ECE 447: Control Systems

goal: frequency-domain analysis of control system performance

(a) transfer matrix (i.e. matrix of transfer functions) $\dot{x} = Ax + Bu$, $y = Cx + Du$ $\rightsquigarrow$ $G_{yu}(s) = C(sI - A)^{-1}B + D$

$u = e^{st} \cdot u_0 \Rightarrow y = e^{st} \cdot G_{yu}(s) \cdot u_0$

(b) Bode plots $u = e^{st} \cdot u_0 \rightsquigarrow y = e^{st} G_{yu}(s) u_0$, $\sin(\omega t) = \operatorname{Im} e^{j\omega t}$

so plotting $\log|G(j\omega)|$, $\angle G(j\omega)$ vs ω shows us how system responds

(c) effect of disturbances $y = \underbrace{\frac{1}{1+PC}}_{} w + \frac{P}{1+PC} v + \frac{PC}{1+PC} r$

so we can use sensitivity $S = \frac{1}{1+PC}$ to assess performance

(d) fundamental limits

(a) transfer matrix [AMv2 Ch 6.3, 9.2] [Nv7 Ch 4.11]

↳ i.e. a matrix of transfer functions relating a vector of inputs to a vector of outputs

- given LTI system in state-space form, $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^o$

know that $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

so $y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$

- let's see what output is produced by $u(t) = e^{st} \cdot u_0$, $s \in \mathbb{C}$, $u_0 \in \mathbb{R}^p$

$$y(t) = Ce^{At}x(0) + \underbrace{C\int_0^t e^{A(t-\tau)} B\left[e^{s\tau} \cdot u_0\right] d\tau}_{= C\left[\int_0^t \underbrace{e^{A(t-\tau)} e^{s\tau}}_{= e^{At} \cdot e^{-A\tau} \cdot e^{sI\tau}} d\tau\right] B u_0} + De^{st} \cdot u_0$$

$$\leftarrow e^{s\tau} \cdot I = e^{sI\tau}$$

$$y(t) = Ce^{At}x(0) + Ce^{At}\left[\underbrace{\int_0^t e^{(sI-A)\tau} d\tau}\right] \cdot B \cdot u_0 + De^{st} \cdot u_0$$

→ evaluate this integral — hint: find expression whose derivative is the integrand $e^{(sI-A)\tau}$

- recall that $\frac{d}{d\tau} e^{(sI-A)\tau} = (sI-A)\, e^{(sI-A)\tau}$

- so $\frac{d}{d\tau}\left[\underbrace{(sI-A)^{-1}} e^{(sI-A)\tau}\right] = e^{(sI-A)\tau}$ ☑

when is this matrix invertible?

$\iff \det(sI-A) \neq 0 \iff s \notin \lambda(A)$, i.e. s isn't an eigenvalue of A

- so $$\int_0^t e^{(sI-A)\tau} d\tau = \left[(sI-A)^{-1} e^{(sI-A)\tau}\right]_{\tau=0}^{\tau=t}$$

$$= (sI-A)^{-1}\left[\underbrace{e^{(sI-A)t}}_{= e^{-At} \cdot e^{sIt} = e^{-At} \cdot e^{st}} - \underbrace{e^{\cancel{0} \cdot I}}_{I}\right]$$

$= u(t)$

hence

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}Bu_0\right)}_{\text{transient response}} + \underbrace{\left[C(sI-A)^{-1}B+D\right]}_{\text{steady-state response}} e^{st} \cdot u_0$$

