4-linearization-and-linearity

[AMv2 Ch 6]

goal: qualitative & quantitative analysis of linear system behavior & relation to nonlinear system behavior

topics:

1°. linear systems

1¹. linearization

1². linearity & time invariance

1³. matrix exponential

1⁴. input/output response

1⁵. frequency response

1°. linear systems

Few physical elements display truly linear characteristics. For example the relation between force on a spring and displacement of the spring is always nonlinear to some degree. The relation between current through a resistor and voltage drop across it also deviates from a straight-line relation. However, if in each case the relation is reasonably *linear, then it will be found that the system behavior will be very close to that obtained by assuming an ideal, linear physical element, and the analytical simplification is so enormous that we make linear assumptions wherever we can possibly do so in good conscience.*

Robert H. Cannon, *Dynamics of Physical Systems*, 1967 [Can03].

↳ so the important question is not "is my system linear?" but rather "is linearity a useful approximation?"

* if it is, you're almost always better off making use of linearity

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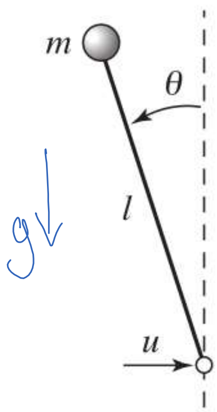
1! linearization

- we'll now see how to relate linear to nonlinear systems via <u>linearization</u>

<u>ex</u>: inverted pendulum



(<u>very</u> simple) model for rocket flight

– previously determined dynamics

$$m l^2 \ddot{\theta} = m g l \sin\theta - \gamma \dot{\theta} + l u \cos\theta$$

have equilibria at:

$$\dot{\theta}_e = 0, \quad \theta_e = k \cdot \pi, \quad k \in \mathbb{Z}$$

→ compute first-order Taylor series about equilibrium $(\theta_e, \dot{\theta}_e) = (k\pi, 0)$

$k\pi$

$$
\begin{aligned}
-ml^2\ddot{\theta} = mgl\sin\theta \quad &\simeq mgl\cdot(\theta - \cancel{\theta_e}^{k\pi}) \\
-\gamma\dot{\theta} \quad &\quad -\gamma\cdot(\dot{\theta} - \cancel{\dot{\theta}_e}^{0}) \\
+ l u \cos\theta \quad &\quad + 0\cdot(\theta - \cancel{\theta_e}^{k\pi}) \quad \left[\frac{\partial}{\partial\theta}\cos\theta\Big|_{\theta=k\pi}\right] = 0 \\
&\quad + l\cdot(u - \cancel{u_e}^{0}) \\
\simeq mgl\cdot(\theta - k\pi) - \gamma\dot{\theta} + lu
\end{aligned}
$$

- more generally, consider nonlinear system

(NL) $\dot{x} = f(x,u)$, $y = h(x,u)$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$

- assume f, h smooth so we can take derivatives w.r.t. x, u

- assume there is an equilibrium

$f(x_e, u_e) = 0$

and consider the (Jacobian) derivatives

$$A = \frac{\partial}{\partial x} f\Big|_{(x_e,u_e)}, \quad B = \frac{\partial}{\partial u} f\Big|_{(x_e,u_e)}$$

$$C = \frac{\partial}{\partial x} h\Big|_{(x_e,u_e)}, \quad D = \frac{\partial}{\partial u} h\Big|_{(x_e,u_e)}$$

$\hookrightarrow \frac{\partial}{\partial x} f$ is the matrix of partial derivatives:

$$\frac{\partial}{\partial x} f = \left[\frac{\partial}{\partial x_j} f_i\right]_{ij} = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \cdots & \partial_{x_n} f_n \end{bmatrix}$$

