# ECE 447: Control Systems

goal: frequency-domain controller synthesis

(a) Nyquist stability criterion if $L = PC$ has no poles in right-half $\mathbb{C}$:
then $\frac{L}{1+L} = \frac{PC}{1+PC}$ is stable $\iff$ $\Omega$ does not encircle $-1 \in \mathbb{C}$

(b) stability margins gain margin $g_m$: distance from $\Omega$ to $-1$ in $|L|$
phase margin $\varphi_m$: distance from $\Omega$ to $-1$ in $\angle L$

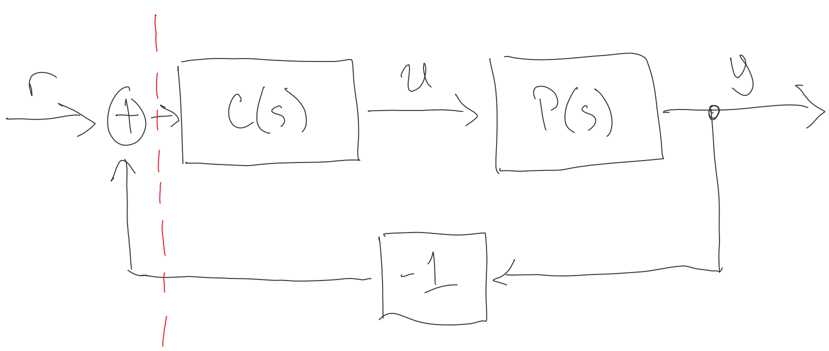
(c) root locus can predict effect of large and small proportional feedback gain using poles, zeros, and #poles − #zeros of process $P$

(d) proportional-integral-derivative (PID)

---

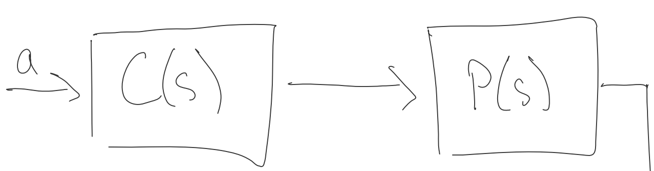
(a) Nyquist stability criterion [AMv2 Ch 10.1, 10.2] [Nv7 Ch 10.3]

- key idea: assess stability, robustness, & sensitivity of closed-loop systems by studying open-loop systems

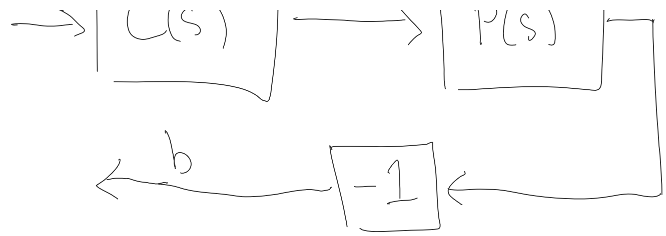



closed-loop system has transfer function

$$G_{yr} = \frac{PC}{1+PC} = \frac{L}{1+L}$$




open-loop system has transfer function




has transfer function $G_{ba} = -PC = -L$

- we'll consider 2 ways the open-loop transfer function tells us about stability of the closed-loop system:

  1°. algebraic observation  2°. thought experiment

1°. algebraic observation: what does $L(s) = P(s)C(s)$ say about

$$G_{yr}(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{L(s)}{1+L(s)} \;?$$

$\longrightarrow$ what happens if $\exists\, s^* \in \mathbb{C}$ s.t. $L(s) = P(s)C(s) = -1$ ?

– then as $s \to s^*$: $|G_{yr}(s)| = \left|\frac{P(s)C(s)}{1+P(s)C(s)}\right| \xrightarrow{s\to s^*} \left|\frac{-1}{1-1}\right| \to \infty$

* practically speaking: system response is <u>unbounded</u> (unstable) for inputs $\simeq e^{s^*t}$

- but practically speaking, we're only concerned with $s = j\omega$, so we're only worried if $\exists\, \omega^* \in \mathbb{R}$ s.t. $L(j\omega^*) = P(j\omega^*)C(j\omega^*) = -1$

2°. thought experiment

$$e^{st} \longrightarrow \boxed{L} \longrightarrow \boxed{-1} \longrightarrow -L(s)e^{st}$$

3. thought experiment $\quad e^{st} \longrightarrow \boxed{L} \longrightarrow \boxed{-1} \longrightarrow -L(s)e^{st}$

what happens when we close feedback loop?

→ what happens to $e^{st}$ if
(i) $|L(s)| < 1$ – attenuated, i.e. $\to 0$
(ii) $|L(s)| > 1$ – amplified, i.e. $\to \infty$
(iii) $|L(s)| = 1$ – sustained

when we close the loop?

