ECE 447: Control Systems

goal: what is a system's state? what is a nonlinear system?

(a) state space $\longleftarrow$ new way of looking at / representing a system using a vector description of quantities of interest and how they interact over time

(b) time: continuous and discrete $\longleftarrow$ we'll consider DE that evolve in continuous time, $\dot{x} = f(x,u)$, or in discrete time, $\tilde{x}^+ = \tilde{f}(\tilde{x},u)$

(c) linear systems $\longleftarrow$ any matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ determine linear state-space system $f(x,u) = Ax + Bu$

(d) nonlinear systems $\longleftarrow$ defined by $\dot{x} = f(x,u)$

(a) state space [AMv2 Ch 3.2] [Nv7 Ch 3.3]

ex: RLC circuit: capacitor charge q & current $\dot{q}$ interact with voltage v over time:

(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$

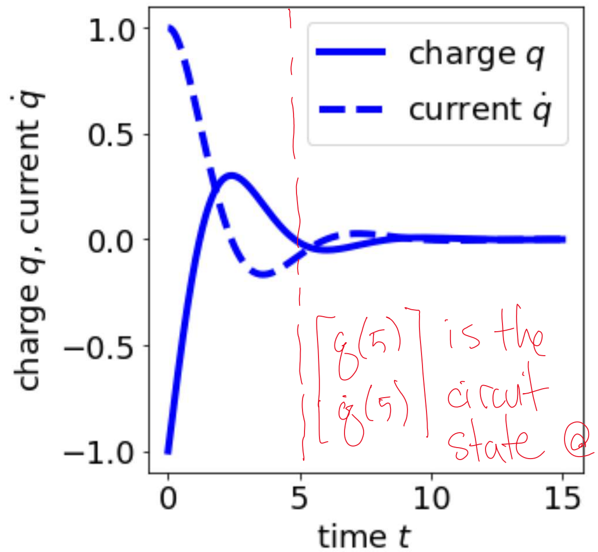
know: given $v : [0,\infty) \to \mathbb{R}$, $: t \mapsto v(t)$, $\underbrace{\text{initial condition } (q(0), \dot{q}(0))}$

then $q(t) = \underbrace{q_0(t)} + \overbrace{q_v(t)}$ — particular response to v

then $\underbrace{q(t) = q_0(t)}_{\text{homogeneous response to ⓪}} + q_v(t)$

∘ the vector $\underbrace{(q(t), \dot{q}(t))} \in \mathbb{R}^2$ is the circuit state at time t

→ if I know ∘ and input $v : [t, \infty)$, $q(\tau)$ determined for $\tau \geq t$

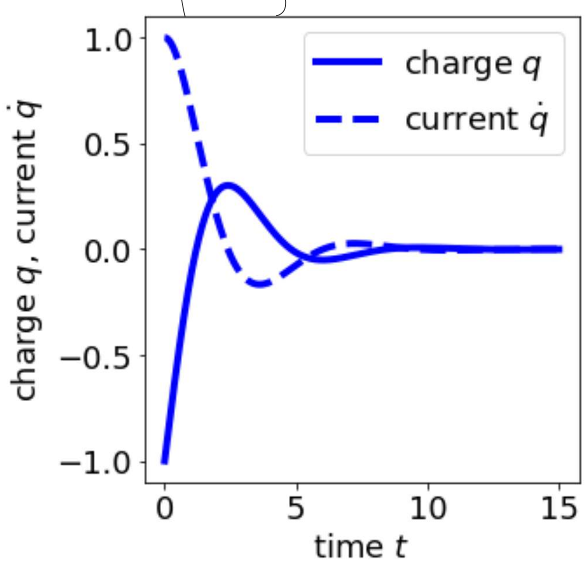


$\begin{bmatrix} q(5) \\ \dot{q}(5) \end{bmatrix}$ is the circuit state @ time $t=5$

