linear quadratic regulation

Lewis et al 6.2

Stengel § 3.7

goal: derive linear-quadratic regulator (LQR), the optimal controller for linear DE

- DT linear quadratic regulation (DT-LQR)

$$\min_u c_\tau(x,u) \text{ s.t. } x_{t+1} = A_t x_t + B_t u_t$$

$$c_\tau(x,u) = \frac{1}{2} x_t^T P_t x_t + \frac{1}{2} \sum_{s=\tau}^{t-1} x_s^T Q_s x_s + u_s^T R_s u_s$$

– we'll use Bellman's principle to determine optimal control:

$$v_\tau^*(x(\tau)) = \min_{u(\tau)\in\mathbb{R}^m} \left[L(\tau, x(\tau), u(\tau)) + c_{\tau+1}^*(x(\tau+1)) \right]$$

→ determine optimal control & value at $\tau = t$

– since $c_t = \frac{1}{2} x_t^T P_t x_t$, $u_t^* = 0$

and $v_t^* = \frac{1}{2} x_t^T P_t x_t$

→ minimize c_{t-1} to determine u_{t-1}^*, v_{t-1}^*:

$$\frac{1}{2} x_{t-1}^T Q_{t-1} x_{t-1} + \frac{1}{2} u_{t-1}^T R_{t-1} u_{t-1} + \frac{1}{2} x_t^T P_t x_t$$

– subbing $x_t = A_{t-1} x_{t-1} + B_{t-1} u_{t-1}$

& differentiating $D_{u_{t-1}} c_{t-1}$:

$u_{t-1}^T R_{t-1} + (A_{t-1}x_{t-1} + B_{t-1}u_{t-1})' P_t B_{t-1}$

- solving $D_{u_{t-1}} c_{t-1} = 0$ yields

$$u_{t-1}^* = -(B_{t-1}^T P_t B_{t-1} + R_{t-1})^{-1} B_{t-1}^T P_t A_{t-1} x_{t-1}$$

- $D^2_{u_{t-1}} c_{t-1} = R_{t-1} + B_{t-1}^T P_t B_{t-1} \succeq 0$,
  so u_{t-1}^* is local min (strict if $\succ 0$)

* optimal control is linear in state!
- for simplicity define

$$K_{t-1} = (B_{t-1}^T P_t B_{t-1} + R_{t-1})^{-1} B_{t-1}^T P_t A_{t-1}$$

- then optimal value v_{t-1}^*:

$$\frac{1}{2} x_{t-1}^T \Big[(A_{t-1} - B_{t-1}K_{t-1})^T P_t (A_{t-1} - B_{t-1}K_{t-1}) + K_{t-1}^T R_{t-1} K_{t-1} + Q_{t-1} \Big] x_{t-1}$$

$$= \frac{1}{2} x_{t-1}^T P_{t-1} x_{t-1}$$

- at $t-2$, we'll perform identical calculation, so optimal policy obtained by recursion:

optimal sol'n to DT-LQR

$$\boxed{\begin{aligned} K_s &= (B_s^T P_{s+1} B_s + R_s)^{-1} B_s^T P_{s+1} A_s \\ u_s^* &= -K_s x_s \\ P_s &= (A_s - B_s K_s)^T P_{s+1} (A_s - B_s K_s) + K_s^T R_s K_s + Q_s \\ v_s^* &= \frac{1}{2} x_s^T P_s x_s \end{aligned}}$$

↳ called a Riccati DE (Jacopo Riccati,

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- consider (CT – LQR)

$$\min_u C_\tau(x,u) \text{ s.t. } \dot{x}_t = A_t x_t + B_t u_t$$

$$J_\tau(x,u) = \frac{1}{2} x_t^T P_t x_t + \frac{1}{2}\int_\tau^t x_s^T Q_s x_s + u_s^T R_s u_s \, ds$$

- can't obviously apply Bellman's principle, since there aren't discrete "stages"

- can try discretizing & applying discrete solution

→ discretize (CT-OCP) with step size $\Delta > 0$ to obtain (DT-OCP)