– then trajectories of the LTI system

(L) $\dot{\xi} = A\xi + B\mu, \quad \eta = C\xi + D\mu$

approximate tr's of (NL) near (x_e, u_e):

if $\|x(s) - x_e\|, \|u(s) - u_e\|$ small

& $\mu(s) = u(s) - u_e$ for all $s \in [0, t]$,

then $\xi(s) \simeq x(s) - x_e$

fact: (L) is stable $\Rightarrow$ (NL) is stable (locally near (x_e, u_e))

$\boxed{\begin{array}{l}\text{* this is a deeply important fact} \\ \text{that justifies application of} \\ \text{linear control theory in the real world!}\end{array}}$

1.2 linearity & time invariance

- a linear time-invariant (LTI) system has the state-space form

$$\dot{x} \text{ or } x^+ = Ax + Bu, \quad y = Cx + Du$$

where: $A \in \mathbb{R}^{n \times n}$ $\quad C \in \mathbb{R}^{q \times n}$
$B \in \mathbb{R}^{n \times p}$ $\quad D \in \mathbb{R}^{q \times p}$

- in a SISO system, $p = q = 1$
  so B is a column vector ($n \times 1$ matrix)
  C is a row vector ($1 \times n$ matrix)
  D is a scalar (1×1 matrix)
- given initial state $x_0 \in \mathbb{R}^n$
  & control input $u : \mathbb{R} \to \mathbb{R}^p$:

* $x_h : \mathbb{R} \to \mathbb{R}^n$ is the homogeneous solution:

$$x_h(0) = x_0, \quad \frac{d}{dt} x_h(t) = A x_h(t)$$

* $x_p : \mathbb{R} \to \mathbb{R}^n$ is the particular solution:

$$x_p(0) = 0, \quad \frac{d}{dt} x_p(t) = A x_p(t) + B u_p(t)$$

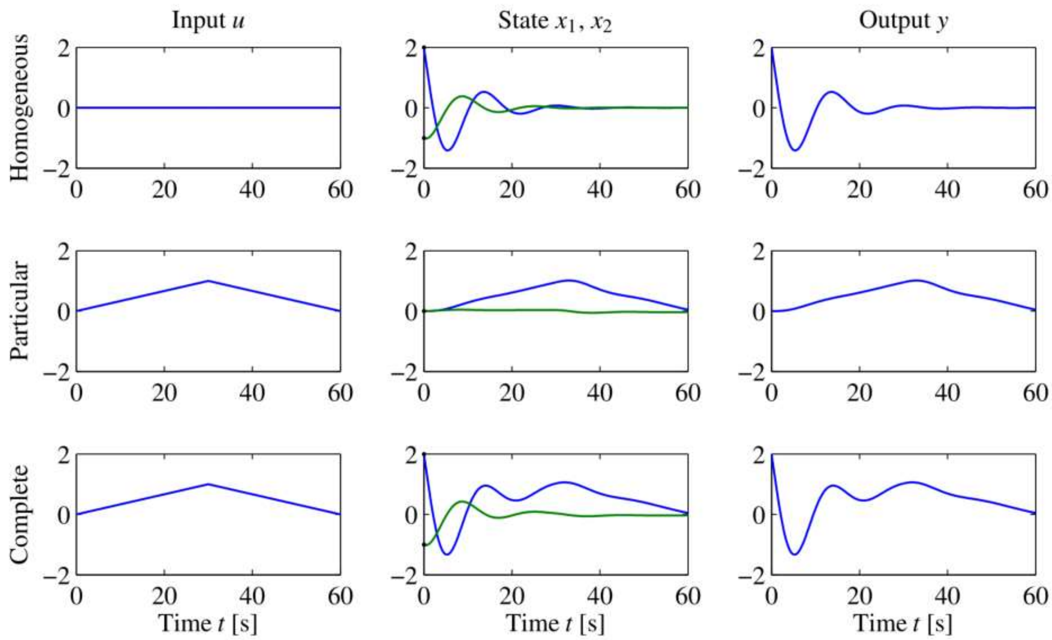


Figure 6.1: Superposition of homogeneous and particular solutions. The first row shows the input, state, and output corresponding to the initial condition response. The second row shows the same variables corresponding to zero initial condition but nonzero input. The third row is the complete solution, which is the sum of the two individual solutions.

