# ECE 447: Control Systems

goal: synthesize stabilizing controllers for state-space systems

(a) state feedback

process model

$$\dot{x} = Ax + Bu$$

$u$ $\qquad$ $x$

$$u = -Kx$$

controller

choose $K$ s.t. $\dot{x} = (A - BK)x$ stable, i.e. $\mathrm{Re}\,\lambda(A - BK) < 0$

(b) state estimation

choose $L$ s.t. $\mathrm{Re}\,\lambda(A + LC) < 0$, i.e. error dynamics $\dot{(x - \hat{x})} = (A + LC)(x - \hat{x})$ are stable, so $\hat{x} \to x$

process model

$u$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$y$

$u$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y})$$
$$\hat{y} = C\hat{x} + Du$$

estimator (observer)

(c) stabilizing controller = (b) state estimation + (a) state feedback

controller

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y})$$
$$\hat{y} = C\hat{x} + Du$$
$$u = -K\hat{x}$$

$u$

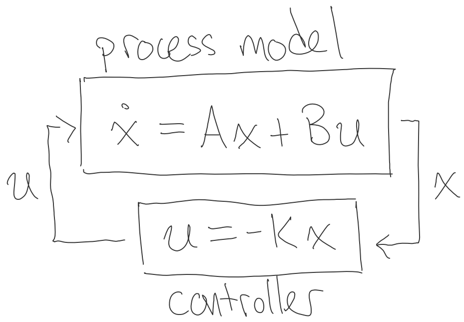
$y$

process

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

assume $A, B, C, D$ are known, but $x$ is <u>not</u>

(a) state feedback [AMv2 Ch 7] [Nv7 Ch 12.2]




- assume given: process model $\dot{x} = Ax + Bu$

→ for now, we'll assume we (i.e. our controller) gets to see the entire state vector $x \in \mathbb{R}^n$

* measuring all voltages/currents in a circuit
positions/velocities in mechanical sys

goal: determine $u$ given $x$ so that $x \to 0$ (i.e. closed-loop system stable)

* if we choose $u$ as a linear function of $x$, $u = -Kx$, $K \in \mathbb{R}^{p \times n}$

then the closed-loop system is linear: $\dot{x} = Ax + Bu$

$$= Ax - BKx = (A - BK)x$$

→ we know how to assess stability:

closed-loop system $\dot{x} = (A - BK)x$ is stable

$x \to 0 \iff \mathrm{Re}\, \lambda(A - BK) < 0$

all eigenvalues of $A - BK$ have negative real part

* our general approach: pole placement / eigenvalue assignment

- if we want the eigenvalues to be $\lambda(A - BK) = \{\lambda_j\}_{j=1}^{n} \subset \mathbb{C}$, we just need to ensure characteristic polynomial

$$\det(sI - (A - BK)) = (s - \lambda_1) \cdot (s - \lambda_2) \cdots (s - \lambda_n) = \prod_{j=1}^{n} (s - \lambda_j)$$

1°. determine $\det(sI - (A - BK))$ — symbolically or numerically

1°. determine $\det(sI-(A-BK))$ — symbolically or numerically

$$= s^n + a_1(K)s^{n-1} + a_2(K)s^{n-2} + \cdots + a_n(K)$$

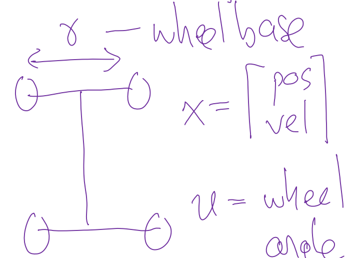
2°. expand $\prod_{j=1}^{n}(s-\lambda_j)$ — symbolically or numerically

$$= s^n + a_1^* s^{n-1} + a_2^* + \cdots + a_n^*$$

3°. solve $a_1(K)=a_1^*,\ a_2(K)=a_2^*,\ \ldots,\ a_n(K)=a_n^*$ for $K \in \mathbb{R}^{p\times n}$

