

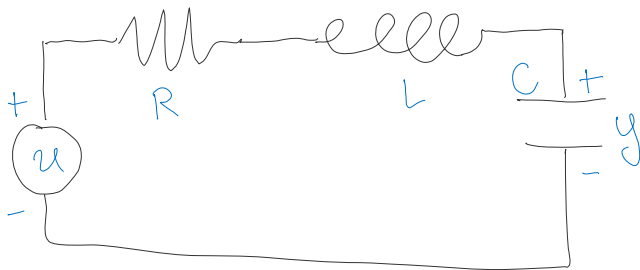
goal: what is a control system?

- (a) what is a system? \leftarrow a mathematical model for a transformation from inputs to outputs
- (b) differential equations (DE) \leftarrow mathematical models that relate inputs/outputs \dot{z} their derivatives at every time
- (c) transfer functions \leftarrow math model that specifies how input e^{st} transforms to output $G(s)e^{st}$
- (d) block diagrams \leftarrow math model of system interconnection
- (e) feedback control \leftarrow design of systems and their interconnections to achieve desired closed-loop transformations

(a) what is a system?

\rightarrow a mathematical model of a transformation from inputs to outputs
 voltage source u cap ~~voltage~~ y
 charge

ex:

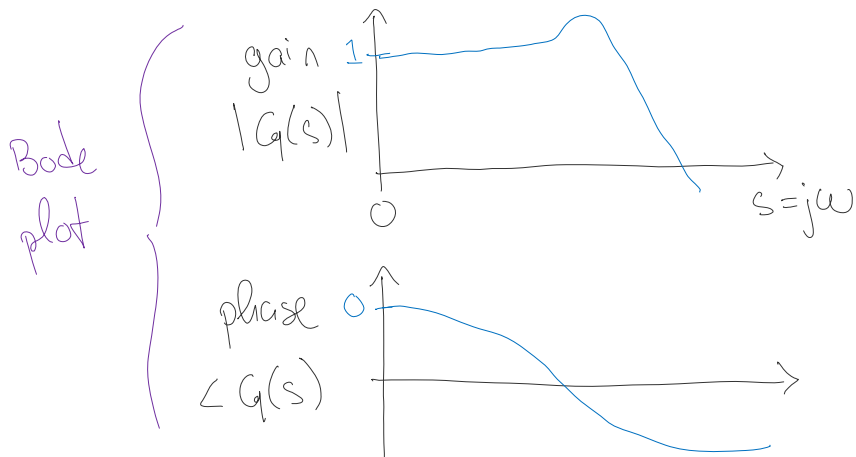


* how does input voltage u transform to output ~~voltage~~ ^{charge} y ?

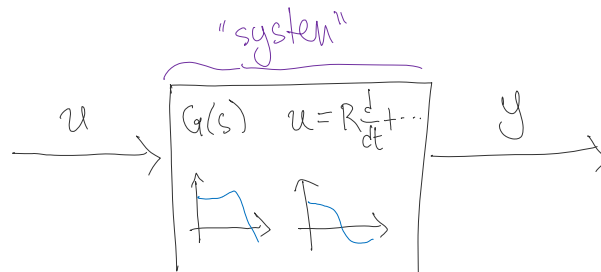
1°. differential equation (DE) KVL $\Rightarrow u = Ri + L \frac{d}{dt} i + \frac{1}{C} y$
 $= R \frac{d}{dt} y + L \frac{d^2}{dt^2} y + \frac{1}{C} y$

$$= R \frac{d}{dt} y + L \frac{d^2}{dt^2} y + \frac{1}{C} y$$

2°. transfer function $u = e^{st} \Rightarrow y = G(s)u = \left(\frac{1}{Ls^2 + Rs + 1/C} \right) u$



3°. block diagram



(b) differential equations (DE) [AMv2 ch2] [Nu7 ch 3,4]

$$(DE) \quad \frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y \quad y - \text{output} \quad t - \text{time}$$

