ECE 447: Control Systems

goal: frequency-domain analysis of control system performance

(a) transfer matrix $\dot{x} = Ax + Bu$, $y = Cx + Du$ $\rightsquigarrow$ $G_{yu}(s) = C(sI-A)^{-1}B + D$

(i.e. matrix of transfer functions) $u = e^{st} \cdot u_0 \Rightarrow y = e^{st} \cdot G_{yu}(s) \cdot u_0$

(b) Bode plots $u = e^{st} \cdot u_0 \rightsquigarrow y = e^{st} G_{yu}(s) u_0$, $\sin(\omega t) = \operatorname{Im} e^{j\omega t}$

so plotting $\log|G(j\omega)|$, $\angle G(j\omega)$ vs ω shows us how system responds

(c) effect of disturbances $y = \underbrace{\frac{1}{1+PC}}_{}w + \frac{P}{1+PC}v + \frac{PC}{1+PC}r$

so we can use sensitivity $S = \frac{1}{1+PC}$ to assess performance

(d) fundamental limits $S = \frac{1}{1+PC} = 1 - \frac{PC}{1+PC} = 1 - T$; $\int_0^\infty \log|S(j\omega)|\,d\omega$ is conserved

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(a) transfer matrix [AMv2 Ch 6.3, 9.2] [Nv7 Ch 4.11]

↳ i.e. a matrix of transfer functions relating a vector of inputs to a vector of outputs

- given LTI system in state-space form, $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$

know that $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

so $y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$

- let's see what output is produced by $u(t) = e^{st} \cdot u_0$, $s \in \mathbb{C}$, $u_0 \in \mathbb{R}^p$

$$y(t) = Ce^{At}x(0) + \underbrace{C\int_0^t e^{A(t-\tau)} B[e^{s\tau} \cdot u_0] d\tau}_{= C\left[\int_0^t \underbrace{e^{A(t-\tau)} e^{s\tau}}_{= e^{At} \cdot e^{-A\tau} \cdot e^{sI\tau}} d\tau\right] B u_0} + De^{st} \cdot u_0$$

$\leftarrow e^{s\tau} \cdot I = e^{sI\tau}$

$$y(t) = Ce^{At}x(0) + Ce^{At}\left[\underbrace{\int_0^t e^{(sI-A)\tau} d\tau}\right] \cdot B \cdot u_0 + De^{st} \cdot u_0$$

$\rightarrow$ evaluate this integral

hint: find expression whose derivative is the integrand $e^{(sI-A)\tau}$

- recall that $\frac{d}{d\tau} e^{(sI-A)\tau} = (sI-A)\, e^{(sI-A)\tau}$

- so $\frac{d}{d\tau}\left[\underbrace{(sI-A)^{-1}}\, e^{(sI-A)\tau}\right] = e^{(sI-A)\tau}$ ☑

when is this matrix invertible?

$\iff \det(sI-A) \neq 0 \iff s \notin \lambda(A)$, i.e. s isn't an eigenvalue of A

- so $\int_0^t e^{(sI-A)\tau} d\tau = \left[(sI-A)^{-1} e^{(sI-A)\tau}\right]_{\tau=0}^{\tau=t}$

$$= (sI-A)^{-1}\left[\underbrace{e^{(sI-A)t}}_{= e^{-At} \cdot e^{sIt} = e^{-At} \cdot e^{st}} - \cancel{e^{0 \cdot I}}^{\,I}\right]$$

$= u(t)$

hence

$$y(t) = \underbrace{Ce^{At}\left(x(0) - (sI-A)^{-1}Bu_0\right)}_{\text{transient response}} + \underbrace{\left[C(sI-A)^{-1}B+D\right]}_{\text{steady-state response}} e^{st}\cdot u_0$$

