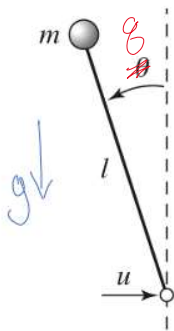


goal: qualitative and quantitative tools to assess a system's stability

- (a) equilibria $\leftarrow (x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$ is an equilibrium for $\dot{x} = f(x, u)$ if $f(x_e, u_e) = 0 = \dot{x}_e$
- (b) characteristic polynomial \leftarrow equilibrium stable $\Leftrightarrow \operatorname{Re} s_k < 0$
 $a(s) = s^n + a_1 s^{n-1} + \dots + a_n$ for all $s_k \in \mathbb{C}$ s.t. $a(s_k) = 0$
- (c) Routh-Hurwitz \leftarrow equilibrium stable \Leftrightarrow [inequalities satisfied on a_1, a_2, \dots, a_n]
- (d) eigenvalues \leftarrow equilibrium stable $\Leftrightarrow \operatorname{Re} \lambda_k < 0$ for all $\dot{x} = Ax + Bu$ $\lambda_k \in \mathbb{C}$ s.t. $\det(\lambda_k I - A) = 0$
- (e) parameter dependence \leftarrow variations in control or design parameters can cause instability — visualize w/ root locus diagram

(a) equilibria [AMv2 Ch 5.3] [Nv7 Ch 2.7, 3.7]

ex: "rocket flight" (really: pendulum)



- state $x = (\theta, \dot{\theta})$ — angle, velocity
- input u — horizontal acceleration of pivot
- (DE) $ml^2 \ddot{\theta} = mgl \sin \theta - \alpha \dot{\theta} + lu \cos \theta$

$$\ddot{\mathbf{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ g/l \sin \theta - \frac{\alpha}{ml^2} \dot{\theta} + \frac{1}{ml} u \cos \theta \end{bmatrix} = f(\mathbf{x}, u)$$

• when $u(t) = u_e$ is constant: $f(\mathbf{x}, u_e) = 0$ if and only if

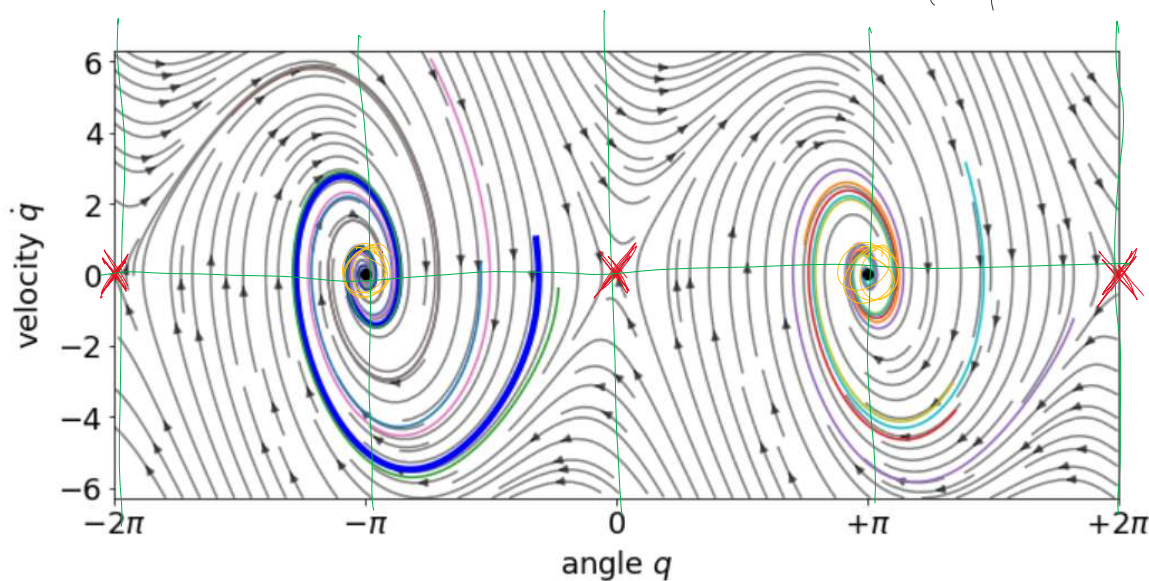
$$\dot{\mathbf{x}} = 0 \Leftrightarrow \dot{\theta} = 0 \text{ and } \ddot{\theta} = 0, \text{ i.e. } \frac{g}{l} \sin \theta = -\frac{1}{ml} u_e \cos \theta$$

