447 Fa19 final exam 10:30--12:20p Thu Dec 12

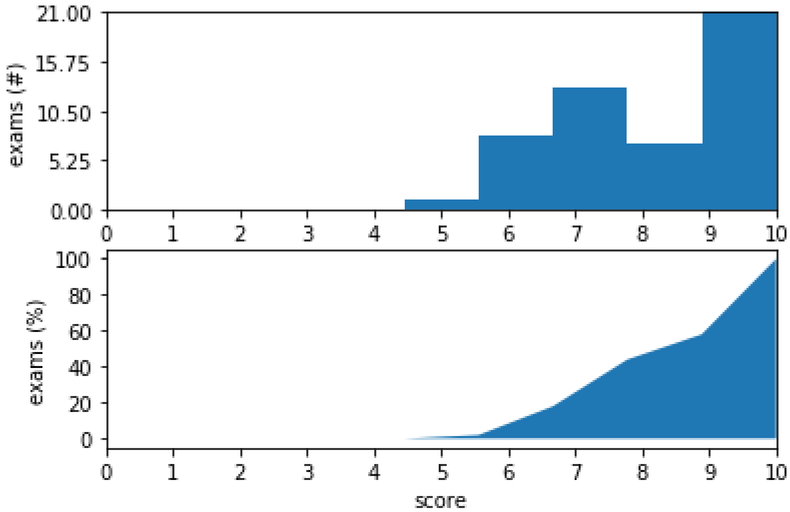
Notes regarding regrade requests:

- ***I will accept regrade requests until midnight (11:59p) Mon Dec 16*** -- I have to submit grades on Tue Dec 17.
- To request a regrade, send me a message using Canvas Conversations with a short explanation of which problem(s) you want regarded, and why you think your solution is equivalent to or equally valid as the one provided.
- ***Note that it is possible your score will decrease after the regrade***, so please be sure you understand the problem and solution before making a request. To help you understand the problems and their solutions before you submit your request, I am happy to answer questions during office hours.

```
import numpy as np
import pylab as plt

scores = np.asarray([5.0, 5.75, 6.0, 6.25, 6.25, 6.5, 6.5, 6.5, 6.5, 6.75,

print('%0.1f <= scores <= %0.1f'%(scores.min(),scores.max()))
print('median score = %0.1f'%np.median(scores))

xlim = (0,10)

plt.figure()
plt.subplot(2,1,1)
h = plt.hist(scores,bins=np.linspace(xlim[0],xlim[1],np.diff(xlim)))
n,s = h[0],h[1]
N = int(np.ceil(h[0].max()))
plt.ylim(0,N)
plt.xticks(np.linspace(xlim[0],xlim[1],np.diff(xlim)+1))
plt.yticks(np.linspace(0,N,(N+2)/4))
plt.xlim(xlim)
plt.ylabel('exams (#)');

plt.subplot(2,1,2)
n *= 100./n.sum()
n = np.hstack((0.,n))
plt.fill_between(s,np.cumsum(n),0*n)
plt.xlim(xlim)
plt.xticks(np.linspace(xlim[0],xlim[1],np.diff(xlim)+1))
plt.yticks(np.linspace(0,100,6))
plt.xlabel('score'); plt.ylabel('exams (%)');
```

ndarray: scores
ndarray with shape (50,)

```
5.0 <= scores <= 10.0
median score = 8.5
```



problem (1.)

