ECE 447: Control Systems

goal: approximate nonlinear system behavior using linear systems

(a) linearization ← $\dot{x} = f(x,u) \simeq A \cdot \delta x + B \cdot \delta u$, $\delta x = x - x_e$

where $A = \partial_x f(x_e, u_e)$, $B = \partial_u f(x_e, u_e)$, $\delta u = u - u_e$

(b) matrix exponential ← $x(t) = e^{At} x(0)$ solves $\dot{x} = Ax$

(homogeneous response) to initial condition

where $e^X = I + X + \frac{1}{2} X^2 + \frac{1}{3} X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$

(c) convolution equation ← $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

(particular response) to control input

solves $\dot{x} = Ax + Bu$

- linear output $y = Cx + Du$ has step response $CA^{-1}e^{At}B - CA^{-1}B + D$

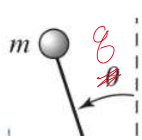
(a) linearization [AMv2 Ch 6.4] [Nv7 Ch 2.11]

Few physical elements display truly linear characteristics. For example the relation between force on a spring and displacement of the spring is always nonlinear to some degree. The relation between current through a resistor and voltage drop across it also deviates from a straight-line relation. However, if in each case the relation is reasonably linear, then it will be found that the system behavior will be very close to that obtained by assuming an ideal, linear physical element, and the analytical simplification is so enormous that we make linear assumptions wherever we can possibly do so in good conscience.

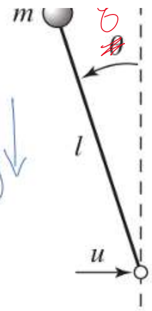
Robert H. Cannon, *Dynamics of Physical Systems*, 1967 [Can03].

→ so the important question is not "is my system linear?" but instead "is linearity a good approximation?"

ex: "rocket flight" (really: pendulum)



- state $x = (\vartheta, \dot{\vartheta})$ – angle, velocity
- input u – horizontal acceleration of pivot



- input u – horizontal acceleration of pivot
- (DE) $m l^2 \ddot{q} = m g l \sin q - \alpha \dot{q} + l u \cos q$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ g/l \sin q - \frac{\alpha}{m l^2} \dot{q} + \frac{1}{m l} u \cos q \end{bmatrix} = f(x, u)$$

- we previously determined that $u_e = 0$ has $x_e = \begin{bmatrix} k \cdot \pi \\ 0 \end{bmatrix} = \begin{bmatrix} q_e \\ \dot{q}_e \end{bmatrix}$, $k \in \mathbb{Z}$

as equilibria $\leadsto$ we will approximate f around (x_e, u_e) using Taylor series:

$\rightarrow$ compute first-order Taylor series of f wrt $\underset{(q,\dot{q})}{x}$ & u @ (x_e, u_e)

– $f((q,\dot{q}), u) = \begin{bmatrix} \dot{q} \\ \ddot{q}(q, \dot{q}, u) \end{bmatrix}$

– $\dot{q} = \dot{q}$ is the first-order Taylor series of $\dot{q}$ wrt $(q, \dot{q})$ & u

$$
\begin{aligned}
– \ddot{q}(q,\dot{q},u) = \; & \frac{g}{l} \sin q && \simeq \frac{g}{l} \cdot (q - \cancel{q_e}^{\,k\pi}) \\
& - \frac{\alpha}{m l^2} \dot{q} && \quad - \frac{\alpha}{m l^2} (\dot{q} - \cancel{\dot{q}_e}^{\,0}) \\
& + \frac{1}{m l} u \cos q && \quad + \frac{1}{m l} \cancel{\cos q_e}^{\,\pm 1} \cdot (u - \cancel{u_e}^{\,0}) \\
& && \quad \cancel{- \frac{1}{m l} u_e \cdot \sin q_e \cdot (q - q_e)} \rightarrow 0
\end{aligned}
$$

$\underline{\text{linear}} \longrightarrow \simeq \frac{g}{l} \cdot (q - k\pi) - \frac{\alpha}{m l^2} \cdot \dot{q} + \frac{1}{m l} \cdot u$

linear $\longrightarrow \simeq \frac{g}{\ell}\cdot(q - k\pi) - \frac{\alpha}{m\ell^2}\cdot\dot{q} + \frac{1}{m\ell}\cdot u$

