HW4 due 5p Fri Nov 13

You are welcome (and encouraged) to work with others, but each individual must submit their own writeup.

You are welcome to use analytical and numerical computational tools; if you do, include the **commented** sourcecode in your submission (e.g. the .ipynb file).

You are welcome to consult websites, textbooks, and other materials; if you do, include a full citation in your writeup (e.g. the .ipynb file).

Important: before you do any work in the Colaboratory notebook, click "File -> Save a copy in Drive ..." and rename the file to something memorable.

0. [preferred name]; [preferred pronouns]

a. Approximately how many hours did you spend on this assignment?

b. Were there specific problems that took much longer than others?

c. What class meeting(s) did you participate in this week?

d. What timezone(s) were you working in this week?

1. step response of a linear system

Consider the scalar DE

$$T\dot{y} + y = u.$$

Assume the input is constant ($u(t) = u_0$).

a. Express $y(t)$ in terms of t, T, $y(0)$, u_0.

b. Show that the *steady-state value* equals the constant input: $y_{ss} = u_0$.

Now assume the initial condition is zero, $y(0) = 0$; in this case the output is termed the *step response*.

c. Show that the *rise time* of the step response (i.e. the time required for the output to go from 10% to 90% of its steady-state value) is approximately $2T$.

d. Show that the *settling time* of the step response (i.e. the time required for the output to reach and stay within 2% of its steady-state value) is approximately $4T$.

Takeaway: *the approximations you established in (c.) and (d.) show what the* ***time constant***

2. matrix exponential

(a.) Show that matrix multiplication does not generally *commute*, i.e. find $A, B \in \mathbb{R}^{n\times n}$ for which $A\,B \neq B\,A$. (*Hint: you should be able to find an example when* $n = 2$.)

Now recall the definition of the *matrix exponential*:

$$\forall A \in \mathbb{R}^{n\times n} : e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

In the remainder of this problem, you'll use this definition directly to establish useful facts.

(b.) Show that every square matrix commutes with its matrix exponential, i.e. $A\,e^A = e^A\,A$ for all $A \in \mathbb{R}^{n\times n}$.

(c.) Given an invertible matrix $T \in \mathbb{R}^{n\times n}$, show that $e^{TAT^{-1}} = Te^A T^{-1}$.

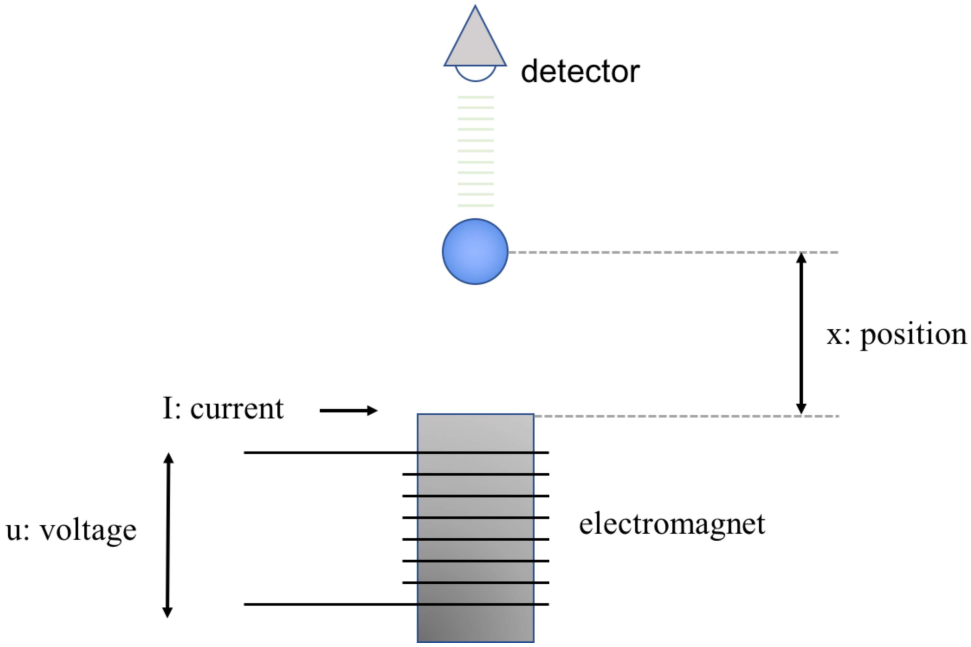
(d.) If $\lambda \in \mathbb{C}$ is an eigenvalue of A with eigenvector $v \in \mathbb{C}^n$, show that $e^{\lambda t}$ is an eigenvalue of e^{At}.

(e.) Suppose that A has n linearly independent eigenvectors so that the matrix $V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ is invertible. Using the preceding parts of the problem, show that $z = V^{-1}x$ yields $z(t) = \sum_{j=1}^n z_j(0)e^{\lambda_j t} v_j$ where $Av_j = \lambda_j v_j$.

Takeaway: these problems show why eigenvalues of A govern stability of $\dot{x} = Ax$ -- if we change coordinates by representing states x using a basis of eigenvectors $z = V^{-1}x$, we find that the dynamics in each eigendirection are determined by the corresponding eigenvalue.

3. linearization of nonlinear system

A steel ball with mass M is levitated under an electromagnet. The input u to the system is the voltage applied to the electromagnet. The output y is the position of the ball (illustrated as x in the diagram, but denoted as q in the equations below) and is measured with a photo-detector. The system is illustrated below.



This system involves dynamic interaction between the ball's position q, velocity $\dot{q}$, and the electromagnet current i:

$$(\mathrm{NL}) \quad \ddot{q} = \frac{C}{M}\left(\frac{i}{q}\right)^2 - g, \; \dot{i} = \frac{1}{L}\left(-Ri + 2C\frac{i\dot{q}}{q^2} + u\right)$$

(For convenience, the positive direction for q is down.)

(a.) Determine the constant input voltage u_e that makes $q_e = 0.05$ an equilibrium position.

(b.) Linearize the nonlinear system (NL) around the equilibrium from (a.) to obtain a linear system (L)

$$(\mathrm{L}) \quad \dot{x} = Ax + Bu, \; y = Cx + Du.$$

(c.) Is the linear system (L) from (b.) stable or unstable?

Suppose proportional feedback $u = -k_P(q - q_e)$ ***is applied to the system.***

(d.) Create a root locus plot for the closed-loop system as parameter k_P varies.

(e.) Can the system be stabilized by proportional feedback?