ECE 447: Control Systems

goal: synthesize stabilizing controllers for state-space systems

(a) state feedback

process model

$$\boxed{\dot{x} = Ax + Bu}$$

u x

$$\boxed{u = -Kx}$$

controller

choose K s.t. $\dot{x} = (A - BK)x$ stable, i.e. $\operatorname{Re} \lambda(A - BK) < 0$

(b) state estimation

process model

$$\xrightarrow{u} \boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}} \xrightarrow{y}$$

$$\xrightarrow{u} \boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \end{aligned}}$$

estimator (observer)

choose L s.t. $\operatorname{Re} \lambda(A + LC) < 0$, i.e. error dynamics $(x - \hat{x})^{\circ} = (A + LC)(x - \hat{x})$ are stable, so $\hat{x} \to x$

(c) stabilizing controller = (b) state estimation + (a) state feedback

controller

$$\boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \\ u &= -K\hat{x} \end{aligned}}$$

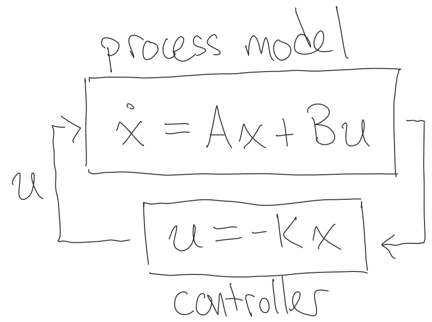
u y

process

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

assume A, B, C, D are known, but x is <u>not</u>

(a) state feedback [AMv2 Ch 7] [Nv7 Ch 12.2]



- assume given: process model $\dot{x} = Ax + Bu$

→ for now, we'll assume we (i.e. our controller) gets to see the entire state vector $x \in \mathbb{R}^n$

* measuring all voltages / currents in a circuit
positions / velocities in mechanical sys

goal: determine u given x so that $x \to 0$ (i.e. closed-loop system stable)

* if we choose u as a linear function of x, $u = -Kx$, $K \in \mathbb{R}^{p \times n}$

then the closed-loop system is linear: $\dot{x} = Ax + Bu$
$= Ax - BKx = (A - BK)x$

→ we know how to assess stability:

closed-loop system $\dot{x} = (A - BK)x$ is stable

$x \to 0 \iff \mathrm{Re}\,\lambda(A - BK) < 0$

all eigenvalues of $A - BK$ have negative real part

* our general approach: pole placement / eigenvalue assignment

- if we want the eigenvalues to be $\lambda(A - BK) = \{\lambda_j\}_{j=1}^{n} \subset \mathbb{C}$,
we just need to ensure characteristic polynomial

$$\det(sI - (A - BK)) = (s - \lambda_1)\cdot(s - \lambda_2)\cdots(s - \lambda_n) = \prod_{j=1}^{n}(s - \lambda_j)$$

1°. determine $\det(sI - (A - BK))$ — symbolically or numerically

1°. determine $\det(sI-(A-BK))$ — symbolically or numerically

$$= s^n + a_1(K)s^{n-1} + a_2(K)s^{n-2} + \cdots + a_n(K)$$

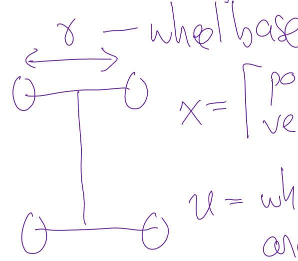
2°. expand $\prod_{j=1}^{n}(s-\lambda_j)$ — symbolically or numerically

$$= s^n + a_1^* s^{n-1} + a_2^* + \cdots + a_n^*$$

3°. solve $a_1(K)=a_1^*,\ a_2(K)=a_2^*, \ldots, a_n(K)=a_n^*$ for $K \in \mathbb{R}^{p\times n}$

— symbolically or numerically

ex: (symbolically) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ [AMv2 Ex 7.4]



$x = \begin{bmatrix} pos \\ vel \end{bmatrix}$

u = wheel angle

1°. determine $\det(sI-(A-BK))$, $K = [k_1 \ \ k_2]$

(recall: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$)

$\rightarrow \det(sI-(A-BK))$ — note: $BK = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}[k_1 \ \ k_2] = \begin{bmatrix} \gamma k_1 & \gamma k_2 \\ k_1 & k_2 \end{bmatrix}$

$$= \det\begin{bmatrix} s+\gamma k_1 & -1+\gamma k_1 \\ k_1 & s+k_2 \end{bmatrix} = (s+\gamma k_1)\cdot(s+k_2) - (\gamma k_1 - 1)\cdot k_1$$

$$= s^2 + (\gamma k_1 + k_2)s + k_1$$

2°. expand $\prod_{j=1}^{n}(s-\lambda_j)$, $\lambda_{\pm} = -\sigma \pm j\omega$, $\sigma > 0$

$$= (s-(-\sigma - j\omega))\cdot(s-(-\sigma+j\omega)) = s^2 + 2\sigma s + \sigma^2 + \omega^2$$