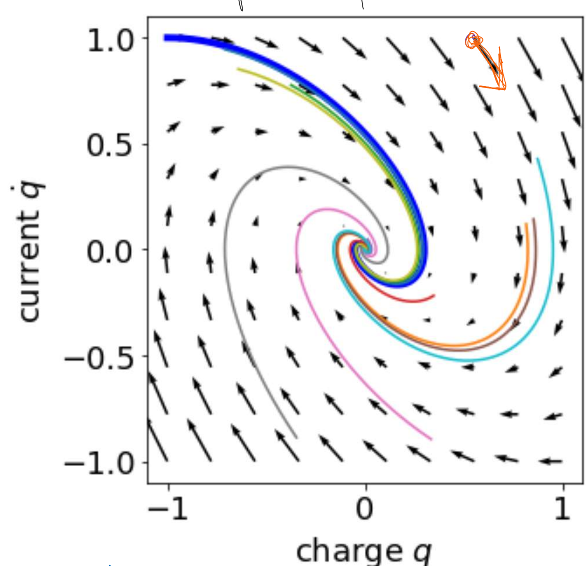
homogeneous response to initial condition / state

$$\begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$$

1°. one trajectory over time

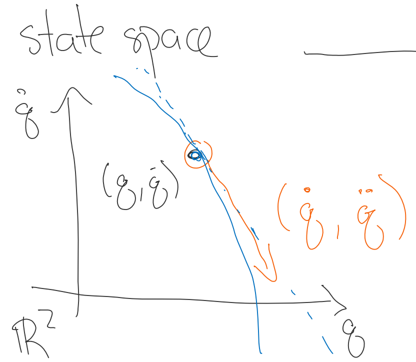


2°. multiple trajectories over time



state space $= \left\{ \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \in \mathbb{R}^2 \right\}$

state space ⟶ instead of a single 2nd-order DE



$\longrightarrow$ instead of a single 2nd-order DE

$$L\ddot q + R\dot q + \frac{1}{C} q = v$$

think about two 1st-order DE

$$\frac{d}{dt}\underbrace{\begin{bmatrix} q \\ \dot q \end{bmatrix}}_{} = \begin{bmatrix} \dot q \\ \ddot q \end{bmatrix} = \begin{bmatrix} \dot q \\ \frac{1}{L}\left(v - R\dot q - \frac{1}{C} q\right) \end{bmatrix}$$

state-space representation for (DE)

more generally: $\frac{d}{dt} x = \dot x = f(x, u)$, $u = v$

where: $\underbrace{x \in \mathbb{R}^n}_{\text{system state}}$, $u \in \mathbb{R}^p$, $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$

$: (x, u) \mapsto f(x, u) = \dot x$

(b) time: continuous and discrete [AMv2 Ch 3.2]

ex: proportional-integral control on a microcontroller/embedded system

- PI $\quad u(t) = k_P e(t) + k_I \int_0^t e(\tau)\, d\tau$

  $\quad = k_P e(t) + k_I x(t)$ where $\dot x(t) = e(t)$ (DE)

  ie $x(t) \in \mathbb{R}^{\cancel{2}}$ is controller state

  and time $t \in \mathbb{R}$ is <u>continuous</u> (ie any real number)

- on an embedded system, microprocessor measure error at <u>discrete</u> instants in time $t = \Delta, 2\Delta, 3\Delta,$ $\Delta > 0$ clock cycle duration

$\longrightarrow$ approximate (DE) as $\frac{1}{\Delta}\left(\tilde x(t+\Delta) - \tilde x(t)\right) \simeq \dot x(t) = e(t)$

ie $\tilde{x}(t+\Delta) = \tilde{x}(t) + \Delta \cdot e(t)$

$\leadsto$ yields a difference equation (DE) $\tilde{x}^+ = e$
whose "solutions" are defined at times $t = k \cdot \Delta$

* to emphasize that $\tilde{x}$ defined only at discrete times, write $\tilde{x}[k]$ to denote $\tilde{x}(k \cdot \Delta)$

- more generally, with $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ denoting state vector

  and $u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in \mathbb{R}^p$ denoting input vector

then state could change in time according to:

1°. differential equation $\frac{d}{dt} x = \dot{x} = f(x, u)$

or

2°. difference equation $\tilde{x}^+ = \tilde{f}(\tilde{x}, u)$

- we'll refer to both (1°) & (2°) as (DE)
and distinguish them notationally by writing:

1°. $x(t)$ for state of differential equation at $\underbrace{\text{time } t \in \mathbb{R}}_{\text{"continuous time"}}$

2°. $\tilde{x}[k]$ for state of difference equation at $\underbrace{\text{time } t = k \cdot \Delta}_{\text{"discrete time"}}$