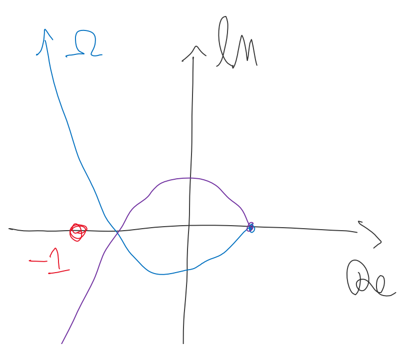
- conclude again that $L(s) = -1$, i.e. $|L(s)| = 1$, $\angle L(s) = \pi = 180°$ is a critical point for $L$ along imaginary axis

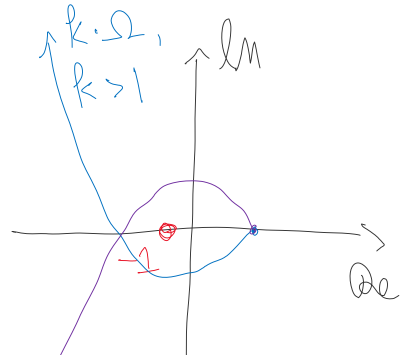
* it turns out that the $\boxed{\text{graph of } L(j\omega) \text{ – Nyquist plot} \quad \Omega = \{L(j\omega) \in \mathbb{C} : -\infty < \omega < +\infty\}}$

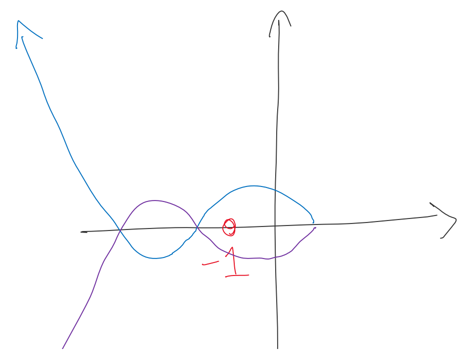
thm: (Nyquist stability criterion) ← application of argument principle

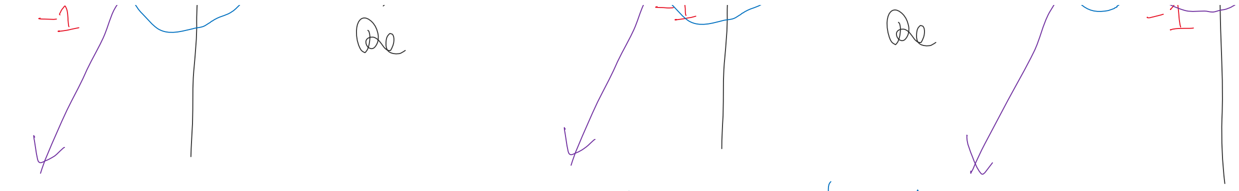
if $L$ has no poles in the right-half plane

then $\frac{L}{1+L} = \frac{PC}{1+PC}$ is stable $\iff$ $\Omega$ does not encircle $-1 \in \mathbb{C}$





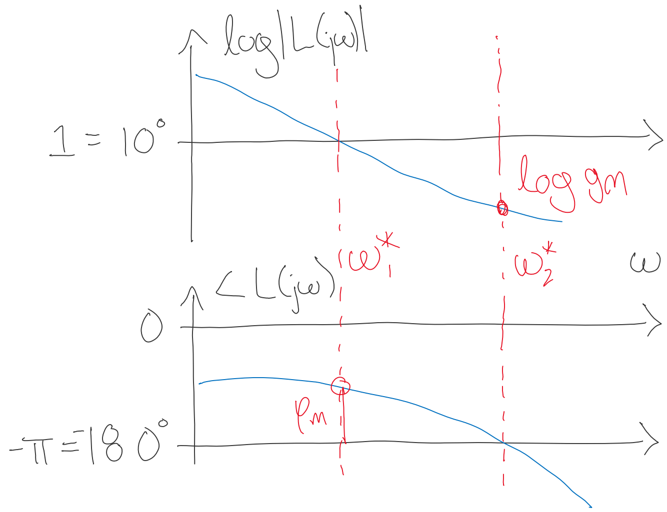

$\Omega$ does not encircle $-1$ $\qquad$ $\Omega$ does encircle $-1$ $\qquad$ $\Omega$ does not encircle $-1$