— symbolically or numerically

ex: (symbolically) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ [AMv2 Ex 7.4]




1°. determine $\det(sI-(A-BK))$, $K = [k_1\ \ k_2]$

(recall: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad-bc$)

→ $\det(sI-(A-BK))$ — note: $BK = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}[k_1\ \ k_2] = \begin{bmatrix} \gamma k_1 & \gamma k_2 \\ k_1 & k_2 \end{bmatrix}$

$$= \det\begin{bmatrix} s+\gamma k_1 & -1+\gamma k_1 \\ k_1 & s+k_2 \end{bmatrix} = (s+\gamma k_1)\cdot(s+k_2) - (\gamma k_1 - 1)\cdot k_1$$

$$= s^2 + (\gamma k_1 + k_2)s + k_1$$

2°. expand $\prod_{j=1}^{n}(s-\lambda_j)$, $\lambda_\pm = -\sigma \pm j\omega$, $\sigma > 0$, $\omega \in \mathbb{R}$

$$= (s-(-\sigma - j\omega))\cdot(s-(-\sigma+j\omega)) = s^2 + 2\sigma s + \sigma^2 + \omega^2$$

3°. → solve $a(K) = a^*$ to determine $k_1, k_2$

* $s^2 + (\gamma k_1 + k_2)s + k_1 = s^2 + 2\sigma s + \sigma^2 + \omega^2$

$\iff k_1^* = \sigma^2 + \omega^2 \quad \gamma k_1^* + k_2^* = 2\sigma$, i.e. $k_2^* = 2\sigma - \gamma k_1^*$
$= 2\sigma - \gamma(\sigma^2 + \omega^2)$

* $u = -Kx,\ K = [k_1^* \ \ k_2^*] \Rightarrow \dot{x} = (A - BK)x$ is stable:
$\lambda(A - BK) = -\sigma \pm j\omega,\ \sigma > 0$

ex: (numerical) $\longrightarrow$ see lecture examples notebook §6

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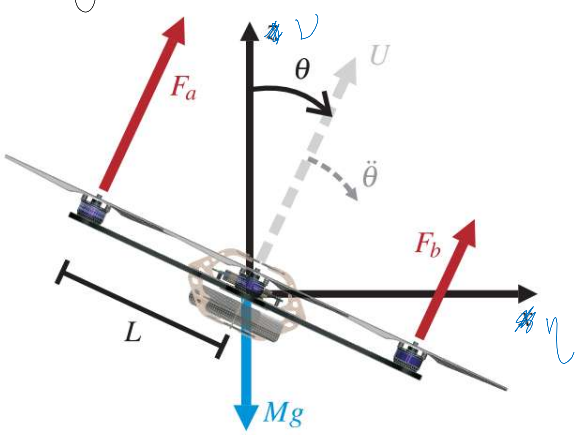
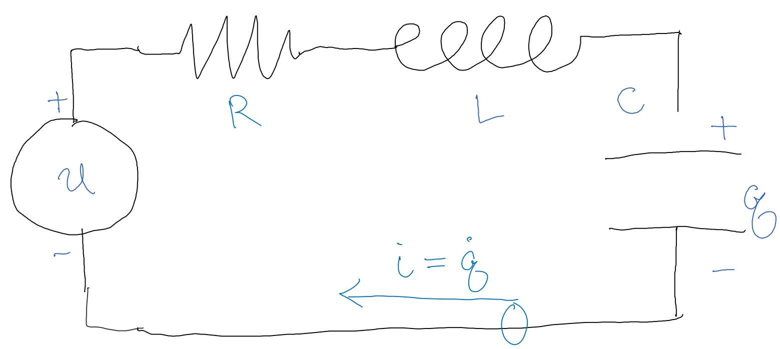
(b) state estimation [AMv2 Ch 8] [Nv7 Ch 12.5]

ex: quadrotor




ex: RLC circuit




$\longrightarrow$ what is the state $x \in \mathbb{R}^n$? $\longrightarrow$ how would you measure each $x_i$?
(what is the sensor? how noisey is it? how expensive?)

