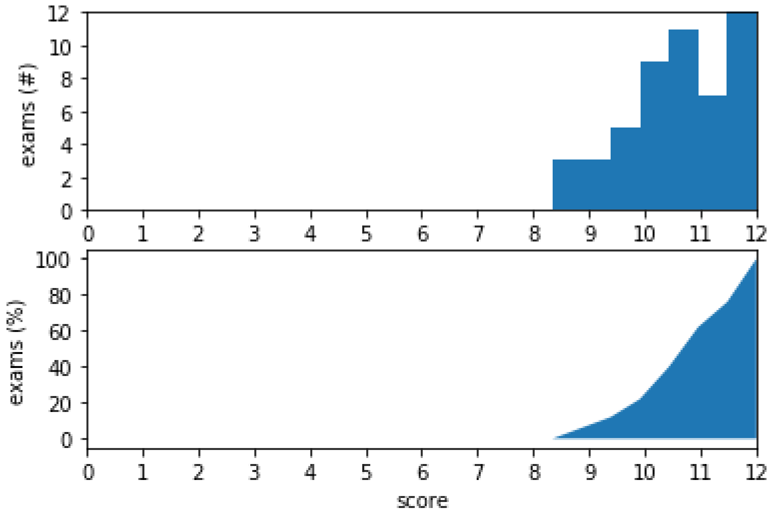
447 Fa19 midterm exam 12:30--2:20p Thu Oct 31

Notes regarding regrade requests:

- ***I will accept regrade requests from Tue Nov 5 to Tue Nov 12*** -- this will give you time to review yo exam and the solutions before requesting a regrade.
- To request a regrade, send me a message using Canvas Conversations with a short explanation which problem(s) you want regarded, and why you think your solution is equivalent to or equally as the one provided.
- ***Note that it is possible your score will decrease after the regrade***, so please be sure you underst problem and solution before making a request. To help you understand the problems and their solutions before you submit your request, I am happy to answer questions during office hours.

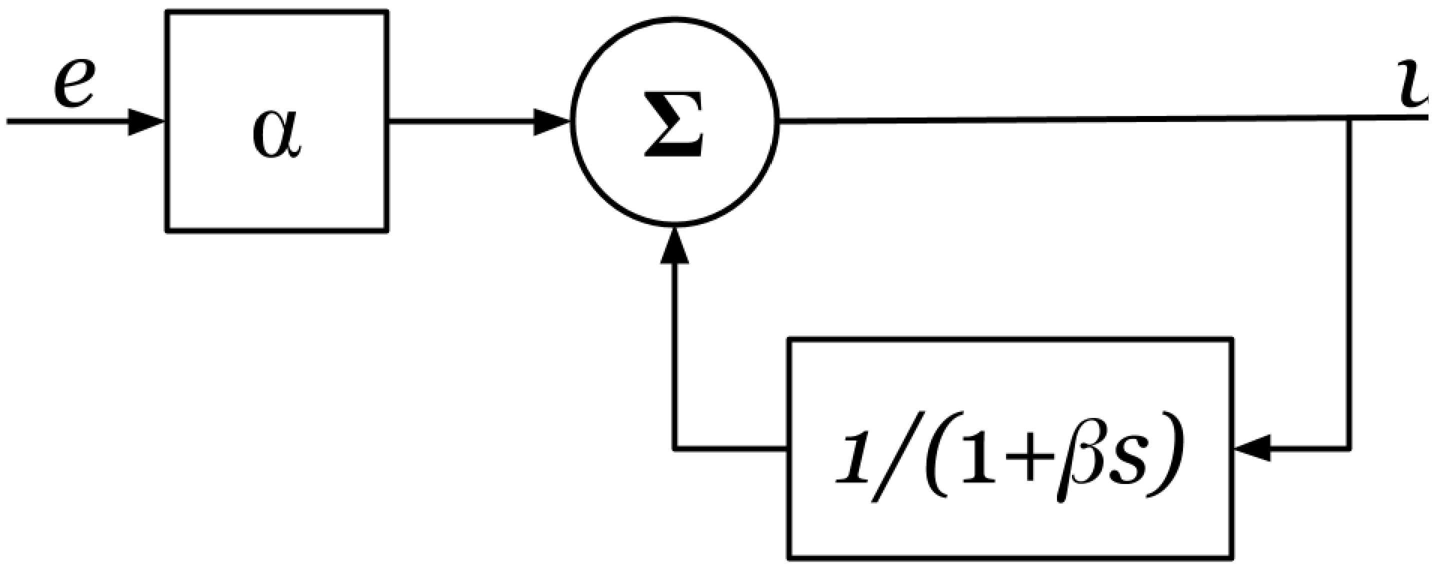
```
import numpy as np
import pylab as plt

scores = np.asarray([8.5, 8.5, 8.75, 9.0, 9.25, 9.38, 9.5, 9.63, 9.75, 9.75, 9.75, 10.0, 10.0

print('%0.1f <= scores <= %0.1f'%(scores.min(),scores.max()))
print('median score = %0.1f'%np.median(scores))

xlim = (0,12)

plt.figure()
plt.subplot(2,1,1)
h = plt.hist(scores,bins=np.linspace(xlim[0],xlim[1],2*np.diff(xlim)))
n,s = h[0],h[1]
N = int(np.ceil(h[0].max()))
plt.ylim(0,N)
plt.xticks(np.linspace(xlim[0],xlim[1],np.diff(xlim)+1))
plt.yticks(np.linspace(0,N,(N+2)/2))
plt.xlim(xlim)
plt.ylabel('exams (#)');

plt.subplot(2,1,2)
n *= 100./n.sum()
n = np.hstack((0.,n))
plt.fill_between(s,np.cumsum(n),0*n)
plt.xlim(xlim)
plt.xticks(np.linspace(xlim[0],xlim[1],np.diff(xlim)+1))
plt.yticks(np.linspace(0,100,6))
plt.xlabel('score'); plt.ylabel('exams (%)');
```

```
8.5 <= scores <= 12.0
median score = 10.5
```



problem 1.