ex: (scalar system)

- consider $\dot{x} = ax + bu, \; y = x$ (DE)

  with $x(0) = x_0, \; u_1 = \alpha \sin \omega_1 t, \; u_2 = \beta \cos \omega_2 t$

- homogeneous solution: $x_h(t) = e^{at} x_0$
- particular solutions:

$$x_{p_1}(t) = -\frac{\alpha}{a^2+\omega_1^2}\left(-\omega_1 e^{at} + \omega_1 \cos \omega_1 t + a \sin \omega_1 t\right)$$

$$x_{p_2}(t) = -\frac{\beta}{a^2+\omega_2^2}\left(a e^{at} - a \cos \omega_2 t_2 + \omega_2 \sin \omega_2 t\right)$$

$\rightarrow$ verify the homogeneous & particular solutions

- by linearity of (DE), applying input $u = u_1 + u_2$ to initial state x_0 yields

$$y(t) = x(t) = x_h(t) + x_{p_1}(t) + x_{p_2}(t)$$

$\rightarrow$ express $x(t)$ in terms of $a, x_0, \omega_1, \omega_2, \alpha, \beta$
& verify the expression satisfies (DE)

1? matrix exponential
(i.e. the homogeneous solution)

- recall the homogeneous solution to scalar LTI DE $\dot{x} = ax$
  is $x(t) = e^{at} x(0)$
  where $e : \mathbb{C} \to \mathbb{C}$ is defined by

$$e^{z} = 1 + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad (\underbrace{k!}_{\text{read as "k factorial"}} = \underbrace{k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1}_{})$$

- this power series converges for every complex number $z \in \mathbb{C}$
- amazingly, this power series makes sense & converges for $X \in \mathbb{C}^{n \times n}$:

$$e^{X} = I + X + \frac{1}{2} X^2 + \frac{1}{3!} X^3 + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} X^k \quad \left(X^0 = I,\ X^2 = X \cdot X,\ X^{k+1} = X \cdot X^k = X^k \cdot X \right)$$

→ noting scalar t commutes with matrix A
$At = tA$, show that $e^{At} = e^{tA}$

- even more amazingly, the derivative rule $\frac{d}{dt} e^{at} = a e^{at}$ generalizes:

→ using definition of e^{At}, show that $\frac{d}{dt} e^{At} = A e^{At}$

$$\Rightarrow \frac{d}{dt} e^{At} = \frac{d}{dt}\left(I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)$$
$$= A + A^2 t + \frac{1}{2}A^3t^2 + \cdots$$
$$= A\cdot\left(I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right)$$
$$= A\cdot \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = A e^{At} \quad !$$

* $x(t) = e^{At}x_0$ is <u>the</u> solution to $\dot{x} = Ax$
w/ initial state $x(0) = e^{A\cdot 0}x_0 = I \cdot x_0 = x_0$

→ unlike scalar case, order is important:
$e^{At} \in \mathbb{R}^{n\times n}$, $x_0 \in \mathbb{R}^{n\times 1}$, so $[e^{At}x_0] \in \mathbb{R}^{n\times 1}$
whereas $[x_0 e^{At}]$ doesn't make sense...