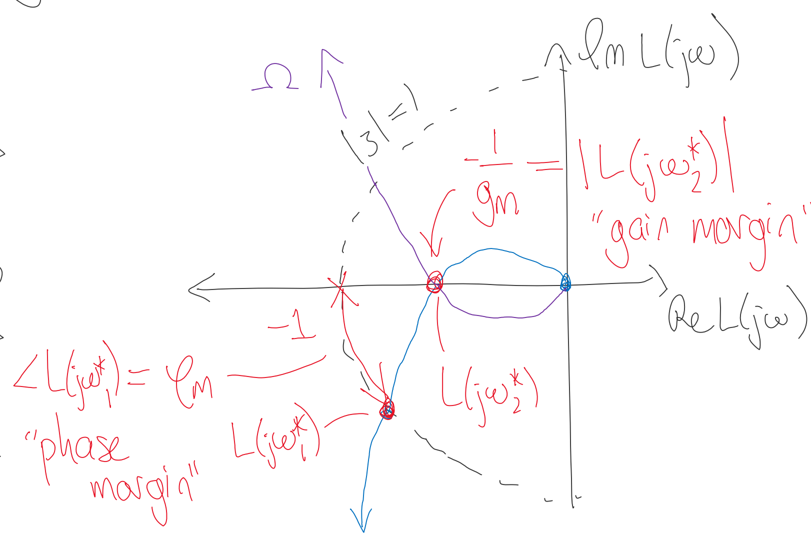
---

(b) stability margins [AMv2 Ch 10.3] [Nv7 Ch 10.7]

- given that a closed-loop system $\frac{PC}{1+PC}$ is stable, $L = PC$

  we can use Nyquist stability criterion to assess robustness:




$\longrightarrow$ use Bode plot of $L$ to sketch Nyquist plot

* what if we know $L = PC$ only approximately, i.e. $\tilde{L} = \tilde{P}\tilde{C} \simeq L$?

  e.g. if we have model uncertainty/inaccuracy in process $\tilde{P} \simeq P$

  e.g. if we have implementation error in controller $\tilde{C} \simeq C$

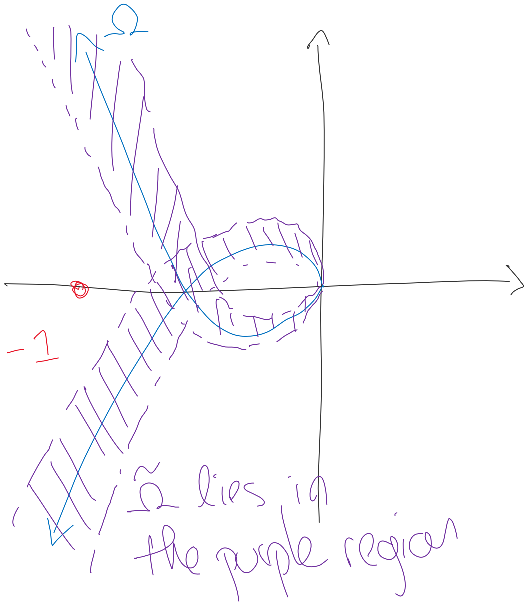
  from components, amplifiers, A2D having errors/tolerances

$\longrightarrow$ Nyquist stability criterion gives a robustness measurement:

  how far is $\Omega$ from $-1 \in \mathbb{C}$?

how far is $\Omega$ from $-1 \in \mathbb{C}$?

* if $\tilde{C} \simeq C$ and $\tilde{P} \simeq P$ then $\tilde{L} = PC \simeq \tilde{P}\tilde{C}$ so $\tilde{\Omega} \simeq \Omega$:




$\rightarrow$ so measuring distance from $\Omega \subset \mathbb{C}$ to $-1 \in \mathbb{C}$ gives a margin of stability:

$g_m$: distance from $\Omega$ to $-1$ if we only change $|L|$

$\varphi_m$: distance from $\Omega$ to $-1$ if we only change $\angle L$

---

(c) root locus [AMv2 Ch 12.5] [Nv7 Ch 9]

- consider a process $P(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$

  that we seek to control using proportional feedback: $C(s) = k > 0$

- then we know the closed-loop transfer function is

$$\frac{PC}{1+PC} = \frac{k\frac{b}{a}}{1 + k\frac{b}{a}} \cdot \frac{a}{a} = \frac{k\,b(s)}{a(s) + k \cdot b(s)}$$

$\rightarrow$ so the closed-loop characteristic polynomial is

$$\tilde{a}(s) = a(s) + k \cdot b(s)$$

$$\tilde{a}(s) = a(s) + k \cdot b(s)$$

* we'll analyze roots of $\tilde{a}$ in two regimes: large & small $k$

1°. small $k > 0$: as $k \to 0$, $\tilde{a} \to a$, so roots of $\tilde{a} \to$ roots of $a$

2°. large $k > 0$ and $s \in \mathbb{C}$: as $k, |s| \to \infty$,

$$\tilde{a}(s) = b(s) \cdot \left( \frac{a(s)}{b(s)} + k \right) \simeq b(s) \cdot \left( \frac{s^{n-m}}{b_0} + k \right)$$

