

goal: qualitative and quantitative tools to assess a system's stability

(a) equilibria  $\leftarrow (x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$  is an equilibrium for  $\dot{x} = f(x, u)$  if  $f(x_e, u_e) = 0 = \dot{x}_e$

(b) characteristic polynomial  $\leftarrow$  equilibrium stable  $\Leftrightarrow \operatorname{Re} s_k < 0$   
 $a(s) = s^n + a_1 s^{n-1} + \dots + a_n$  for all  $s_k \in \mathbb{C}$  s.t.  $a(s_k) = 0$

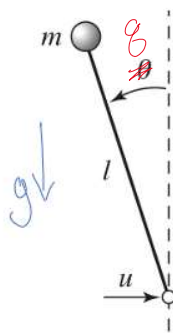
(c) Routh-Hurwitz  $\leftarrow$  equilibrium stable  $\Leftrightarrow$  [inequalities satisfied on  $a_1, a_2, \dots, a_n$ ]

(d) eigenvalues  $\leftarrow$  equilibrium stable  $\Leftrightarrow \operatorname{Re} \lambda_k < 0$  for all  $\dot{x} = Ax + Bu$   $\lambda_k \in \mathbb{C}$  s.t.  $\det(\lambda_k I - A) = 0$

(e) parameter dependence  $\leftarrow$  variations in control or design parameters can cause instability — visualize w/ root locus diagram

(a) equilibria [AMv2 Ch 5.3] [Nv7 Ch 2.7, 3.7]

ex: "rocket flight" (really: pendulum)



- state  $x = (\theta, \dot{\theta})$  — angle, velocity
- input  $u$  — horizontal acceleration of pivot
- (DE)  $ml^2 \ddot{\theta} = mgl \sin \theta - \alpha \dot{\theta} + lu \cos \theta$

$$\ddot{\mathbf{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ g/l \sin \theta - \frac{\alpha}{ml^2} \dot{\theta} + \frac{1}{ml} u \cos \theta \end{bmatrix} = f(\mathbf{x}, u)$$

• when  $u(t) = u_e$  is constant:  $f(\mathbf{x}, u_e) = 0$  if and only if

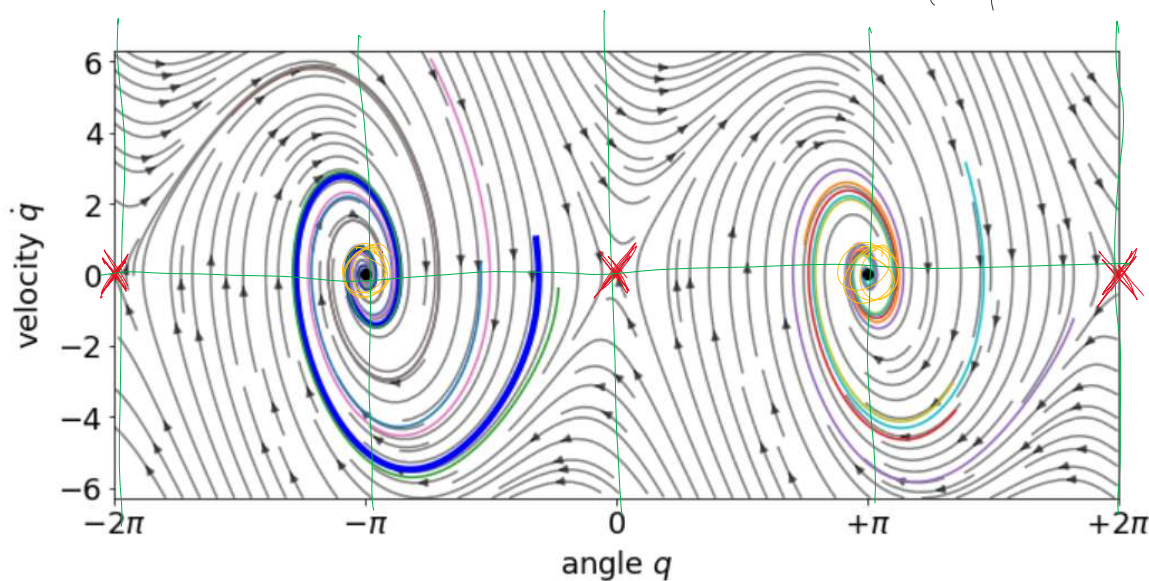
$$\dot{\mathbf{x}} = 0 \Leftrightarrow \dot{\theta} = 0 \text{ and } \ddot{\theta} = 0, \text{ i.e. } \frac{g}{l} \sin \theta = -\frac{1}{ml} u_e \cos \theta$$

$$\Leftrightarrow \dot{\theta}_e = 0 \text{ and } \tan \theta_e = \frac{-u_e}{mg}$$