- the solution is obviously linear in x_0

(since multiplication by matrix e^{At} is linear)

ex: 6.2 (double integrator)

- consider $\ddot{q} = u, \quad y = q$
- with $x = (q, \dot{q})$ we have

$$\frac{d}{dt} x = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= Ax + b$$

- noting $A^2 = 0$ (so $A^k = 0$ for $k \geq 2$)

  we compute $e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

  and conclude homogeneous solution is

$$x_h(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix}$$

$$= \begin{bmatrix} q(0) + t\,\dot{q}(0) \\ \dot{q}(0) \end{bmatrix}$$

↳ agrees with intuition from physics: in the absence of forcing, a mass will continue at constant speed

ex: 6.3 (mechanical oscillator)

- somewhat more generally, consider a spring-mass with no damping:

$$\ddot{q} + \omega_0^2 q = u$$

- with $x = (q, \dot{q}/\omega_0)$ we have

$$\frac{d}{dt} x = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$= Ax + b$$

- $e^{At} = \begin{bmatrix} \cos \omega_0 t & \sin \omega_0 t \\ -\sin \omega_0 t & \cos \omega_0 t \end{bmatrix}$

$\rightarrow$ verify this formula via differentiation

[bonus: verify this formula by substituting power series expressions for sin & cos]

- including damping,

$$\ddot{q} + 2\zeta\omega_0 \dot{q} + \omega_0^2 q = u$$

we have $A = \begin{bmatrix} -\zeta\omega_0 & \omega_d \\ -\omega_d & -\zeta\omega_0 \end{bmatrix}$

- assuming $|\zeta| < 1$ yields

$$e^{At} = e^{-\zeta\omega_0 t} \begin{bmatrix} \cos \omega_d t & \sin \omega_d t \\ -\sin \omega_d t & \cos \omega_d t \end{bmatrix}$$

where $\omega_d = \omega_0 \sqrt{1-\zeta^2}$

→ verify this formula via differentiation

→ read [AMv2 "Eigenvalues and Modes"]

↳ interesting discussion of eigenvalues, eigenvectors, and coordinate choice

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14. input/output response

- consider the state-space LTI system

$$\dot{x} = Ax + Bu$$

<u>fact</u>: given initial state $x(0)$, input u:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$$

↳ termed the <u>convolution equation</u>

→ verify this formula via differentiation

- if $y = Cx + Du$ then $y(t)$ has 2 parts:

$$y(t) = C e^{At} x(0) + \int_0^t C e^{A(t-\tau)} B u(\tau)\, d\tau + Du(t)$$

$$+ Du(t)$$

$\underbrace{\qquad\qquad}_{\text{homogeneous response to initial condition}}$ $\underbrace{\qquad\qquad\qquad\qquad}_{\text{particular response to input}}$

- let's examine the response to unit step $\sigma(t) = [t \geq 0]$, termed <u>step response</u>, assuming $x(0) = 0$, A invertible:

→ use the convolution formula to compute step response (evaluate the integral)

$$\int_0^t Ce^{A(t-\tau)} B\sigma(\tau)\,d\tau + D\sigma(t)$$

$$= C\left[\int_0^t e^{A(t-\tau)}\,d\tau\right] B + D \quad \text{for } t \geq 0$$

$$= C\left[-A^{-1}e^{A(t-\tau)}\right]_{\tau=0}^{\tau=t} B + D$$

$$= C\left[-A^{-1}e^{A\cdot 0} + A^{-1}e^{At}\right] B + D$$

$$= \underbrace{CA^{-1}e^{At}B}_{\substack{\text{transient}\\ \text{input response}}} \underbrace{- CA^{-1}B + D}_{\substack{\text{steady-state}\\ \text{input response}}} \qquad e^{A\cdot 0} = I$$

- note: if A is stable

(i.e. all eigenvalues have negative real part)
then transient $\to 0$ as $t \to \infty$