(c) linear systems [AMv2 Ch 3.2] [Nv7 Ch 3.3]

ex: RLC circuit: capacitor charge q & current $\dot{q}$ interact with voltage v over time:

$$\text{(DE)} \quad L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$$

∘ with $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{1}{L}\left(v - R\dot{q} - \frac{1}{C} q\right) \end{bmatrix} = \overbrace{f(x,u)}^{\in \mathbb{R}^2}$, $\overbrace{u}^{\in \mathbb{R}} = v$

note: f is linear: $f(x,u) = Ax + Bu$

→ verify

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}}_{\in \mathbb{R}^{2\times 2}} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/L \end{bmatrix}}_{\in \mathbb{R}^{2\times 1}} v$$

∘ more generally, any $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n \times p}$ defines a linear system in state-space form $\dot{x} = Ax + Bu$

ex: given $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = u$

∘ choosing $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d^{n-1}}{dt^{n-1}} y \\ \frac{d^{n-2}}{dt^{n-2}} y \\ \vdots \\ \frac{d}{dt} y \\ \vdots \end{bmatrix} \in \mathbb{R}^n$

$\in \mathbb{R}^{n\times 1}$ $\in \mathbb{R}^{n\times 1}$

$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} \quad \begin{bmatrix} \frac{d}{dt} y \\ y \end{bmatrix}$$

yields

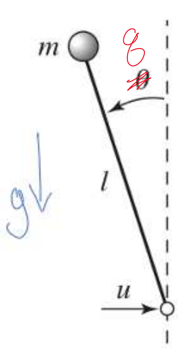
$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \overbrace{\begin{bmatrix} -a_1 x_1 - a_2 x_2 \cdots - a_n x_n + u \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix}}^{\in \mathbb{R}^{n\times 1}} = \overset{\overset{\in \mathbb{R}^{n\times 1}}{\sim}}{f(x,u)}$$

$$= Ax + Bu = \underbrace{\left[\begin{array}{c|c|c|c|c} -a_1 & -a_2 & \cdots & \cdots & -a_n \\ \hline 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right]}_{= A \in \mathbb{R}^{n\times n}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}^{\in \mathbb{R}^{n\times 1}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}}_{= B \in \mathbb{R}^{n\times 1}} u$$

$$A = \begin{array}{cc} \begin{array}{cc} n-1 & 1 \end{array} & \\ \left[\begin{array}{c|c} -a^T & \\ \hline I_{n-1} & 0_{n-1\times 1} \end{array}\right] & \begin{array}{c} 1 \\ n-1 \end{array} \end{array}$$

(d) nonlinear systems [AMv2 Ch 3.1] [Nv7 Ch 3.7]

ex: "rocket flight" (really: pendulum)



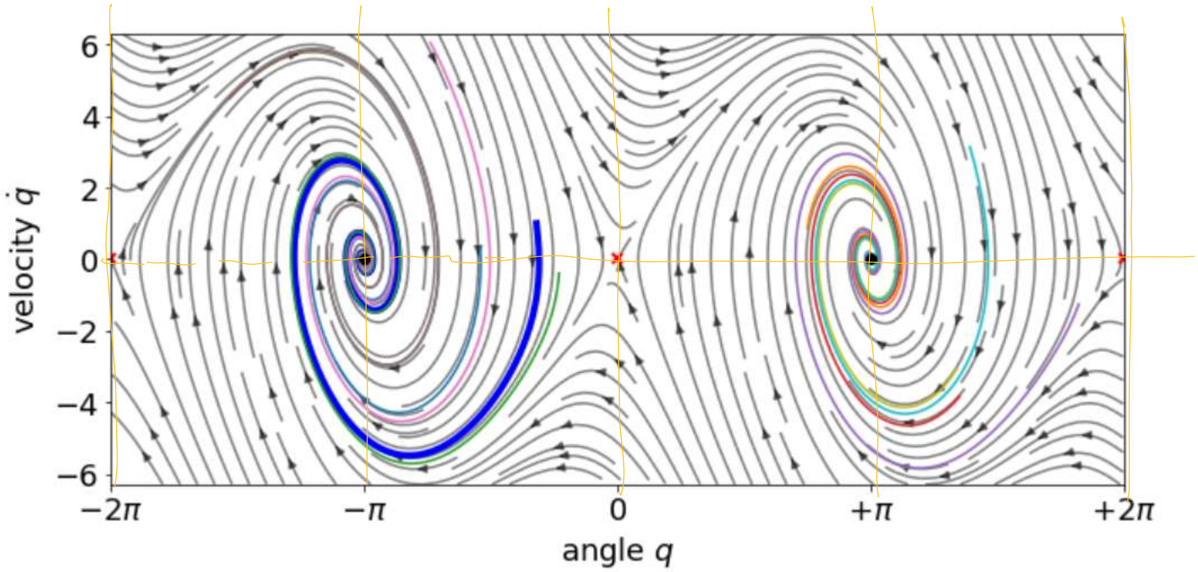
- state $x = (q, \dot{q})$ – angle, velocity
- input u – horizontal acceleration of pivot
- (DE) $m l^2 \ddot{q} = mgl \sin q - \alpha \dot{q} + l u \cos q$

