ECE 447: Control Systems

goal: what is a control system?

(a) what is a system? ← a mathematical model for a transformation from inputs to outputs

(b) differential equations (DE) ← mathematical models that relate inputs/outputs & their derivatives at every time

(c) transfer functions ← math model that specifies how input e^{st} transforms to output $G(s)e^{st}$

(d) block diagrams ← math model of system interconnection

(e) feedback control ← design of systems and their interconnections to achieve desired closed-loop transformations

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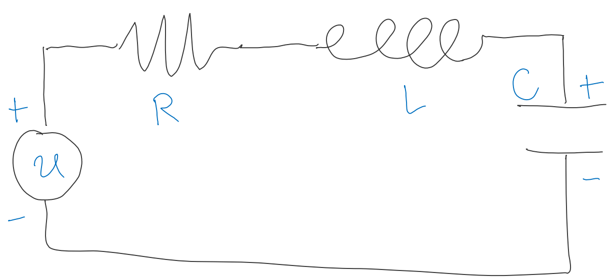
(a) what is a system?

→ a mathematical model of a transformation from inputs to outputs

voltage source u cap ~~voltage~~ charge y

ex:



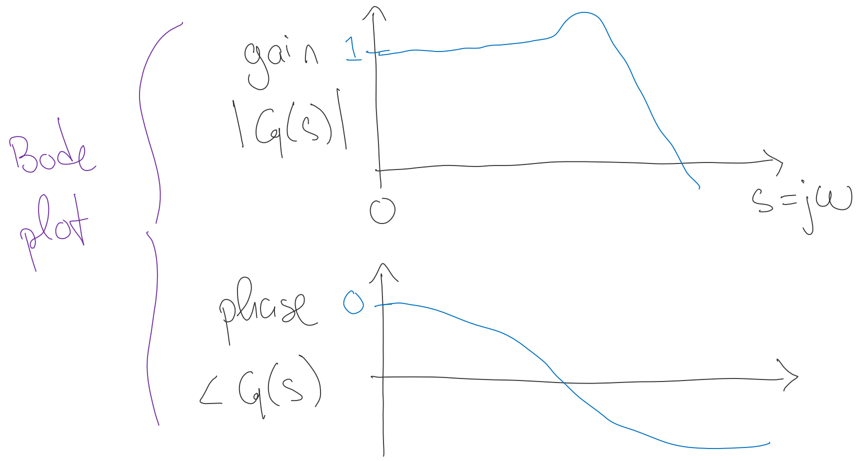
* how does input voltage u transform to output ~~voltage~~ charge y?

1°. differential equation (DE) KVL ⇒ $u = Ri + L\frac{d}{dt}i + \frac{1}{C}y$

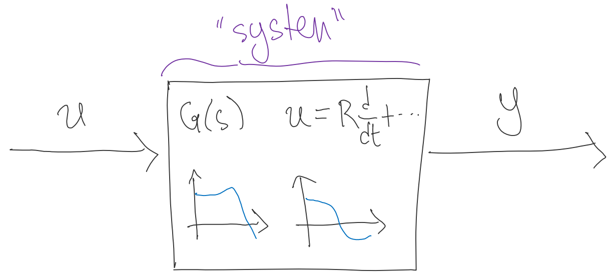
$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{C}y$$

$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{C}y$$

2°. transfer function $u = e^{st} \Rightarrow y = G(s)u = \left(\frac{1}{Ls^2 + Rs + 1/C}\right)u$



3°. block diagram



(b) differential equations (DE) [AMv2 Ch 2] [Nu7 Ch 3,4]

(DE) $\frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y$ $\quad$ y – output $\quad$ t – time

$$= b_1\frac{d^{n-1}}{dt^{n-1}}u + \cdots + b_n u$$ $\quad$ u – input

where $\{a_k\}_{k=1}^n, \{b_\ell\}_{\ell=1}^n \subset \mathbb{R}$

◦ notice: (DE) is specified by two polynomial expressions:

"characteristic" $\rightarrow$ 1°. $a(\boxed{s}) = \boxed{s}^n + a_1\boxed{s}^{n-1} + \cdots + a_n$

"characteristic" polynomial $\longrightarrow$ 1°. $a(\boxed{s}) = \boxed{s}^n + a_1 \boxed{s}^{n-1} + \cdots + a_n$