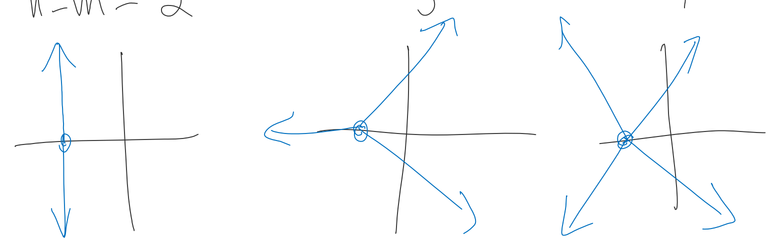
* assuming $n > m$, so $P$ is strictly proper, ie causal,

the roots of $\tilde{a}(s) \longrightarrow \begin{cases} \text{roots of } b(s) \\ \underline{\text{and}} \ \sqrt[n-m]{-b_0 k} \end{cases}$

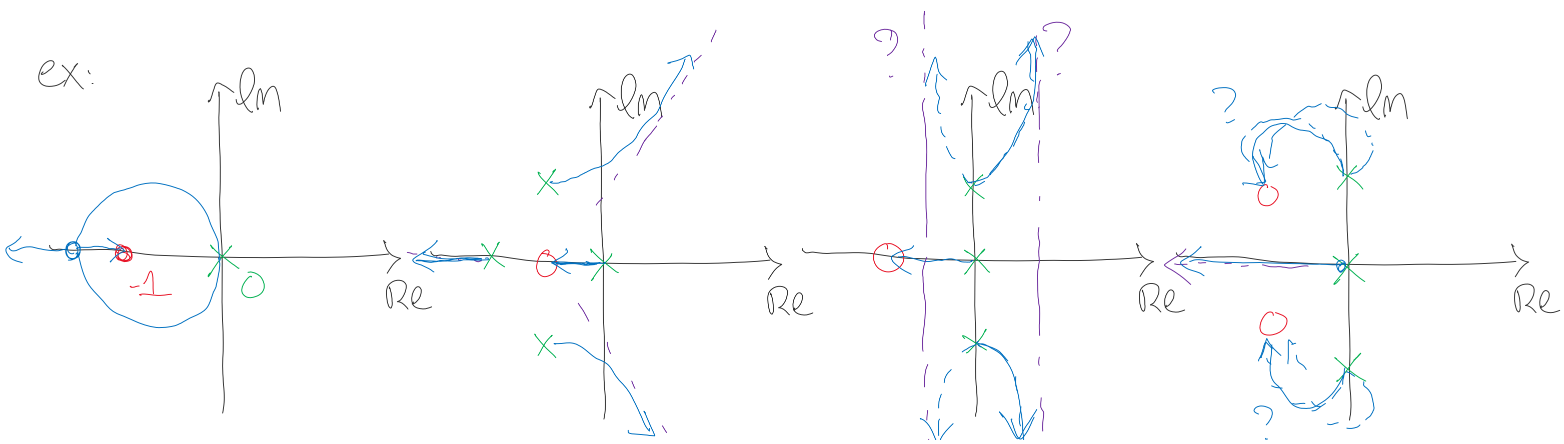
$\longrightarrow$ so as $k, |s| \to \infty$ the closed-loop poles converge to:

zeros of $P$ (i.e. roots of $b(s)$) or $(n-m)$-th "roots of unity"

$n - m = 2$      $= 3$      $= 4$



ex:



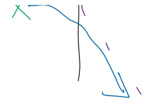
$P = \frac{s+1}{s^2}$

poles: $2 @ 0 \in \mathbb{C}$

zeros: $1 @ -1 \in \mathbb{C}$

$n-m: 2-1=1$

* know system stable for all $k>0$ large



$P = \frac{s+1}{s(s+2)(s^2+2s+4)}$

poles: $1 @ 0$
$1 @ -2$
$2 @ -1 \pm j$

zeros: $1 @ -1$

$n-m = 4-1 = 3$

* know system is unstable for $k>0$ too large

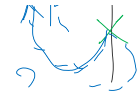
$P = \frac{s+1}{s(s^2+1)}$

poles: $1 @ 0$
$2 @ \pm j$

zeros: $1 @ -1$

$n-m = 3-1 = 2$

* system is unstable for all $k>0$



$P = \frac{s^2+2s+2}{s(s^2+1)}$

poles: $1 @ 0$
$2 @ \pm j$

zeros: $-1 \pm j\omega$

$n-m = 3-2 = 1$

* know $k>0$ large will stabilize system

---

(d) proportional-integral-derivative (PID) [AMv2 Ch 11]
[Nv7 Ch 9.4]