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ g/\ell \sin q - \frac{\alpha}{m\ell^2}\dot{q} + \frac{1}{m\ell} u \cos q \end{bmatrix} = f(x, u)$$

"equilibria"

* when input is zero ($u = 0$), $x(0) = \begin{bmatrix} q(0) \\ \dot{q}(0) \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \Rightarrow \dot{x}(0) = 0$

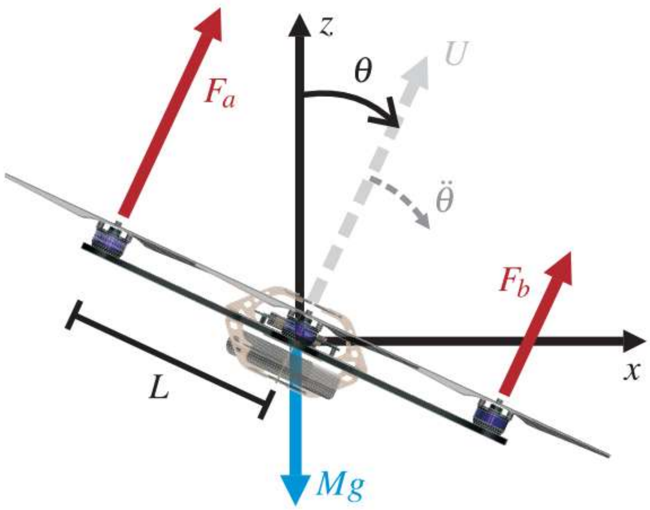
$\Rightarrow x(t) = 0$



ex: quadrotor

A Simple Learning Strategy for High-Speed Quadrocopter Multi-Flips

Sergei Lupashin, Angela Schöllig, Michael Sherback, Raffaello D'Andrea



$$M\ddot{z} = (F_a + F_b + F_c + F_d)\cos\theta - Mg \tag{1}$$
$$M\ddot{x} = (F_a + F_b + F_c + F_d)\sin\theta \tag{2}$$
$$I_{yy}\ddot{\theta} = L(F_a - F_b)\,, \tag{3}$$

$\eta = x$ (horizontal)
$\nu = z$ (vertical)

$$M\ddot{\eta} = F\sin\theta$$
$$M\ddot{\nu} = -Mg + F\cos\theta$$
$$I\ddot{\theta} = \tau$$

where $F = F_a + F_b + F_c + F_d$ is the net thrust from 4 rotors

$\tau = L(F_a - F_b)$ is the net torque around roll axis

- with $q = (\eta, \nu, \theta) \in \mathbb{R}^3$ denoting positions

$\dot{q} = \frac{d}{dt} q = (\dot{\eta}, \dot{\nu}, \dot{\theta}) \in \mathbb{R}^3$ denoting velocities,

the state is $x = (q, \dot{q}) \in \mathbb{R}^6$, input is $u = (F, \tau) \in \mathbb{R}^2$

so dynamics are $\dot{x} = \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q}(x,u) \end{bmatrix} = f(x,u)$

where $\ddot{q}(x,u) = \begin{bmatrix} F/M \sin\theta \\ -g + F/M \cos\theta \\ \tau/I \end{bmatrix}$

$$f : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}^6$$
$$: (x, u) \mapsto f(x, u)$$