- more generally, for nonlinear system (NL) $\dot{x} = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

with equilibrium $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$ s.t. "$\dot{x}_e$" $= f(x_e, u_e) = 0$

then
$$\dot{x} = f(x,u) \simeq \cancel{f(x_e,u_e)}^{\,0} + \boxed{\frac{\partial}{\partial x} f(x_e,u_e)(x - x_e) + \frac{\partial}{\partial u} f(x_e,u_e)\cdot(u - u_e)} + \underbrace{O(\|x - x_e\|^2) + O(\|u - u_e\|^2)}_{\text{"higher-order" terms}}$$

our "linearization"

where $\underbrace{\frac{\partial}{\partial x} f}_{} = \left[\frac{\partial}{\partial x_j} f_i\right]_{i,j} = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_n} f_1 \\ \partial_{x_1} f_2 & \cdots & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \partial_{x_1} f_n & \cdots & \cdots & \partial_{x_n} f_n \end{bmatrix}$

"Jacobian" derivative

ie an $n \times n$ matrix, $\in \mathbb{R}^{n\times n}$

$\underbrace{\frac{\partial}{\partial u} f}_{} = \left[\frac{\partial}{\partial u_\ell} f_i\right]_{i,\ell} = \begin{bmatrix} \partial_{u_1} f_1 & \partial_{u_2} f_1 & \cdots & \partial_{u_p} f_1 \\ \vdots & \ddots & & \vdots \\ & & \ddots & \vdots \\ \partial_{u_1} f_1 & \cdots & \cdots & \partial_{u_p} f_n \end{bmatrix}$

$\in \mathbb{R}^{n \times p}$

and $\frac{\partial}{\partial x} f(x_e,u_e) = \left.\frac{\partial}{\partial x} f\right|_{\substack{x = x_e \\ u = u_e}}$, $\frac{\partial}{\partial u} f(x_e,u_e) = \left.\frac{\partial}{\partial u} f\right|_{\substack{x = x_e \\ u = u_e}}$

* so if we let $\delta x = x - x_e$, $\delta u = u - u_e$

we have (L) $\delta\dot{x} \simeq A\cdot\delta x + B\cdot\delta u$, $A = \frac{\partial}{\partial x} f(x_e,u_e)$

$B = \frac{\partial}{\partial u} f(x_e,u_e)$

ie (NL) is approximately equal to (L)

i.e. (NL) is approximately equal to (L), $B = \frac{\partial}{\partial u} f(x_e, u_e)$

which is a linear time-invariant system!

(b) matrix exponential [AMv2 Ch 6.2] [Nv7 Ch 4.11 & Appendix I]

- recall the homogeneous solution to scalar (DE) $\dot{y} + ay = 0$ are, with $x = y$ be state of (DE), $x(t) = e^{-at} x(0)$

where $e : \mathbb{C} \to \mathbb{C}$ is defined by a power series

$$: z \mapsto 1 + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

and satisfies $\frac{d}{dt} e^{-at} = -a e^{at}$

($k! = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1$) is read as "k factorial"

* the power series converges for every $z \in \mathbb{C}$

$\rightarrow$ amazingly, it also makes sense for $X \in \mathbb{C}^{n \times n}$:

well-defined b/c matrix mult. is associative

$$e : \mathbb{C}^{n\times n} \to \mathbb{C}^{n \times n}$$

$$\exp = \; : X \to I + X + \frac{1}{2} \overbrace{X \cdot X}^{X^2} + \frac{1}{3!} \overbrace{X \cdot X \cdot X}^{X^3} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

* this power series converges for every $X \in \mathbb{C}^{n \times n}$ (!) $= e^X = \exp(X)$

$\rightarrow$ show that $\frac{d}{dt} e^{At} = A e^{At}$, $A \in \mathbb{C}^{n\times n}$, $t \in \mathbb{R}$

using definition of e^{At} as a power series:

- $\frac{d}{dt} e^{At} = \frac{d}{dt} \left(\sum_{k=0}^{\infty} \frac{1}{k!} (At)^k \right) = \frac{d}{dt} \left(I + At + \frac{1}{2} \underline{(At)} \cdot \underline{(At)} + \cdots \right)$

$-\ \frac{d}{dt} e^{At} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{1}{k!}(At)^k\right) = \frac{d}{dt}\left(I + At + \frac{1}{2}\underbrace{(At)\cdot(At)} + \cdots\right)$

$A\cdot t\cdot A\cdot t = A\cdot A\cdot t\cdot t = A^2 t^2$

$= 0 + A + A^2 t + \frac{1}{2}A^3 t^2 + \cdots$

$\frac{d}{dt}\left(\frac{1}{k!}(At)^k\right) = \frac{d}{dt}\left(\frac{1}{k!}A^k \cdot t^k\right)$

$= A\cdot\left(I + A + \frac{1}{2}A^2t^2 + \cdots\right)$

$= \frac{1}{(k-1)!}A^k t^{k-1}$

$= A\cdot\sum_{k=0}^{\infty}\frac{1}{k!}A^k t^k = A e^{At}$ ☑ !