* assuming A stable: $\Rightarrow e^{At} \to 0$ as $t \to \infty$

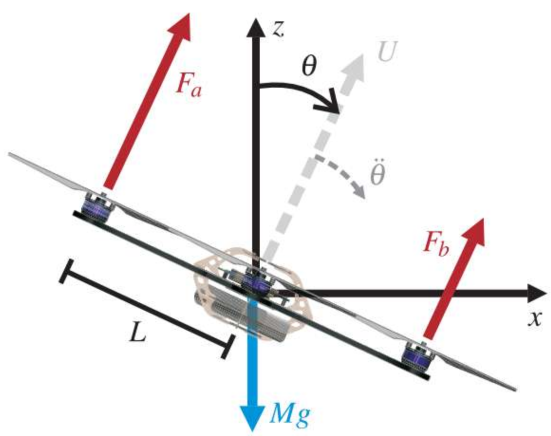
transient response $\to 0$ as $t \to \infty$

steady-state response $G_{yu}(s) \in \mathbb{R}^{o \times p}$

takeaway: given $\dot{x} = Ax + Bu$, $y = Cx + Du$ get $G_{yu}(s) = C(sI-A)^{-1}B + D$
transfer matrix
(ie matrix of transfer functions)

ex: quadrotor

inputs $u = \begin{bmatrix} F_a \\ F_b \end{bmatrix} \in \mathbb{R}^2$ outputs: $\begin{bmatrix} x \\ z \\ \theta \end{bmatrix} = y \in \mathbb{R}^3$



$$G_{yu} = \begin{bmatrix} G_{xF_a} & G_{xF_b} \\ G_{zF_a} & G_{zF_b} \\ G_{\theta F_a} & G_{\theta F_b} \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

so if $u(t) = e^{st}\cdot u_0 \in \mathbb{R}^2$ then $y(t) = e^{st}\cdot G_{yu}(s)\cdot u_0 \in \mathbb{R}^3$

(b) Bode plots [AMv2 Ch 9.6] [Nv7 Ch 10.1]

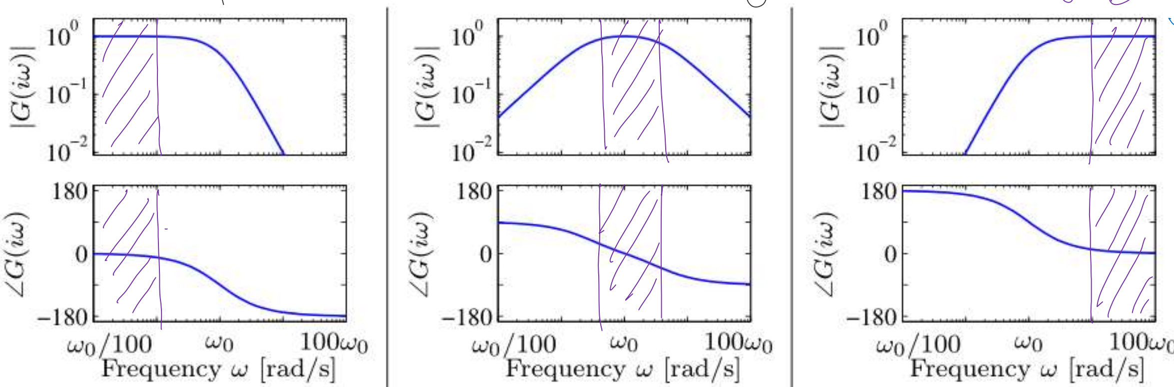
- since a complex exponential input $e^{st}\cdot u_0$ yields complex exponential output $e^{st}\cdot G_{yu}(s)\cdot u_0$

and $\sin(\omega t) = \mathrm{Im}\, e^{j\omega t}$, $\omega \in \mathbb{R}$ and $\cos(\omega t) = \mathrm{Re}\, e^{j\omega t}$

then evaluating / visualizing $G_{yu}(j\omega)$ tells us how system responds to any input $\in \mathbb{C}$ physically-realizable (i.e. sin or cos

- a Bode plot is a visualization of $|G_{yu}(j\omega)|$, $\angle G_{yu}(j\omega)$ vs ω

ex: Bode plots for second-order systems "pass" band



$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(a) Low-pass filter

$$G(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(b) Band-pass filter

$$G(s) = \frac{s^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(c) High-pass filter

Figure 9.17: Bode plots for low-pass, band-pass, and high-pass filters. The upper plots are the gain curves and the lower plots are the phase curves. Each system passes frequencies in a different range and attenuates frequencies outside of that range.