- $x_{s+\Delta} = \bar{A}_s x_s + \bar{B}_s u_s$,

any convergent discretization scheme will satisfy this approximation

$$\left.\bar{A}_s = I + \Delta A_s + O(\Delta^2), \bar{B}_s = \Delta B_s + O(\Delta^2)\right\}$$

$$\bar{C}_\tau = \frac{1}{2} x_t^T P x_t + \frac{1}{2}\sum_{s=\tau}^{t-1} x_s^T \bar{Q}_s x_s + u_s^T \bar{R}_s u_s$$

$$\left.\bar{Q}_s = \Delta Q_s + O(\Delta^2), \bar{R}_s = \Delta R_s + O(\Delta^2)\right\}$$

- optimal solution to (DT-OCP):

$$K_s = (\bar{B}_s^T P_{s+\Delta} \bar{B}_{s+\Delta} + \bar{R}_{s+\Delta})^{-1} \bar{B}_{s+\Delta}^T P_{s+\Delta} \bar{A}_{s+\Delta}$$

$$u_s^* = -K_s x_s$$

$$P_s = (\bar{A}_s - \bar{B}_s K_s)^T P_{s+\Delta} (\bar{A}_s - \bar{B}_s K_s) + K_s^T \bar{R}_s K_s + \bar{Q}_s$$

$$c_s^* = \frac{1}{2} x_s^T P_s x_s$$

$\to$ determine K_s to 0-th order in Δ

- since $\bar{A}_s = I + \Delta A_s + O(\Delta^2)$, $\bar{B}_s = B_s + O(\Delta^2)$

$$\Rightarrow \bar{B}_{s+\Delta}^T P_{s+\Delta} \bar{B}_{s+\Delta} + \bar{R}_{s+\Delta} = \Delta R_s + O(\Delta^2)$$

$$\Rightarrow K_s = R_s^{-1} B_s^T P_s + O(\Delta)$$

$\to$ determine P_s to 1-st order in Δ

$$(\bar{A}_s - \bar{B}_s K_s)^T P_{s+\Delta} (\bar{A}_s - \bar{B}_s K_s)$$

$$= (I + \Delta(A_s - B_s K_s))^T P_{s+\Delta} (I + \Delta(A_s - B_s K_s))$$

$$= P_{s+\Delta} + \Delta(A_s - B_s K_s)^T P_{s+\Delta} + \Delta P_{s+\Delta}(A_s - B_s K_s) + O(\Delta^2)$$

$$= P_{s+\Delta} + \Delta A_s^T P_{s+\Delta} + \Delta P_{s+\Delta} A_s - 2\Delta P_s B_s R_s^{-1} B_s^T P_s + O(\Delta^2)$$

$\to$ compute $\mathring{P}_s = \lim_{\Delta \to 0} \frac{1}{\Delta}(P_{s+\Delta} - P_s)$

$$\frac{-1}{\Delta}(P_{s+\Delta} - P_s) = A_s^T P_{s+\Delta} + P_{s+\Delta} A_s - P_s B_s R_s^{-1} B_s^T P_s + Q_s + O(\Delta)$$

$$\Rightarrow \mathring{P}_s = -(A_s^T P_s + P_s A_s - P_s B_s R_s^{-1} B_s^T P_s + Q_s)$$

- to summarize: optimal sol'n to (CT-LQR)

Riccati DE

$$\boxed{\begin{aligned} &\mathring{P}_s = -(A_s^T P_s + P_s A_s - P_s B_s R_s^{-1} B_s^T P_s + Q_s) \\ &K_s = R_s^{-1} B_s^T P_s, \quad u_s^* = -K_s x_s, \quad v_s^* = \frac{1}{2} x_s^T P_s x_s \end{aligned}}$$

$$\boxed{\begin{aligned} &\dot{P}_s = -\left(A_s^T P_s + P_s A_s - P_s B_s R_s^{-1} B_s^T P_s + Q_s\right) \\ &K_s = R_s^{-1} B_s^T P_s , \quad u_s^* = -K_s x_s , \quad v_s^* = \tfrac{1}{2} x_s^T P_s x_s \end{aligned}}$$