Translate the following model into state-space form using 3-dimensional state vector $x = (q, \dot{q}, \ell) \in \mathbb{R}^3$ and linearize about the origin,

$$m\ddot{q} = \kappa(\ell - 2q) + \mu(\ell - 2q)^2 + u,\ \beta\dot{\ell} = -\kappa(\ell - q) - \alpha\cos(\ell - q),$$

i.e. determine function f and matrices A, B such that $\dot{x} = f(x, u) \approx Ax + Bu$ near $x_e = 0$, $u_e = 0$.

solution

Dividing through by m and β, we obtain that:

$$\boxed{f(x,u) = \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \\ \dot{\ell} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{1}{m}(\kappa(\ell - 2q) + \mu(\ell - 2q)^2 + u) \\ \frac{1}{\beta}(\kappa(\ell - q) - \alpha\cos(\ell - q)) \end{bmatrix}}$$

To obtain matrices A, B, we can compute the Jacobians $A = \frac{df}{dx}, B = \frac{df}{du}$ and evaluate at the equilibrium point $(0, 0, 0)$ to obtain

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-2\kappa}{m} & 0 & \frac{\kappa}{m} \\ \frac{\kappa}{\beta} & 0 & \frac{-\kappa}{\beta} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}}$$

Notes:

- 0.25 points earned for correct NL equation f
- 0.25 points earned for correct B
- 0 to 0.5 points earned for correctly computing Jacobian and plugging in equilibrium point for A

problem (2.)

Consider the following LTI system with an output,

$$\dot{x} = Ax = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} x,\ y = Cx = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Determine the output feedback matrix L that places all poles of the error dynamics $\dot{e} = (A - LC)e$ at $-1 \in \mathbb{C}$ using the fact that the characteristic polynomial of A is $\chi(s) = s^3 + a_1 s^2 + a_2 s + a_3$.

Hint: you do not need to compute the determinant of a 3×3 matrix to solve this problem.

solution

Since we want all poles to be at -1, we know that the characteristic polynomial of $A - LC$ should match the coefficients of the characteristic polynomial $\chi(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$.

We can also compute

$$\tilde{A} = A - LC = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -(a_1 + \ell_1) & 1 & 0 \\ -(a_2 + \ell_2) & 0 & 1 \\ -(a_3 + \ell_3) & 0 & 0 \end{bmatrix}$$

We can see that $\tilde{A}$ has the same form as A, and as such, we know that the characteristic polynomial of $\tilde{A}$ is $\chi'(s) = s^3 + (a_1 + \ell_1)s^2 + (a_2 + \ell_2)s + (a_3 + \ell_3)$.

Matching the coefficients of $\chi(s)$ to $\chi'(s)$, we can see that $a_1 + \ell_1 = 3$, $a_2 + \ell_2 = 3$, $a_3 + \ell_3 = 1$, and therefore,

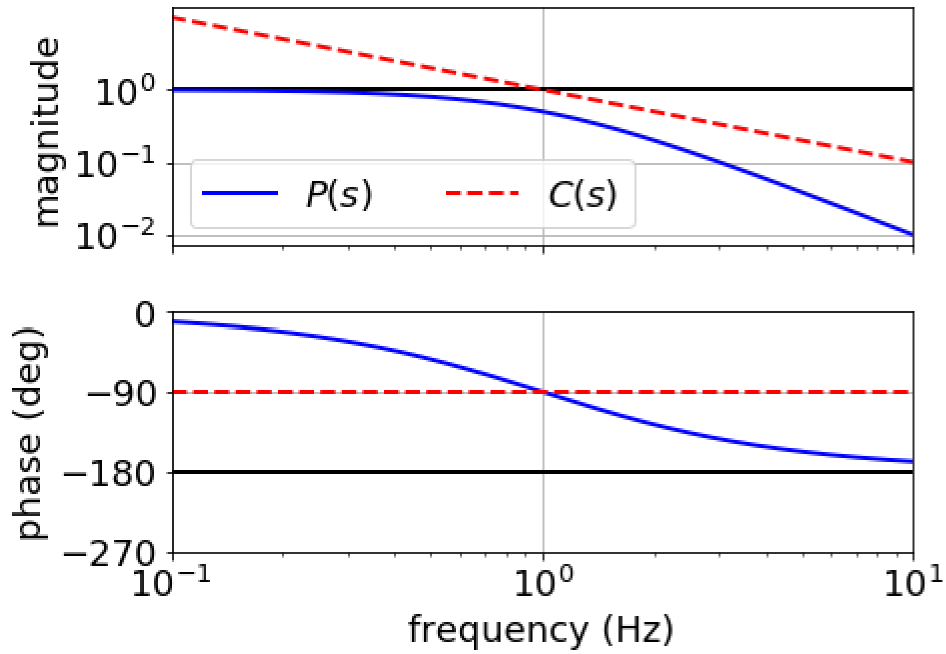
$$\boxed{L = \begin{bmatrix} 3 - a_1 \\ 3 - a_2 \\ 1 - a_3 \end{bmatrix}}$$

