s) = \dfrac{G_1(s)}{1 + G_1(s)G_2(s)}$

– note the convention: G_{yu} is transfer function from u to y

→ derive formulae for (a), (b), (c) by letting $u = e^{st}$ and solving for y/u

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2: effects of feedback

◦ there are many uses & types of feedback;

- there are many uses & types of feedback; we'll focus on these important cases:

  2¹. disturbance attenuation

  2². unmodeled dynamics

  2³. reference tracking

* read [AMv2 ch 2] to learn about other uses & types of feedback

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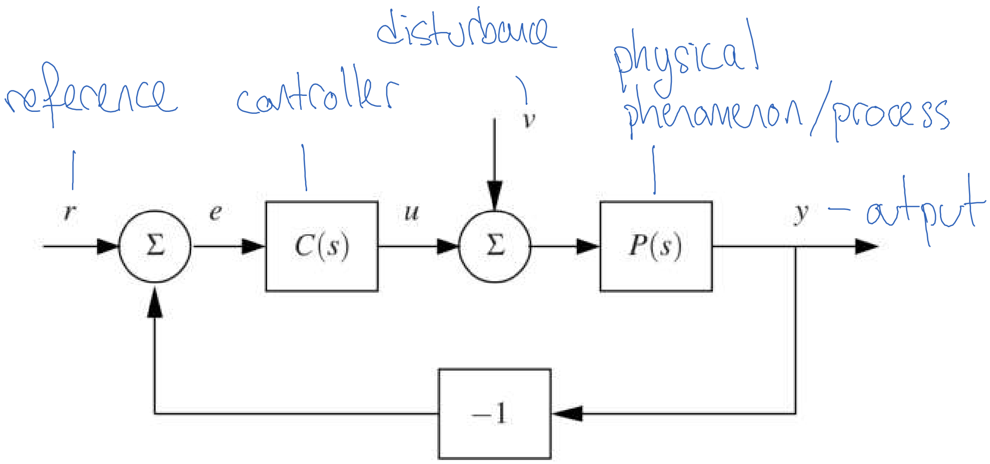
2¹. disturbance attenuation [AMv2 Ch 2.3]

- consider the block diagram



- for simplicity, consider $r = 0$ (reference tracking studied in later section)

- transfer function from v to y satisfies

$$y = P(v + Ce) \qquad e = r - y$$

$$= P(v + C(r - y)) \qquad r = 0$$

$$= P(v - Cy)$$

$$\Longleftrightarrow (1 + PC)y = Pv$$

$$\Longleftrightarrow y = \frac{P}{1+PC} v$$

i.e. $G_{yv}(s) = \dfrac{P(s)}{1 + P(s)C(s)}$

ex: for simplicity, consider $P(s) = \dfrac{b}{s+a}$,

i.e. $\dot{y} + ay = bu$, $a, b > 0$

- velocity of car;
- angular velocity of flywheel;
- temperature of a mass;
- liquid in a reservoir

$\rightarrow$ in this case:

r - desired velocity

u - throttle / gas pedal

y - car velocity

v - road slope, headwind, ...

a - air resistance, wheel friction, ...

b - conversion from force to accel

- try proportional control: $u = k_p e$

- try <u>proportional control</u>: $u = k_p e$

  i.e. $C(s) = k_p$, and hence

$$G_{yv} = \frac{P}{1+PC} = \frac{b/_{s+a}}{1 + bk_p/_{s+a}} = \frac{b}{s+(a+bk_p)}$$

  – so disturbance & output satisfy (DE)

$$\dot{y} + (a + bk_p)y = bv$$

  – this (closed-loop) system is stable if all roots of characteristic polynomial

$$a(s) = s + (a + bk_p)$$ are negative,

  i.e. if $a + bk_p > 0$

  – in this case, constant disturbance v_0 (e.g. slope of hill) yields

$$y \to y_0 = G_{yv}(0) = \frac{b}{a+bk_p} v_0$$

  with time constant $T = 1/_{a+bk_p}$

  * without feedback ($k_p = 0$), $y \to bv_0/a$ at rate $T_0 = 1/a$ so feedback both

  i) attenuates disturbance $\left(\frac{b}{a+bk_p} < \frac{b}{a}\right)$

  ii) speeds convergence $\left(T = \frac{1}{a+bk_p} < \frac{1}{a} = T_0\right)$

- try <u>proportional-integral control</u>:

$u(t) = k_p e(t) + k_I \int^t \ldots$

$$u(t) = k_P e(t) + k_I \int_0^t e(\tau)\,d\tau$$

– differentiating: $\dot{u} = k_P \dot{e} + k_I e$

so transfer function $C(s) = \dfrac{k_P s + k_I}{s}$

$$= k_P + k_I/s$$

– closing the loop yields

$$G_{yv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a + bk_P)s + bk_I}$$

→ verify this expression for G_{yv}

– as a (DE)

$$\ddot{y} + (a + bk_P)\dot{y} + bk_I y = b\dot{v}$$

* constant disturbance $v = v_0$ yields zero steady-state error, $G_{yv}(0) = 0$!

