8-frequency-domain

goal: tools for analysis using transfer functions, Nyquist / Bode plots

1°. frequency domain analysis

1¹. Nyquist stability criterion [AMv2 Ch 10.1, 10.2] [Nv7 ch 10.3]

1². stability margins [AMv2 Ch 10.3] [Nv7 Ch 10.7]

1³. sensitivity functions [AMv2 ch 12.1, 12.2] [Nv7 not covered]

* general comment: these techniques were developed before we had cheap computers, so there are many graphing heuristics that are traditionally taught;
→ we'll rely on computers to graph, but still extract intuition

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1°. frequency domain analysis

- key idea: determine stability, robustness, & sensitivity of closed-loop systems by studying open-loop system

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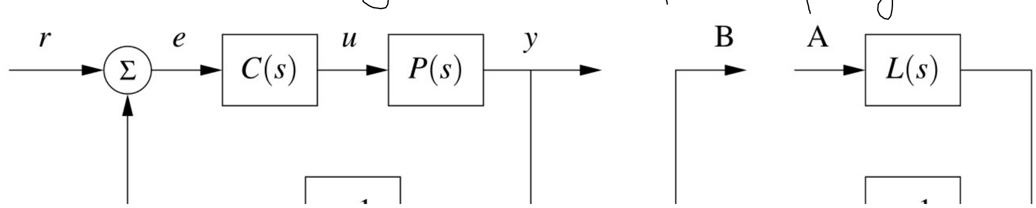
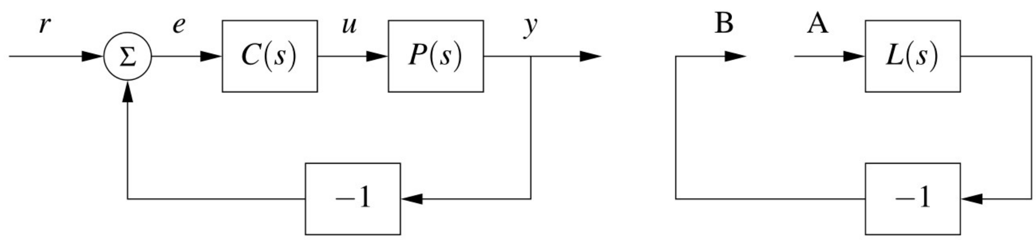
1¹. Nyquist stability criterion

- consider the following closed- and open-loop systems





(a) Closed loop system (b) Open loop system

Figure 10.1: The loop transfer function. The stability of the feedback system (a) can be determined by tracing signals around the loop. Letting $L = PC$ represent the loop transfer function, we break the loop in (b) and ask whether a signal injected at the point A has the same magnitude and phase when it reaches point B.

- we know the transfer function from r to y is

$$G_{yr} = \frac{PC}{1+PC} = \frac{n_p(s)\,n_c(s)}{d_p(s)\,d_c(s) + n_p(s)\,n_c(s)} \qquad P = \frac{n_p}{d_p}, \quad C = \frac{n_c}{d_c}$$

so that stability is determined by the characteristic polynomial

$$d_p(s)\,d_c(s) + n_p(s)\,n_c(s)$$

which we can analyze / synthesize in principle (though laboriously)

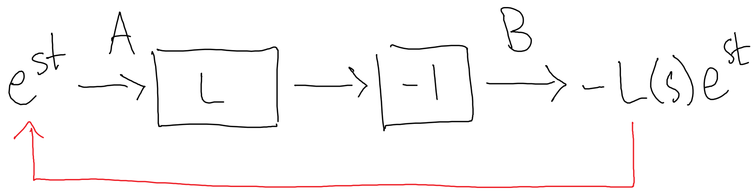
- we'll now learn how the <u>loop transfer function</u>

$$L(s) = P(s)\,C(s)$$

can be used for analysis / synthesis ($L = PC$ is simpler algebraically than $G_{yr} = \frac{PC}{1+PC}$)

∘ thought experiment: what does $L(s)$

tell us about how pure exponential signal e^{st} input at A is transformed when it reaches B?



→ suppose $\angle L(s) = 180°$ and we close the loop from B to A;

what happens to e^{st} as $t \to \infty$ if $|L(s)| < 1$, $|L(s)| > 1$, $|L(s)| = 1$?