$$\Leftrightarrow \dot{\theta}_e = 0 \text{ and } \tan \theta_e = \frac{-u_e}{mg}$$

* in particular, when $u_e = 0$,

$$\dot{\mathbf{x}} = 0 \Leftrightarrow \dot{\theta}_e = 0, \theta_e = k\pi,$$

$$k \in \{\dots, -2, -1, 0, +1, +2, \dots\}$$



○ - stable

✗ - unstable

• in a nonlinear system $\ddot{\mathbf{x}} = f(\mathbf{x}, u)$,

(x_e, u_e) s.t. $f(x_e, u_e) = 0$ are termed equilibria

takeaways: 1°. $x(0) = x_e, u(t) = u_e \Rightarrow x(t) = x_e$

takeaways: 1°. $x(0) = x_e, u(t) = u_e \Rightarrow x(t) = x_e$

ie equilibrium point $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$

determines equilibrium trajectory $x: \mathbb{R} \rightarrow \mathbb{R}^n$
 $: t \mapsto x(t) = x_e$

2°. we'll derive techniques to assess stability of

x_e , that is, whether trajectories

converge to x_e — x_e is stable

or diverge from x_e — x_e is unstable

(b) characteristic polynomial [lec 01 b & c] [AMv2 ch2] [Nv7 ch 2,3,4]

• recall that for a linear system in DE & TF form:

differential equation (DE) & transfer function (TF)

$$\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y$$

$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u + \dots + b_n u$$

$$\frac{b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n} = G(s)$$

• input $u(t) = e^{st}$ yields output $y(t) = \sum_{k=1}^n C_k e^{s_k t} + G(s) e^{st}$

where $\{s_k\}_{k=1}^n$ are the roots of characteristic polynomial

$$\subset \mathbb{C}$$

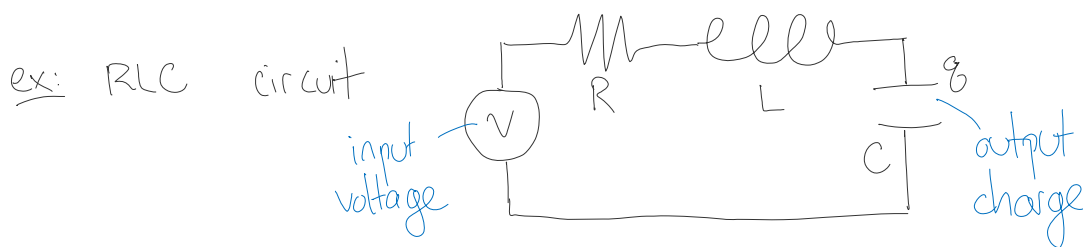
$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

ie $\{s_k\}_{k=1}^n = \{s \in \mathbb{C} : a(s) = 0\} \leftarrow$ all and only the
roots of $a(s)$

ie $\{s_k\}_{k=1}^n = \{s \in \mathbb{C} : a(s)=0\}$ \leftarrow all and only the complex numbers $s \in \mathbb{C}$ that make $a(s)=0$

• for (y_e, u_e) to be an equilibrium, $u(t) = u_e = u_e \cdot e^{0 \cdot t}$
 $y(t) = y_e = G(0) \cdot u_e$

• for (y_e, u_e) to be stable, $e^{s_k t} \rightarrow 0$ as $t \rightarrow \infty$
 ie $\operatorname{Re} s_k < 0$ for all $k \in \{1, \dots, n\}$