* note: $G(s) = \dfrac{b_1(s)\, b_2(s)}{a_1(s)\, a_2(s)}$

$$\Rightarrow \log|G| = \log|b_1| + \log|b_2| - \log|a_1| - \log|a_2|$$

$$\Rightarrow \angle G = \angle b_1 + \angle b_2 - \angle a_1 - \angle a_2$$

ex: (spring-mass-damper / RLC circuit) i.e. "passive" elements

$$m\ddot{q} + c\dot{q} + kq = u \text{ or } L\ddot{q} + R\dot{q} + \tfrac{1}{C}q = v$$

$m, c, k > 0$
$R, L, C > 0$

$$\rightsquigarrow G(s) = \frac{1}{ms^2 + cs + k} \quad \text{or} \quad = \frac{1}{Ls^2 + Rs + 1/C}$$

$\rightsquigarrow \quad G(s) = \dfrac{1}{ms^2 + cs + k} \quad \text{or} \quad = \dfrac{1}{Ls^2 + Rs + 1/C}$

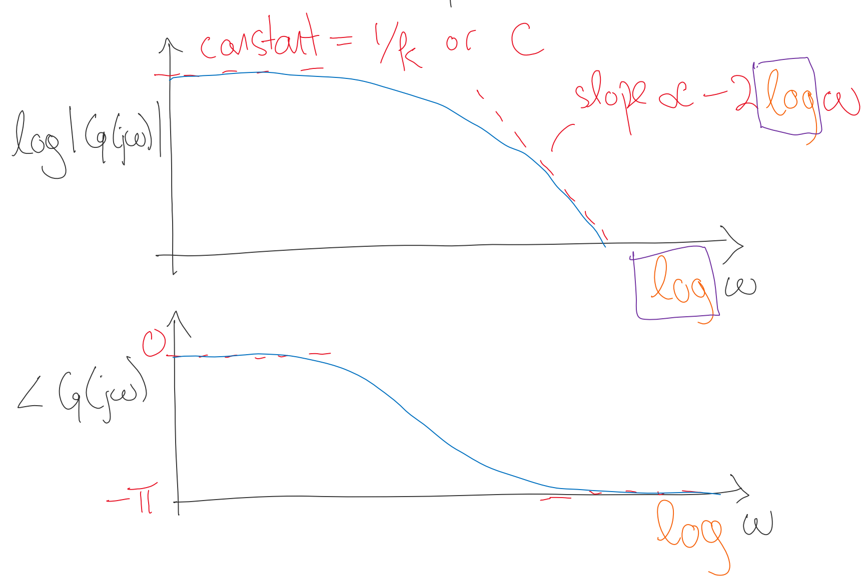
note: as $s \to 0$, $G(s) \to 1/k$ or C, i.e. $|G(s)| \to$ constant

$\angle G(s) \to 0$

as $s \to \infty$, $G(s) \to 1/ms^2$ or $1/Ls^2$, i.e. $\log|G(j\omega)| \propto -2\omega$

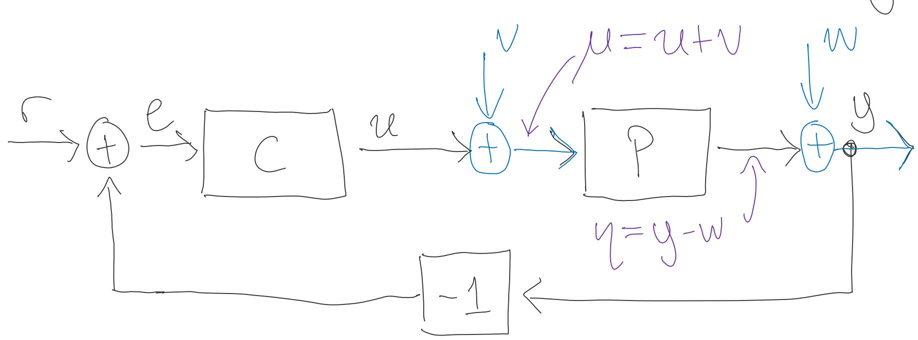
$\angle G(s) \to \pm\pi$

so we can sketch Bode plot:



(c) effect of disturbances [AMv2 Ch 12.1] [Nv7 Ch 7.5]

- consider negative feedback block diagram with disturbances:



r – reference; ideally r → y

v – input disturbance, eg actuator noise,

w – output disturbance, eg sensor noise

* assuming input disturbance v and output disturbance w

* assuming input disturbance v and output disturbance w are independent of other signals in the diagram, they affect y linearly:

→ derive transfer functions G_{yr}, G_{yv}, G_{yw}
so that $y = G_{yr}\cdot r + G_{yv}\cdot v + G_{yw}\cdot w$

– $y = w + \eta = w + P\mu = w + P(v+u) = w + Pv + PCe$

$= w + Pv + PC(r-y)$

$\iff \underbrace{(I+PC)}_{\text{* assuming invertible}} y = w + Pv + PCr$

$$\iff y = \underbrace{(I+PC)^{-1}}_{G_{yw}} w + \underbrace{(I+PC)^{-1}P}_{G_{yv}} v + \underbrace{(I+PC)^{-1}PC}_{G_{yr}} r$$

in single-input/single-output case: $= \overbrace{\frac{1}{1+PC}}^{G_{yw}} w + \overbrace{\frac{P}{1+PC}}^{G_{yv}} v + \overbrace{\frac{PC}{1+PC}}^{G_{yr}} r$

* the "ideal" transformations are different for each transformation:

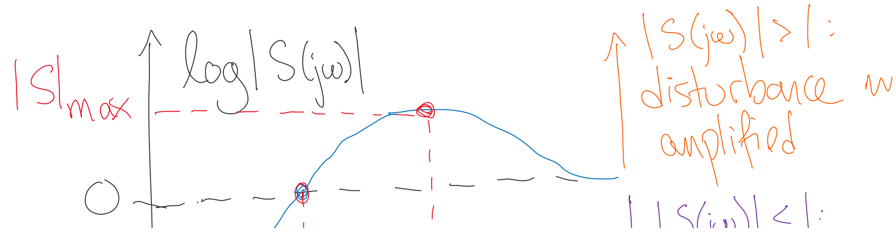
– ideally $G_{yr} \simeq I$ so that $y = G_{yr}\cdot r \simeq r$

– ideally $G_{yv}, G_{yw} \simeq 0$ so that v & w don't affect y

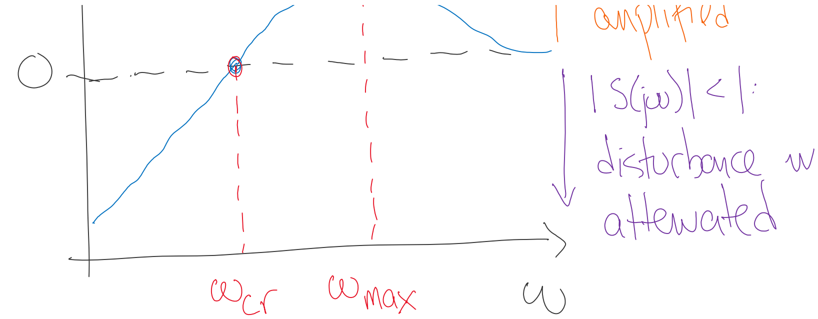
(we'll see that we can't achieve all "ideals" at the same time)