- given $\angle L(s) = 180°$, then e^{st} and $-L(s)e^{st}$ have the same phase (due to negative feedback)

* so we conclude that initial signal e^{st}:
  - attenuates ($\to 0$) if $|L(s)| < 1$
  - amplifies ($\to \infty$) if $|L(s)| > 1$
  - sustains ($= e^{st}$) if $|L(s)| = 1$

- conclude that $L(s) = -1$ (i.e. $\angle L(s) = 180°$, $|L(s)| = 1$) is a <u>critical point</u> for the (open-)loop transfer function L

→ next we'll see that the way the graph of $L(j\omega)$ encircles $-1 \in \mathbb{C}$ . . .

tells us a lot about closed-loop stability

- it turns out that the graph of $L(j\omega) = \Omega$

termed the Nyquist plot

(where $L(s) = P(s)C(s)$ is the (open-) loop transfer function)

tells us about (closed-loop) stability

thm: (Nyquist stability criterion, general)

suppose the (open-) loop transfer function $L(s) = P(s)\,C(s)$ has P poles in the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and the graph of $L(j\omega)$,

$$\Omega = \{L(j\omega) : -\infty < \omega < \infty\},$$

encircles the critical point $-1 \in \mathbb{C}$, N times.

$\rightarrow$ then the closed-loop transfer function $\frac{PC}{1+PC} = \frac{L}{1+L}$

has $Z = N + P$ poles in the right half-plane (RHP)

included for completeness;

$\boxed{\text{NOT NEEDED FOR HW/EXAM}}$

cor: (Nyquist stability criterion, simplified)

if L has no poles in the RHP,

then: $\frac{L}{1+L} = \frac{PC}{1+PC}$ is stable

$\Longleftrightarrow$ Ω does not encircle $-1 \in \mathbb{C}$

this is the result you should study / include in your note sheet

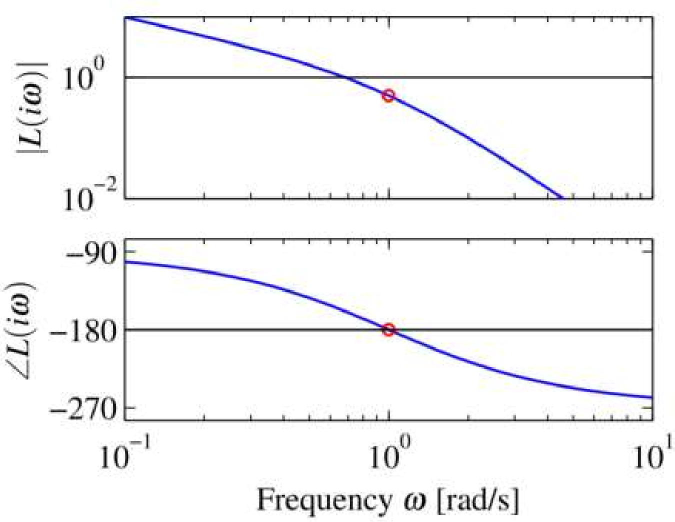
$1+PC$

$\iff \Omega$ does not encircle $-1 \in \mathbb{C}$

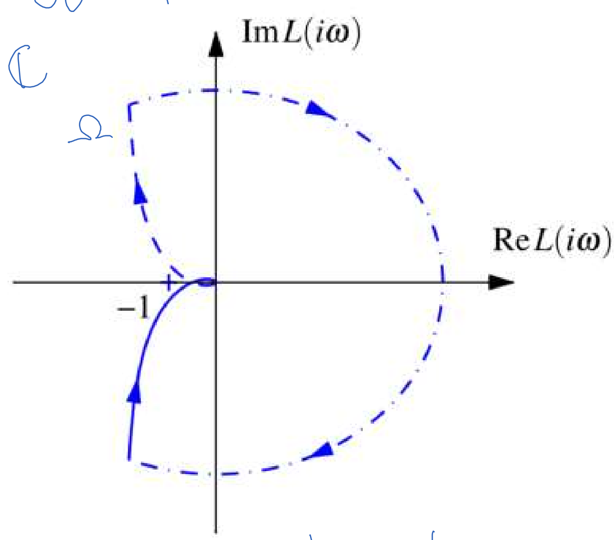
" $\{L(j\omega) : -\infty < \omega < \infty\} \subset \mathbb{C}$

your note sheet

→ sketch the Nyquist plot (i.e. graph Ω) of transfer function L using Bode plot (what can you say about stability of closed-loop $\frac{L}{1+L}$?)



– Nyquist plot:



– since Ω doesn't encircle $-1 \in \mathbb{C}$, the Nyquist stability criterion implies $\frac{L}{1+L}$ is (asymptotically) stable

* conclude that Bode / Nyquist plot of (open-)loop transfer function $L(s) = P(s)C(s)$ determines stability of closed-loop $\frac{L}{1+L} = \frac{PC}{1+PC}$

1°. stability margins

- in addition to providing a graphical tool to determine stability, Nyquist's stability criterion gives a graphical tool to determine robustness !