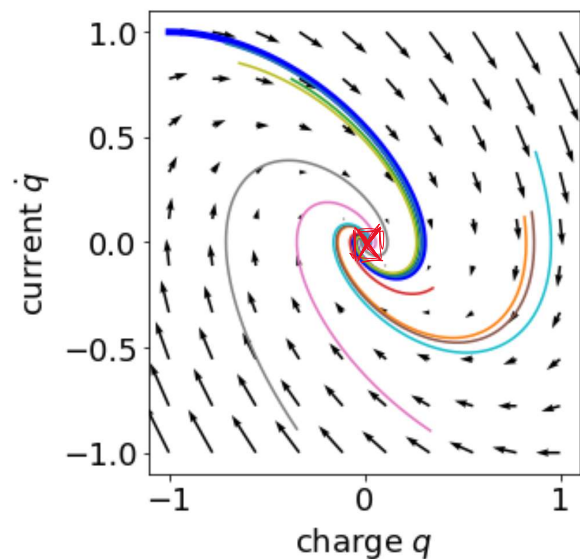
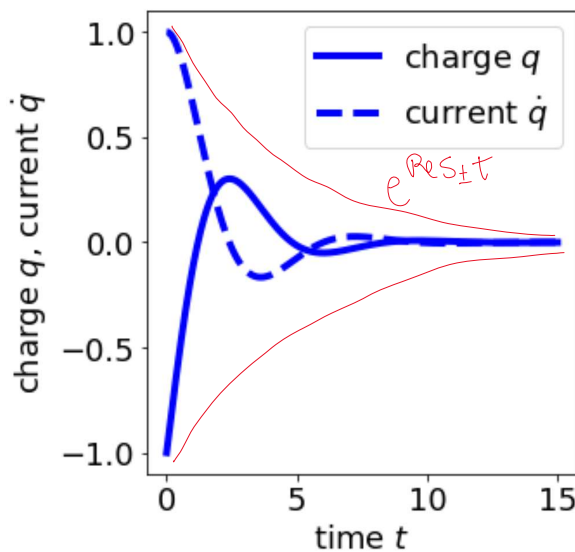
(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C}q = V$

(TF) $G(s) = \frac{b(s)}{a(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$

\hookrightarrow characteristic polynomial
 $a(s) = Ls^2 + Rs + \frac{1}{C}$

roots
 $s_{\pm} = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$

\rightarrow assuming $R, L, C > 0$, $\operatorname{Re} s_{\pm} < 0$



(c) Routh - Hurwitz [AMv2 Ch 2.2] [Nv7 Ch 6.2]

• a linear time-invariant system with characteristic polynomial $a(s)$ is stable if $\text{Re } s_k < 0$ for all $s_k \in \mathbb{C}$ s.t. $a(s_k) = 0$

* if $a(s) = (s - s_1) \cdot (s - s_2) \cdot \dots \cdot (s - s_n) \leftarrow$ factored form then it's easy to verify $\text{Re } s_k < 0$ for all $k \in \{1, \dots, n\}$

* if $a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$,

it's hard to determine roots $\{s_k\}_{k=1}^n$, so we'd like another way to assess stability

→ Routh (1831 - 1907) & Hurwitz (1859 - 1919)

provide necessary & sufficient criteria for stability using only the coefficients $\{a_k\}_{k=1}^n$ (not $\{s_k\}_{k=1}^n$)

roots of $a(s)$ have if and [some algebraic conditions]

roots of $a(s)$ have negative real part if and only if $\left[\text{some algebraic conditions on } \{a_k\}_{k=1}^n \text{ are satisfied} \right]$

\Longleftrightarrow

$s^2 + a_1 s + a_2$	$a_1, a_2 > 0$
$s^3 + a_1 s^2 + a_2 s + a_3$	$a_1, a_2, a_3 > 0, \quad a_1 a_2 > a_3$
$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$	$a_1, a_2, a_3, a_4 > 0, \quad a_1 a_2 > a_3, \quad a_1 a_2 a_3 > a_1^2 a_4 + a_3^2$
\vdots	\vdots