Determine the transfer function G_{ue} from e to u for the following block diagram:



solution 1:

From the block diagram algebra, we get:

$u = \alpha e + \frac{u}{\beta + s}$

$\Leftrightarrow u(1 - \frac{1}{1+\beta s}) = \alpha e$

$$\Leftrightarrow G_{ue} = \frac{u}{e} = \frac{\alpha}{1 - \frac{1}{1+\beta s}} = \frac{\alpha(1+\beta s)}{\beta s} = \frac{\alpha}{\beta s} + \alpha s.$$

You may recognize this as the transfer function for a proportional-integral controller – this ***positive fe*** block diagram is one way to implement this type of controller.

Notes:

- -0.125 points for minor algebraic error in final answer
- -0.25 points if the input variable e is present in expression for G_{ue}
- -0.5 points for input over output instead of output over input (i.e. $G_{eu} = \frac{e}{u}$ instead of $G_{ue} = \frac{u}{e}$

problem 2.

Translate the following model into linear state-space form using 3-dimensional state vector $x = (q, \dot{q}, \ell) \in \mathbb{R}^3$,

$$m\ddot{q} = \kappa(\ell - 2q) + u,\ \beta\dot{\ell} = -\kappa(\ell - q).$$

i.e. determine matrices A, B such that $\dot{x} = Ax + Bu$.

solution 2.

Rewriting the two equations, we get,

$$\ddot{q} = \frac{(\kappa(\ell - 2q) + u)}{m},\ \dot{\ell} = \frac{-\kappa(\ell - q)}{\beta}.$$

Writing the state space equation in the form $\dot{x} = Ax + Bu$, we get:

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{2\kappa}{m} & 0 & +\frac{\kappa}{m} \\ +\frac{\kappa}{\beta} & 0 & -\frac{\kappa}{\beta} \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \\ l \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} u = Ax + bu.$$

We can sanity check the result verifying shape of A and B matrices. Since we have 3 states and 1 inpu $A \in \mathbb{R}^{3\times3}$, $B \in \mathbb{R}^{3\times1}$

Notes:

- -0.25 points for substitution errors (e.g. if rows are swapped)
- -0.125 points for each algebra error

problem 3.

Consider the following process model:

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^2 + b_2 s}{a_0 s^3 + a_1 s^2 + a_2 s + a_3}$$

where b_1, b_2 and a_0, a_1, a_2, a_3 are nonzero parameters.

subproblem 3a.

If a constant input is applied to P, what will the output be?

solution 3a.

The output obtained from a constant input is given by the ***static*** (or ***DC***) ***gain*** $P(0) = 0$.

Notes:

- 0.5 points earned for correctly stating that the output will be constant, even if the constant dete
  is incorrect.
- 0.5 points earned for correctly determining that the constant output will be zero because $P(0)$

subproblem 3b.

Suppose $a_0 = 10$, $a_1 = 1$, $a_2 = 2$; what range of values for a_3 ensure P is stable?

solution 3b.

Applying the Routh-Hurwitz stability criteria for a third-order system, we find the following inequalities
be satisfied:

$$\frac{a_1}{a_0}, \frac{a_2}{a_0}, \frac{a_3}{a_0} > 0, \ \frac{a_1}{a_0}\frac{a_2}{a_0} > \frac{a_3}{a_0}.$$

Substituting $a_0 = 10$, $a_1 = 1$, $a_2 = 2$ we find $0 < a_3 < \frac{2}{10}$.

Notes:

- 0.5 points earned for correctly stating the Routh-Hurwitz stability criteria for a third-order charac
  polynomial.
- 0 to 0.5 points earned depending on the correctness and completeness of the application of the
  Hurwitz stability criteria.

problem 4.

Assume the state-space linear system $\dot{x} = Ax$ is stable.

subproblem 4a.

If $\lambda \in \mathbb{C}$ is an eigenvalue of A, what must be true about the real part of λ?

solution 4a.

The real part of λ must be ***negative***.

subproblem 4b.

Suppose T is an invertible matrix with the same number of rows and columns as A. Is it possible for linear system $\dot{z} = TAT^{-1}z$ to be ***unstable***? Why or why not?

solution 4b.

Given that $\dot{x} = Ax$ is stable, then it is ***not possible*** for $\dot{z} = TAT^{-1}z$ to be unstable.

There are several ways to reason about this:

1. You may recall or prove that A and TAT^{-1} have the same eigenvalues:

$$Av = \lambda v \implies TAT^{-1}(Tv) = TAv = \lambda Tv.$$

2. You may reason geometrically about the effect of change-of-coordinates on phase portraits fro problem 2(f,g) on homework 3 since $z(t) = Tx(t)$.
3. You may use the fact from the midterm review that $e^{TAT^{-1}} = Te^{A}T^{-1}$, so $z(t) = e^{TAT^{-1}t}z(0) = Te^{At}T^{-1}z(0) \to 0$ as $t \to \infty$ since $e^{At} \to 0$ as $t \to \infty$.

Notes:

- 0.5 points earned for stating that $\dot{z} = TAT^{-1}z$ is stable.
- 0 to 0.5 points earned depending on the correctness and completeness of the explanation.