01-control-systems

ECE 447: Control Systems

goal: what is a control system?

(a) what is a system? ← a mathematical model for a transformation from inputs to outputs

(b) differential equations (DE) ← mathematical models that relate inputs/outputs & their derivatives at every time

(c) transfer functions ← math model that specifies how input e^{st} transforms to output $G(s)e^{st}$

(d) block diagrams ← math model of system interconnection

(e) feedback control ← design of systems and their interconnections to achieve desired closed-loop transformations

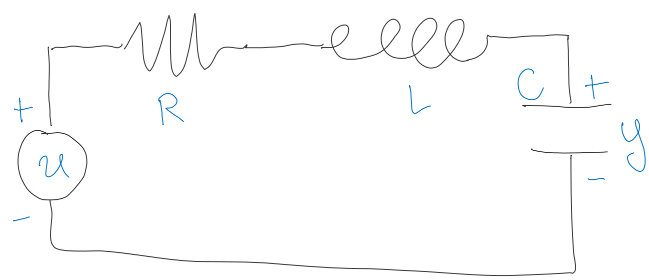
---

(a) what is a system?

→ a mathematical model of a transformation from inputs to outputs

voltage source u — cap ~~voltage~~ charge y

ex:



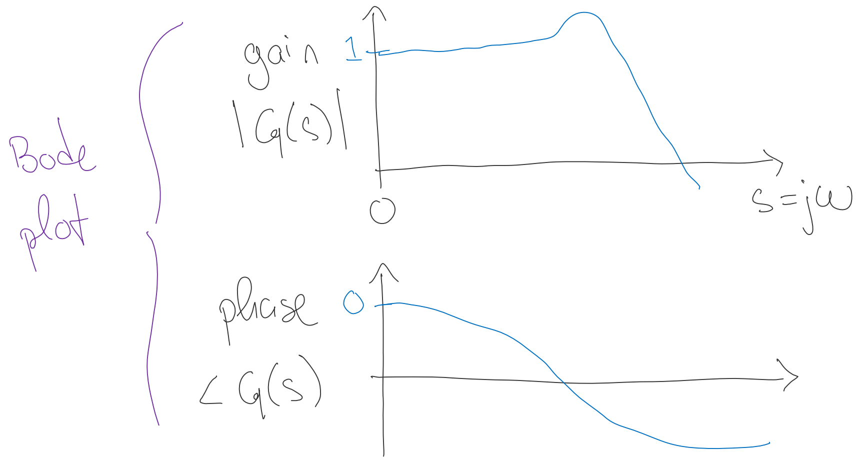
* how does input voltage u transform to output ~~voltage~~ charge y?

1°: differential equation (DE) KVL ⇒ $u = Ri + L\frac{d}{dt}i + \frac{1}{C}y$

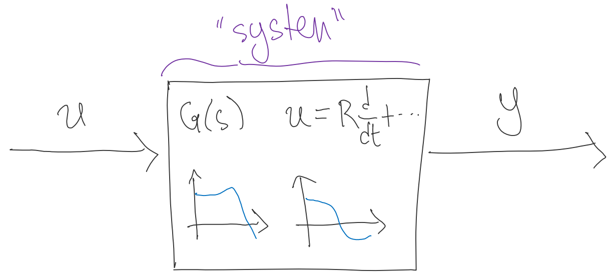
$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{C}y$$

$$= R\frac{d}{dt}y + L\frac{d^2}{dt^2}y + \frac{1}{C}y$$

2°. transfer function $u = e^{st} \Rightarrow y = G(s)u = \left(\dfrac{1}{Ls^2 + Rs + 1/C}\right)u$



3°. block diagram



(b) differential equations (DE) [AMv2 Ch 2] [Nv7 Ch 3,4]

(DE) $\dfrac{d^n}{dt^n}y + a_1\dfrac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y$ $\quad$ y - output $\quad$ t - time

$\qquad = b_1\dfrac{d^{n-1}}{dt^{n-1}}u + \cdots + b_n u$ $\quad$ u - input

where $\{a_k\}_{k=1}^{n}, \{b_\ell\}_{\ell=1}^{n} \subset \mathbb{R}$

∘ notice: (DE) is specified by two polynomial expressions:

"characteristic" $\longrightarrow$ 1°. $a(\boxed{s}) = \boxed{s}^n + a_1\boxed{s}^{n-1} + \cdots + a_n$

"characteristic" polynomial $\longrightarrow$ 1°. $a(\boxed{s}) = \boxed{s}^n + a_1 \boxed{s}^{n-1} + \cdots + a_n$

2°. $b(\boxed{s}) = \cancel{\boxed{s}^n} \cancel{+} b_1 \boxed{s}^{n-1} + \cdots + b_n$

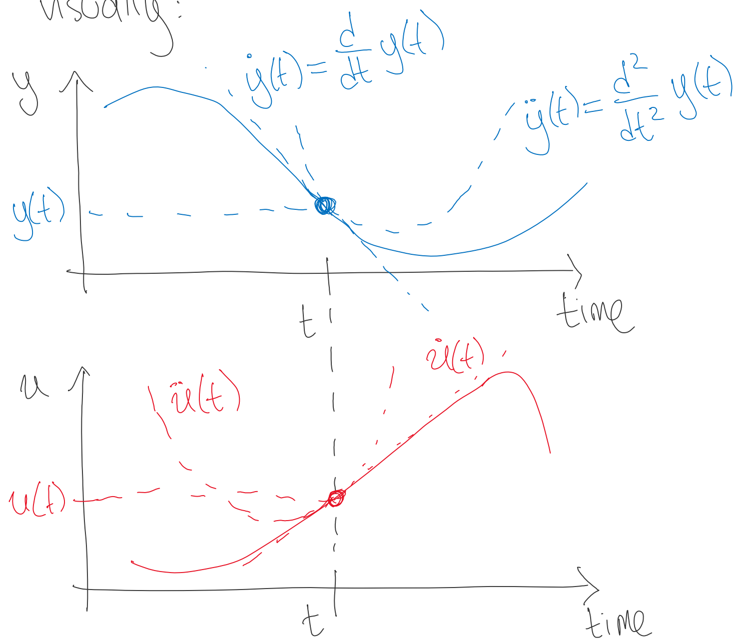
* we'll see that these polynomials govern input/output behavior of (DE)

- a "solution" to (DE) is a pair of signals (u, y)

$$u : \mathbb{R} \longrightarrow \mathbb{R} \qquad y : \mathbb{R} \longrightarrow \mathbb{R}$$
$$: t \longmapsto u(t) \qquad : t \longmapsto y(t)$$

that satisfy (DE) at all $t \in \mathbb{R}$: for all times!

visually:



algebraically: $\forall t \in \mathbb{R}$:

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_n y(t)$$
$$= b_1 \frac{d^{n-1}}{dt^{n-1}} u(t) + \cdots + b_n u(t)$$

fact: every solution to (DE) is a linear combination (ie sum) of:

1°. homogeneous solution (where $u = 0$)

2°. particular solution (where $u \neq 0$)

1°. homogeneous solution: when $u = 0$, $\frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_n y = 0$

1°. homogeneous solution: when $u=0$, $\frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = 0$

y is a linear combination of (complex) exponentials:

$$y(t) = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t} = \sum_{k=1}^{n} c_k e^{s_k t}$$

where $\{s_k\}_{k=1}^n \subset \mathbb{C}$ are the <u>roots</u> of characteristic polynomial $a(s)$ $\Big\}$ ie $a(s_k)=0$

* recall: n-th order polynomial a has no more than n roots (and no fewer than 1 root) eg $\square^n = 0$

and $\{c_k\}_{k=1}^n \subset \mathbb{C}$ are determined by <u>initial condition</u>

$$\{y(0), \dot{y}(0), \ddot{y}(0), \ldots, \frac{d^{n-1}}{dt^{n-1}}y(0)\} = \left\{\frac{d^k}{dt^k}y(0)\right\}_{k=0}^{n-1}$$

(c) transfer functions [AMv2 Ch 2] [Nu7 Ch 2]

- in contrast to DE, transfer functions characterize how a specific class of input signals are transformed by a system

- when $u(t) = e^{st}$, $s \notin \{s_k\}_{k=1}^n$ ie s is not a root of characteristic polynomial $a(\square)$

know $y(t) = G(s)e^{st}$, some $G(s)\in\mathbb{C}$

$\frac{d^k}{dt^k}u(t) = s^k e^{st}$

$\frac{d^k}{dt^k}y(t) = s^k G(s) e^{st}$

$\rightarrow$ verify that (u,y) satisfy (DE)