2°. $b(\boxed{s}) = \boxed{s}^n + b_1 \boxed{s}^{n-1} + \cdots + b_n$

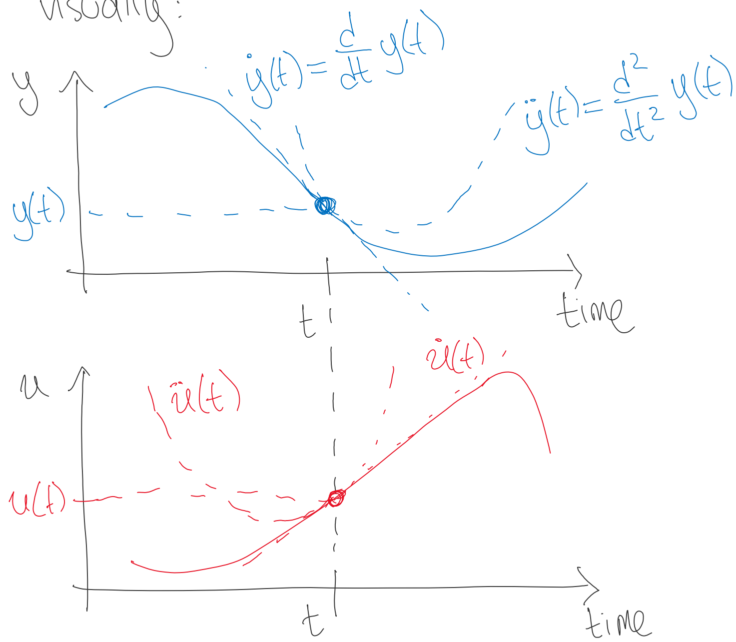
* we'll see that these polynomials govern input/output behavior of (DE)

- a "solution" to (DE) is a pair of signals (u, y)

$$u : \mathbb{R} \to \mathbb{R} \qquad y : \mathbb{R} \to \mathbb{R}$$
$$: t \mapsto u(t) \qquad : t \mapsto y(t)$$

that satisfy (DE) at all $t \in \mathbb{R}$: for all times!

visually:



algebraically: $\forall t \in \mathbb{R}$:

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t)$$
$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)$$

fact: every solution to (DE) is a linear combination (ie sum) of:

1°. homogeneous solution (where $u = 0$)

2°. particular solution (where $u \neq 0$)

1°. homogeneous solution: when $u = 0$, $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$

1°. homogeneous solution: when $u=0$, $\frac{d^n}{dt^n}y + a_1 \frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = 0$

y is a linear combination of (complex) exponentials:

$$y(t) = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t} = \sum_{k=1}^{n} c_k e^{s_k t}$$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the roots of characteristic polynomial $a(s)$ ie $a(s_k)=0$

* recall: n-th order polynomial a has no more than n roots (and no fewer than 1 root) eg $\square^n = 0$

and $\{c_k\}_{k=1}^n \subset \mathbb{C}$ are determined by initial condition

$$\{y(0), \dot{y}(0), \ddot{y}(0), \ldots, \frac{d^{n-1}}{dt^{n-1}}y(0)\} = \left\{\frac{d^k}{dt^k}y(0)\right\}_{k=0}^{n-1}$$

(c) transfer functions [AMv2 Ch 2] [Nu7 Ch 2]

- in contrast to DE, transfer functions characterize how a specific class of input signals are transformed by a system

- when $u(t) = e^{st}$, $s \notin \{s_k\}_{k=1}^n$ ie s is not a root of characteristic polynomial $a(\square)$

know $y(t) = G(s)e^{st}$, some $G(s) \in \mathbb{C}$

$\rightarrow$ verify that (u, y) satisfy (DE)

$$- \quad \frac{d}{dt}u(t) = s e^{st}, \quad \frac{d^2}{dt^2}u(t) = s^2 e^{st}, \quad \ldots, \quad \frac{d^{n-1}}{dt^{n-1}}u(t) = s^{n-1}e^{st}$$

- $\frac{d}{dt} y(t) = s\, G(s) e^{st}, \quad \ldots, \quad \frac{d^n}{dt^n} y(t) = s^n G(s) e^{st}$

- substituting into (DE): $(s^n + a_1 s^{n-1} + \cdots + a_n) G(s) \cancel{e^{st}}$
$= (b_1 s^{n-1} + \cdots + b_n) \cancel{e^{st}}$

* so if $\underbrace{G(s)}_{\text{termed the transfer function}} = \dfrac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \dfrac{b(s)}{a(s)}$ then (u, y) satisfy (DE)

(b & c) differential equation & transfer function

$$\underbrace{\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = b_1 \frac{d^{n-1}}{dt^{n-1}} u + \cdots + b_n u}_{\text{differential equation}} \qquad \underbrace{\frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}_{\text{transfer function}} = G(s)$$

- exponential input $u(t) = e^{st}$ to a linear time-invariant system yields exponential output $y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{1^\circ} + \underbrace{G(s) e^{st}}_{2^\circ}$

1°: homogeneous response to initial condition $\left\{ \frac{d^k}{dt^k} y(0) \right\}_{k=0}^{n-1}$