Notes:

- 0.5 points for correctly determining that the characteristic polynomial for $A - LC$ is equal to $\chi'(s) = s^3 + (a_1 + \ell_1)s^2 + (a_2 + \ell_2)s + (a_3 + \ell_3)$.
- 0.5 points for determining that because the poles should be at -1, the characteristic polynomial we want to obtain for $\tilde{A}$ should be $\chi = (s+1)^3$

problem (3.)

Consider the following ***Bode plots*** of ***process*** $P(s)$ and ***controller*** $C(s)$ transfer functions.



Apply the ***Nyquist stability criterion*** to the open-loop transfer function $L(s) = P(s)C(s)$ to determine whether the closed-loop system is stable.

solution

The key to this problem is recalling the way Bode plots combine when two transfer functions are multiplied:

$$|L(s)| = |P(s)| \cdot |C(s)|, \ \angle L(s) = \angle P(s) + \angle C(s).$$

Using these relationships, it is clear from inspection that the Nyquist plot of L only crosses the negative real axis at $\omega = 1$Hz (where $\angle L(j\omega) = \angle P(j\omega) + \angle C(j\omega) = 180^\circ$), and that the critical point $-1 \in \mathbb{C}$ is not encircled since the magnitude of L is smaller that 1 at this point (since $|C(j\omega)| = 1$ and $|P(j\omega)|$ < 1).

Since the critical point $-1 \in \mathbb{C}$ is not encircled, the Nyquist stability criterion implies the closed-loop system is stable.

Notes:

- To receive credit on this problem, it was necessary to use the Bode plot relationships above (either algebraically or graphically) and apply the Nyquist stability criterion to the transfer function L (not P or C).

problem (4.)

Recall that the ***sensitivity*** $S(s)$ and ***complementary sensitivity*** $T(s)$ transfer functions for a process $P(s)$ and controller $C(s)$ in the standard negative feedback interconnection are

$$S(s) = \frac{1}{1 + P(s)C(s)},\ T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}.$$

subproblem (a.)

Show that zeros of the sensitivity function are poles of the process or controller.

solution

Letting $P = \frac{n_P}{d_P}, C = \frac{n_C}{d_C}$ we find that

$$S = \frac{1}{1 + PC} = \frac{1}{1 + \frac{n_P n_C}{d_P d_C}} = \frac{d_P d_C}{d_P d_C + n_P n_C},$$

so the zeros of S are the zeros of d_P or d_C, which are the poles of P or C.

Another accepted solution is to observe that $S(z) = 0$ requires $1 + P(z)C(z) = \infty$, i.e. $P(z) = \infty$ or $C(z) = \infty$, i.e. z must be a pole of P or C to be a zero of S.

subproblem (b.)

Suppose the process has a purely imaginary pole at $j\omega$, and that neither the process nor controller has a zero on the imaginary axis.

Determine $T(j\omega)$.

solution

Using the result from (a.) (regardless of whether the solution provided for (a.) is correct), we know that $S(j\omega) = 0$ since $j\omega$ is a pole of P.

Since $S + T \equiv 1$, it must be the case that $T(j\omega) = 1$.

Notes:

- To receive credit on this problem, it was necessary to use the relationship $S + T \equiv 1$ to provide a numerical value for $T(j\omega)$; it was not sufficient to substitute $j\omega$ into the given expression for T.

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ndarray: scores

ndarray with shape (50,)
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