ECE 447: Control Systems

goal: what is a system's state? what is a nonlinear system?

(a) state space ← new way of looking at/representing a system using a vector description of quantities of interest and how they interact over time

(b) time: continuous and discrete ← we'll consider DE that evolve in continuous time, $\dot{x} = f(x,u)$, or in discrete time, $\tilde{x}^+ = \tilde{f}(\tilde{x},u)$

(c) linear systems ← any matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ determine linear state-space system $f(x,u) = Ax + Bu$

(d) nonlinear systems ← defined by $\dot{x} = f(x,u)$

(a) state space [AMv2 Ch 3.2] [Nv7 Ch 3.3]

ex: RLC circuit: capacitor charge q & current $\dot{q}$ interact with voltage v over time:

(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$

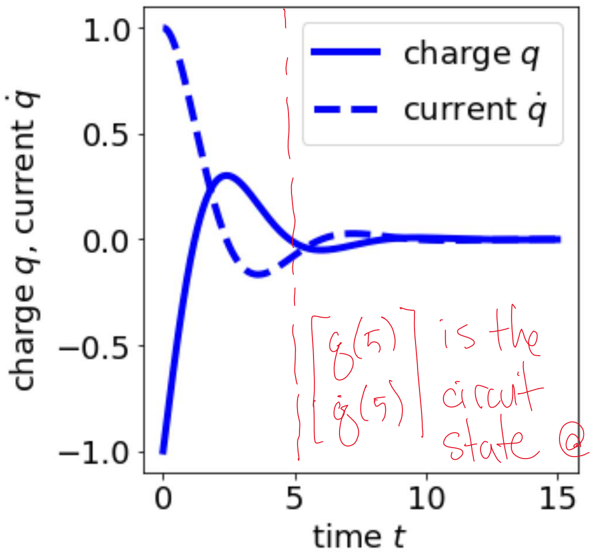
know: given $v : [0,\infty) \to \mathbb{R}$,
$: t \mapsto v(t)$
initial condition $(q(0), \dot{q}(0))$

then $q(t) = q_0(t) + q_v(t)$ particular response to v

then $q(t) = \underbrace{q_0(t)}_{\text{homogeneous response to}} + q_v(t)$

- the vector $\underbrace{(q(t), \dot{q}(t))} \in \mathbb{R}^2$ is the circuit state at time t

→ if I know · and input $v: [t, \infty)$, $q(\tau)$ determined for $\tau \geq t$



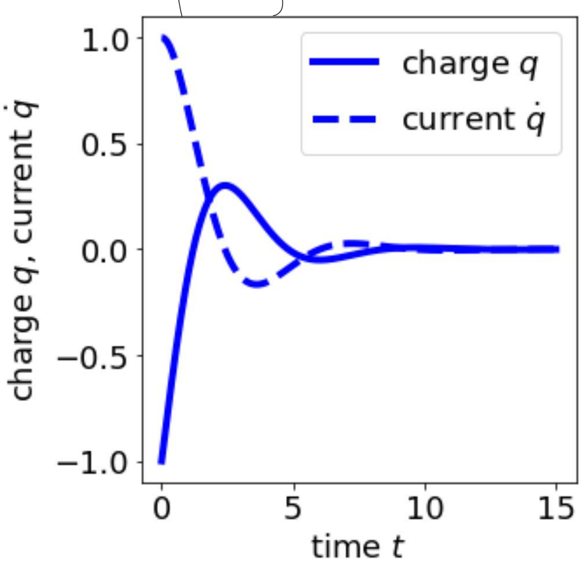
$\begin{bmatrix} q(5) \\ \dot{q}(5) \end{bmatrix}$ is the circuit state @ time $t=5$