* assuming A stable: $\Rightarrow e^{At} \to 0$ as $t \to \infty$

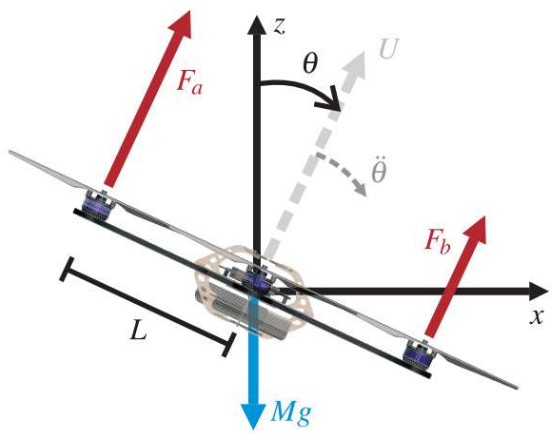
transient response $\to 0$ as $t \to \infty$

steady-state response $G_{yu}(s) \in \mathbb{R}^{o \times p}$

takeaway: given $\dot{x} = Ax + Bu$, $y = Cx + Du$ get $G_{yu}(s) = C(sI-A)^{-1}B + D$ transfer matrix (ie matrix of transfer functions)

ex: quadrotor

inputs $u = \begin{bmatrix} F_a \\ F_b \end{bmatrix} \in \mathbb{R}^2$ outputs: $\begin{bmatrix} x \\ z \\ \theta \end{bmatrix} = y \in \mathbb{R}^3$



$$G_{yu} = \begin{bmatrix} G_{xF_a} & G_{xF_b} \\ G_{zF_a} & G_{zF_b} \\ G_{\theta F_a} & G_{\theta F_b} \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

so if $u(t) = e^{st} \cdot u_0 \in \mathbb{R}^2$ then $y(t) = e^{st} \cdot G_{yu}(s) \cdot u_0 \in \mathbb{R}^3$

(b) Bode plots [AMv2 Ch 9.6] [Nv7 Ch 10.1]

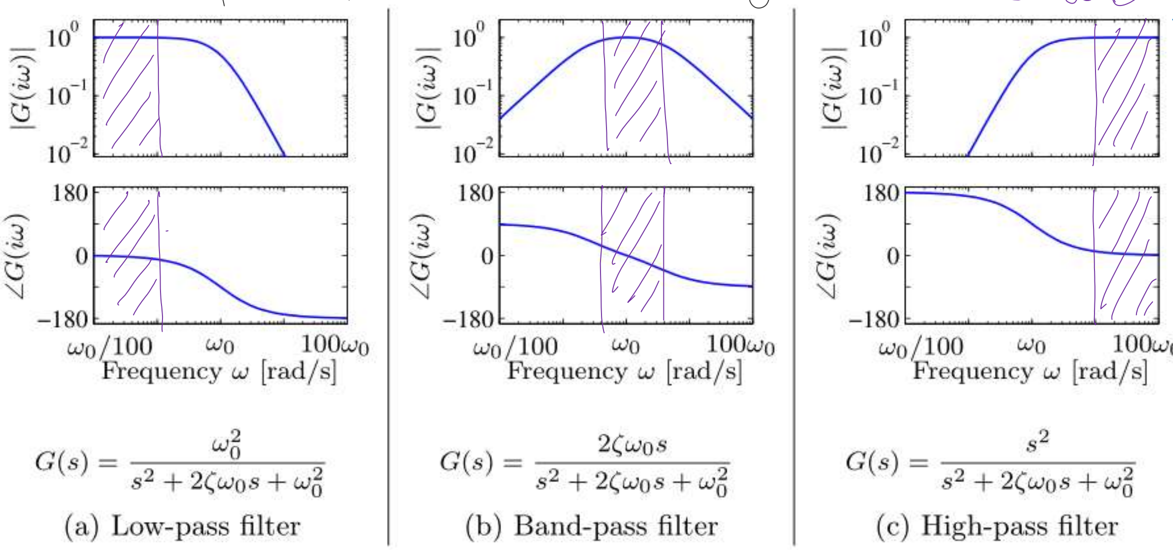
- since a complex exponential input $e^{st} \cdot u_0$ yields complex exponential output $e^{st} \cdot G_{yu}(s) \cdot u_0$

and $\sin(\omega t) = \operatorname{Im} e^{j\omega t}$, $\omega \in \mathbb{R}$

then evaluating / visualizing $G_{yu}(j\omega)$ tells us how system responds to any input $\in \mathbb{C}$