$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u + \dots + b_n u \quad u - \text{input}$$

where $\{a_k\}_{k=1}^n, \{b_k\}_{k=1}^n \subset \mathbb{R}$

notice: (DE) is specified by two polynomial expressions:

"characteristic" \rightarrow 1°. $a(s) = s^n + a_1 s^{n-1} + \dots + a_n$

"characteristic" polynomial \rightarrow 1°. $a(s) = s^n + a_1 s^{n-1} + \dots + a_n$

2°. $b(s) = \cancel{s^n} \cancel{+} b_1 s^{n-1} + \dots + b_n$

* we'll see that these polynomials govern input/output behavior of (DE)

• a "solution" to (DE) is a pair of signals (u, y)

$$u: \mathbb{R} \rightarrow \mathbb{R}$$

$$: t \mapsto u(t)$$

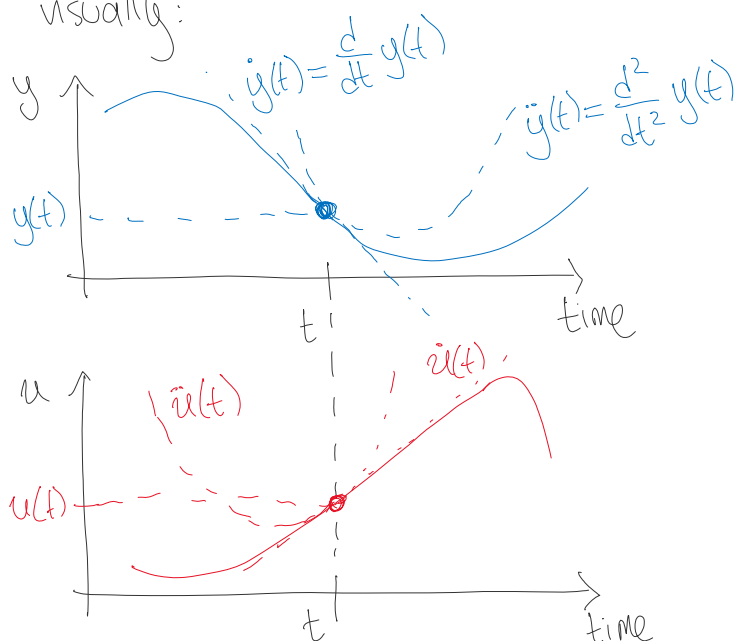
$$y: \mathbb{R} \rightarrow \mathbb{R}$$

$$: t \mapsto y(t)$$

that satisfy (DE) at all $t \in \mathbb{R}$:

for all times!

visually:



algebraically: $\forall t \in \mathbb{R}$:

$$\begin{aligned} & \frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \dots + a_n y(t) \\ &= b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \dots + b_n u(t) \end{aligned}$$

fact: every solution to (DE) is a linear combination (ie sum) of:

1°. homogeneous solution (where $u=0$)

2°. particular solution (where $u \neq 0$)

1°. homogeneous solution: when $u=0$, $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y = 0$

1° homogeneous solution: when $u=0$, $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y = 0$

y is a linear combination of (complex) exponentials:

$$y(t) = C_1 e^{s_1 t} + \dots + C_n e^{s_n t} = \sum_{k=1}^n C_k e^{s_k t}$$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the roots of } ie $a(s_k) = 0$
characteristic polynomial $a(s)$

* recall: n -th order polynomial a has no more than n roots
(and no fewer than 1 root) eg $\square^n = 0$

and $\{C_k\}_{k=1}^n \subset \mathbb{C}$ are determined by initial condition
 $\{y(0), \dot{y}(0), \ddot{y}(0), \dots, \frac{d^{n-1}}{dt^{n-1}} y(0)\} = \left\{ \frac{d^k}{dt^k} y(0) \right\}_{k=0}^{n-1}$

(c) transfer functions [AMv2 ch 2] [Nv7 ch 2]

• in contrast to DE, transfer functions characterize how
a specific class of input signals are transformed by a system