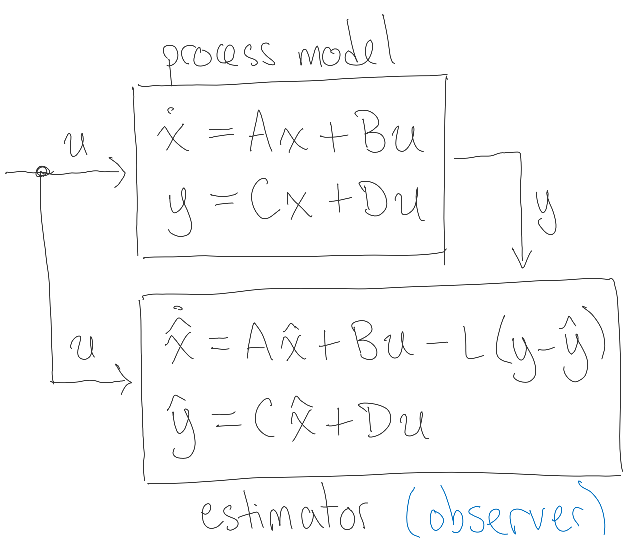
| | quadrotor | | circuit | |
|---|---|---|---|---|
| state: | positions | velocities | voltages | currents |
| sensor: | GPS | gyrometer | voltmeter | ammeter |

sensor: GPS, camera(s), LIDAR/RADAR, gyrometer, accelerometer | voltmeter, ammeter --- E&M ---

* different states are harder/more expensive to measure

* it would be great if we could only measure a subset of states (e.g. positions, voltages) and estimate the rest (velocities, currents)

→ we will develop control system techniques for state estimation




- $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^o$

* $\overset{o}{\cancel{q}} < n$, i.e. we're measuring some (not all) of the states

<u>ex:</u> positions (not velocities)

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad y = q = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{} x + 0 \cdot u = Cx + \overset{0}{\cancel{D}}u$$

- given process model $\dot{x} = Ax + Bu$, $y = Cx + Du$ (i.e. given $A, B, C, D$)

and assuming we know $u : [0, \infty) \to \mathbb{R}^p$, $y : [0, \infty) \to \mathbb{R}^o$
(b/c we choose $u$) (b/c we measure $y$)

we construct another LTI system called an <u>estimator</u> (or <u>observer</u>):

$$\dot{\hat{x}} = A\hat{x} + Bu - L(y - \hat{y}) \quad \text{where } L \in \mathbb{R}^{n \times o} \text{ is an}$$

$$\dot{\hat{x}} = A\hat{x} + Bu - L(\underbrace{y-\hat{y}}_{\text{output error}})$$
$$\hat{y} = C\hat{x} + Du$$

where $L \in \mathbb{R}^{n\times o}$ is an output error feedback matrix

→ to see why this works, determine the dynamics of $e = x - \hat{x}$ (your answer should be of the form $\dot{e} = M \cdot e$)

- $e = x - \hat{x} \Rightarrow \dot{e} = \dot{x} - \dot{\hat{x}}$

$$= (Ax + \cancel{Bu}) - (A\hat{x} + \cancel{Bu} - L(\overbrace{y - \hat{y}}^{=(Cx+\cancel{Du})-(C\hat{x}+\cancel{Du})}))$$

$$= Ax - A\hat{x} + L(Cx - C\hat{x})$$

$$= A(\underbrace{x-\hat{x}}_{=e}) + LC(\underbrace{x-\hat{x}}_{=e}) = (A+LC)e$$

* so if $L \in \mathbb{R}^{n\times o}$ is chosen such that $\mathrm{Re}\,\lambda(A+LC) < 0$ then $\dot{e} = (A+LC)e$ is <u>stable</u>! i.e. $\underline{e = x - \hat{x} \to 0}$ !

