HW6 due 5p Fri Nov 20

You are welcome (and encouraged) to work with others, but each individual must submit their own writeup.

You are welcome to use analytical and numerical computational tools; if you do, include the **commented** sourcecode in your submission (e.g. the .ipynb file).

You are welcome to consult websites, textbooks, and other materials; if you do, include a full citation in your writeup (e.g. the .ipynb file).

Important: before you do any work in the Colaboratory notebook, click "File -> Save a copy in Drive ..." and rename the file to something memorable.

0. [preferred name]; [preferred pronouns]

a. Approximately how many hours did you spend on this assignment?

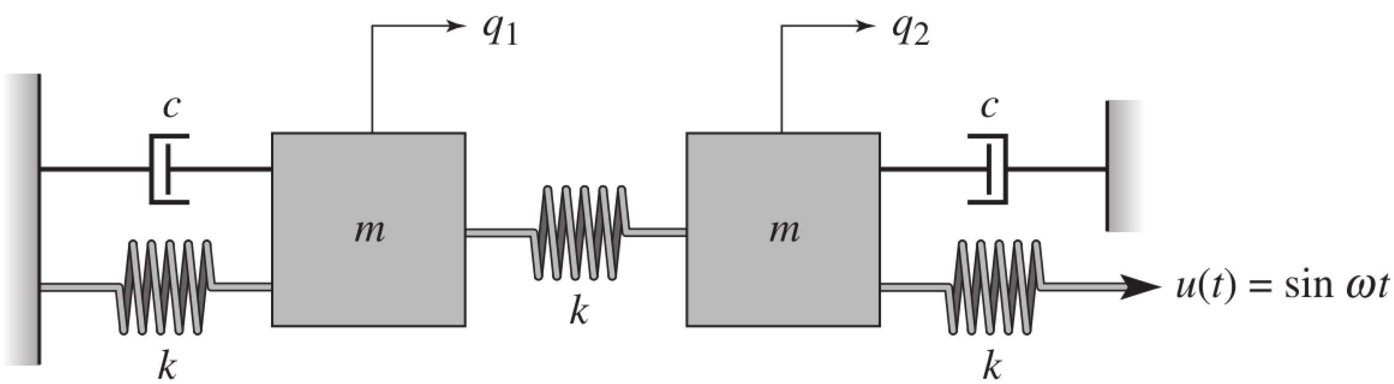
b. Were there specific problems that took much longer than others?

c. What class meeting(s) did you participate in this week?

d. What timezone(s) were you working in this week?

1. spring-mass-damper a deux redux

Consider the following spring-mass-damper system from hw3:



The input to this system is the sinusoidal motion of the end of rightmost spring. Applying Newton's laws to determine the forces acting on both masses, we find two coupled second-order DE that model the system's dynamics:

$$m\ddot{q}_1 = -c\dot{q}_1 - kq_1 + k(q_2 - q_1),$$
$$m\ddot{q}_2 = -c\dot{q}_2 + k(u - q_2) - k(q_2 - q_1).$$

Defining $x = (q_1, \dot{q}_1, q_2, \dot{q}_2)$, we can transcribe the two coupled second-order DE obtained above into one first-order DE in matrix/vector form $\dot{x} = Ax + Bu$:

$$\dot{x} = \begin{bmatrix} \dot{q}_1 \\ \ddot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{m} & -\frac{c}{m} & +\frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ +\frac{k}{m} & 0 & -\frac{2k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k}{m} \end{bmatrix} u = Ax + Bu.$$

Consider the alternative choice of state vector $z = (p_1, \dot{p}_1, p_2, \dot{p}_2)$ where $p_1 = \frac{1}{2}(q_1 + q_2)$, $p_2 = \frac{1}{2}(q_1 - q_2)$.

a. Determine the matrix $T \in \mathbb{R}^{4 \times 4}$ such that $z = Tx$.

b. Determine the matrix $T^{-1} \in \mathbb{R}^{4 \times 4}$ such that $x = T^{-1} z$. (***Hint:*** solve the equations $p_1 = \frac{1}{2}(q_1 + q_2), p_2 = \frac{1}{2}(q_1 - q_2)$ for q_1 and q_2 in terms of p_1 and p_2.)

c. Compute $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TB$ and verify that $\dot{z} = \widetilde{A}z + \widetilde{B}u$ by comparing with the solution to problem (1d.) from homework 3.

d. Verify that A has the same eigenvalues as $\widetilde{A}$ using parameters $m = 250$, $k = 50$, $c = 10$.