• when $u(t) = e^{st}$, $s \notin \{s_k\}_{k=1}^n$ ie s is not a root of
characteristic polynomial $a(\square)$

know $y(t) = G(s)e^{st}$, some $G(s) \in \mathbb{C}$ $\frac{d^k}{dt^k} u(t) = s^k e^{st}$

→ verify that (u, y) satisfy (DE)

$$- \frac{d}{dt} u(t) = s e^{st}, \quad \frac{d^2}{dt^2} u(t) = s^2 e^{st}, \quad \dots, \quad \frac{d^{n-1}}{dt^{n-1}} u(t) = s^{n-1} e^{st}$$

$$- \frac{d}{dt} u(t) = \dots, \quad \frac{d^2}{dt^2} u(t) = \dots, \quad \dots, \quad \frac{d^{n-1}}{dt^{n-1}} u(t) = \dots$$

$$- \frac{d}{dt} y(t) = s G(s) e^{st}, \quad \dots, \quad \frac{d^n}{dt^n} y(t) = s^n G(s) e^{st}$$

$$- \text{substituting into (DE): } (s^n + a_1 s^{n-1} + \dots + a_n) G(s) e^{st} = (b_1 s^{n-1} + \dots + b_n) e^{st}$$

* so if $G(s) = \frac{b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{b(s)}{a(s)}$ then (u, y) satisfy (DE)

↳ termed the transfer function

(b & c) differential equation

↔

transfer function

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y = b_1 \frac{d^{n-1}}{dt^{n-1}} u + \dots + b_n u$$

$$\frac{b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = G(s)$$

• exponential input $u(t) = e^{st}$ to a linear time-invariant system

$$\text{yields exponential output } y(t) = \underbrace{\sum_{k=1}^n C_k e^{s_k t}}_{1^\circ} + \underbrace{G(s) e^{st}}_{2^\circ}$$

1°: homogeneous response to initial condition $\left\{ \frac{d^k}{dt^k} y(0) \right\}_{k=0}^{n-1}$

2°: particular response to input signal u

2: particular response to input signal u

note: $\{s_k\}_{k=1}^n$ are the roots of characteristic polynomial $a(s)$

fact: since a_k 's are real, the roots of $a(s)$ are:

real

or

complex-conjugate pairs

↓

real exponential $e^{s_k t}$

↓

$$s_k = \sigma \pm j\omega \in \mathbb{C}, \quad \sigma, \omega \in \mathbb{R}$$

yields complex exponential

$$e^{s_k t} = e^{\sigma t} \cos(\omega t) \pm j e^{\sigma t} \sin(\omega t)$$

→ plot $e^{s_k t}$ vs t when:

$$s_k < 0; \quad s_k = 0; \quad s_k > 0$$

→ plot $\operatorname{Re} e^{s_k t}$ vs t ;
how does plot vary with σ, ω ?

*note: signal decays asymptotically to zero if and only if $\operatorname{Re} s_k < 0$
(\Leftrightarrow)

ex: RLC circuit



$$(DE) \quad L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$$

↳ characteristic polynomial

$$a(s) = Ls^2 + Rs + \frac{1}{C}$$

→ compute roots of a

$$D \pm \sqrt{D^2 - 4AC}$$

$$(TF) \quad G(s) = \frac{b(s)}{a(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