3°. $\longrightarrow$ solve $a(K) = a^*$ to determine k_1, k_2

* $s^2 + (\gamma k_1 + k_2)s + k_1 = s^2 + 2\sigma s + \sigma^2 + \omega^2$

$\iff k_1^* = \sigma^2 + \omega^2 \quad \gamma k_1^* + k_2^* = 2\sigma$, i.e. $k_2^* = 2\sigma - \gamma k_1^*$

$= 2\sigma - \gamma(\sigma^2 + \omega^2)$

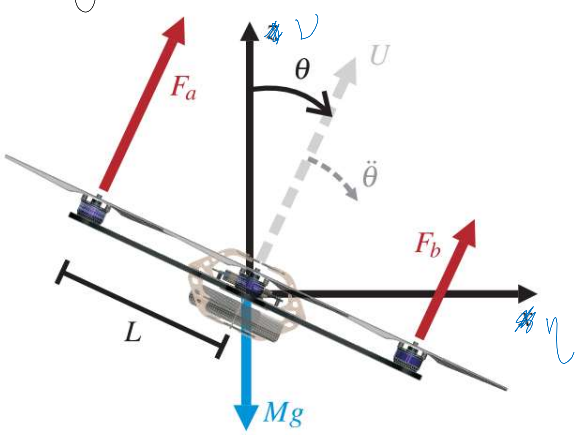
* $u = -Kx, \quad K = [k_1^* \;\; k_2^*] \Rightarrow \dot{x} = (A - BK)x$ is stable:

$\lambda(A - BK) = -\sigma \pm j\omega, \; \sigma > 0$

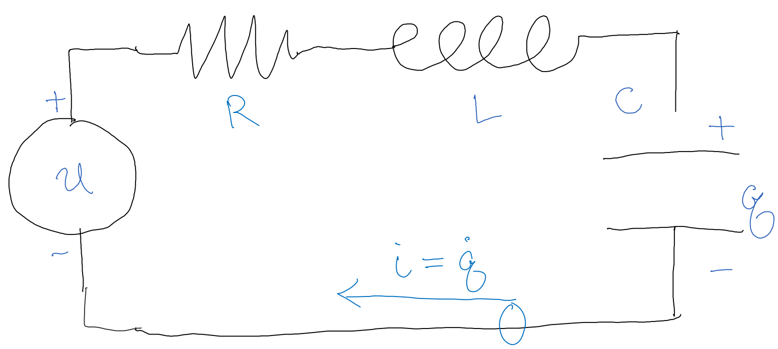
ex: (numerical) → see lecture examples notebook § 6

(b) state estimation [AMv2 Ch 8] [Nv7 Ch 12.5]

ex: quadrotor



ex: RLC circuit



→ what is the state $x \in \mathbb{R}^n$? → how would you measure each x_i?
(what is the sensor? how noisey is it? how expensive?)

	quadrotor		circuit	
state:	positions	velocities	voltages	currents
sensor:	GPS	gyrometer	voltmeter	ammeter

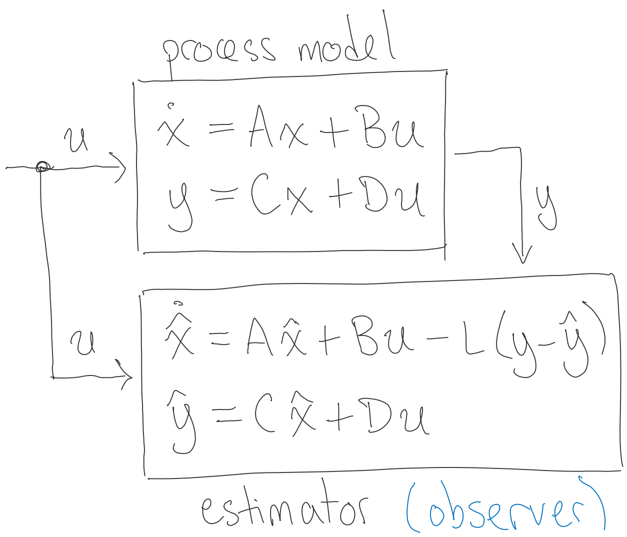
sensor: GPS, camera(s), LIDAR/RADAR, gyrometer, accelerometer | voltmeter, ammeter

--- E&M ---

* different states are harder/more expensive to measure

* it would be great if we could only measure a subset of states (e.g. positions, voltages) and estimate the rest (velocities, currents)