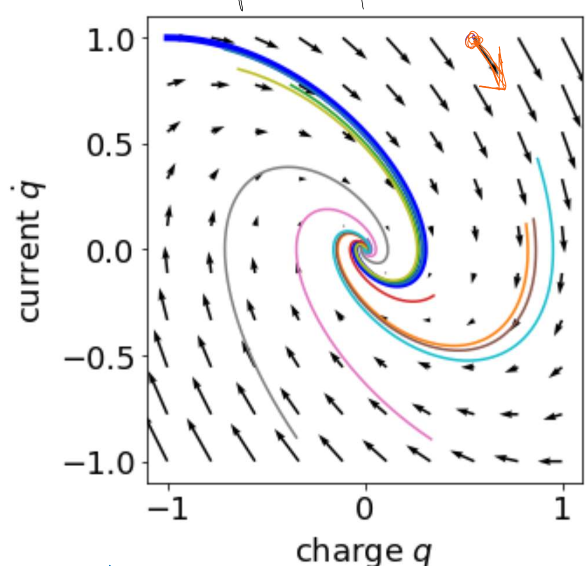
homogeneous response to initial condition / state

$$\begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

1°. one trajectory over time

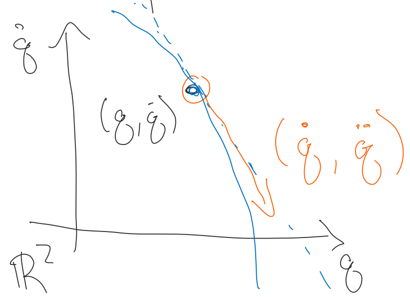
2°. multiple trajectories over time





state space $= \left\{ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^2 \right\}$

state space ⟶ instead of a single 2nd-order DE state space $\longrightarrow$ instead of a single 2nd-order DE



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$$

think about two 1st-order DE

$$\frac{d}{dt}\underbrace{\begin{bmatrix} q \\ \dot{q} \end{bmatrix}}_{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{1}{L}\left(v - R\dot{q} - \frac{1}{C}q\right) \end{bmatrix}$$

state-space representation for (DE)

more generally: $\frac{d}{dt}x = \dot{x} = f(x,u)$, $u = v$

where: $\underbrace{x \in \mathbb{R}^n}_{\text{system state}}$, $u \in \mathbb{R}^p$, $f : \mathbb{R}^n \times \mathbb{R}^p \longrightarrow \mathbb{R}^n$
$: (x,u) \longmapsto f(x,u) = \dot{x}$

(b) time: continuous and discrete [AMv2 Ch 3.2]

<u>ex</u>: proportional-integral control on a microcontroller/embedded system

- PI $u(t) = k_P e(t) + k_I \int_0^t e(\tau)\,d\tau$

$$= k_P e(t) + k_I x(t) \quad \text{where } \dot{x}(t) = e(t) \quad \text{(DE)}$$

ie $x(t) \in \mathbb{R}^2$ is controller state

and time $t \in \mathbb{R}$ is <u>continuous</u> (ie any real number)

- on an embedded system, microprocessor measure error at <u>discrete</u> instants in time $t = \Delta, 2\Delta, 3\Delta,$ $\Delta > 0$ clock cycle duration

$\longrightarrow$ approximate (DE) as $\frac{1}{\Delta}\left(\tilde{x}(t+\Delta) - \tilde{x}(t)\right) \simeq \dot{x}(t) = e(t)$

ie $\tilde{x}(t+\Delta) = \tilde{x}(t) + \Delta \cdot e(t)$

$\rightsquigarrow$ yields a difference equation (DE) $\tilde{x}^+ = e$
whose "solutions" are defined at times $t = k \cdot \Delta$

* to emphasize that $\tilde{x}$ defined only at discrete times,
write $\tilde{x}[k]$ to denote $\tilde{x}(k \cdot \Delta)$

- more generally, with $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ denoting state vector

and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in \mathbb{R}^p$ denoting input vector

then state could change in time according to:

1°. differential equation $\frac{d}{dt} x = \dot{x} = f(x, u)$

or

2°. difference equation $\tilde{x}^+ = \tilde{f}(\tilde{x}, u)$

- we'll refer to both (1°) & (2°) as (DE)

and distinguish them notationally by writing:

1°. $x(t)$ for state of differential equation at $\overbrace{\text{time } t \in \mathbb{R}}^{\text{"continuous time"}}$

2°. $\tilde{x}[k]$ for state of difference equation at $\underbrace{\text{time } t = k \cdot \Delta}_{\text{"discrete time"}}$

(c) linear systems [AMv2 Ch 3.2] [Nv7 Ch 3.3]

ex: RLC circuit: capacitor charge q & current $\dot{q}$ interact with voltage v over time:

(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$

∘ with $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{1}{L}\left(v - R\dot{q} - \frac{1}{C} q\right) \end{bmatrix} = \overbrace{f(x,u)}^{\in \mathbb{R}^2}$, $\overbrace{u = v}^{\in \mathbb{R}}$

note: f is linear: $f(x,u) = Ax + Bu$

→ verify $\left\{ \; = \underbrace{\begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}}_{\in \mathbb{R}^{2\times 2}} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/L \end{bmatrix}}_{\in \mathbb{R}^{2\times 1}} v \right.$

∘ more generally, any $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ defines a linear system in state-space form $\dot{x} = Ax + Bu$

ex: given $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = u$

∘ choosing $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x \end{bmatrix} = \begin{bmatrix} \frac{d^{n-1}}{dt^{n-1}} y \\ \frac{d^{n-2}}{dt^{n-2}} y \\ \vdots \\ \frac{d}{dt} y \\ \cdot \end{bmatrix} \in \mathbb{R}^n$

$\in \mathbb{R}^n$

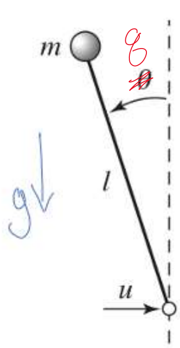
$$\begin{bmatrix} \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \quad \begin{bmatrix} \vdots \\ \frac{d}{dt} y \\ y \end{bmatrix}$$

yields
$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 x_1 - a_2 x_2 \cdots - a_n x_n + u \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} = \underbrace{f(x,u)}_{\in \mathbb{R}^n}$$

$$= Ax + Bu = \underbrace{\begin{bmatrix} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ \hline 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}}_{= A \in \mathbb{R}^{n\times n}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{= B \in \mathbb{R}^{n\times 1}} u$$

(d) nonlinear systems

ex: "rocket flight" (really: pendulum)