• if $u(t) = e^{st}$, $|s|$ small

$$\text{then } y(t) = G(s)u(t) \approx C u(t)$$

→ compute roots of

$$s_{\pm} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

$$\approx C u(t)$$

so if $s=0$, $u(t)=1$,
 $y(t)=C$

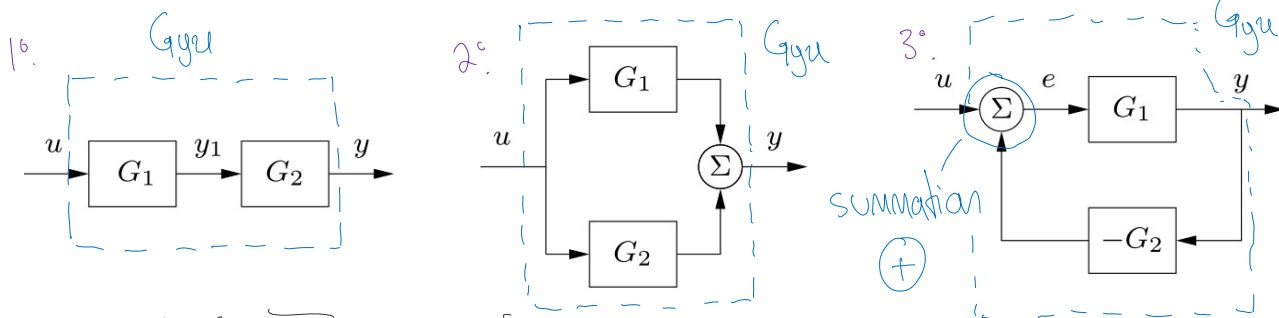
→ is $\text{Re } s_{\pm} < 0$? ☒

– if $R^2 - 4L/C \geq 0$ then $\text{Re } s_{\pm} < 0$
 (assuming $R, L, C > 0$)

– also $\text{Re } s_{\pm} < 0$ if $R^2 - 4L/C < 0$

(d) block diagrams

• particularly useful for modeling interconnections between systems



• each $\boxed{G_1}, \boxed{G_2}$ (blocks) is a system

• each $\xrightarrow{u}, \xrightarrow{y}, \rightarrow$ (arrows) is a signal

$$1^\circ: y = G_2 y_1 = \underbrace{G_2 G_1}_{G_{yu}} u = G_{yu} \cdot u$$

$$2^\circ: y = G_1 u + G_2 u = (G_1 + G_2) u = G_{yu} u$$

$$3^\circ: y = G_1 e = G_1 (u - G_2 y) = G_1 u - G_1 G_2 y$$

bad place! ☒

$$\Leftrightarrow u + G_1 G_2 u = G_1 u \quad \rightarrow \text{assuming } \boxed{G_1 G_2 = -1}$$

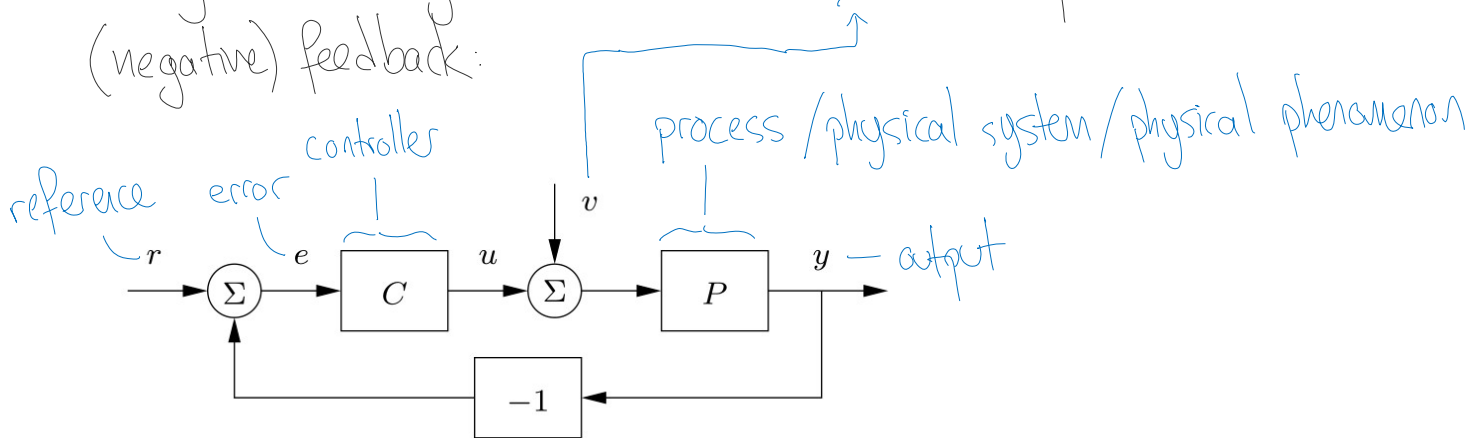
$$\Leftrightarrow y + G_1 G_2 y = G_1 u \quad \rightarrow \text{assuming } \boxed{G_1 G_2 = -1}$$

$$\Leftrightarrow (1 + G_1 G_2) y = G_1 u \stackrel{*}{\Leftrightarrow} y = \frac{G_1}{1 + G_1 G_2} u = G_{yr} \cdot u$$

