[AMv2 Ch 9]

goal: frequency-domain tools and concepts for analysis & control

1°. frequency-domain modeling

1¹. transfer function of an LTI system

1². block diagrams

1³. poles & zeros

1⁴. Bode plot

1°. frequency domain modeling

- key idea: represent LTI system by how it responds (at steady-state) to pure sinusoidal signals
  - particularly convenient / powerful to analyze complex feedback

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1¹. transfer function of LTI system

consider the response of LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

to input signal $u(t)$

<u>fact:</u> $$\underbrace{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{convolution equation}}$$

<u>intuition:</u> input $u(\tau)$ applied at time τ contributes response $e^{A(t-\tau)}Bu(\tau)$ to "initial condition" $\tilde{x}(\tau) = Bu(\tau)$

<u>thus:</u> $$y(t) = Cx(t) + Du(t)$$

$$= \underbrace{Ce^{At}x(0)}_{\text{homogeneous response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{particular response}}$$

- now consider response to pure exponential $u(t) = e^{st}$

[recall $\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$

$\sin(\omega t) = \frac{1}{2}(e^{j\omega t} - e^{-j\omega t})$

so e^{st} is a mathematical convenience that so e^{st} is a mathematical convenience that enables us to easily determine response to actual inputs $\cos(\omega t)$, $\sin(\omega t)$

- $$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)} B e^{s\tau} d\tau + De^{st}$$
$$= Ce^{At}x(0) + Ce^{At}\left[\int_0^t e^{(sI-A)\tau} d\tau\right] B + De^{st}$$

→ what expression has $e^{(sI-A)\tau}$ as its derivative?
(what restriction do you need to place on $s \in \mathbb{C}$?)

$$y(t) = Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \left[C(sI-A)^{-1}B + D\right]e^{st}$$

- if A is stable, then
$$y \to \left(C(sI-A)^{-1}B + D\right)e^{st}$$
$$= G_{yu}(s)u$$

→ what choice of initial state $x(0)$ ensures $y = G(s)u$?

- $x(0) = (sI-A)^{-1}B$

* G_{yu} is the <u>transfer function</u> from input u to output y

(aside: the transfer function idea extends to (some) nonlinear systems, systems w/ delay, and partial differential equations)

- this formula gives a procedure for determining the transfer function G_{yu} from the matrices A, B, C, D
  - since input & output spaces have been specified, there is a unique G_{yu} corresponding to A, B, C, D;

    the reverse is not true!

- since the state space is not specified for a given G_{yu}, there are many possible state space <u>realizations</u>:
  - suppose $G(s) = C(sI - A)^{-1}B + D$

- let $z = Tx$, T invertible

→ determine $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$

s.t. $\dot{z} = \tilde{A} z + \tilde{B} u$

$y = \tilde{C} z + \tilde{D} u$

- $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$, $\tilde{D} = D$

→ compute $\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$

- $$\begin{aligned} \tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\ &= CT^{-1}(sTT^{-1} - TAT^{-1})^{-1}TB + D \\ &= C(sI - A)^{-1}B + D \\ &= G(s) \end{aligned}$$

* conclude that transfer functions don't depend on the choice of state space

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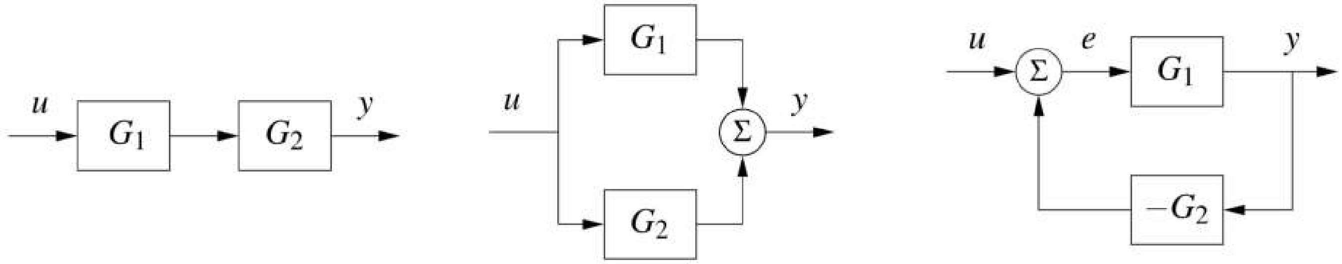
12. block diagrams