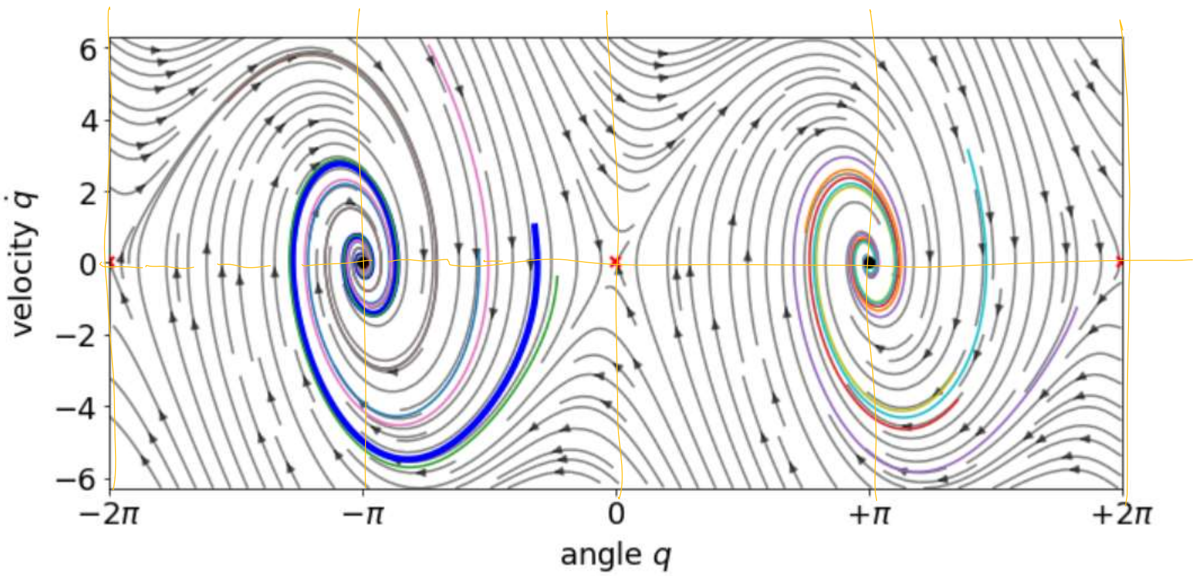


- state $x = (q, \dot{q})$ – angle, velocity
- input u – horizontal acceleration of pivot
- (DE) $ml^2 \ddot{q} = mgl \sin q - \alpha \dot{q} + lu \cos q$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q \end{bmatrix} = f(x,u)$$

"equilibria"

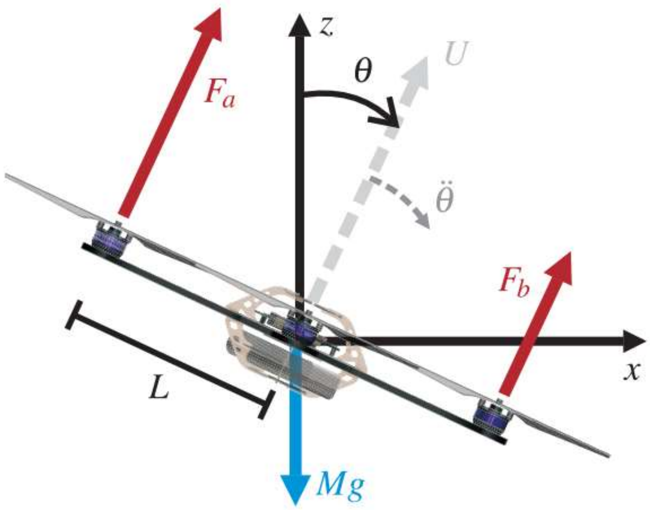
* when input is zero ($u=0$), $x(0) = \begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \Rightarrow \dot{x}(0) = 0$
$\Rightarrow x(t) = 0$



ex: quadrotor

A Simple Learning Strategy for High-Speed Quadrocopter Multi-Flips

Sergei Lupashin, Angela Schöllig, Michael Sherback, Raffaello D'Andrea



$$\begin{aligned} M\ddot{z} &= (F_a + F_b + F_c + F_d)\cos\theta - Mg && (1)\\ M\ddot{x} &= (F_a + F_b + F_c + F_d)\sin\theta && (2)\\ I_{yy}\ddot{\theta} &= L(F_a - F_b)\,, && (3) \end{aligned}$$

$\eta = x$ (horizontal)
$\nu = z$ (vertical)

$$M\ddot{\eta} = F\sin\theta$$

$$M\ddot{\nu} = -Mg + F\cos\theta$$

$$I\ddot{\theta} = \tau$$

where $F = F_a + F_b + F_c + F_d$ is the net thrust from 4 rotors

$\tau = L(F_a - F_b)$ is the net torque around roll axis

- with $q = (\eta, \nu, \theta) \in \mathbb{R}^3$ denoting positions

$\dot{q} = \frac{d}{dt} q = (\dot{\eta}, \dot{\nu}, \dot{\theta}) \in \mathbb{R}^3$ denoting velocities,

the state is $x = (q, \dot{q}) \in \mathbb{R}^6$, input is $u = (F, \tau) \in \mathbb{R}^2$

so dynamics are $\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q}(x,u) \end{bmatrix} = f(x,u)$

where $\ddot{q}(x,u) = \begin{bmatrix} F/M \sin\theta \\ -g + F/M \cos\theta \\ \tau/I \end{bmatrix}$

$$f : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^6$$
$$: (x, u) \longmapsto f(x, u)$$