(f) feedback control

• a control system is an interconnection between a physical system P and a controller C — our goal is to design C

* today we'll design C to do disturbance rejection via (negative) feedback:



* we want to tune transformation G_{yv}

$$y = P(v + u) = Pv + PCe = Pv + PC(r - y)$$

$$\Leftrightarrow \underbrace{y + PCy}_{=(1+PC)y} = Pv + PCr$$

$$\Leftrightarrow y = \frac{P}{1+PC} v + \frac{PC}{1+PC} r = G_{yv} \cdot v + G_{yr} \cdot r$$

ex: consider $P(s) = \frac{b}{s+a} \Leftrightarrow \dot{y} + ay = bx$, $a, b > 0$

• interpreted as a model for velocity of a car:

• interpreted as a model for velocity of a car:

r - desired velocity v - road slope; headwind

u - throttle / gas pedal a - air resistance; wheel friction

y - car velocity b - conversion from throttle to accel

* try two different controllers: 1°: proportional 2°: proportional-integral

1°: proportional control: $u = k_p e$ i.e. $C(s) = k_p$

→ determine transfer function G_{yv}

$$- G_{yv} = \frac{P}{1+PC} = \frac{b/sta}{1 + b/sta \cdot k_p} \cdot \frac{sta}{sta} = \frac{b}{s + (a + b \cdot k_p)}$$

• this (closed-loop) system is stable (ie $v \xrightarrow{G_{yv}} y$ doesn't "blow up")

\Leftrightarrow all roots of characteristic polynomial

$a(s) = s + (a + b \cdot k_p)$ are negative, ie $(a + k_p \cdot b) > 0$

• in this case, constant disturbance $v(t) = v_0$ (slope of hill)

yields $y \rightarrow y_0 = G_{yv}(0) \cdot v_0 = \frac{b}{a + k_p \cdot b} v_0$

* by increasing $k_p > 0$, steady-state error y_0 decreases

2°: proportional-integral: $u(t) = k_p e(t) + k_I \int_0^t e(\tau) d\tau$

→ determine transfer function $C(s)$ that yields ↗

$$- C(s) = k_p + \frac{1}{s} k_I$$

$$C(s) = \frac{1}{s} \quad \text{or} \quad \frac{1}{s^2}$$

→ determine transfer function G_{yv}

$$- G_{yv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a+bk_p)s + bk_I}$$

• equivalently, $\ddot{y} + (a+bk_p)\dot{y} + bk_I y = b\dot{v}$

* constant disturbance $v=v_0$ yields zero steady-state error: $G_{yv}(0)=0$

• characteristic polynomial $a(s) = s^2 + (a+bk_p)s + bk_I$

- suppose we want roots $-\sigma_d \pm j\omega_d$ ($d = \text{"desired"}$)

ie characteristic polynomial $(s + \sigma_d + j\omega_d) \cdot (s + \sigma_d - j\omega_d)$
 $= s^2 + 2\sigma_d s + (\sigma_d^2 + \omega_d^2)$

→ yields time-domain homogeneous response $e^{-\sigma_d t} \sin(\omega_d t)$
 ie damped oscillations $e^{-\sigma_d t} \cos(\omega_d t)$

→ determine k_p, k_I so that $-\sigma_d \pm j\omega_d$ are roots of $a(s)$

- $(a+bk_p) = 2\sigma_d \Leftrightarrow k_p = \frac{2\sigma_d - a}{b}$

- $bk_I = (\sigma_d^2 + \omega_d^2) \Leftrightarrow k_I = \frac{\sigma_d^2 + \omega_d^2}{b}$