- a Bode plot is a visualization of $|G_{yu}(j\omega)|$, $\angle G_{yu}(j\omega)$ vs ω

ex: Bode plots for second-order systems "pass" band



we usually use a logarithmic axis for gain but not phase

← also log axis for frequency

Figure 9.17: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

* note: $$G(s) = \frac{b_1(s)\, b_2(s)}{a_1(s)\, a_2(s)}$$

$$\Rightarrow \log|G| = \log|b_1| + \log|b_2| - \log|a_1| - \log|a_2|$$

$$\Rightarrow \angle G = \angle b_1 + \angle b_2 - \angle a_1 - \angle a_2$$

ex: (spring-mass-damper / RLC circuit)

$$m\ddot{q} + c\dot{q} + kq = u \quad \text{or} \quad L\ddot{q} + R\dot{q} + \tfrac{1}{C}q = v$$

$$\rightsquigarrow G(s) = \frac{1}{ms^2 + cs + k} \quad \text{or} \quad = \frac{1}{Ls^2 + Rs + 1/C}$$

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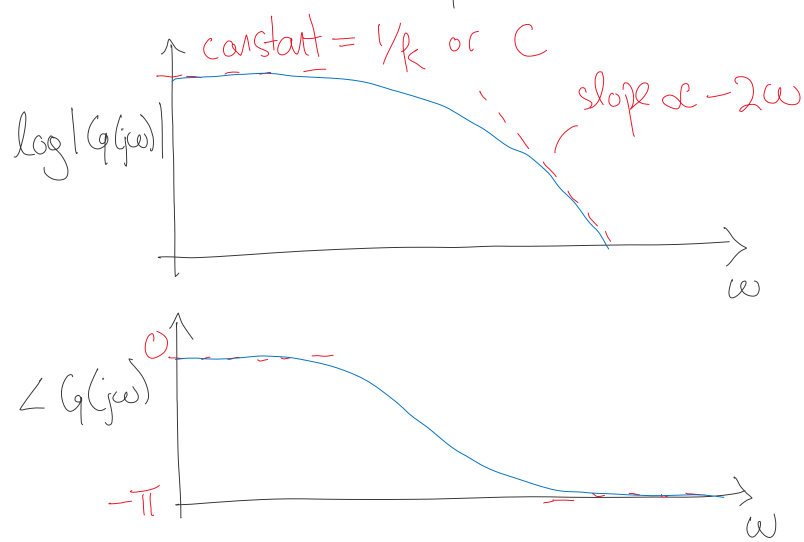
note: as $s \to 0$, $G(s) \to 1/k$ or C, i.e. $|G(s)| \to$ constant

$\angle G(s) \to 0$

as $s \to \infty$, $G(s) \to 1/ms^2$ or $1/Ls^2$, i.e. $\log|G(j\omega)| \propto -2\omega$

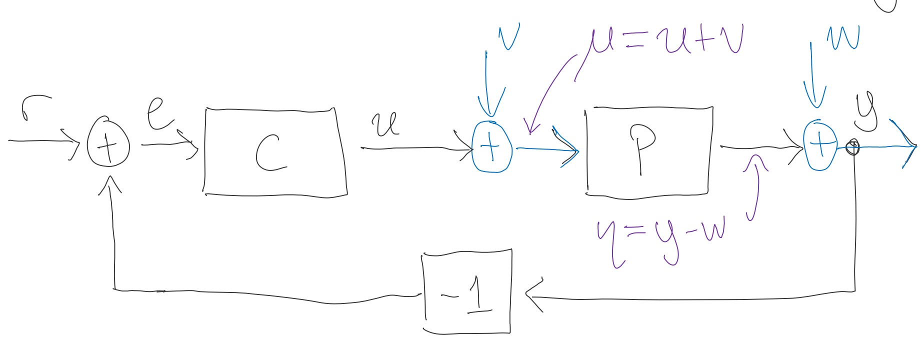
$\angle G(s) \to \pm\pi$

so we can sketch Bode plot:



(c) effect of disturbances [AMv2 Ch 12.1] [Nv7 Ch 7.5]

- consider negative feedback block diagram with disturbances:



r – reference; ideally $r \to y$

v – input disturbance, eg actuator noise,

w – output disturbance, eg sensor noise

* assuming input disturbance v and output disturbance w

* assuming <u>input disturbance</u> v and <u>output disturbance</u> w are independent of other signals in the diagram, they affect y linearly:

→ derive transfer functions G_{yr}, G_{yv}, G_{yw} so that $y = G_{yr} \cdot r + G_{yv} \cdot v + G_{yw} \cdot w$

– $y = w + \eta = w + Pu = w + P(v + u) = w + Pv + PCe$

$= w + Pv + PC(r - y)$

$\Longleftrightarrow \underbrace{(I + PC)}_{\text{* assuming invertible}} y = w + Pv + PCr$

$\Longleftrightarrow y = \underbrace{(I+PC)^{-1}}_{G_{yw}} w + \underbrace{(I+PC)^{-1}P}_{G_{yv}} v + \underbrace{(I+PC)^{-1}PC}_{G_{yr}} r$

in single-input/single-output case: $= \overbrace{\frac{1}{1+PC}}^{G_{yw}} w + \overbrace{\frac{P}{1+PC}}^{G_{yv}} v + \overbrace{\frac{PC}{1+PC}}^{G_{yr}} r$

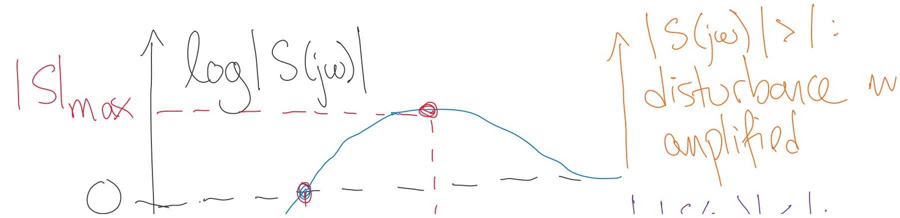
* the "ideal" transformations are different for each transformation:

– ideally $G_{yr} \simeq I$ so that $y = G_{yr} \cdot r \simeq r$

– ideally $G_{yv}, G_{yw} \simeq 0$ so that v & w don't affect y

(we'll see that we can't achieve all "ideals" at the same time)

∘ in terms of Bode plot:



let $S = G_{yw}$ be

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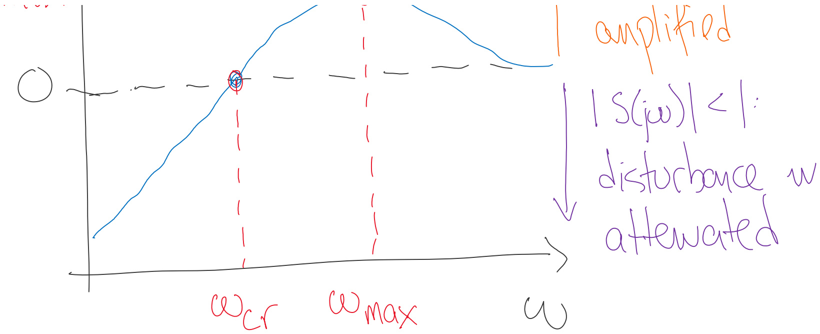
sensitivity of closed-loop system



- performance of closed-loop system can be assessed using peak sensitivity $|S|_{max}$ or crossover frequency ω_{cr}

(d) fundamental limits [AM v2 Ch 12.1, 14.2]