\* in particular, when  $u_e = 0$ ,

$$\dot{\mathbf{x}} = 0 \Leftrightarrow \dot{\theta}_e = 0, \theta_e = k\pi,$$

$$k \in \{\dots, -2, -1, 0, +1, +2, \dots\}$$



○ - stable

✗ - unstable

• in a nonlinear system  $\ddot{\mathbf{x}} = f(\mathbf{x}, u)$ ,

$(x_e, u_e)$  s.t.  $f(x_e, u_e) = 0$  are termed equilibria

takeaways: 1°.  $x(0) = x_e, u(t) = u_e \Rightarrow x(t) = x_e$

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ie equilibrium point  $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$

determines equilibrium trajectory  $x: \mathbb{R} \rightarrow \mathbb{R}^n$   
 $: t \mapsto x(t) = x_e$

2°. we'll derive techniques to assess stability of

$x_e$ , that is, whether trajectories  
converge to  $x_e$  —  $x_e$  is stable

or diverge from  $x_e$  —  $x_e$  is unstable

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(b) characteristic polynomial [lec 01 b & c] [AMv2 ch2] [Nv7 ch 2,3,4]

• recall that for a linear system in DE & TF form:

differential equation (DE) & transfer function (TF)

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y$$

$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u + \dots + b_n u$$

$$\frac{b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = G(s)$$

• input  $u(t) = e^{st}$  yields output  $y(t) = \sum_{k=1}^n C_k e^{s_k t} + G(s) e^{st}$

where  $\{s_k\}_{k=1}^n$  are the roots of characteristic polynomial

$$\subset \mathbb{C}$$

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

ie  $\{s_k\}_{k=1}^n = \{s \in \mathbb{C} : a(s) = 0\} \leftarrow$  all and only the  
roots of  $a(s)$

ie  $\{s_k\}_{k=1}^n = \{s \in \mathbb{C} : a(s)=0\}$   $\leftarrow$  all and only the complex numbers  $s \in \mathbb{C}$  that make  $a(s)=0$

• for  $(y_e, u_e)$  to be an equilibrium,  $u(t) = u_e = u_e \cdot e^{0 \cdot t}$

$$y(t) = y_e = G(0) \cdot u_e$$

• for  $(y_e, u_e)$  to be stable,  $e^{s_k t} \rightarrow 0$  as  $t \rightarrow \infty$

ie  $\operatorname{Re} s_k < 0$  for all  $k \in \{1, \dots, n\}$

ex: RLC circuit



$$(DE) \quad L\ddot{q} + R\dot{q} + \frac{1}{C}q = v$$

$$(TF) \quad G(s) = \frac{b(s)}{a(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

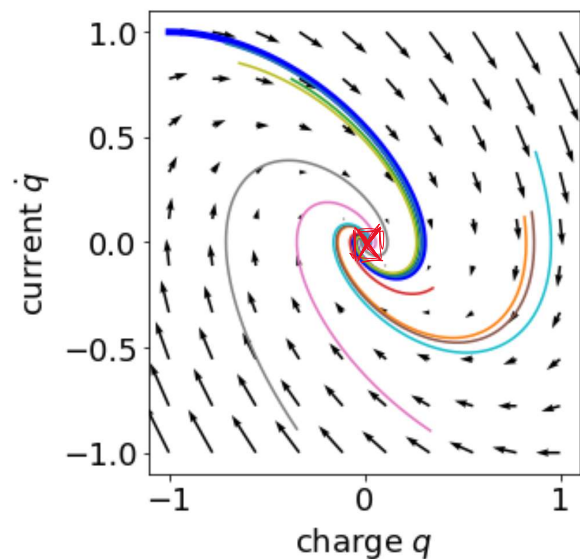
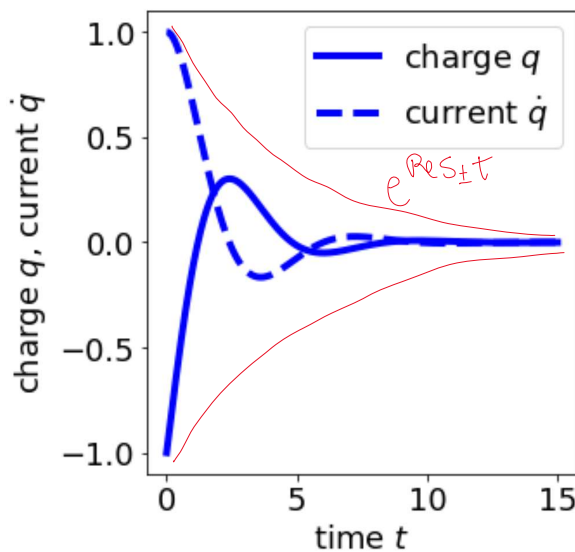
$\hookrightarrow$  characteristic polynomial

$$a(s) = Ls^2 + Rs + \frac{1}{C}$$

roots

$$s_{\pm} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

$\rightarrow$  assuming  $R, L, C > 0$ ,  $\operatorname{Re} s_{\pm} < 0$



(c) Routh - Hurwitz [AMv2 Ch 2.2] [Nv7 Ch 6.2]