* in this class, we'll only consider characteristic polynomials of degree 4 or fewer

(d) eigenvalues [AMv2 Ch 5.3] [Nv7 Ch 6.5]

• consider a linear system in state-space form:

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^p, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}$$

• an equilibrium (x_e, u_e) with $u(t) = u_e$ satisfies

$$0 = \dot{x}_e = Ax_e + Bu_e = 0 \Leftrightarrow Ax_e = -Bu_e = b_e \in \mathbb{R}^n$$

i.e. the set of equilibria is determined by a linear equation

• when $u_e = 0$ then $\underbrace{x_e = 0}_{\text{"origin"}}$ is always an equilibrium

• in either case, stability of an equilibrium (x_e, u_e) is determined by the set of eigenvalues of A :

$$\lambda(A) = \{s \in \mathbb{C} \mid \exists v \in \mathbb{C}^n: \underbrace{Av = sv}, v \neq 0\}$$

ie A simply "scales" v by s

$$= \{s \in \mathbb{C} \mid \det(sI - A) = 0\}$$

$\hookrightarrow \det: \mathbb{C}^{n \times n} \rightarrow \mathbb{C}$ is the determinant

* recall that $\det(sI - A)$ is a polynomial in s w/ degree n
 $= a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = a(s)$

\hookrightarrow termed the characteristic polynomial of A

• to see how eigenvalues determine stability, consider diagonal A :

$$\dot{x} = Ax = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \det(sI - A) = \det \begin{bmatrix} s - \lambda_1 & 0 \\ 0 & s - \lambda_2 \end{bmatrix}$$

* recall: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = (s - \lambda_1)(s - \lambda_2) = 0 \Leftrightarrow s \in \{\lambda_1, \lambda_2\}$

\rightarrow what are eigenvectors associated with λ_1, λ_2 ?

$$\dot{x}_1 = \lambda_1 x_1 \leftarrow \text{doesn't depend on } x_2$$

$$\dot{x}_2 = \lambda_2 x_2 \leftarrow \text{" " } x_1$$

$$\text{so } \left. \begin{matrix} x_1(t) = e^{\lambda_1 t} x_1(0) \\ x_2(t) = e^{\lambda_2 t} x_2(0) \end{matrix} \right\} \text{ so } x_1, x_2 \rightarrow 0 \text{ (ie stable)}$$

$$\text{so } \begin{cases} x_1(t) = e^{\lambda_1 t} x_1(0) \\ x_2(t) = e^{\lambda_2 t} x_2(0) \end{cases} \quad \left\{ \begin{array}{l} \text{so } x_1, x_2 \rightarrow 0 \text{ (ie stable)} \\ \lambda_1, \lambda_2 < 0 \end{array} \right.$$

• another special case: $\dot{x} = Ax = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

→ determine eigenvalues of A

$$- \det(sI - A) = \det \begin{bmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{bmatrix} = (s - \sigma)^2 + \omega^2 = 0$$

lecture error

→ $\lambda_{\pm} = \sigma \pm j\omega$ are roots of characteristic polynomial

* thus $x_1(t) = e^{\sigma t} (x_1(0) \cos \omega t + x_2(0) \sin \omega t)$

$$x_2(t) = e^{\sigma t} (-x_1(0) \sin \omega t + x_2(0) \cos \omega t)$$

→ what condition(s) on σ and/or ω ensure system is stable?