$*$ we conclude that $x(t) = e^{At}\cdot x(0)$ solves $\dot{x} = Ax$, $x \in \mathbb{R}^n$!

$\rightarrow$ compare with $y(t) = \sum_{k=1}^{n} c_k e^{s_k t}$, $\{c_k\}_{k=1}^{n}$ are determined by the initial condition $\left\{\frac{d^k}{dt^k}y(0)\right\}_{k=0}^{n-1}$

- $x(t) = e^{At}x(0)$ solves $\dot{x} = Ax$

where $e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3}X^3 + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}X^k$

ex: $\ddot{q} = u$ (double integrator)

- with $x = \begin{bmatrix} q \\ \dot{q}\end{bmatrix}$ we have $\frac{d}{dt}x = \begin{bmatrix}\dot{q} \\ \ddot{q}\end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}}^{A}\overbrace{\begin{bmatrix} q \\ \dot{q}\end{bmatrix}}^{x} + \overbrace{\begin{bmatrix} 0 \\ 1\end{bmatrix}}^{B}\overset{u}{u}$

$= Ax + Bu$

- noting that $A\cdot A = A^2 = 0$ ($\Rightarrow A^k = A^{k-2}\cdot A^2 = 0,\ k > 2$)

so we compute $e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1\end{bmatrix}$

$\rightarrow$ homogeneous solution $\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} = x(t) = e^{At} x(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix}$

agrees with physical / signal intuition: in the absence of forcing, a mass moves at constant speed $\longrightarrow = \begin{bmatrix} q(0) + t \cdot \dot{q}(0) \\ \dot{q}(0) \end{bmatrix}$

ex: spring-mass w/ no damping: $\ddot{q} + \omega^2 q = u$

- with $x = \begin{bmatrix} q \\ \dot{q}/\omega \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q}/\omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q}/\omega \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$= Ax + Bu$

$\rightarrow$ verify $e^{At} = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$ via differentiation

- so $\begin{bmatrix} q(t) \\ \frac{\dot{q}(t)}{\omega} \end{bmatrix} = x(t) = e^{At} x(0) = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix} \begin{bmatrix} q(0) \\ \frac{\dot{q}(0)}{\omega} \end{bmatrix}$

if $\left(q(0), \frac{\dot{q}(0)}{\omega}\right) = (1, 0)$ then $\begin{bmatrix} q(t) \\ \frac{\dot{q}(t)}{\omega} \end{bmatrix} = \begin{bmatrix} \cos \omega t \\ -\sin \omega t \end{bmatrix}$

ex: $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \longleftarrow$ recall $\lambda(A) = \sigma \pm j\omega$

$\rightarrow$ verify $e^{At} = e^{\sigma t}\begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$ by differentiation

$$\Rightarrow \quad x(t) = e^{At}x(0) = e^{\sigma t} \cdot \begin{bmatrix} x_1(0)\cos\omega t + x_2(0)\sin\omega t \\ -x_1(0)\sin\omega t + x_2(0)\cos\omega t \end{bmatrix}$$

(c) convolution equation [AMv2 Ch 6.3] [Nv7 Ch 4.11 & App. I]

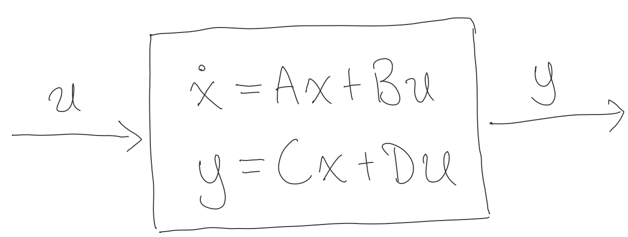
- consider the state-space LTI system $\dot{x} = Ax + Bu$

fact: given $x(0) \in \mathbb{R}^n$ and $u : [0,t] \to \mathbb{R}^p$,

$$x(t) = \underbrace{e^{At}x(0)}_{\text{homogeneous response to } x(0)} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{particular response to } u} \longleftarrow \text{convolution equation}$$