• a linear time-invariant system with characteristic polynomial  $a(s)$  is stable if  $\text{Re } s_k < 0$  for all  $s_k \in \mathbb{C}$  s.t.  $a(s_k) = 0$

\* if  $a(s) = (s - s_1) \cdot (s - s_2) \cdot \dots \cdot (s - s_n) \leftarrow \text{factored form}$  then it's easy to verify  $\text{Re } s_k < 0$  for all  $k \in \{1, \dots, n\}$

\* if  $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$ ,

it's hard to determine roots  $\{s_k\}_{k=1}^n$ , so we'd like another way to assess stability

→ Routh (1831-1907) & Hurwitz (1859-1919)

provide necessary & sufficient criteria for stability using only the coefficients  $\{a_k\}_{k=1}^n$  (not  $\{s_k\}_{k=1}^n$ )

roots of  $a(s)$  have if and [some algebraic conditions]

roots of  $a(s)$  have negative real part if and only if  $\left[ \text{some algebraic conditions on } \{a_k\}_{k=1}^n \text{ are satisfied} \right]$

$\iff$

$s^2 + a_1 s + a_2$	$a_1, a_2 > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0, \quad a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0, \quad a_1 a_2 > a_3, \quad a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$
$\vdots$	$\vdots$

\* in this class, we'll only consider characteristic polynomials of degree 4 or fewer

## (d) eigenvalues

• consider a linear system in state-space form:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}$$

• an equilibrium  $(x_e, u_e)$  with  $u(t) = u_e$  satisfies

$$0 = \dot{x}_e = Ax_e + Bu_e = 0 \iff Ax_e = Bu_e = b_e \in \mathbb{R}^n$$

i.e. the set of equilibria is determined by a linear equation

• when  $u_e = 0$  then  $\underbrace{x_e = 0}_{\text{"origin"}}$  is always an equilibrium

• in either case, stability of an equilibrium  $(x_e, u_e)$  is determined by the set of eigenvalues of  $A$ :

$$\lambda(A) = \{s \in \mathbb{C} \mid \exists v \in \mathbb{C}^n: \underbrace{Av = sv}, v \neq 0\}$$

ie  $A$  simply "scales"  $v$  by  $s$

$$= \{s \in \mathbb{C} \mid \det(sI - A) = 0\}$$

$\hookrightarrow \det: \mathbb{C}^{n \times n} \rightarrow \mathbb{C}$  is the determinant

\* recall that  $\det(sI - A)$  is a polynomial in  $s$  w/ degree  $n$   
 $= a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = a(s)$

$\hookrightarrow$  termed the characteristic polynomial of  $A$

• to see how eigenvalues determine stability, consider diagonal  $A$ :

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \det(sI - A) = \det \begin{bmatrix} s - \lambda_1 & 0 \\ 0 & s - \lambda_2 \end{bmatrix}$$

\* recall:  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = (s - \lambda_1)(s - \lambda_2) = 0 \Leftrightarrow s \in \{\lambda_1, \lambda_2\}$

$\rightarrow$  what are eigenvectors associated with  $\lambda_1, \lambda_2$ ?

$\dot{x}_1 = \lambda_1 x_1 \leftarrow$  doesn't depend on  $x_2$

$\dot{x}_2 = \lambda_2 x_2 \leftarrow$  " "  $x_1$

so  $x_1(t) = e^{\lambda_1 t} x_1(0)$  } so  $x_1, x_2 \rightarrow 0$  (ie stable)  
 $\quad \quad \quad \uparrow$

$$\text{so } \begin{cases} x_1(t) = e^{\lambda_1 t} x_1(0) \\ x_2(t) = e^{\lambda_2 t} x_2(0) \end{cases} \quad \left\{ \begin{array}{l} \text{so } x_1, x_2 \rightarrow 0 \text{ (ie stable)} \\ \lambda_1, \lambda_2 < 0 \end{array} \right.$$

• another special case:  $\dot{x} = Ax = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ determine eigenvalues of  $A$

$$- \det(sI - A) = \det \begin{bmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{bmatrix} = (s - \sigma)^2 + \omega = 0$$

→  $\lambda_{\pm} = \sigma \pm j\omega$  are roots of characteristic polynomial

\* thus  $x_1(t) = e^{\sigma t} (x_1(0) \cos \omega t + x_2(0) \sin \omega t)$

$$x_2(t) = e^{\sigma t} (-x_1(0) \sin \omega t + x_2(0) \cos \omega t)$$

→ what condition(s) on  $\sigma$  and/or  $\omega$  ensure system is stable?

