control barrier functions (CBF)

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Control Barrier Functions: Theory and Applications

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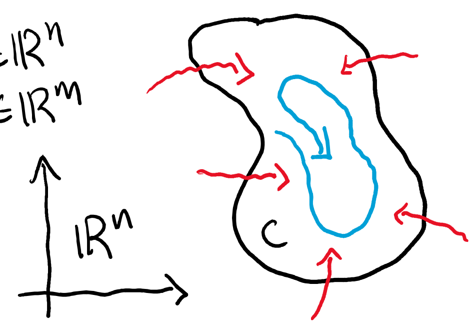
def: safety for $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$
means choosing $u(x)$ to make
safe set C forward-invariant
(& , ideally, asymptotically stable)



idea: choose $u(x)$ to "point inward" along boundary ∂C

- let C be defined by function $h : \mathbb{R}^n \to \mathbb{R}$
  s.t. $C = \{x : h(x) \geq 0\}$, $\partial C = \{x : h(x) = 0\}$

def: an extended class K function $\alpha : \mathbb{R} \to \mathbb{R}$
is strictly increasing and $\alpha(0) = 0$

def: h is a control barrier function (CBF) if def: h is a control barrier function (CBF) if

$$\exists \alpha : \sup_u \big(\underbrace{L_f h(x) + L_g h(x) u}_{\mathring{h} = Dh \cdot f + Dh \cdot g \cdot u} \big) \geq -\alpha(h(x))$$

thm: if h is a CBF and $Dh(x) \neq 0$ for $x \in \partial C$

then any Lipschitz $u : \mathbb{R}^n \to \mathbb{R}^m$

satisfying $u(x) \in \{ \mu : L_f h(x) + L_g h(x) \mu + \alpha(h(x)) \geq 0 \}$

renders C safe (ie forward-invariant)

furthermore, if u renders C safe, then $h|_C$ is a CBF

◦ one may ask: how to choose u? there turns out to be a simple answer if we allow ourselves to solve a convex optimization problem:

- given nominal policy $\bar{u} : \mathbb{R}^n \to \mathbb{R}^m$, define u by

$$u(x) = \arg\min_{\mu} \frac{1}{2} \| \mu - \bar{u}(x) \|^2$$
$$\text{s.t. } L_f h(x) + L_g h(x) \mu \geq \alpha(h(x))$$

* this quadratic program has a closed-form solution!

→ solve this QP

- rewrite as $\min_{\mu} \frac{1}{2} \| \mu - \mu_0 \|^2$ s.t. $A\mu \geq b$

- Lagrangian is $L(\mu, \lambda) = \| \mu - \mu_0 \|^2 + \lambda (A\mu - b)$

- $\frac{1}{2}(\mu - \mu_0)^T (\mu - \mu_0) = \frac{1}{2} \mu^T \mu - \mu_0^T \mu + \frac{1}{2} \mu_0^T \mu_0$

$\Rightarrow D_\mu L = \mu^T - \mu_0^T + \lambda A = 0 \Leftrightarrow \mu = A^T \lambda^T + \mu_0$

$$-D_\lambda L = A\mu - b = 0 \iff AA^T\lambda^T + A\mu_0 - b = 0$$

$$\iff \lambda^T = (AA^T)^{-1}(b - A\mu_0)$$

$\Rightarrow$ solution is

$$\mu^* = \begin{cases} A^T(AA^T)^{-1}(b - Ax_0) + \mu_0, & A\mu^* = b \\ \mu_0, & \text{else} \end{cases}$$

* Lipschitz continuous !

<u>takeaway</u>: CBF framework provides an optimization-based approach to safety for nonlinear systems