$-\ \frac{d}{dt}u(t) = s e^{st}$, $\frac{d^2}{dt^2}u(t) = s^2 e^{st}$, $\ldots$, $\frac{d^{n-1}}{dt^{n-1}}u(t) = s^{n-1}e^{st}$

- $\frac{d}{dt}u(t) = s\,e^{st}$, $\frac{d^2}{dt^2}u(t) = s^2 e^{st}$, ..., $\frac{d^{n-1}}{dt^{n-1}}$

- $\frac{d}{dt}y(t) = s\,G(s)e^{st}$, ..., $\frac{d^n}{dt^n}y(t) = s^n G(s) e^{st}$

- substituting into (DE): $(s^n + a_1 s^{n-1} + \cdots + a_n)\,G(s)\cancel{e^{st}}$
$= (b_1 s^{n-1} + \cdots + b_n)\cancel{e^{st}}$

* so if $\underbrace{G(s)}_{\text{termed the transfer function}} = \dfrac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} = \dfrac{b(s)}{a(s)}$ then (u,y) satisfy (DE)

(b & c) differential equation & transfer function

$$\underbrace{\frac{d^n}{dt^n}y + a_1\frac{d^{n-1}}{dt^{n-1}}y + \cdots + a_n y = b_1\frac{d^{n-1}}{dt^{n-1}}u + \cdots + b_n u}_{\text{differential equation}} \qquad \underbrace{\frac{b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}_{\text{transfer function}} = G(s)$$

- exponential input $u(t) = e^{st}$ to a linear time-invariant system yields exponential output $y(t) = \underbrace{\sum_{k=1}^{n} c_k e^{s_k t}}_{1^\circ} + \underbrace{G(s)e^{st}}_{2^\circ}$

1°: homogeneous response to initial condition $\left\{\frac{d^k}{dt^k}y(0)\right\}_{k=0}^{n-1}$

2°: particular response to input signal u

2°: particular response to input signal u

note: $\{s_k\}_{k=1}^{n}$ are the roots of characteristic polynomial $a(s)$

fact: since a_k's are real, the roots of $a(s)$ are:

real or complex-conjugate pairs

real → real exponential $e^{s_k t}$

→ plot $e^{s_k t}$ vs t when:

$s_k < 0$; $s_k = 0$; $s_k > 0$

complex-conjugate pairs → $s_k = \sigma \pm j\omega \in \mathbb{C}$, $\sigma, \omega \in \mathbb{R}$

yields complex exponential

$$e^{s_k t} = e^{\sigma t}\cos(\omega t) \pm j e^{\sigma t}\sin(\omega t)$$

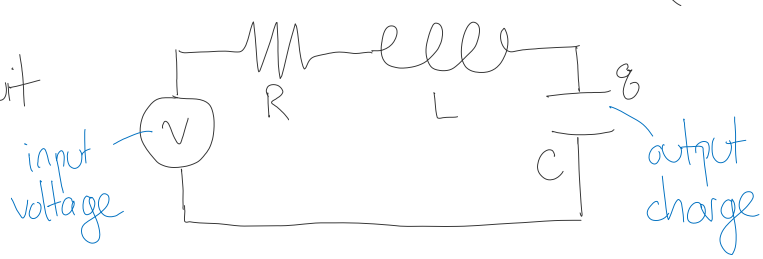
→ plot $\operatorname{Re} e^{s_k t}$ vs t;

how does plot vary with σ, ω?

*note: signal decays asymptotically to zero $\underline{\text{if and only if}}$ $\operatorname{Re} s_k < 0$

$(\Longleftrightarrow)$

ex: RLC circuit



(DE) $L\ddot{q} + R\dot{q} + \frac{1}{C} q = v$

↳ characteristic polynomial

$$a(s) = Ls^2 + Rs + \frac{1}{C}$$

→ compute roots of a

(TF) $G(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{Ls^2 + Rs + \frac{1}{C}}$

• if $u(t) = e^{st}$, $|s|$ small

then $y(t) = G(s)\,u(t)$

$\simeq C\,u(t)$

→ complex roots of ...