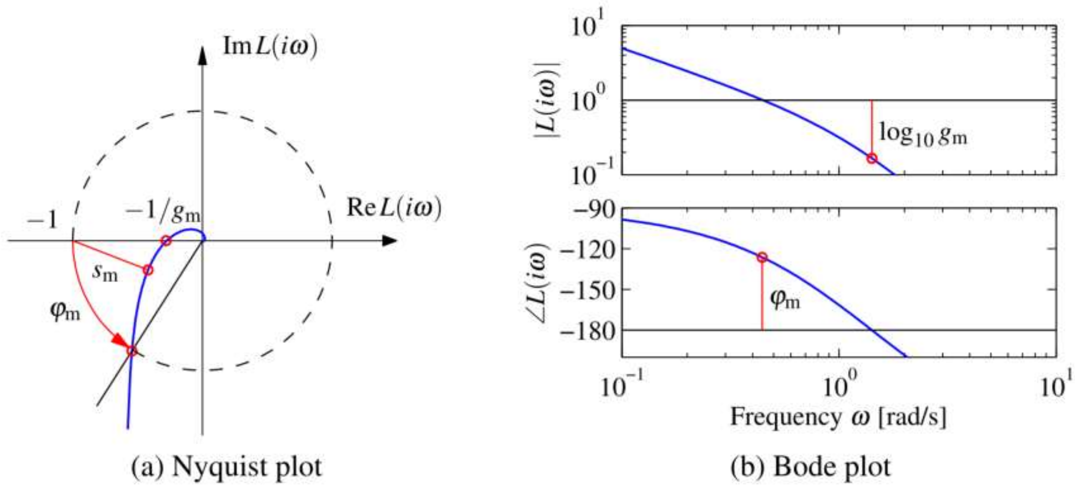


Figure 10.11: Stability margins for a third-order loop transfer function $L(s)$. The Nyquist plot (a) shows the stability margin, s_m, the gain margin g_m, and the phase margin φ_m. The stability margin s_m is the shortest distance to the critical point -1. The gain margin corresponds to the smallest increase in gain that creates an encirclement, and the phase margin is the smallest change in phase that creates an encirclement. The Bode plot (b) shows the gain and phase margins.

- stability margin s_m = distance from Ω to $-1 \in \mathbb{C}$
- gain margin $g_m = \begin{pmatrix} \text{distance from } \Omega \text{ to } -1 \in \mathbb{C} \\ \text{restricted to real axis} \end{pmatrix}^{-1}$
- phase margin φ_m = distance from Ω to $-1 \in \mathbb{C}$

restricted to rotation of Ω

– it turns out that

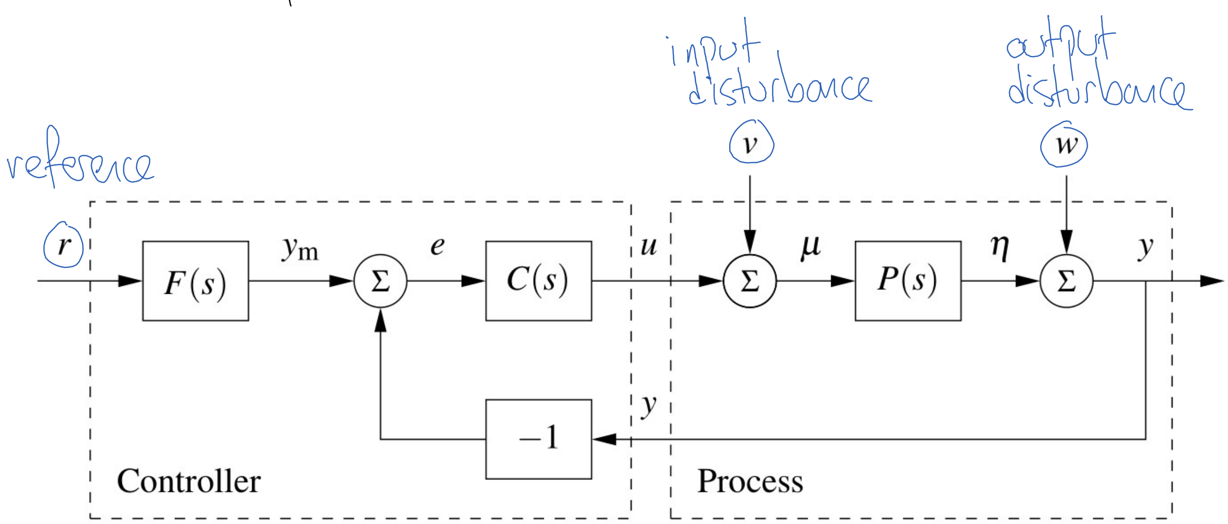
$$g_m \geq \frac{1}{1 - s_m}, \quad \varphi_m \geq 2 \sin^{-1}\left(\frac{s_m}{2}\right)$$

→ how would you determine (i.e. approximate) s_m, g_m, φ_m using computational tools?