note: for LTI systems, M_p, T_r, T_s are independent of step size

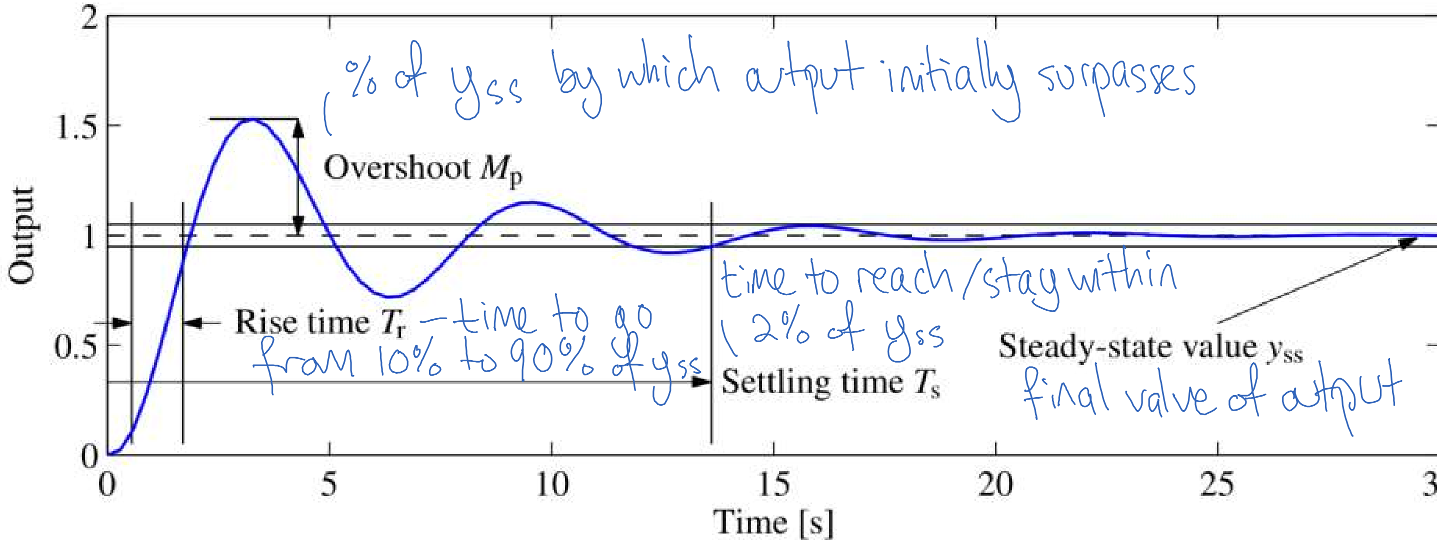

Figure 6.9: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

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15. frequency response

- let's consider system response to
  $u(t) = \cos \omega t = \frac{1}{2}(e^{j\omega t} - e^{-j\omega t})$

- since system is linear, consider e^{st}, $s = \pm j\omega$

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Be^{s\tau}d\tau + De^{st}$$

$$= Ce^{At}x(0) + Ce^{At}\int_0^t e^{(sI-A)\tau}Bd\tau + De^{st}$$

- so long as $s = \pm j\omega \notin \lambda(A)$,
  $sI - A$ is invertible

→ why is this true?
(Hint: recall $\lambda(A) = \{ z \in \mathbb{C} : \det(zI - A) = 0 \}$)

so we can re-use calculation from (1⁵):

$$y(t) = Ce^{At}x(0) + C(sI-A)^{-1}e^{st}B - Ce^{At}(sI-A)^{-1}B + De^{st}$$

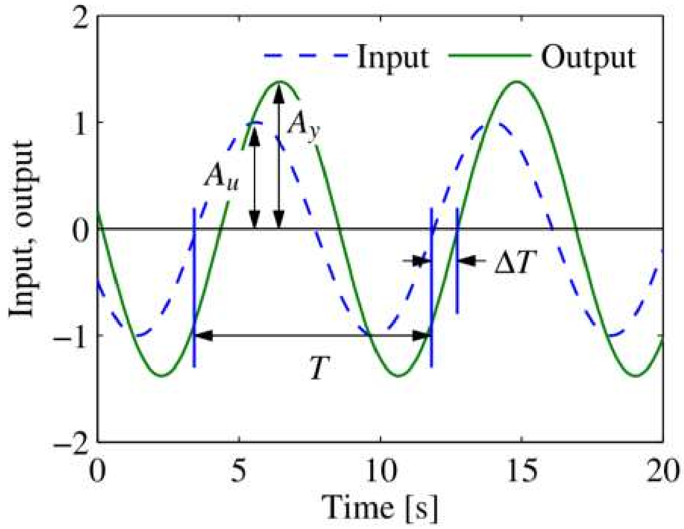
$$= \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{transient} \to 0 \text{ if } A \text{ stable}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)e^{st}}_{\text{steady-state}}$$

