HW3 due 5p Fri Oct 30 2020

You are welcome (and encouraged) to work with others, but each individual must submit their own writeup.

You are welcome to use analytical and numerical computational tools; if you do, include the **commented** sourcecode in your submission (e.g. the .ipynb file).

You are welcome to consult websites, textbooks, and other materials; if you do, include a full citation in your writeup (e.g. the .ipynb file).

Important: before you do any work in the Colaboratory notebook, click "File -> Save a copy in Drive ..." and rename the file to something memorable.

0. [preferred name]; [preferred pronouns]

a. Approximately how many hours did you spend on this assignment?

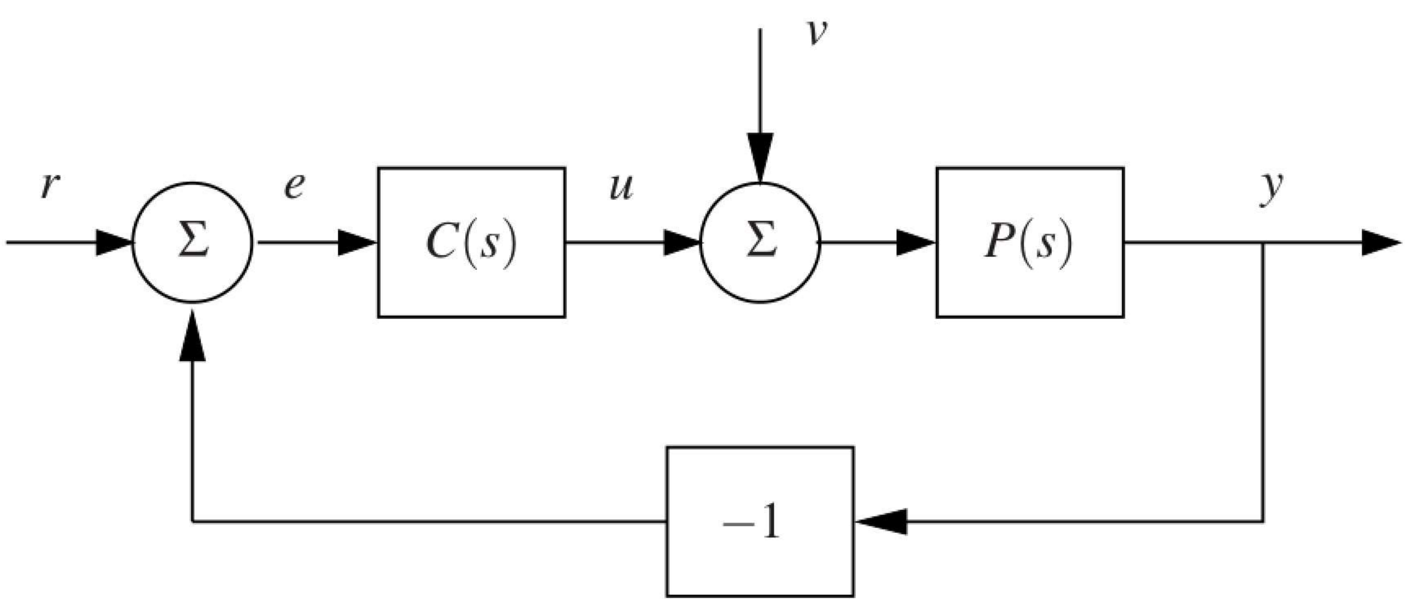
b. Were there specific problems that took much longer than others?

c. What class meeting(s) did you participate in this week?

d. What timezone(s) were you working in this week?

1. proportional-integral control

This problem considers the following standard negative feedback block diagram.



Purpose: you will analyze and synthesize a ***proportional-integral*** controller

$$C(s) = k_P + k_I/s \iff u(t) = k_P e(t) + k_I \int_0^t e(\tau) d\tau$$

for the purpose of ***reference tracking***. Specifically, you'll apply this controller to a first-order process model $P(s) = b/(s+a)$ to obtain the closed-loop transfer function G_{yr}.