– see lecture notebook for solution

1.3 sensitivity functions

- we now return to the general feedback diagram with 3 external inputs:
  - r : reference
  - v : input disturbance
  - w : output disturbance



– note that the process input μ and output η aren't what our controller commands (u)

or measures (y)

y	u	e	μ	η	
$\frac{PCF}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{F}{1+PC}$	$\frac{CF}{1+PC}$	$\frac{PCF}{1+PC}$	r
$\frac{P}{1+PC}$	$\frac{-PC}{1+PC}$	$\frac{-P}{1+PC}$	$\frac{1}{1+PC}$	$\frac{P}{1+PC}$	v
$\frac{1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-1}{1+PC}$	$\frac{-C}{1+PC}$	$\frac{-PC}{1+PC}$	w

* we're particularly concerned with how input & output disturbances v, w map to controller input & output u, y

- neglecting signs, we're focused on:

$S = \frac{1}{1+PC}$ sensitivity $\quad T = \frac{PC}{1+PC}$ complementary sensitivity

→ note: $S + T = \frac{1+PC}{1+PC} = 1$, thus name makes sense

$PS = \frac{P}{1+PC}$ input sensitivity $\quad CS = \frac{C}{1+PC}$ output sensitivity

→ suppose (open-) loop transfer function

$L(s) = P(s)\,C(s) \to 0$ as $s \to \infty$

(i.e. L is strictly proper),

what can you say about how:

a) high-frequency input disturbance affects output

b) high-frequency output disturbance affects input

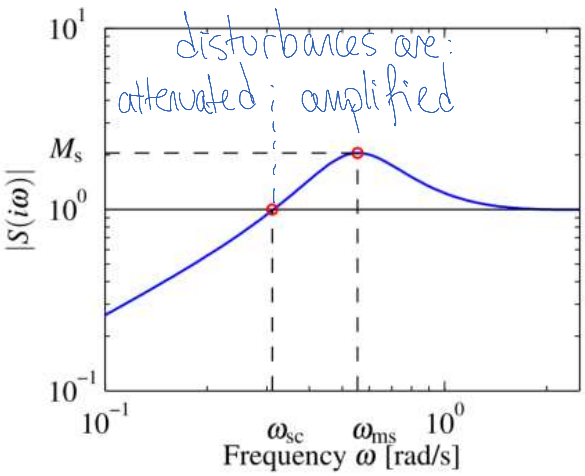
?

- the sensitivity transfer functions

  S, T, PS, CS

  can be used to assess (or specify) performance of a closed-loop system
  - for example, the maximum gain M_s of the sensitivity function S is related to the stability margin s_m

    via $M_s = \frac{1}{s_m}$



(a) Gain curves

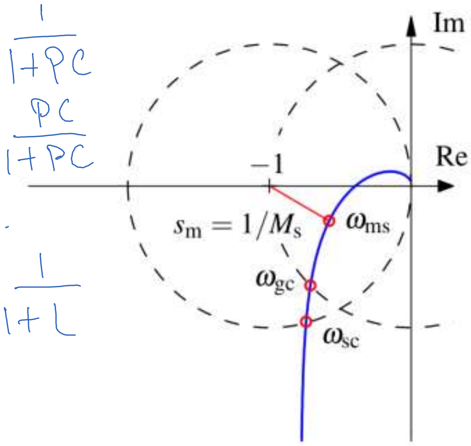
note:

$S = \frac{1}{1+PC}$

$L = \frac{PC}{1+PC}$

so...

$S = \frac{1}{1+L}$



(b) Nyquist plot

  - specifications may be based on <u>peak gain</u> or corresponding frequency ω_{ms}, <u>crossover frequency</u> ω_{sc} (smallest freq for which gain equals one (1)), <u>bandwidth</u>, ...

→ how would you measure these performance specifications empirically?

(suppose you have access to signals r, u, y
in the block diagram, i.e. you can measure
and/or alter additively)