04 -- Mon Oct 25

ECE 447: Control Systems (Fall 2021)

Prof: Sam Burden TA: Sat Singh

today: ☑ logistics: HW 3 due Fri Oct 29

exam 1 assigned Fri Oct 29 ⟶ due Fri Nov 5

HW 4 due Fri Nov 12

☑ tutorial

☑ break

☐ office hour

Prof Burden TODO: ☐ HW3 p1b specify $\sigma_d > 0$, ω_d anything

☐ quad linearization

☐ HW solutions

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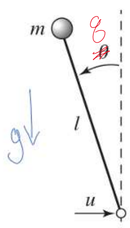
tutorial on linearization

ex: "rocket flight" (really: pendulum)



- state $x = (\vartheta, \dot\vartheta)$ – angle, velocity
- input u – horizontal acceleration of pivot
- (DE) $m\ell^2 \ddot\vartheta = mg\ell \sin\vartheta - \alpha\dot\vartheta + \ell u \cos\vartheta$

$$\dot x = \begin{bmatrix} \dot\vartheta \\ \ddot\vartheta \end{bmatrix} = \begin{bmatrix} \dot\vartheta \\ g/\ell \sin\vartheta - \frac{\alpha}{m\ell^2}\dot\vartheta + \frac{1}{m\ell} u \cos\vartheta \end{bmatrix} = f(x,u)$$

- we previously determined that $u_e = 0$ has $x_e = \begin{bmatrix} k\cdot\pi \\ 0 \end{bmatrix} = \begin{bmatrix} \vartheta_e \\ \dot\vartheta_e \end{bmatrix}$ $k \in \mathbb{Z}$

as equilibria ⇝ we will approximate f

- for nonlinear system (NL) $\dot x = f(x,u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

with equilibrium $(x_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^p$ s.t. "$\dot x_e$" $= f(x_e, u_e) = 0$

$$\dot x = f(x,u) \simeq \overset{0}{\cancel{f(x_e, u_e)}} + \boxed{\frac{\partial}{\partial x} f(x_e, u_e)(x - x_e)} + O(\|x - x_e\|^2)$$

$$\dot{x} = f(x,u) \simeq \cancel{f(x_e,u_e)}^{\,0} + \boxed{\frac{\partial}{\partial x} f(x_e,u_e)(x-x_e) + \frac{\partial}{\partial u} f(x_e,u_e)\cdot(u-u_e)} + \underbrace{O(\|x-x_e\|^2) + O(\|u-u_e\|^2)}_{\text{"higher-order" terms}}$$

our "linearization"

2 ingredients for linearization:

1°. $\dot{x} = f(x,u) = \begin{bmatrix} \dot{q} \\ g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$

"the dynamics"

2°. $f(x_e,u_e) = 0 \qquad x_e = \begin{bmatrix} k\cdot\pi \\ 0 \end{bmatrix} = \begin{bmatrix} q_e \\ \dot{q}_e \end{bmatrix}, \quad u_e = 0$ ($k \in \mathbb{Z}$)

"an equilibrium"

2 steps for linearization:

1°. compute $\frac{\partial f_i}{\partial x_j}$ for all $i,j \in \{1,\dots,n\}$, $x \in \mathbb{R}^n$

" " $\frac{\partial f_i}{\partial u_k}$ for all $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$, $u \in \mathbb{R}^m$

$f_1(x,u) = x_2 \Rightarrow \quad \frac{\partial f_1}{\partial x_1} = 0 \quad \frac{\partial f_1}{\partial x_2} = 1 \quad \frac{\partial f_1}{\partial u} = 0$

$f_1((q,\dot{q}),u) = \dot{q} \Rightarrow \partial_q f_1 = 0, \quad \partial_{\dot{q}} f_1 = 1 \quad \partial_u f_1 = 0$

$f_2(x,u) = g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q$

$\Rightarrow \frac{\partial f_2}{\partial q} = \frac{g}{\ell}\cos q - \frac{u}{m\ell}\sin q$

$\frac{\partial f_2}{\partial \dot{q}} = -\frac{\alpha}{m\ell^2} \qquad \frac{\partial f_2}{\partial u} = \frac{1}{m\ell}\cos q$