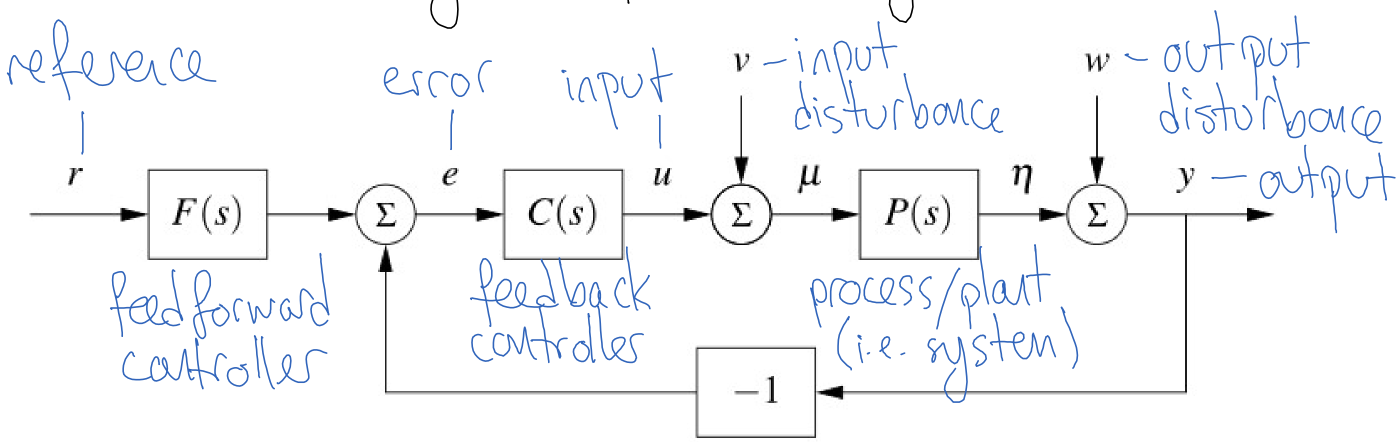
- we've seen examples of block diagram representations & algebra



(a) $G_{yu}(s) =$ [redacted] (b) $G_{yu}(s) =$ [redacted] (c) $G_{yu}(s) =$ [redacted]

→ derive expressions for G_{yu} in terms of G_1 & G_2

- now consider the general feedback diagram



→ derive an expression for e in terms of r, v, w, i.e. find G_{er}, G_{ev}, G_{ew} s.t. $e = G_{er} r + G_{ev} v + G_{ew} w$

$$- e = \underbrace{\frac{F}{1+PC}}_{G_{er}} r - \underbrace{\frac{P}{1+PC}}_{G_{ev}} v - \underbrace{\frac{1}{1+PC}}_{G_{ew}} w$$

- we'll investigate each of these transfer functions

& how they influence performance in coming weeks

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13. poles & zeros

- consider the transfer function

$$G(s) = \frac{b(s)}{a(s)}$$

  - G is a rational function, that is, a ratio of polynomials a, b
  - roots of a are termed poles, roots of b are termed zeros
  - difference between order of a & that of b termed relative degree; G is proper if ≥ 0, strictly proper if > 0
- let's relate these concepts to LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$
  - poles $= \lambda(A)$, i.e. eigenvalues of A
  - zeros are complex numbers $s \in \mathbb{C}$ s.t. $u(t) = u_0 e^{st}$ yields $y(t) = 0$,

$\rightarrow$ substitute this input/output pair into LTI DE with $x(t) = x_0 e^{st}$

to find condition that gives a zero

* if state is fully actuated (B square)
or fully measured (C square)
then there are no zeros

- since poles & zeros govern system behavior, often visualize them with a pole-zero diagram