-  $\sigma < 0$  ie  $\text{Re } \lambda_{\pm} = \sigma < 0$

(e) parameter dependence

• models of process  $P$  and controller  $C$  have parameters

that can vary: 1°. control parameters like  $k_p, k_I$   
can be chosen by us

2°. design parameters  $R, L, C, m, J, k, \gamma$   
can be "chosen" by others / environment

• we can explicitly represent parameter dependence:



- we can explicitly represent parameter dependence:

$$\dot{x} = f(x, u; \mu) \leftarrow \text{semicolon (";")} \text{ indicates } \mu \text{ doesn't vary in time}$$

- \* since equilibrium  $(x_e, u_e)$  satisfies  $0 = \dot{x}_e = f(x_e, u_e; \mu)$ , the equilibrium generally varies with parameters:  $x_e(\mu)$

→ in linear systems,  $\dot{x} = A(\mu)x$ ,

equilibrium won't move:  $0 = A(\mu) \cdot 0 = 0 \rightarrow$  we're interested in  $x_e = 0$

- to assess how parameter variations affect stability,

visualize root locus diagram  $\lambda(A(\mu)) : \mathbb{R} \Rightarrow \mathbb{C}$   
of char. poly.  $\leftarrow$  plot, i.e. graph  $\rightarrow$  graph of eigenvalues assume  $\in \mathbb{R} : \mu \mapsto \{\lambda_k\}_{k=1}^n$

by evaluating / plotting eigenvalues of  $A(\mu)$  / roots of characteristic polynomial of  $A(\mu)$

ex: proportional-integral control of first-order system

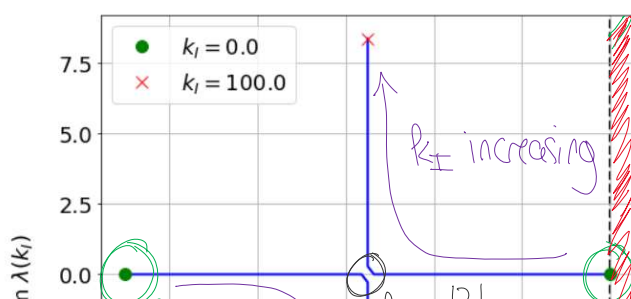
$$P(s) = \frac{b}{s+a} \quad C(s) = k_p + \frac{1}{s} k_I \quad G_{gv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a+bk_p)s + bk_I}$$

- choosing  $a=1$ ,  $b=1$ ,  $k_p=10$ , and varying  $k_I = \mu \in \mathbb{R}$

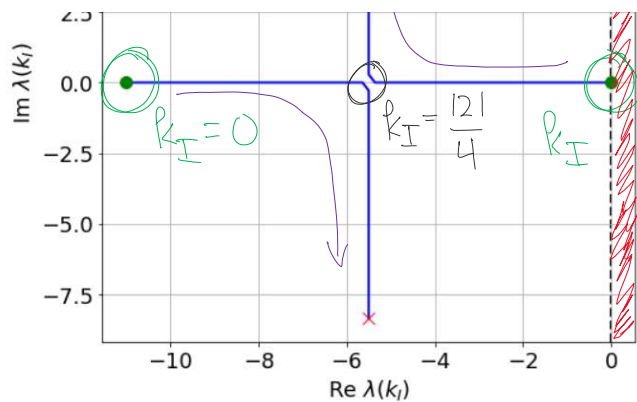
roots of char. poly. are

$$-\frac{11}{2} \pm \frac{1}{2} \sqrt{121 - 4k_I}$$

- plotting these in complex



- plotting these in complex plane as  $k_I$  varies yields root locus diagram:



- \* this analysis suggests  $k_I$  can be arbitrarily large, but that isn't physically realistic — as  $k_I$  increases, the frequency of oscillations (imaginary part of roots) increases, which can excite unmodeled dynamics

- include "unmodeled" dynamics in process with time constant  $T \ll \frac{1}{a}$  yields  $P(s) = \frac{b}{(s+a)(1+sT)}$

$$\text{so } G_{\text{sys}} = \frac{P}{1+PC} = \frac{bs}{Ts^3 + (1+aT)s^2 + (a+bk_p)s + bk_I}$$

- using parameters  $a=1$ ,  $b=1$ ,  $k_p=10$ ,  $T = \frac{1}{10}$

and plotting root locus diagram:

- \* importantly, two roots leave the left-half complex plane when  $k_I > 121.1$ , so the system is unstable when  $k_I$  is large