2°. evaluate derivatives @ equilibrium & populate Jacobian matrices

$$\frac{\partial}{\partial x} f(x_e,u_e) = \begin{bmatrix} \partial_q f_1 & \partial_{\dot{q}} f_1 \\ \partial_q f_2 & \partial_{\dot{q}} f_2 \end{bmatrix}\Bigg|_{\substack{x = x_e = \begin{bmatrix}\pi \\ 0\end{bmatrix} \\ u = u_e = 0}} = \begin{bmatrix} 0 & 1 \\ -g/\ell & -\frac{\alpha}{m\ell^2} \end{bmatrix} = A$$

$$\frac{\partial}{\partial u} f(x_e,u_e) = \begin{bmatrix} \partial_u f_1 \\ \partial_u f_2 \end{bmatrix}\Bigg|_{\substack{x = x_e = \begin{bmatrix}\pi \\ 0\end{bmatrix} \\ u = u_e = 0}} = \begin{bmatrix} 0 \\ -\frac{1}{m\ell} \end{bmatrix} = B$$

$$\frac{\partial}{\partial u} f(x_e, u_e) = \begin{bmatrix} \partial_u f_1 \\ \partial_u f_2 \end{bmatrix} \Bigg|_{\substack{x = x_e = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \\ u = u_e = 0}} = \begin{bmatrix} 0 \\ -\frac{1}{m\ell} \end{bmatrix} = B$$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q \end{bmatrix}$$

$$\simeq A(x - x_e) + B(u - u_e)$$

$$= \begin{bmatrix} \dot{q} - \cancel{\dot{q}_e}^{\,0} \\ -\frac{g}{\ell}(q - q_e) - \frac{\alpha}{m\ell^2}(\dot{q} - \cancel{\dot{q}_e}^{\,0}) \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{m\ell}(u - \cancel{u_e}^{\,0}) \end{bmatrix}$$

RLC circuit

Consider the following differential equation (DE) model of a series RLC circuit

$$L\ddot{q} + R\dot{q} + q/C = u$$

where q denotes the charge on the capacitor, (R, L, C) denote the (resistor, inductor, capacitor) parameters, and u denotes a series voltage source.

Letting $x = (q, \dot{q}) \in \mathbb{R}^2$ denote the state of the circuit, we can rewrite (DE) in state-space form as $\dot{x} = f(x, u)$ where

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ (-R\dot{q} - q/C + u)/L \end{bmatrix} = f((q, \dot{q}), u) = f(x, u).$$

1°. $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\frac{R}{L}\dot{q} - \frac{q}{LC} + \frac{1}{L} u \end{bmatrix} = f(x, u)$

2°. $f(x_e, u_e) = 0 \quad x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad u_e = 0$

1°. $\partial_x f = \begin{bmatrix} \partial_q f_1 & \partial_{\dot{q}} f_1 \\ \partial_q f_2 & \partial_{\dot{q}} f_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} = A$

$\partial_u f = \begin{bmatrix} \partial_u f_1 \\ \partial_u f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = B$

2°. $\dot{x} = f(x, u) \simeq A x + B u$

actually "=" in this case — f is linear

quadrotor

Consider the simplified vertical-plane quadrotor model

$$M\ddot{\eta} = F\sin\theta,$$
$$M\ddot{\nu} = -Mg + F\cos\theta,$$
$$I\ddot{\theta} = \tau$$

where (η, ν) denote the quadrotor (horizontal, vertical) position and θ denotes the quadrotor's rotation, (M, I) denote quadrotor (mass, inertia), g is acceleration due to gravity, and (F, τ) denote the net (thrust, torque) applied by the spinning rotors.

With $q = (\eta, \nu, \theta) \in \mathbb{R}^3$ denoting positions and $\dot{q} = \frac{d}{dt}q = (\dot{\eta}, \dot{\nu}, \dot{\theta}) \in \mathbb{R}^3$ denoting velocities, the state vector is $x = (q, \dot{q}) \in \mathbb{R}^6$, the input vector is $u = (F, \tau) \in \mathbb{R}^2$, and the state-space model is

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q}(x,u) \end{bmatrix} = f(x,u),$$

where $\ddot{q} : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$\ddot{q}(x,u) = \begin{bmatrix} \frac{F}{M}\sin\theta \\ -g + \frac{F}{M}\cos\theta \\ \frac{\tau}{I} \end{bmatrix}.$$

1°.

2°. $q_e = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$, $\dot{q}_e = 0$

$u_e = \begin{bmatrix} F_e \\ \tau_e \end{bmatrix} = \begin{bmatrix} M\cdot g \\ 0 \end{bmatrix}$

$q \in \mathbb{R}^3 \Rightarrow \dot{q} \in \mathbb{R}^3 \Rightarrow f(x,u) = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} \in \mathbb{R}^6$

$$\begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 & \cdots & \partial_{x_6} f_1 \\ \vdots & & & \vdots \\ \partial_{x_1} f_6 & \partial_{x_2} f_6 & \cdots & \partial_{x_6} f_6 \end{bmatrix} = \partial_x f \in \mathbb{R}^{6\times 6} \qquad \partial_u f = \begin{bmatrix} \partial_{u_1} f_1 & \partial_{u_2} f_1 \\ \vdots & \vdots \\ \partial_{u_1} f_1 & \partial_{u_2} f_1 \end{bmatrix} \in \mathbb{R}^{6\times 2}$$