- representing steady-state response as

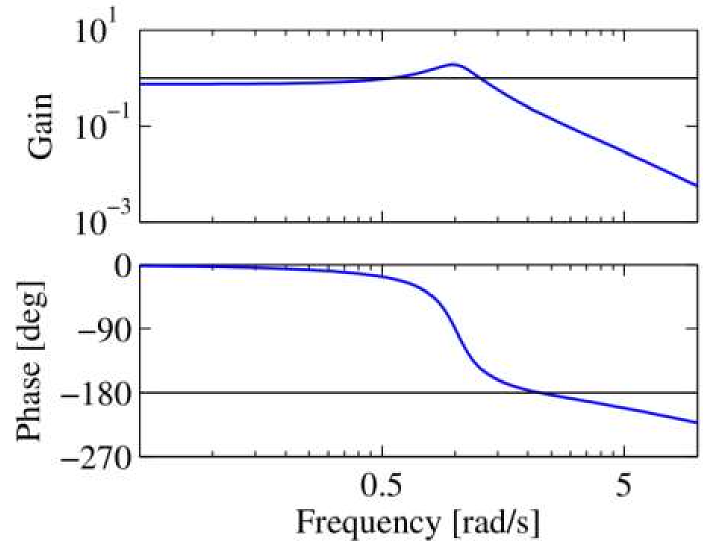
$$y_{ss}(t) = \left(C(sI-A)^{-1}B + D\right)e^{st}$$

$$= \underbrace{G(s)}_{\text{* this is the LTI system's transfer function!}} e^{st}$$

- recall: magnitude $|G(s)|$ termed gain
  angle $\angle G(s)$ termed phase
- if $\angle G(s)$ positive → output leads input
  " " negative → " lags "

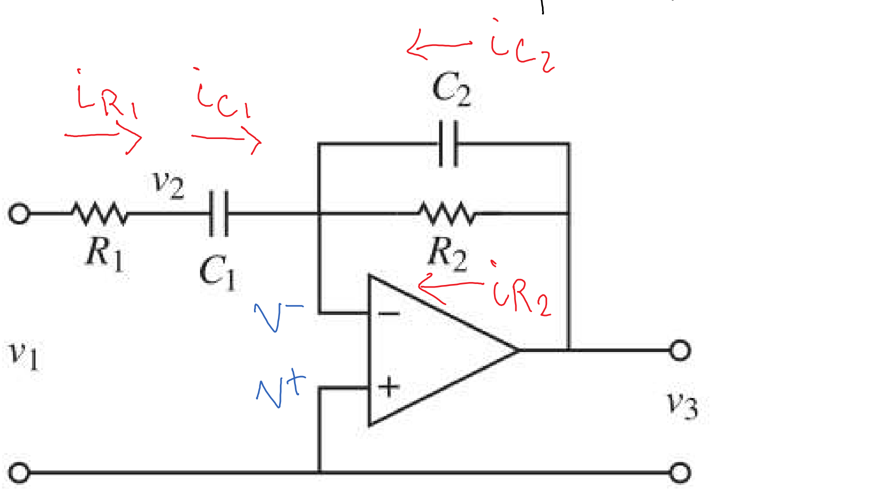


(a) Input/output response



(b) Frequency response

ex: 6.8 active band-pass filter



* recall: sum of currents at any node must be zero

→ derive two DE involving v_2, v_3 assuming $v^+ = v^-$

(see [AMv2 Ch 4.3] for discussion of when this is a valid assumption]

