ECE 447: Control Systems

goal: qualitative and quantitative tools to assess a system's stability

(a) equilibria $\leftarrow$ $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$ is an equilibrium for $\dot{x} = f(x,u)$ if $f(x_e, u_e) = 0 = $ "$\dot{x}_e$"

(b) characteristic polynomial $\leftarrow$ equilibrium stable $\Leftrightarrow$ $\mathrm{Re}\, s_k < 0$ for all $s_k \in \mathbb{C}$ s.t. $a(s_k) = 0$

$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$

(c) Routh-Hurwitz $\leftarrow$ equilibrium stable $\Leftrightarrow$ [inequalities satisfied on $a_1, a_2, \ldots, a_n$]

(d) eigenvalues $\leftarrow$ equilibrium stable $\Leftrightarrow$ $\mathrm{Re}\, \lambda_k < 0$ for all $\lambda_k \in \mathbb{C}$ s.t. $\det(\lambda_k \cdot I - A) = 0$

$\dot{x} = Ax + Bu$

(e) parameter dependence $\leftarrow$ variations in control or design parameters can cause instability — visualize w/ root locus diagram

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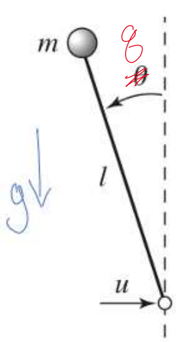
(a) equilibria [AMv2 Ch 5.3] [Nv7 Ch 2.7, 3.7]

ex: "rocket flight" (really: pendulum)



- state $x = (q, \dot{q})$ — angle, velocity
- input u — horizontal acceleration of pivot
- (DE) $m l^2 \ddot{q} = mgl \sin q - \alpha \dot{q} + l u \cos q$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q \end{bmatrix} = f(x, u)$$

- when $u(t) = u_e$ is constant: $f(x, u_e) = 0$ if and only if

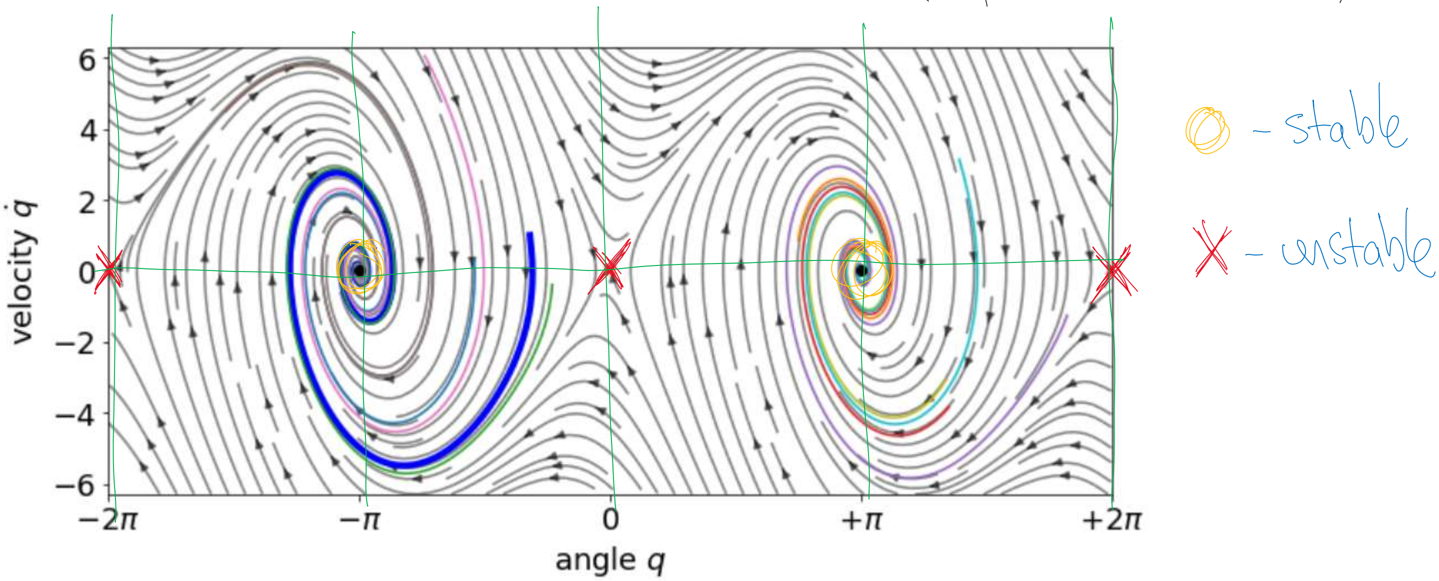
$$\dot{x} = 0 \iff \dot{q} = 0 \text{ and } \ddot{q} = 0, \text{ ie. } \frac{g}{\ell}\sin q = \frac{-1}{m\ell} u_e \cos q$$