Takeaway: a linear system's dynamics can be represented using different choices of state variables (also termed ***coordinates***), which changes the "A" and "B" matrices. However, the eigenvalues of the A's -- hence, the system's stability properties -- aren't affected by the change-of-coordinates (also termed a ***similarity transform***).

2. state and output feedback

Purpose: apply analytical and computational techniques to observe and control the state of a system.

Consider the following model of a robot arm that consists of a rigid rod of length ℓ attached to a point mass m at one end and a rotational joint at the other end,

$$m\ell^2 \ddot{\theta} = mg\ell \sin\theta - \gamma\dot{\theta} + \tau,$$

where γ is a coefficient of friction for the rotational joint and τ is a torque applied by a motor attached to the rotational joint.

With state $x = (\theta, \dot{\theta})$ and input $u = \tau$, linearizing the dynamics around the vertical equilibrium $x_e = (\theta_e, \dot{\theta}_e) = (0, 0)$, $u_e = \tau_e = 0$ yields the linear control system

$$\dot{\delta x} = A\delta x + B\delta u$$

where $x \approx x_e + \delta x$, $u \approx u_e + \delta u$,

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{\gamma}{m\ell^2} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{m\ell^2} \end{bmatrix}.$$

a. Design a linear state-feedback law $\delta u = -K\delta x$ that gives a closed-loop characteristic polynomial $s^2 + 2\zeta\omega s + \omega^2$ that corresponds to a second-order system that is ***overdamped*** ($\zeta = 3/2$) and has ***natural frequency*** $\omega = 2$Hz.

b. Verify that the `ctrl.place` command gives the same answer as (a.) when you use parameter values $m = 1$ kg, $\ell = 1$ m, $g = 9.81$ m sec^{-2}, $\gamma = 1$.

c. Implement a simulation of the closed-loop system using parameter values from (b.) to verify that the closed-loop dynamics are as expected.

Now suppose you have to control the system using a time-of-flight sensor attached to the mass so that the output is

$$y = \ell \sin\theta.$$

d. Linearize the output equation at the vertical equilibrium, that is, evaluate

$$C = \left.\frac{\partial y}{\partial x}\right|_{x_e,u_e}, \; D = \left.\frac{\partial y}{\partial u}\right|_{x_e,u_e}$$

so that $y \approx \delta y = C\delta x + D\delta u$ near the equilibrium.

e. Design a linear observer

$$\dot{\delta\widehat{x}} = A\delta\widehat{x} + Bu - L(\delta y - \delta\hat{y}), \; \delta\hat{y} = C\delta\widehat{x} + Du$$

so that the error dynamics $\dot{e} = (A - LC)e$ has characteristic polynomial $s^2 + 2\xi\sigma s + \sigma^2$ that corresponds to a second-order system that is ***underdamped*** ($\xi = 1/2$) and has ***natural frequency*** $\sigma = 10$Hz

f. Verify that the `ctrl.place` command gives the same answer as (d.) when you use parameter values from (b.).

(***Note:*** when using `ctrl.place` to design an observer, you must transpose the A and B matrices and negate and transpose the result: `L = -ctrl.place(A.T,C.T,roots).T`.)

g. Implement a simulation of the observer system using parameter values from (b.) to verify that the error dynamics are as expected when the real system is at equilibrium (i.e. $\delta x = 0$, $\delta u = 0$).

h. Implement a simulation where you use the observer from (g.) to obtain the state estimate $\delta\widehat{x}$ that you then use in the controller from (c.) to simultaneously control the observer and system.

Bonus: Implement a simulation where you apply the observer from (g.) and controller from (c.) to the ***nonlinear*** system (nonlinear dynamics and nonlinear output equation).