– $s_\pm = \dfrac{-R \pm \sqrt{R^2 - 4L/C}}{2L}$

$\simeq C\,u(t)$

so if $s = 0$, $u(t) = 1$, $y(t) = C$

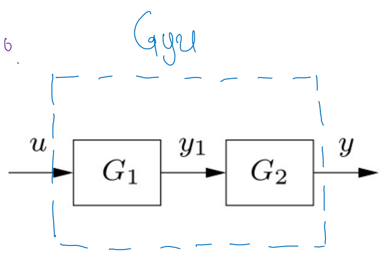
→ is $\mathrm{Re}\, s_\pm < 0$? ☑

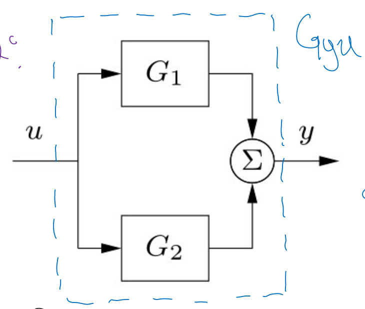
– if $R^2 - 4L/C \geq 0$ then $\mathrm{Re}\, s_\pm < 0$ (assuming $R, L, C > 0$)

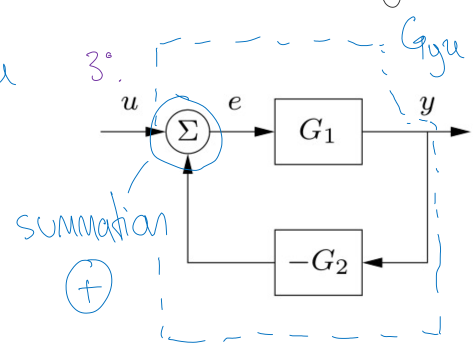
– also $\mathrm{Re}\, s_\pm < 0$ if $R^2 - 4L/C < 0$

(d) block diagrams

∘ particularly useful for modeling interconnections between systems

1°. 

2°. 

3°. 

∘ each $\boxed{G_1}$, $\boxed{G_2}$ (blocks) is a system

∘ each $\xrightarrow{u}$, $\xrightarrow{y}$, $\longrightarrow$ (arrows) is a signal

1°. $y = G_2 y_1 = \underbrace{G_2 G_1}_{G_{yu}} u = G_{yu} \cdot u$

2°. $y = G_1 u + G_2 u = (G_1 + G_2) u = G_{yu} u$

3°. $y = G_1 e = G_1 (u - G_2 y) = G_1 u - G_1 G_2 y$ bad place!

$\iff u + G_1 G_2 u = G_1 u$ → assuming $G_1 G_2 = -1$

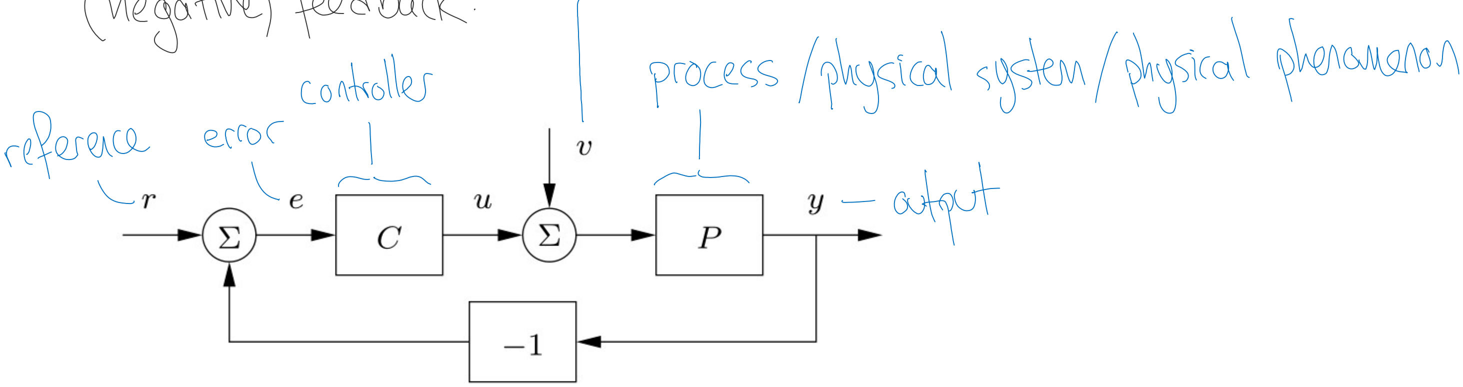
$\iff y + G_1 G_2 y = G_1 u$ $\quad$ ↳ assuming $G_1 G_2 \neq -1$