◦ in terms of Bode plot:

let $S = G_{yw}$ be



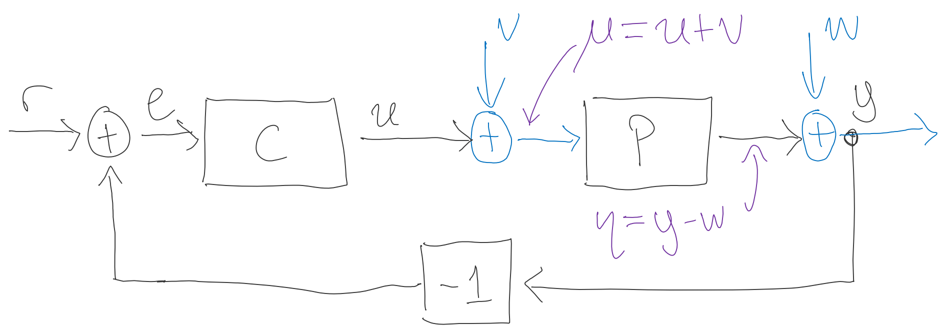
let $S = G_{yw}$ be
sensitivity of closed-loop system

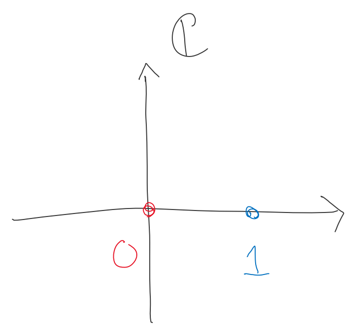


- performance of closed-loop system can be assessed using peak sensitivity $|S|_{max}$ or crossover frequency ω_{cr}

(d) fundamental limits [AMv2 Ch 12.1, 14.2]

- consider the negative feedback block diagram with disturbances





- recall that

$$y = G_{yr} \cdot r + G_{yv} \cdot v + G_{yw} \cdot w$$
$$= \frac{PC}{1+PC} \cdot r + \frac{P}{1+PC} \cdot v + \frac{1}{1+PC} \cdot w$$

and we want: 1°. disturbances rejected, i.e. $G_{yv}, G_{yw} \simeq 0$

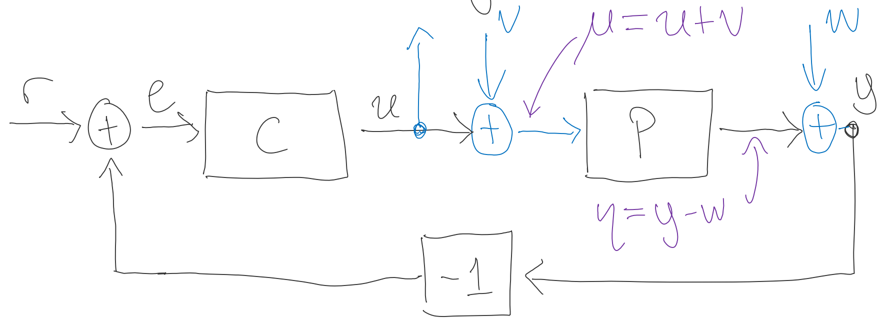
2°. references tracked, i.e. $G_{yr} \simeq 1$

- since $G_{yw} + G_{yr} = \frac{1}{1+PC} + \frac{PC}{1+PC} = \frac{1+PC}{1+PC} = 1$

so $G_{yr} \simeq 1 \iff G_{yw} \simeq 0$, which seems great!

→ but this happy coincidence is misleading...

- return to block diagram and consider effect of disturbances v, w on input u



→ determine G_{ur}, G_{uv}, G_{uw} s.t. $u = G_{ur} \cdot r + G_{uv} \cdot v + G_{uw} \cdot w$

- $u = C(r - y) = Cr - C(w + P\mu) = Cr - Cw - CP(v + u)$

$$\Leftrightarrow (1 + PC)u = Cr - PCv - Cw$$

$$\Leftrightarrow \quad u = \frac{C}{1+PC} r - \frac{PC}{1+PC} v - \frac{C}{1+PC} w$$

$$= G_{ur} \cdot r + G_{uv} \cdot v + G_{uw} \cdot w$$

recall: $y = G_{yr} \cdot r + G_{yv} \cdot v + G_{yw} \cdot w$

$$= \frac{PC}{1+PC} \cdot r + \frac{P}{1+PC} \cdot v + \frac{1}{1+PC} \cdot w$$

observe: $G_{yw} - G_{uv} = \dfrac{1}{1+PC} + \dfrac{PC}{1+PC} = \boxed{\dfrac{1+PC}{1+PC} = 1}$

but we want to reject disturbances, i.e. $\boxed{G_{yw}, G_{uv} \simeq 0}$

we can't have both !

