system-level synthesis (SLS)

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Review article

System level synthesis

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- consider the DT-LTV system

$$x_{t+1} = A_t x_t + B_t u_t + w_t, \quad x_t \in \mathbb{R}^n, \; u_t \in \mathbb{R}^p$$

- suppose $u_t = K_t(x_0, x_1, \cdots, x_t)$, i.e. causal feedback, and let K_t be linear
- define

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_T \end{bmatrix}, \quad u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_T \end{bmatrix}, \quad w = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_T \end{bmatrix}$$

$$K = \begin{bmatrix} K^{0,0} & 0 & \cdots & 0 \\ K^{1,1} & K^{1,0} & & \vdots \\ \vdots & & \ddots & 0 \end{bmatrix} \quad \text{so } u_t = K_t \cdot x$$

K_1

$$\begin{bmatrix} K^{1,1} & K^{1,0} & & \vdots \\ \vdots & & \ddots & 0 \\ K^{T,T} & \cdots & K^{T,1} & K^{T,0} \end{bmatrix} \quad K_1$$

and $$A = \begin{bmatrix} A_0 & & & \\ & A_1 & & 0 \\ 0 & & \ddots & A_{T-1} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} B_0 & & & \\ & B_1 & & 0 \\ 0 & & \ddots & B_{T-1} & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & & & \\ I & 0 & & 0 \\ 0 & I & & \\ 0 & & \ddots & \\ & & I & 0 \end{bmatrix}$$

* note: A, B, K, Z are all block lower-triangular

- now the dynamics over the entire time horizon are

$$x = ZAx + ZBu + w = Z(A+BK)x + w$$

- rearranging: $x = (I - Z(A+BK))^{-1} w$

$$u = \underbrace{K(I - Z(A+BK))^{-1}} w$$

→ verify that ( ) is $\underbrace{\text{block lower-triangular}}_{\text{"BLT"}}$

1°. product of BLT is BLT

2°. inverse of BLT is BLT

- we can write $\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} w$ with Φ_x, Φ_u BLT

thm 2.1:

1°. $\begin{bmatrix} I - ZA & -ZB \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I$ for all system responses

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1°. $[I - zA \quad -zB] \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I$ for all system responses

2°. if Φ_x, Φ_u are BLT then $K = \Phi_u \Phi_x^{-1}$ is controller

pf: (1°.) $\Phi_x = (I - z(A+BK))^{-1}$, $\Phi_u = K(I - z(A+BK))^{-1}$

so $(I-zA)\Phi_x - zB\Phi_u = I$ $\longrightarrow$ easy to verify

(2°.) note that $\Phi_x^{t,0} = I$ so inverse exists

- $(I - z(A + B\Phi_u\Phi_x^{-1}))^{-1} = \left[((I-zA)\Phi_x - zB\Phi_u)\Phi_x^{-1}\right]^{-1}$

$= \Phi_x\left[(I-zA)\Phi_x - zB\Phi_u\right]^{-1} = \Phi_x$

so $x = \Phi_x w$

- $u = \Phi_u \Phi_x^{-1} x = \Phi_u \Phi_x^{-1} \Phi_x w = \Phi_u w$ ∎

- so what? now the system response is parameterized by Φ_x & Φ_u so closed-loop dynamics become <u>linear</u> constraints in an optimal control problem

<u>ex</u>: (LQR) let $w \sim \mathcal{N}(0, \Sigma_w)$ be iid

$$\min_{x_t, u_t} \sum_{t=0}^{T} \mathbb{E}\left[x_t^T Q x_t + u_t^T R_t u_t\right]$$

s.t. $x_{t+1} = Ax_t + Bu_t + w_t$

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can be rewritten

$$\begin{cases} \min_{\Phi_x, \Phi_u} \left\| \begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \Sigma_W^{1/2} \right\|_F^2 \\ \text{s.t. } \begin{bmatrix} I - ZA & -ZB \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} = I \end{cases}$$

→ convex quadratic program in Φ_x, Φ_u!

aside: if you're worried about inverting Φ_x, fear not!

$$u_t = \sum_{\tau=1}^{T} \Phi_u(t,\tau)\hat{w}_{t-\tau}, \quad \hat{x}_{t+1} = \sum_{\tau=1}^{T-1} \Phi_x(t+1,\tau+1)\hat{w}_{t-\tau},$$

$\hat{w}_t = x_{t+1} - \hat{x}_{t+1}$ implements $u = Kx = \Phi_u \Phi_x^{-1} x$

takeaway: SLS allows reformulation of optimal (robust) control problems — more amenable to constrained/distributed settings