03 -- Mon Oct 18

ECE 447: Control Systems (Fall 2021)

Prof: Sam Burden TA. Sat Singh

this week: * HW2 due Fri Oct 22

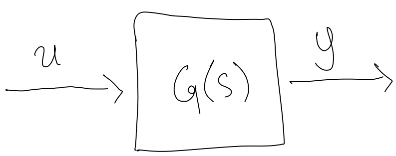
- ☑ HW 3 assigned → due Fri Oct 29
- ☐ week 3 lecture material
- ☐ break
- ☐ Office "Hour"

Prof Burden TODO:

---

tutorial on roots, eigenvalues, and characteristic polynomials

* why is stability governed by roots of characteristic polynomial



$$u(t) = e^{st} \Rightarrow y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{\text{homogeneous}} + \underbrace{G(s)\, e^{st}}_{\text{particular}}$$

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$\Updownarrow$$

time domain $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$ ← RHS is zero for homogeneous response

Q: does there exist (non-zero) signal y that satisfies this eqn?

A: YES! but: only solutions are (linear combinations of)

$y(t) = e^{s_k t}$ where s_k is a root of characteristic polynomial $s^n + a_1 s^{n-1} + \dots + a_n$

* eqn is satisfied only when s_k is a root!

why? $\frac{d}{dt} e^{s_k t} = s_k e^{s_k t}$, so $\frac{d^2}{dt^2} e^{s_k t} = \frac{d}{dt} s_k e^{s_k t} = s_k^2 e^{s_k t}$

so $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \dots + a_n y = (s_k^n + a_1 s_k^{n-1} + \dots + a_n) y = 0$

ex: cruise control $m\dot{v} = F = -\beta v + \tau$

v – velocity (output)

τ – throttle (input)

$\tau \to \boxed{G(s)} \to v$ $\qquad G(s) = \frac{1}{ms + \beta}$

$m\dot{v} = -\beta v + \tau \iff m\dot{v} + \beta v = \tau \overset{\mathcal{L}}{\longleftrightarrow} (ms + \beta) v = \tau$

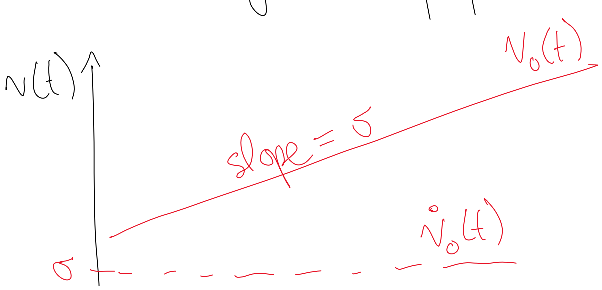
$\iff v = \frac{1}{ms + \beta} \tau = G(s) \tau$

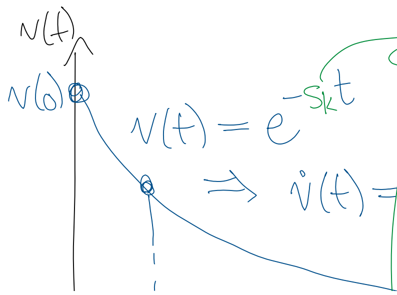
two cases:

1°. (homogeneous) $\tau \equiv 0$ but $v(0) = 25$ mph $\Rightarrow \dot{v}(t) \propto v(t)$ $= -\frac{\beta}{m} v(t)$

then we want $v : [0, T] \to \mathbb{R}$ s.t. $\forall t \geq 0 : m\dot{v}(t) = -\beta v(t)$

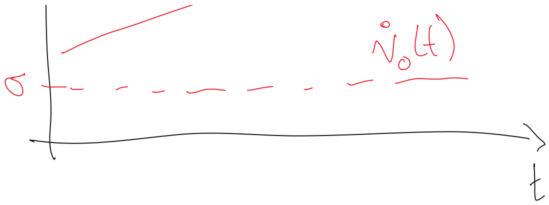
Q: what signal is proportional to its time derivative at all times?

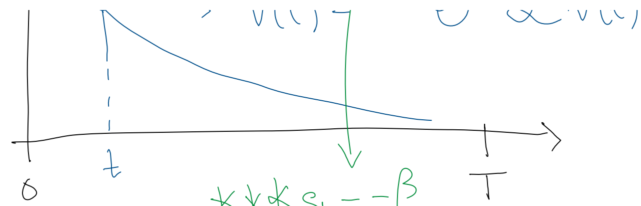




$v(t) = e^{-s_k t}$ $\qquad s_k = -\frac{\beta}{m}$

$\Rightarrow \dot{v}(t) = - \; e^{-s_k t} \propto v(t)$





$*** \; s_k = \frac{-\beta}{m}$ is the root of $ms + \beta$

2°. (particular) $\tau \neq 0$ $\quad \tau(t) = e^{st} \Longrightarrow y(t) = c_1 e^{s_1 t} + G(s) e^{st}$

suppose I apply a sinusoidal input $\tau_{\sin}(t) = \sin(t)$

then I'll represent $\sin(t)$ as a linear combination of complex exponentials, $\sin(t) = \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \underbrace{\alpha}_{= \frac{1}{2i}} e^{it} + \underbrace{\gamma}_{= -\frac{1}{2i}} e^{-it}$

so particular response is $\alpha G(i) e^{it} + \gamma G(-i) e^{-it}$

---

tutorial on roots, eigenvalues, and characteristic polynomials

$\xrightarrow{u} \boxed{G(s)} \xrightarrow{y} \qquad u(t) = e^{st} \Longrightarrow y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{\text{homogeneous}} + \underbrace{G(s) e^{st}}_{\text{particular}}$

* why is stability governed by roots of characteristic polynomial?

time domain $\quad \frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0 \leftarrow$ homogeneous response

freq. domain $\quad (s^n + a_1 s^{n-1} + \cdots + a_n) y = 0 \leftarrow$ does there exist nonzero y s.t. this equation is zero?

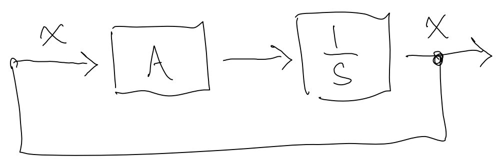
* YES!, but: only solutions are $y(t) = e^{s_k t}$

where s_k are roots of $\underbrace{s^n + a_1 s^{n-1} + \cdots + a_n}$

characteristic polynomial — denominator of $G(s)$

* why is stability governed by eigenvalues?

time domain $\quad \dot{x} = Ax$



freq domain $\quad sx = Ax \iff (sI - A)x = 0 \leftarrow$ does there exist nonzero x s.t. this eqn is zero?

* YES! but: only solutions are $x(t) = e^{s_k t} v_k$

where $A v_k = s_k v_k \leftarrow s_k, v_k$ are eigval / eigvec pair

i.e. s_k is a root of $\underbrace{\det(sI - A)}$

characteristic polynomial

of