We seek to tune the controller parameters k_P, k_I to obtain complex-conjugate roots $-\sigma_d \pm j\omega_d$ for the closed-loop system, i.e. we want the characteristic polynomial of G_{yr} to be

$$(s + \sigma_d + j\omega_d)(s + \sigma_d - j\omega_d) = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2.$$

a. By matching coefficients that multiply the same power of the variable s in the actual and desired characteristic polynomials, express k_P and k_I in terms of σ_d, ω_d, a, b.

b. Validate your result from (a) using a numerical simulation of the closed-loop system with parameters $a = 1, b = 1$: choose σ_d, ω_d, determine the corresponding k_P, k_I, and plot the system's ***step response***, that is, the output corresponding to a reference $r(t)$ that is equal to zero for $t < 0$ and equal to 1 for $t \geq 0$.

According to the preceding analyses, the convergence rate can be made arbitrarily fast (i.e. σ_d can be made arbitrarily large) by making the controller parameters k_P, k_I large. In practice, such *high-gain feedback* can excite *unmodeled dynamics* and lead to instability.

To see how this can happen, suppose unmodeled dynamics in $P(s)$ (e.g. dynamics of sensors, actuators, vibratory modes, electromagnetic coupling, etc.) have time constant $T > 0$ (assume $-1/T \ll -a$, so we were initially justified in neglecting these dynamics in our model because they are stable and converge much faster than the original modeled dynamics). Including these dynamics yields the process model

$$P(s) = \frac{b}{(s+a)(1+sT)}.$$

c. Determine the characteristic polynomial of the closed-loop transfer function G_{yr}.

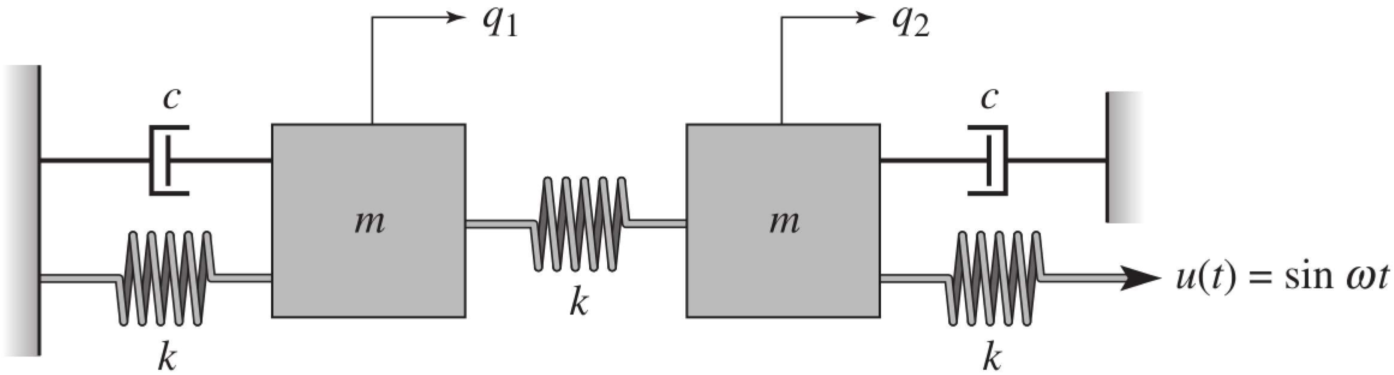
d. Apply the *Routh-Hurwitz stability criterion* to determine algebraic conditions that must be satisfied by a, b, k_P, k_I, T for the closed-loop system to be stable.

Conclude that σ_d is limited by T, i.e. the convergence rate of a PI controller applied to a first-order process is limited by the characteristic time constant of unmodeled dynamics.

e. Validate your result from (d) using a numerical simulation of the closed-loop system with $a = 1$, $b = 1, T = 0.1$: choose k_P, k_I such that the Routh-Hurwitz criteria are (i) satisfied (so the closed-loop system is stable) and (ii) violated (so the closed-loop system is unstable), and provide plots showing the step response in these two cases.