$$\iff \dot{q}_e = 0 \text{ and } \tan q_e = \frac{-u_e}{mg}$$

* in particular, when $u_e = 0$,

$$\dot{x} = 0 \iff \dot{q}_e = 0,\ q_e = k\pi,$$

$$k \in \{\ldots, -2, -1, 0, +1, +2, \ldots\}$$



- in a nonlinear system $\dot{x} = f(x, u)$,

(x_e, u_e) s.t. $f(x_e, u_e) = 0$ are termed <u>equilibria</u>

takeaways: 1°. $x(0) = x_e,\ u(t) = u_e \Rightarrow x(t) = x_e$

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ie equilibrium point $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$ determines equilibrium trajectory $x : \mathbb{R} \to \mathbb{R}^n$
$: t \mapsto x(t) = x_e$

2°: we'll derive techniques to assess stability of x_e, that is, whether trajectories
converge to x_e — x_e is stable
or diverge from x_e — x_e is unstable

(b) characteristic polynomial [lec 01 b & c] [AMv2 Ch 2] [Nv7 Ch 2,3,4]

- recall that for a linear system in DE & TF form:

differential equation (DE) & transfer function (TF)

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = b_1 \frac{d^{n-1}}{dt^{n-1}} u + \cdots + b_n u$$

$$\frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = G(s)$$

- input $u(t) = e^{st}$ yields output $y(t) = \sum_{k=1}^{n} c_k e^{s_k t} + G(s) e^{st}$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the roots of characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

ie $\{s_k\}_{k=1}^n = \{ s \in \mathbb{C} : a(s) = 0 \}$ ← all and only the ie $\{s_k\}_{k=1} = \{ s \in \mathbb{C} : a(s) = 0 \}$ ← all and only the complex numbers $s \in \mathbb{C}$ that make $a(s) = 0$

- for (y_e, u_e) to be an equilibrium, $u(t) = u_e = u_e \cdot \cancel{e^{0 \cdot t}}^{\,1}$

$$y(t) = y_e = G(0) \cdot u_e$$

- for (y_e, u_e) to be stable, $e^{s_k t} \longrightarrow 0$ as $t \to \infty$

ie $\operatorname{Re} s_k < 0$ for all $k \in \{1, \dots, n\}$

ex: RLC circuit

(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = V$ (TF) $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{Ls^2 + Rs + \frac{1}{C}}$