<u>ex:</u> (vehicle steering) $\dot{x} = Ax + Bu$, $y = Cx$ $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} \gamma \\ 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$

→ determine state estimate error dynamics matrix $A + LC$ and the characteristic polynomial

- $L \in \mathbb{R}^{2\times 1}$, $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, $A + LC = \begin{bmatrix} l_1 & 1 \\ l_2 & 0 \end{bmatrix}$

$- \det(sI - (A+LC)) = s^2 - \ell_1 s - \ell_2$

- we want to choose $L$ (i.e. $\ell_1, \ell_2$) s.t. $\mathrm{Re}\,\lambda(A+LC) < 0$

1°. characteristic polynomial $\det(sI - (A+LC)) = s^2 - \ell_1 s - \ell_2$

2°. want: $(s+\varsigma)^2 = s^2 + 2\varsigma s + \varsigma^2$, $\varsigma > 0$

3°. matching coefficients: $\ell_2 = -\varsigma^2$, $\ell_1 = -2\varsigma$

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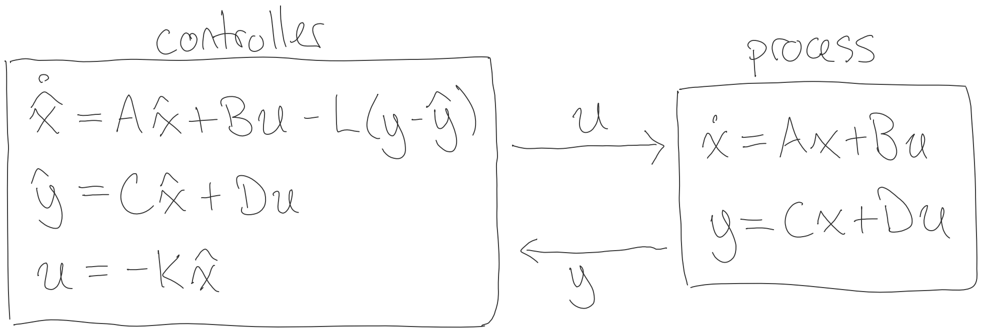
(c) stabilizing controller [AMv2 Ch 8] [Nv7 Ch 12.5]




* assume $A, B, C, D$ given and $K, L$ are chosen such that:

$$\mathrm{Re}\,\lambda(A - BK) < 0 \quad \text{and} \quad \mathrm{Re}\,\lambda(A + LC) < 0$$

$\longrightarrow$ already saw that $e = x - \hat{x} \Rightarrow \dot{e} = (A+LC)e$ regardless of the input signal $u$

$\longrightarrow$ determine dynamics of $x$ when $u = -K\hat{x}$

$\longrightarrow$ determine dynamics of $x$ when $u = -K\hat{x}$
(substitute to write $\dot{x}$ in terms of $x$ & $e$, not $\hat{x}$)

$$- \dot{x} = Ax + Bu = Ax - BK\hat{x}$$
$$= Ax - BK(x-e)$$
$$= (A-BK)x - BKe$$

$e = x - \hat{x}$
so $\hat{x} = x - e$

$\circ$ with $\bar{x} = \begin{bmatrix} x \\ e \end{bmatrix} \Rightarrow \dot{\bar{x}} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & -BK \\ 0 & A+LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \bar{A}\bar{x}$

$*$ note: $\det(sI - \bar{A}) = \det(sI - (A-BK)) \cdot \det(sI - (A+LC))$

$\longrightarrow$ so $\mathrm{Re}\,\lambda(A-BK) < 0$ & $\mathrm{Re}\,\lambda(A+LC) < 0 \Rightarrow \mathrm{Re}\,\lambda(\bar{A}) < 0$

$\Rightarrow$ $\underline{x \text{ \& } e \to 0}$ ! i.e. the combined state estimator & state feedback controller stabilizes both systems