→ there's a tradeoff

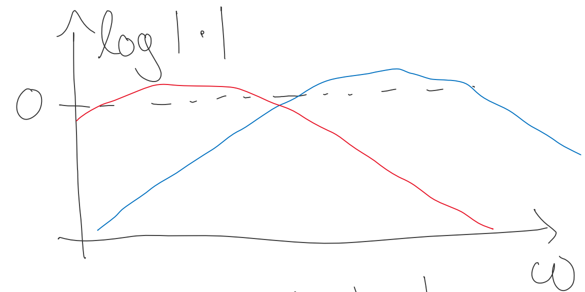
- in many systems, frequency content of : ex: cruise control

∘ in many systems, frequency content of: ex: cruise control

input disturbance v is low ⟵ wind pushing on car

output disturbance w is high ⟵ sensor noise

so if we look at Bode plots:



which means we can design

$S = \frac{1}{1+PC} = G_{yw}$ (sensitivity)

$T = 1 - S = \frac{PC}{1+PC} = -G_{uw}$

(complementary sensitivity)

to design frequency-dependent (complementary) sensitivity

∘ it turns out there are limits on how we can reshape S:

thm: (Bode integral formula / argument principle)

$$\int_0^\infty \log|S(j\omega)|\,d\omega \geq \text{constant, independent of } C$$

$$= \pi \cdot \Sigma \{ \operatorname{Re} p \mid p \text{ is a pole of } P \text{ in right-half plane} \}$$

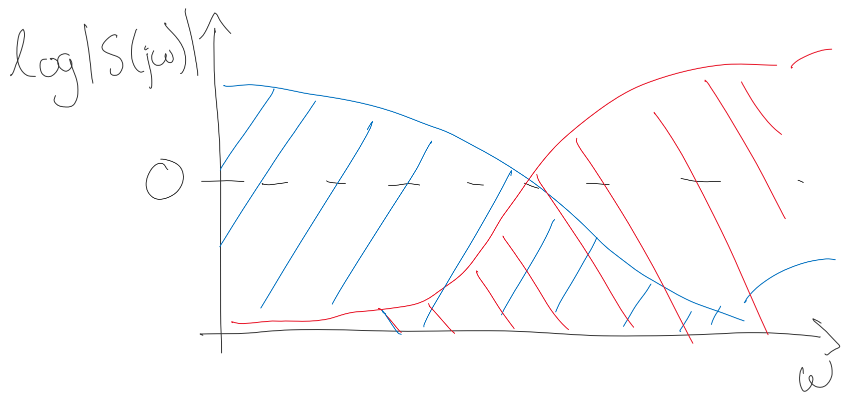
$$= 0 \text{ if process is stable}$$

* since $\log|S(j\omega)| < 0 \iff |S(j\omega)| < 1 \iff$ disturbance attenuated

any range of frequencies where controller attenuates disturbance

... controller amplifies disturbance

any range of frequencies where controller <u>attenuates</u> disturbance is compensated by range where controller <u>amplifies</u> disturbance

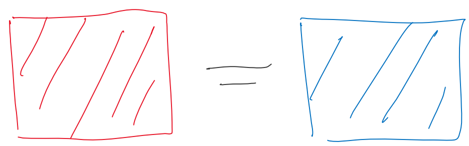


controlled: $S = \frac{1}{1+PC}$

$C \neq 1$

uncontrolled: $S = \frac{1}{1+P}$,

$C = 1$

thm:  ← we can't create or destroy "area" – only shift it