2°: particular response to input signal u

note: $\{s_k\}_{k=1}^{n}$ are the roots of characteristic polynomial $a(s)$

fact: since a_k's are real, the roots of $a(s)$ are:

real or complex-conjugate pairs

real $\downarrow$ real exponential $e^{s_k t}$

→ plot $e^{s_k t}$ vs t when:

$s_k < 0$; $s_k = 0$; $s_k > 0$

complex-conjugate pairs $\downarrow$ $s_k = \sigma \pm j\omega \in \mathbb{C}$, $\sigma, \omega \in \mathbb{R}$

yields complex exponential

$$e^{s_k t} = e^{\sigma t}\cos(\omega t) \pm j e^{\sigma t}\sin(\omega t)$$

→ plot $\operatorname{Re} e^{s_k t}$ vs t;

how does plot vary with σ, ω?

*note: signal decays asymptotically to zero <u>if and only if</u> $\operatorname{Re} s_k < 0$

($\Longleftrightarrow$)

<u>ex:</u> RLC circuit

(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$

↳ characteristic polynomial

$$a(s) = Ls^2 + Rs + \frac{1}{C}$$

→ compute roots of a

$$- \quad s_\pm = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

(TF) $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{Ls^2 + Rs + \frac{1}{C}}$

- if $u(t) = e^{st}$, $|s|$ small

then $y(t) = G(s)\,u(t)$

$\simeq C\,u(t)$

so if $s = 0$, $u(t) = 1$,

$y(t) = C$

$2L$

$y(t) = C$

→ is $\operatorname{Re} s_\pm < 0$? ☑

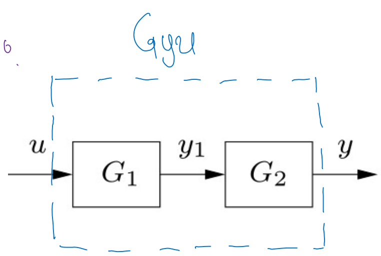
– if $R^2 - 4L/C \geq 0$ then $\operatorname{Re} s_\pm < 0$
  (assuming $R, L, C > 0$)

– also $\operatorname{Re} s_\pm < 0$ if $R^2 - 4L/C < 0$

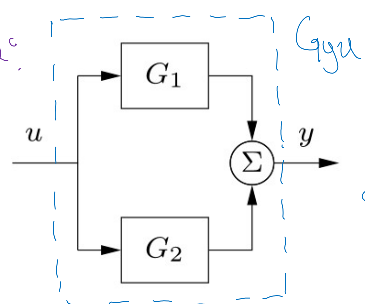
(d) block diagrams

∘ particularly useful for modeling interconnections between systems

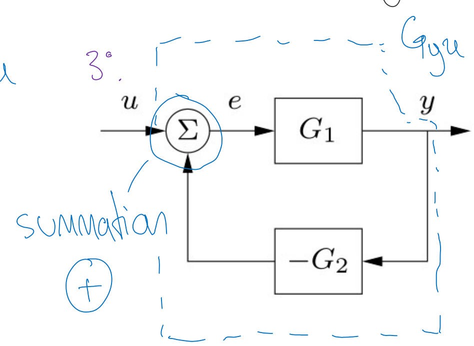
1°.



2°.



3°.



∘ each $\boxed{G_1}$, $\boxed{G_2}$ (blocks) is a system

∘ each $\xrightarrow{u}$, $\xrightarrow{y}$, $\longrightarrow$ (arrows) is a signal

1°. $y = G_2 y_1 = \underbrace{G_2 G_1}_{G_{yu}} u = G_{yu} \cdot u$

2°. $y = G_1 u + G_2 u = (G_1 + G_2) u = G_{yu} u$

3°. $y = G_1 e = G_1(u - G_2 y) = G_1 u - G_1 G_2 y$ bad place!