- $\sigma < 0$ ie $\text{Re } \lambda_{\pm} = \sigma < 0$

(e) parameter dependence [AMv2 ch 5.5] [Nv7 ch 8]

• models of process P and controller C have parameters that can vary:

- 1°. control parameters like k_p, k_I can be chosen by us
- 2°. design parameters R, L, C, m, J, k, γ can be "chosen" by others / environment

• we can explicitly represent parameter dependence:

- we can explicitly represent parameter dependence:

$$\dot{x} = f(x, u; \mu) \leftarrow \text{semicolon (";")} \text{ indicates } \mu \text{ doesn't vary in time}$$

- * since equilibrium (x_e, u_e) satisfies $0 = \dot{x}_e = f(x_e, u_e; \mu)$, the equilibrium generally varies with parameters: $x_e(\mu)$

→ in linear systems, $\dot{x} = A(\mu)x$,

equilibrium won't move: $0 = A(\mu) \cdot 0 = 0 \rightarrow$ we're interested in $x_e = 0$

- to assess how parameter variations affect stability,

visualize root locus diagram $\lambda(A(\mu)) : \mathbb{R} \Rightarrow \mathbb{C}$
of char. poly. \leftarrow plot, i.e. graph \rightarrow graph of eigenvalues assume $\in \mathbb{R} : \mu \mapsto \{\lambda_k\}_{k=1}^n$

by evaluating / plotting eigenvalues of $A(\mu)$ / roots of characteristic polynomial of $A(\mu)$

ex: proportional-integral control of first-order system

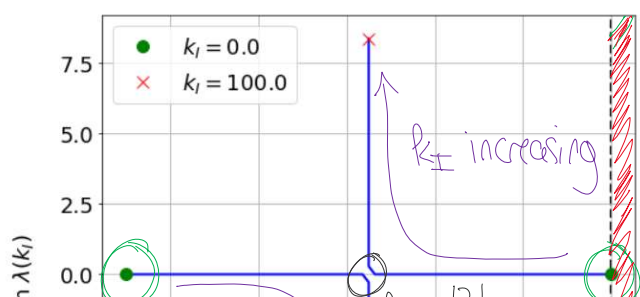
$$P(s) = \frac{b}{s+a} \quad C(s) = k_p + \frac{1}{s} k_I \quad G_{gv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a+bk_p)s + bk_I}$$

- choosing $a=1$, $b=1$, $k_p=10$, and varying $k_I = \mu \in \mathbb{R}$

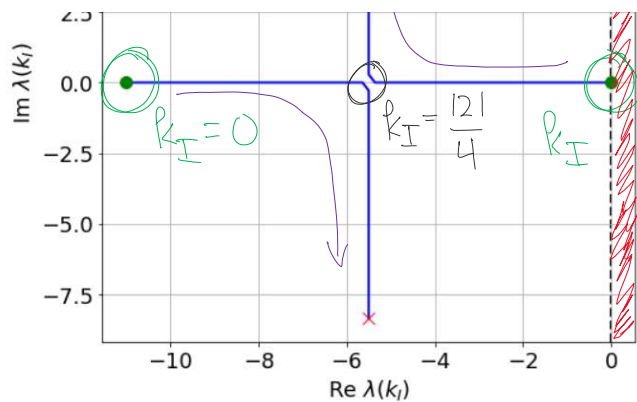
roots of char. poly. are

$$-\frac{11}{2} \pm \frac{1}{2} \sqrt{121 - 4k_I}$$

- plotting these in complex



- plotting these in complex plane as k_I varies yields root locus diagram:



* this analysis suggests k_I can be arbitrarily large, but that isn't physically realistic — as k_I increases, the frequency of oscillations (imaginary part of roots) increases, which can excite unmodeled dynamics

- include "unmodeled" dynamics in process with time constant $T \ll \frac{1}{a}$ yields $P(s) = \frac{b}{(s+a)(1+sT)}$

$$\text{so } G_{\text{yn}} = \frac{P}{1+PC} = \frac{bs}{Ts^3 + (1+aT)s^2 + (a+bk_p)s + bk_I}$$

- using parameters $a=1$, $b=1$, $k_p=10$, $T = \frac{1}{10}$

and plotting root locus diagram:

* importantly, two roots leave the left-half complex plane when $k_I > 121.1$, so the system is unstable when k_I is large