2. spring-mass-damper a deux

Consider the following spring-mass-damper system:



The input to this system is the sinusoidal motion of the end of the rightmost spring. Applying Newton's laws to determine the forces acting on both masses, we find two coupled second-order DE that model the system's dynamics:

$$m\ddot{q}_1 = -c\dot{q}_1 - kq_1 + k(q_2 - q_1),$$
$$m\ddot{q}_2 = -c\dot{q}_2 + k(u - q_2) - k(q_2 - q_1).$$

change-of-coordinates

Purpose: observe how a clever change-of-coordinates can simplify analysis of a complex system's dynamics. ***Note:*** you ***do not*** need to solve the DEs in (a.--d.).

a. Combine the two second-order DE above to obtain one fourth-order DE with u as the input and q_1 as the output. (***Note:*** this DE is hard to solve -- since the characteristic polynomial is fourth-order, you can't easily solve for the roots needed to obtain the ***homogeneous solution***.)

b. Rewrite the original two second-order DE in terms of $p_1 = \frac{1}{2}(q_1 + q_2)$ and $p_2 = \frac{1}{2}(q_1 - q_2)$. (***Note:*** you should now have two second-order DE that are decoupled, that is, they can be solved independently; two second-order DE are ***much*** easier to solve than the one fourth-order DE you obtained in (a.).)

c. Translate the original two DE to matrix/vector form using state vector $x = (q_1, \dot{q}_1, q_2, \dot{q}_2)$ (i.e. determine matrices A, B such that $\dot{x} = Ax + Bu$).

d. Translate the two DE from (b.) to matrix/vector form using state vector $z = (p_1, \dot{p}_1, p_2, \dot{p}_2)$. Noting that the system's "A" matrix is block-diagonal, determine $A_1, A_2 \in \mathbb{R}^{2\times 2}$ and $B_1, B_2 \in \mathbb{R}^{2\times 1}$ such that

$$\dot{z} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} z + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u.$$

Observe: in matrix/vector form, decoupling between subsystems manifests with block-diagonal structure in the "A" matrix.

Bonus (1 point for correctness): determine the matrix T such that $z = Tx$.

resonance

Purpose: use computational tools to investigate a complex system's steady-state behavior in the time- and frequency- domain.

e. Setting $m = 250$, $k = 50$, $c = 10$, plot the motion of the first and second masses in response to an input motion $u = a\sin(\omega t)$ with $\omega = 1$ rad/sec and $a = 1$ cm. How long does it take for the system to reach 2% of its steady-state oscillation amplitude starting from the origin?

f. Plot the steady-state amplitude of the motion of the first mass as a function of the frequency of the input, ω. (***Hint:*** you can either derive the transfer function from your answer to (a.) above, or you can run many (~100) simulations to create this plot; experiment with different ranges of ω until you find a range that demonstrates two resonant frequencies (sharp peaks) in the plot. You may find it helpful to use the `numerical_simulation` function defined and applied in the [lecture examples notebook].)

3. parametric stability

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 4 \end{bmatrix} u.$$

open-loop system

Purpose: assess stability of an uncontrolled ("open-loop") system in state-space form using eigenvalues.

First, we'll consider the *open-loop* linear system's "A" matrix,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}.$$

a. What are the eigenvalues of A? How did you determine them?

b. Is this system stable (all eigenvalues have negative real part) or unstable (one or more eigenvalues has zero or positive real part)?

closed-loop system

Purpose: assess stability of a controlled ("closed-loop") system in state-space form using eigenvalues as a function of a parameter.

Now we'll consider the effect of the output feedback $u = \begin{bmatrix} -k & 0 \end{bmatrix} x$, which results in the following ***closed-loop*** "A" matrix for the linear system:

$$A = \begin{bmatrix} k & 1 \\ -4k & -3 \end{bmatrix}.$$

c. What are the eigenvalues of A? How did you determine them?

d. Plot the eigenvalues of A in the complex plane for $k \geq 0$; annotate the plot with several values of k.

Takeaway: this kind of plot is termed a ***root locus*** -- the eigenvalues are the *roots* of the characteristic polynomial $a(s)$ for A, and their plot is the *locus* of points that satisfy the homogenous equation $a(s) = 0$. Later we'll learn more techniques for predicting and interpreting a linear system's root locus plot. We'll pay particular attention to whether the eigenvalues lie in the left-half-plane, i.e. whether the system is stable.