→ what assumption is needed on k_P, k_I?

– to ensure closed-loop system is stable & performance is desirable, we can tune k_P, k_I:

– the characteristic equation of (DE) is

$$a_{cl}(s) = s^2 + (a + bk_P)s + b$$

↳ cl = "closed loop"

d = "desired"

– if we desire complex-conjugate roots $\underbrace{-\sigma_d \pm j\omega_d}$,

- if we desire complex-conjugate roots $-\sigma_d \pm j\omega_d$,

$$(s+\sigma_d+j\omega_d)(s+\sigma_d-j\omega_d) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2$$

yielding solutions $e^{-\sigma_d t}\sin(\omega_d t)$, $e^{-\sigma_d t}\cos(\omega_d t)$ to the homogeneous equation, i.e. damped oscillations

- matching coeff's in char. poly.'s yields

$$k_p = \frac{2\sigma_d - a}{b}, \quad k_I = \frac{\sigma_d^2 + \omega_d^2}{b}$$

* let's try a common (and clever) parameterization

$$\underbrace{\omega_c = \sqrt{\sigma_d^2 + \omega_d^2}}_{\text{natural frequency}}, \quad \underbrace{\zeta_c = \frac{\sigma_d}{\omega_c}}_{\text{damping ratio}}$$

→ this may seem arbitrary – its utility will become more clear as we learn more

- $|\zeta_c| \leq 1$ determines response shape,
  ω_c " " speed

- now $G_{yv}(s) = \dfrac{P(s)}{1+P(s)C(s)} = \dfrac{bs}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}$

- conclude that disturbances attenuated if the gain $|G_{yv}(j\omega)|$ small for all ω

$$|G_{yv}(j\omega)| \simeq \begin{cases} b\omega/\omega_c, & \omega \text{ small} \\ b/\omega, & \omega \text{ large} \end{cases}$$

note: $\max_\omega |G_{yv}(j\omega)| = |G_{yv}(j\omega_c)|$

note: $\max_\omega |G_{yv}(j\omega)| = |G_{yv}(j\omega_c)|$

$$= b/(2\zeta_c\omega_c)$$

so ω_c large attenuates disturbances

2°. unmodeled dynamics i.e. disturbance attenuation [AMv2 Ch 2.4]

- preceding analysis suggests performance can be arbitrarily good, but reality is not the same as our simple model...
  - since $k_P = (2\zeta_c\omega_c - a)/b$, $k_I = \omega_c^2/b$, large $\omega_c \rightsquigarrow$ high performance, but large k_P, k_I
  - suppose unmodeled dynamics of sensors, actuators, etc have time constant $T > 0$

    so $P(s) = \dfrac{b}{(s+a)(1+sT)}$
  - now the closed-loop characteristic poly. is

    $a_{cl} = s(s+a)(1+sT) + k_P s + k_I$

    $= s^3 T + s^2(1+aT) + 2\zeta_c\omega_c s + \omega_c^2$

    $\rightarrow$ R-H implies closed-loop system stable if

    $\omega_c^2 T < 2\zeta_c\omega_c(1+aT) \iff \omega_c T < 2\zeta_c(1+aT)$

* conclude that ω_c is limited by T, i.e.
characteristic time constant of unmodeled dynamics generally limit achievable performance
→ although we rely on simple models for control design, the resulting controller must always be sanity-checked / validated on physical system

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2^3. reference tracking [AMv2 Ch2.5]

- now consider the case where reference $r \neq 0$, e.g. cruise control, satellite tracking
  - suppose plant/process is first-order, $P(s) = \frac{b}{s+a}$
    controller is proportional-integral, $C(s) = \frac{k_P s + k_I}{s}$
  - for simplicity, neglect disturbance: $v = 0$
  - block diagram algebra yields
    $$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{b k_P s + b k_I}{s^2 + (a + b k_P) s + b k_I}$$
  - since $G_{yr}(0) = 1$, then r constant $\Rightarrow y = r$ (assuming system is stable)
  - to check stability, apply R-H to closed-loop char. poly. $a_{cl}(s) = s^2 + (a + b k_P) s + b k_I$
  - identifying coefficients with $s^2 + 2\zeta_c \omega_c s + \omega_c^2$ yields $k_P = \frac{2\zeta_c \omega_c - a}{b}$, $k_I = \frac{\omega_c^2}{b}$

yields $k_P = \dfrac{2\zeta_c\omega_c - a}{b}$, $k_I = \dfrac{\omega^2}{b}$

- with this parameterization,

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{(2\zeta_c\omega_c - a)s + \omega_c^2}{s^2 + 2\zeta_c\omega_c s + \omega_c^2}$$

* if $s = j\omega$ and $|\omega| < \omega_c$, then $G_{yr}(s) \simeq 1$ so ω_c limits <u>bandwidth</u> of references that can be tracked accurately

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