$(i_{R_1} - i_{C_1} = 0)$ $\quad \frac{v_1 - v_2}{R_1} - C_1 \dot{v}_2 = 0$

$(i_{C_1} + i_{C_2} + i_{R_2} = 0)$ $\quad C_2 \dot{v}_3 + C_1 \dot{v}_2 + \frac{v_3}{R_2} = 0$

- with v_2, v_3 as states:

$$\dot{v}_2 = \frac{v_1 - v_2}{R_1 C_1} \qquad \dot{v}_3 = \frac{-v_3}{R_2 C_2} - \frac{v_1 - v_2}{R_1 C_2}$$

- with $x = (v_2, v_3)$, $u = v_1$, $y = v_3$:

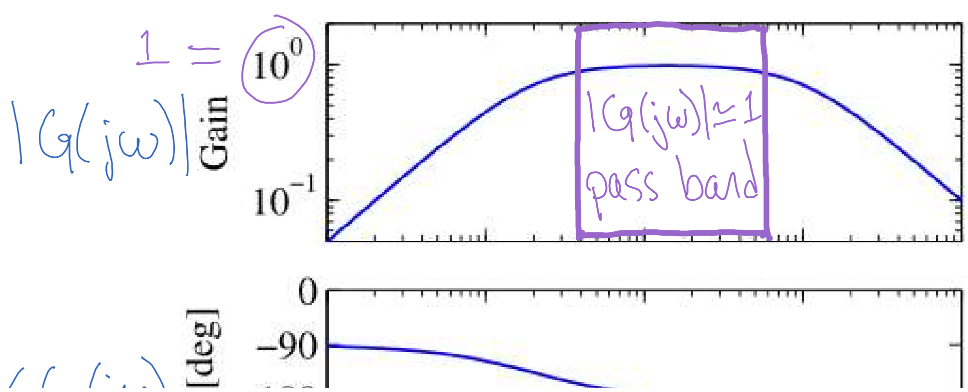
$$\dot{x} = \begin{bmatrix} -1/R_1C_1 & 0 \\ 1/R_1C_2 & -1/R_2C_2 \end{bmatrix} x + \begin{bmatrix} 1/R_1C_1 \\ -1/R_1C_2 \end{bmatrix} u$$

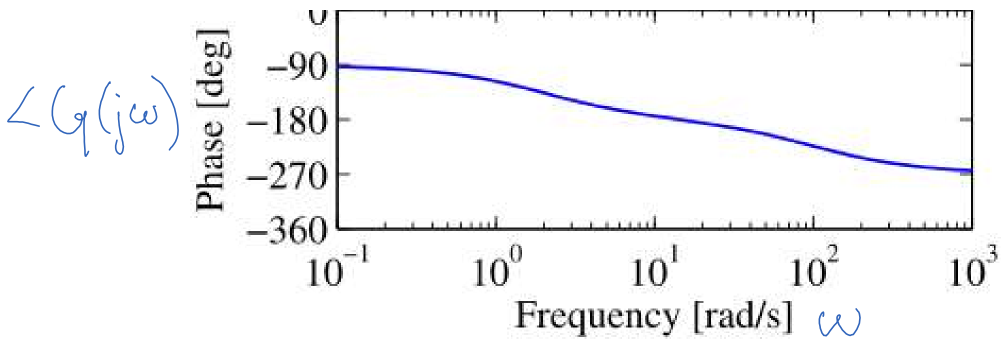
$$= Ax + bu$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x = Cx \qquad (D = 0)$$

- computing the frequency response:

$$G(s) = C(sI - A)^{-1} b + D = -\frac{R_2}{R_1} \frac{R_1 C_1 s}{(1 + R_1 C_1 s)(1 + R_2 C_2 s)}$$





(b) Frequency response