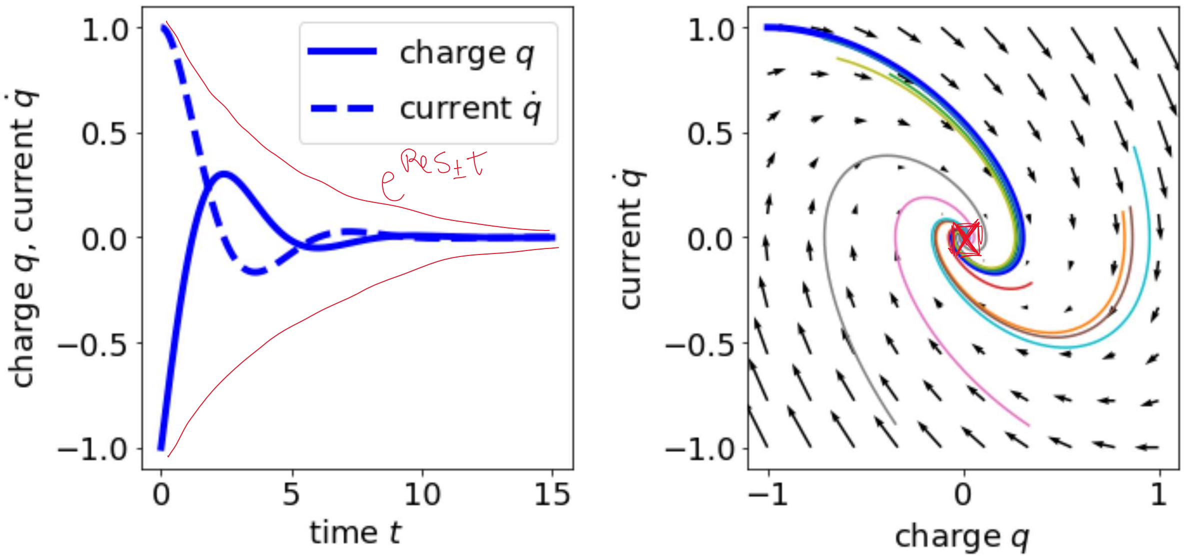
↳ characteristic polynomial

$$a(s) = Ls^2 + Rs + \frac{1}{C}$$

roots

$$s_{\pm} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$$

→ assuming $R, L, C > 0$, $\operatorname{Re} s_{\pm} < 0$



(c) Routh - Hurwitz [AMv2 Ch 2.2] [Nv7 Ch 6.2]

- a linear time-invariant system with characteristic polynomial $a(s)$ is stable if $\mathrm{Re}\, s_k < 0$ for all $s_k \in \mathbb{C}$ s.t. $a(s_k) = 0$

* if $a(s) = (s - s_1)\cdot(s - s_2)\cdot \cdots \cdot (s - s_n)$ ← factored form
then it's easy to verify $\mathrm{Re}\, s_k < 0$ for all $k \in \{1, \dots, n\}$

* if $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$,
it's hard to determine roots $\{s_k\}_{k=1}^n$, so we'd like another way to assess stability

→ Routh (1831 - 1907) & Hurwitz (1859 - 1919)
provide necessary & sufficient criteria for stability using only the coefficients $\{a_k\}_{k=1}^n$ (not $\{s_k\}_{k=1}^n$)

roots of $a(s)$ have if and [some algebraic conditions] roots of $a(s)$ have negative real part $\underset{\text{if and only if}}{\iff}$ [some algebraic conditions on $\{a_k\}_{k=1}^n$ are satisfied]

$s^2 + a_1 s + a_2$	$a_1, a_2 > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0$, $a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0$, $a_1 a_2 > a_3$, $a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$
$\vdots$	$\vdots$

* in this class, we'll only consider characteristic polynomials of degree 4 or fewer

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(d) eigenvalues [AMv2 Ch 5.3] [Nv7 Ch 6.5]

- consider a linear system in state-space form:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^p, \ A \in \mathbb{R}^{n\times n}, \ B \in \mathbb{R}^{n\times p}$$

- an equilibrium (x_e, u_e) with $u(t) = u_e$ satisfies

$$\text{"}\dot{x}_e\text{"} = Ax_e + Bu_e = 0 \iff Ax_e = Bu_e = b_e \in \mathbb{R}^n$$

i.e. the set of equilibria is determined by a linear equation

- when $u_e = 0$ then $\underbrace{x_e = 0}_{\text{"origin"}}$ is <u>always</u> an equilibrium

- in either case, stability of an equilibrium (x_e, u_e) is determined by the set of <u>eigenvalues</u> of A:

$$\lambda(A) = \{ s \in \mathbb{C} \mid \exists v \in \mathbb{C}^n : \underbrace{Av = sv}, \ v \neq 0 \}$$

ie A simply "scales" v by s

$$= \{ s \in \mathbb{C} \mid \underbrace{\det}(sI - A) = 0 \}$$

↳ $\det : \mathbb{C}^{n \times n} \to \mathbb{C}$ is the <u>determinant</u>

* recall that $\det(sI - A)$ is a polynomial in s w/ degree n

$$= a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = a(s)$$

↳ termed the <u>characteristic polynomial</u> of A

- to see how eigenvalues determine stability, consider diagonal A:

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \det(sI - A) = \det \begin{bmatrix} s - \lambda_1 & 0 \\ 0 & s - \lambda_2 \end{bmatrix}$$

* recall: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ $\quad = (s - \lambda_1)(s - \lambda_2) = 0 \Leftrightarrow s \in \{\lambda_1, \lambda_2\}$

→ what are eigenvectors associated with λ_1, λ_2?

$$\Leftrightarrow \quad \begin{aligned} \dot{x}_1 &= \lambda_1 x_1 \quad \leftarrow \text{doesn't depend on } x_2 \\ \dot{x}_2 &= \lambda_2 x_2 \quad \leftarrow \text{" " } x_1 \end{aligned}$$

so $x_1(t) = e^{\lambda_1 t} x_1(0)$ $\}$ so $x_1, x_2 \to 0$ (ie stable)

$\Uparrow$

so $\left.\begin{aligned} x_1(t) &= e \;\; x_1(0) \\ x_2(t) &= e^{\lambda_2 t} x_2(0) \end{aligned}\right\}$ so $x_1, x_2 \to 0$ (ie stable) $\Updownarrow$ $\lambda_1, \lambda_2 < 0$

∘ another special case: $\dot{x} = Ax = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ determine eigenvalues of A

– $\det(sI - A) = \det \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix} = (s-\sigma)^2 + \omega = 0$

⇝ $\lambda_\pm = \sigma \pm j\omega$ are roots of characteristic polynomial

∗ thus $x_1(t) = e^{\sigma t}(x_1(0)\cos\omega t + x_2(0)\sin\omega t)$

$x_2(t) = e^{\sigma t}(-x_1(0)\sin\omega t + x_2(0)\cos\omega t)$

→ what condition(s) on σ and/or ω ensure system is stable?

– $\sigma < 0$ ie $\mathrm{Re}\,\lambda_\pm = \sigma < 0$

(e) parameter dependence [AMv2 Ch 5.5] [Nv7 Ch 8]

∘ models of process P and controller C have parameters that can vary:

1°. control parameters like k_P, k_I can be chosen by us

2°. design parameters R, L, C, m, J, k, γ can be "chosen" by others / environment

∘ we can explicitly represent parameter dependence:

◦ we can explicitly represent parameter dependence:

$\dot{x} = f(x, u; \mu)$ ⟵ semicolon (";") indicates μ doesn't vary in time

* since equilibrium (x_e, u_e) satisfies $0 = "\dot{x}_e" = f(x_e, u_e; \mu)$, the equilibrium generally varies with parameters: $x_e(\mu)$

⟶ in linear systems, $\dot{x} = A(\mu)x$,

equilibrium won't move: $"\dot{0}" = A(\mu) \cdot 0 = 0$ ⟶ we're interested in $x_e = 0$

◦ to assess how parameter variations affect stability, visualize root locus diagram $\lambda(A(\mu)) : \mathbb{R} \rightrightarrows \mathbb{C}$

of char. poly. ↲ ↳ plot, i.e. graph; assume $\in \mathbb{R}$; $: \mu \mapsto \{\lambda_k\}_{k=1}^{n}$

graph of eigenvalues

by evaluating / plotting eigenvalues of $A(\mu)$ / roots of characteristic polynomial of $A(\mu)$

ex: proportional-integral control of first-order system

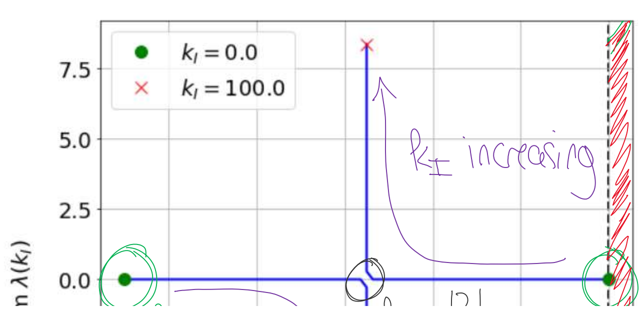
$$P(s) = \frac{b}{s+a} \qquad C(s) = k_P + \frac{1}{s} k_I \qquad G_{yv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a + bk_P)s + bk_I}$$