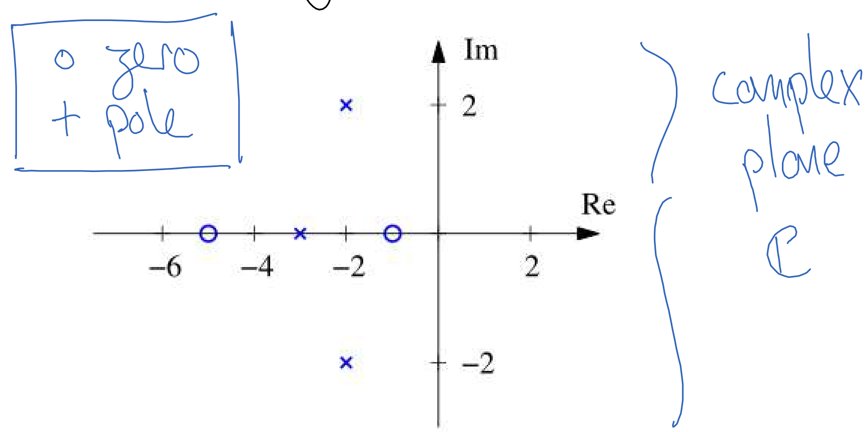

Figure 9.9: A pole zero diagram for a transfer function with zeros at -5 and -1 and poles at -3 and $-2\pm 2j$. The circles represent the locations of the zeros, and the crosses the locations of the poles. A complete characterization requires we also specify the gain of the system.

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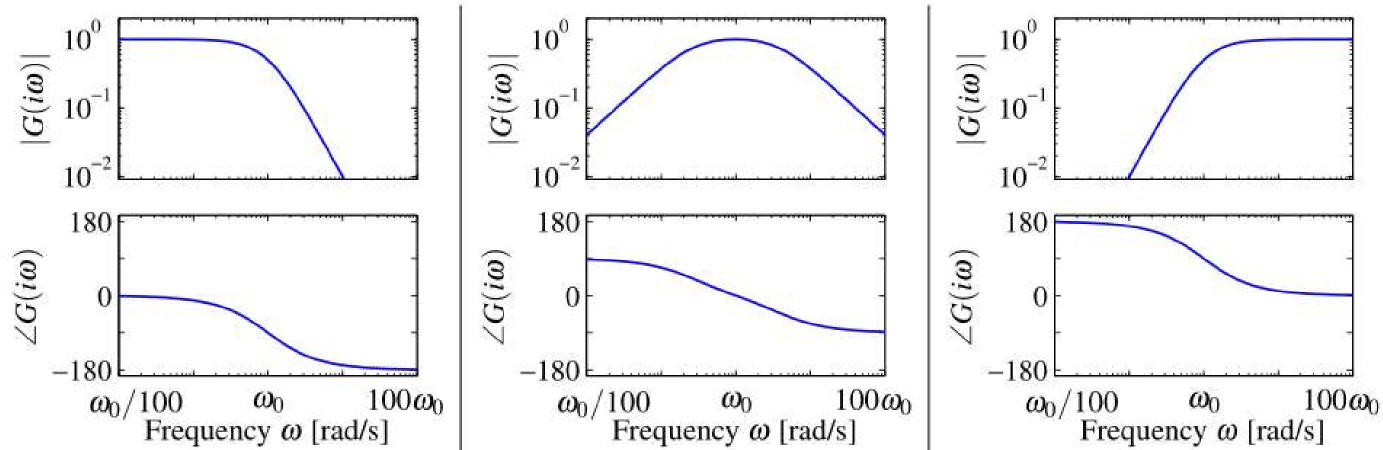
14. Bode plot

- since a complex exponential input $e^{j\omega t}$ yields output $G(s)e^{j\omega t} = |G(j\omega)|\, e^{j\omega(t + \angle G(j\omega))}$

it's useful to visualize

gain $|G(j\omega)|$ and phase $\angle G(j\omega)$
as functions of frequency ω:



$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Figure 9.17: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.