$\rightarrow$ verify this formula by differentiation
(recall Leibniz's formula for differentiating an integral)

- consider $\dot{x} = Ax + Bu$ with
output $y = Cx + Du$
$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^o$



so that $y(t) = Cx(t) + Du(t)$

$$= Ce^{At}x(0) + \int^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + \overbrace{Du(t)}^{\text{"feedthrough"}}$$

$$= \underbrace{Ce^{At}x(0)}_{\text{homogeneous output}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{particular output}}$$

- let's examine the response to unit step $\mu(\tau) = \begin{cases} 1, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}$ i.e. the <u>step response</u>, when $x(0) = 0$

$$\int_0^t Ce^{A(t-\tau)}B\mu(\tau)\,d\tau + D\mu(t) \quad (= 0 \text{ when } t<0, \text{ so assume } t \geq 0\text{:})$$

assuming $t \geq 0$: $\quad = C\int_0^t e^{A(t-\tau)}\,d\tau\, B + D$

assuming A invertible: $\quad = C\left[-A^{-1}e^{A(t-\tau)}\right]_{\tau=0}^{\tau=t} B + D$

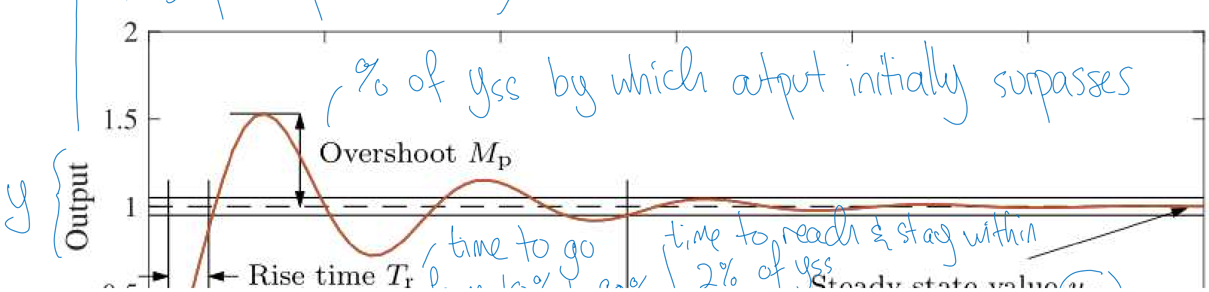
note: $e^0 = I + \cancel{0 + \frac{1}{2}0^2 \ldots}$ $\quad = C\left(-A^{-1}\cancel{e^{A\cdot 0}}^{I} + A^{-1}e^{At}\right)B + D$

$$= \underbrace{CA^{-1}e^{At}B}_{\substack{\text{transient} \\ \text{step response}}} - \underbrace{CA^{-1}B + D}_{\substack{\text{steady-state} \\ \text{step response}}}$$

- assuming A stable, i.e. all eigenvalues have negative real part:

$$\lim_{t\to\infty} e^{At} \to 0, \text{ so transient} \to 0 \text{ as } t \to \infty$$

→ step response to μ as above



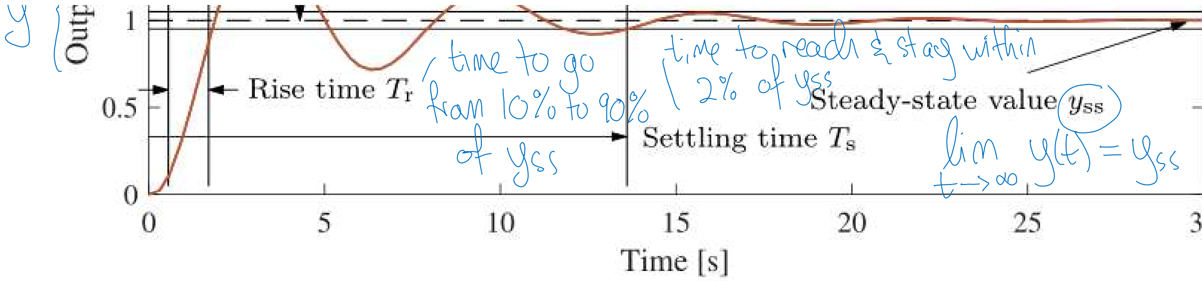


Figure 6.9: Sample step response. The rise time, overshoot, settling time, and steady-state value give the key performance properties of the signal.

note: M_p, T_r, T_s are independent of step size