$\iff (1 + G_1 G_2) y = G_1 u \overset{*}{\iff} y = \dfrac{G_1}{1 + G_1 G_2} u = G_{yu} \cdot u$

(f) feedback control

- a control system is an interconnection between a physical system P and a controller C — our goal is to design C

* today we'll design C to do disturbance rejection via (negative) feedback:



* we want to tune transformation G_{yv}

$$y = P(v + u) = Pv + PCe = Pv + PC(r - y)$$

$$\iff \underbrace{y + PCy}_{= (1 + PC) y} = Pv + PCr$$

$$\iff y = \frac{P}{1 + PC} v + \frac{PC}{1 + PC} r = G_{yv} \cdot v + G_{yr} \cdot r$$

ex: consider $P(s) = \dfrac{b}{s + a} \iff \dot{y} + a y = b u$, $\quad a, b > 0$

- interpreted as a model for velocity of a car:

- interpreted as a model for velocity of a car:

  r – desired velocity $\qquad$ v – road slope; headwind

  u – throttle / gas pedal $\qquad$ a – air resistance; wheel friction

  y – car velocity $\qquad$ b – conversion from throttle to accel

* try two different controllers: 1°. proportional 2°. proportional-integral

1°. proportional control: $u = k_p e$ i.e. $C(s) = k_p$

→ determine transfer function G_{yv}

– $G_{yv} = \dfrac{P}{1+PC} = \dfrac{b/s+a}{1 + b/s+a \cdot k_p} \cdot \dfrac{s+a}{s+a} = \dfrac{b}{s+(a+b\cdot k_p)}$

- this (closed-loop) system is stable (ie $v \overset{G_{yv}}{\longmapsto} y$ doesn't "blow up")

  $\iff$ all roots of characteristic polynomial

  $$a(s) = s + (a + b\cdot k_p) \text{ are negative, ie } (a + k_p \cdot b) > 0$$

- in this case, constant disturbance $v(t) = v_0$ (slope of hill)

  yields $y \to y_0 = G_{yv}(0) \cdot v_0 = \dfrac{b}{a+k_p \cdot b} v_0$

  * by increasing $k_p > 0$, steady-state error y_0 decreases

2°. proportional-integral: $u(t) = k_p e(t) + k_I \int_0^t e(t)\, d\tau$

→ determine transfer function $C(s)$ that yields

– $C(s) = k_p + \frac{1}{s} k_I = \dfrac{k_p s + k_I}{s}$

- $C(s) = k_P + \frac{1}{s} k_I = \frac{k_P s + k_I}{s}$

→ determine transfer function G_{yv}

- $G_{yv} = \frac{P}{1+PC} = \frac{bs}{s^2 + (a+bk_P)s + bk_I}$

∘ equivalently, $\ddot{y} + (a+bk_P)\dot{y} + bk_I y = b\dot{v}$

* constant disturbance $v = v_0$ yields <u>zero</u> steady-state error: $G_{yv}(0) = 0$

∘ characteristic polynomial $a(s) = s^2 + (a+bk_P)s + bk_I$

- suppose we want roots $-\sigma_d \pm j\omega_d$ (d = "desired")

  ie characteristic polynomial $(s + \sigma_d + j\omega_d)\cdot(s + \sigma_d - j\omega_d)$

$$= s^2 + 2\sigma_d s + (\sigma_d^2 + \omega_d^2)$$

  ⇝ yields time-domain homogeneous response $e^{-\sigma_d t}\sin(\omega_d t)$

  ie damped oscillations $e^{-\sigma_d t}\cos(\omega_d t)$

→ determine k_P, k_I so that $\sigma_d \pm j\omega_d$ are roots of $a(s)$

- $(a + bk_P) = 2\sigma_d \iff k_P = \frac{2\sigma_d - a}{b}$
- $bk_I = (\sigma_d^2 + \omega_d^2) \iff k_I = \frac{\sigma_d^2 + \omega_d^2}{b}$