$\iff y + G_1 G_2 y = G_1 u$ → assuming $G_1 G_2 = -1$

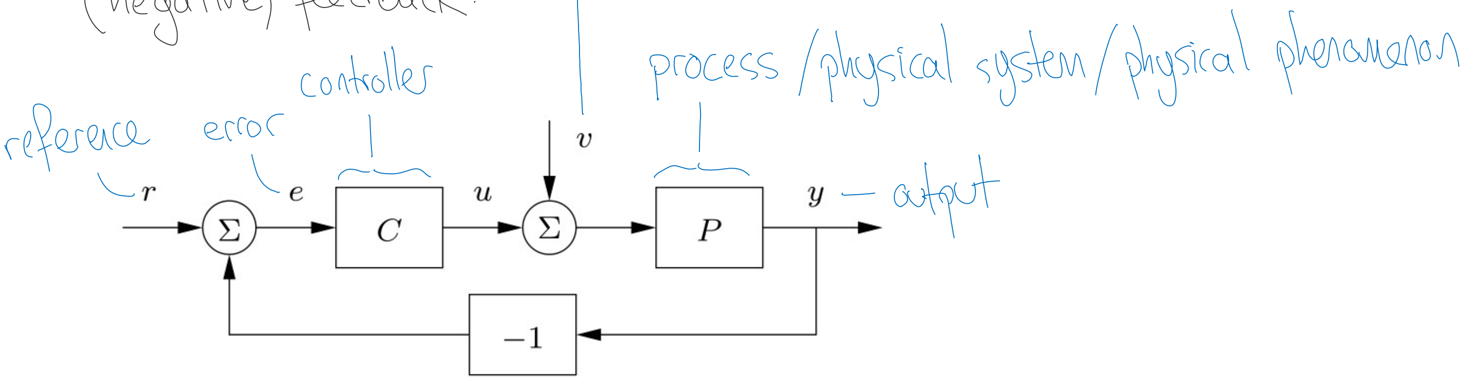
$\iff (1 + G_1 G_2) y = G_1 u \overset{*}{\iff} y = \dfrac{G_1}{1 + G_1 G_2} u = G_{yu} \cdot u$

$$\iff (1+G_1G_2)\,y = G_1 u \overset{*}{\iff} y = \frac{G_1}{1+G_1G_2}u = G_{yu}\cdot u$$

(f) feedback control

- a control system is an interconnection between a physical system P and a controller C — our goal is to design C

* today we'll design C to do disturbance rejection via (negative) feedback:



* we want to tune transformation G_{yv}

$$y = P(v+u) = Pv + PCe = Pv + PC(r-y)$$

$$\iff \underbrace{y + PCy}_{=(1+PC)\,y} = Pv + PCr$$

$$\iff y = \frac{P}{1+PC}v + \frac{PC}{1+PC}r = G_{yv}\cdot v + G_{yr}\cdot r$$

ex: consider $P(s) = \frac{b}{s+a} \iff \dot{y} + ay = bu$, $a,b>0$

- interpreted as a model for velocity of a car:

r – desired velocity $\qquad$ v – road slope; headwind

r – desired velocity $\qquad$ v – road slope; headwind

u – throttle / gas pedal $\qquad$ a – air resistance; wheel friction

y – car velocity $\qquad$ b – conversion from throttle to accel

* try two different controllers: 1°. proportional 2°. proportional-integral

1°. proportional control: $u = k_P e$ i.e. $C(s) = k_P$

→ determine transfer function G_{yv}

– $$G_{yv} = \frac{P}{1+PC} = \frac{b/s+a}{1 + b/s+a \cdot k_P} \cdot \frac{s+a}{s+a} = \frac{b}{s + (a + b \cdot k_P)}$$

- this (closed-loop) system is stable (ie $v \overset{G_{yv}}{\longmapsto} y$ doesn't "blow up")

$\Leftrightarrow$ all roots of characteristic polynomial

$$a(s) = s + (a + b \cdot k_P) \text{ are negative, i.e. if } (a + b \cdot k_P) > 0$$

- in this case, constant disturbance $v(t) = v_0$ (slope of hill)

yields $y \longrightarrow y_0 = G_{yv}(0) = \frac{b}{a + k_P b} v_0$

* by increasing $k_P > 0$, steady-state error y_0 decreases

2°. proportional-integral: $C(s) = k_P + \frac{1}{s} k_I$

ie $u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau$

→ determine transfer function G_{yv}

– $$G_{yv} = \frac{P}{\ldots} = \frac{bs}{\ldots}$$

- $G_{yv} = \dfrac{P}{1+PC} = \dfrac{bs}{s^2 + (a+bk_p)s + bk_I}$

- equivalently, $\ddot{y} + (a+bk_p)\dot{y} + bk_I y = b\dot{v}$

* constant disturbance $v = v_0$ yields zero steady-state error, $G_{yv}(0) = 0$

- characteristic polynomial $a(s) = s^2 + (a+bk_p)s + bk_I$

  - suppose we want roots $-\sigma_d \pm j\omega_d$ – d = "desired"

    ie characteristic polynomial $(s+\sigma_d+j\omega_d)(s+\sigma_d-j\omega_d)$

    $= s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2$

    $\rightsquigarrow$ yields time-domain homogeneous response $e^{-\sigma_d t}\sin(\omega_d t)$, $e^{-\sigma_d t}\cos(\omega_d t)$

    ie damped oscillations

  - match coefficients between polynomials to find:

$$k_p = \frac{2\sigma_d - a}{b}, \quad k_I = \frac{\sigma_d^2 + \omega_d^2}{b}$$