◦ choosing $a = 1$, $b = 1$, $k_P = 10$, and varying $k_I = \mu \in \mathbb{R}$

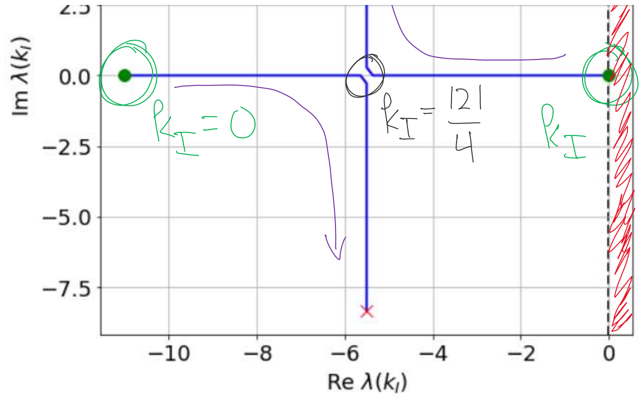
roots of char. poly. are

$$-\frac{11}{2} \pm \frac{1}{2}\sqrt{121 - 4k_I}$$

◦ plotting these in complex



◦ plotting these in complex plane as k_I varies yields root locus diagram:



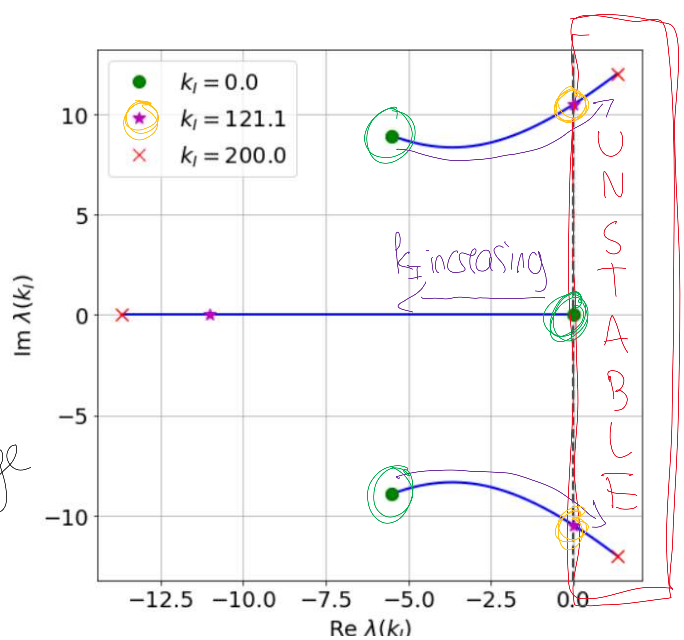
* this analysis suggests k_I can be arbitrarily large, but that isn't physically realistic — as k_I increases, the frequency of oscillations (imaginary part of roots) increases, which can excite unmodeled dynamics

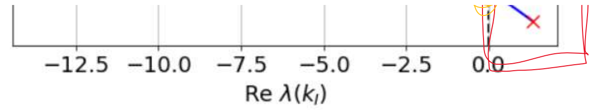
◦ include "unmodeled" dynamics in process with time constant $T \ll \frac{1}{a}$ yields $P(s) = \frac{b}{(s+a)(1+sT)}$

so $G_{yv} = \frac{P}{1+PC} = \frac{bs}{Ts^3 + (1+aT)s^2 + (a+bk_p)s + bk_I}$

◦ using parameters $a=1$, $b=1$, $k_p = 10$, $T = \frac{1}{10}$ and plotting root locus diagram:

* importantly, two roots leave the left-half complex plane when $k_I > 121.1$, so the system is unstable when k_I is large




−12.5 −10.0 −7.5 −5.0 −2.5 0.0
Re $\lambda(k_I)$