→ we will develop control system techniques for state estimation



- $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$

* $q < n$, i.e. we're measuring some (not all) of the states

ex: positions (not velocities)

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad y = q = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{} x + 0 \cdot u$$
$$= Cx + \cancel{D}^{\,0} u$$

- given process model $\dot{x} = Ax + Bu$, $y = Cx + Du$ (i.e. given A, B, C, D)

and assuming we know $u : [0,\infty) \to \mathbb{R}^p$, $y : [0,\infty) \to \mathbb{R}^q$
(b/c we choose u) (b/c we measure y)

we construct another LTI system called an <u>estimator</u> (or <u>observer</u>):

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y}) \qquad \text{where } L \in \mathbb{R}^{n \times q} \text{ is an}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\underbrace{y - \hat{y}}_{\text{output error}})$$
$$\hat{y} = C\hat{x} + Du$$

where $L \in \mathbb{R}^{n \times o}$ is an output error feedback matrix

→ to see why this works, determine the dynamics of $e = x - \hat{x}$ (your answer should be of the form $\dot{e} = M \cdot e$)

- $e = x - \hat{x} \Rightarrow \dot{e} = \dot{x} - \dot{\hat{x}}$

$$= (Ax + \cancel{Bu}) - (A\hat{x} + \cancel{Bu} - L(y - \hat{y}))$$

$$y - \hat{y} = (Cx + \cancel{Du}) - (C\hat{x} + \cancel{Du})$$

$$= Ax - A\hat{x} + L(Cx - C\hat{x})$$
$$= A\underbrace{(x - \hat{x})}_{=e} + LC\underbrace{(x - \hat{x})}_{=e} = (A + LC)e$$

* so if $L \in \mathbb{R}^{n \times o}$ is chosen such that $\operatorname{Re}\lambda(A + LC) < 0$ then $\dot{e} = (A + LC)e$ is <u>stable</u>! i.e. $\underline{e = x - \hat{x} \to 0}$!

<u>ex:</u> (vehicle steering) $\quad \dot{x} = Ax + Bu \quad y = Cx$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$$
$$C = [1 \; 0], \quad D = 0$$

→ determine state estimate error dynamics matrix $A + LC$ and the characteristic polynomial

- $L \in \mathbb{R}^{2 \times 1}$, $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, $\quad A + LC = \begin{bmatrix} l_1 & 1 \\ l_2 & 0 \end{bmatrix}$

$-\ \det(sI - (A+LC)) = s^2 - \ell_1 s - \ell_2$

- we want to choose L (i.e. ℓ_1, ℓ_2) s.t. $\mathrm{Re}\,\lambda(A+LC) < 0$

1°. characteristic polynomial $\det(sI - (A+LC)) = s^2 - \ell_1 s - \ell_2$

2°. want: $(s+\rho)^2 = s^2 + 2\rho s + \rho^2$, $\rho > 0$

3°. matching coefficients: $\ell_2 = -\rho^2$, $\ell_1 = -2\rho$

(c) stabilizing controller [AMv2 Ch 8] [Nv7 Ch 12.5]

controller

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y})$$
$$\hat{y} = C\hat{x} + Du$$
$$u = -K\hat{x}$$

$\xrightarrow{u}$ process; process $\xrightarrow{y}$ controller

process

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

* assume A, B, C, D given and K, L are chosen such that:

$$\mathrm{Re}\,\lambda(A - BK) < 0 \quad \text{and} \quad \mathrm{Re}\,\lambda(A + LC) < 0$$

$\rightarrow$ already saw that $e = x - \hat{x} \Rightarrow \dot{e} = (A+LC)e$ regardless of the input signal u

$\rightarrow$ determine dynamics of x when $u = -K\hat{x}$

→ determine dynamics of x when $u = -K\hat{x}$
(substitute to write $\dot{x}$ in terms of x & e, not $\hat{x}$)

- $\dot{x} = Ax + Bu = Ax - BK\hat{x}$
$= Ax - BK(x - e)$
$= (A - BK)x - BKe$

$e = x - \hat{x}$
so $\hat{x} = x - e$

∘ with $\bar{x} = \begin{bmatrix} x \\ e \end{bmatrix} \Rightarrow \dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \bar{A}\bar{x}$

* note: $\det(sI - \bar{A}) = \det(sI - (A - BK)) \cdot \det(sI - (A + LC))$

→ so $\mathrm{Re}\,\lambda(A - BK) < 0$ & $\mathrm{Re}\,\lambda(A + LC) < 0 \Rightarrow \mathrm{Re}\,\lambda(\bar{A}) < 0$

$\Rightarrow$ x & $e \to 0$! i.e. the combined state estimator & state feedback controller stabilizes both systems