trajectory optimization

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goal: necessary & sufficient conditions for optimality in optimal control problems (OCP)

- recall DT optimal control problem (DT-OCP)

$$\min_{u} c(x,u) \quad \text{s.t.} \quad x(s+1) = F(s, x(s), u(s))$$

$$c(x,u) = \ell(t, x(t)) + \sum_{s=0}^{t-1} \mathcal{L}(s, x(s), u(s))$$

- Bellman's principle reduces this to solving a sequence of NLP:

  if we let $v_s^*(x(s))$ denote optimal cost from state $x(s)$ at time s,

$$v_s^*(x(s)) = \min_{\substack{u(s,x(s)) \\ \in \mathbb{R}^m}} \left[\mathcal{L}(s, x(s), u(s, x(s))) + v_{s+1}^*(x(s+1)) \right]$$

*note: yields optimal <u>policy</u> $u^*: [0,t] \times \mathbb{R}^n \to \mathbb{R}^m$

$\to$ solving even 1 stage of v_s^* usually impossible

(main exception is linear dyn, quadratic cost, which we'll consider in next lecture)

- what if we instead try to solve DT-OCP for a given initial cond $x(0) \in \mathbb{R}^n$ to obtain optimal input $u^*: [0,t] \to \mathbb{R}^m$?

$\to$ show that (DT-OCP) can be rewritten as

$$\min_{\bar{u} \in \mathbb{R}^{\bar{m}}} \bar{c}(\bar{x}, \bar{u}) \text{ s.t. } \bar{f}(\bar{x}, \bar{u}) = 0$$

(i.e. determine $\bar{u}, \bar{x}, \bar{c}, \bar{f}$)

- recalling that $(u: [0,t] \to \mathbb{R}^m) \in (\mathbb{R}^m)^{[0,t-1]}$, let $\bar{u} \in \mathbb{R}^{tm}$ be $[\bar{u}]_s = u(s)$, and similarly $[\bar{x}]_s = x^u(s)$ where x^u is the traj generated by u

- then $\bar{c}(\bar{x}, \bar{u}) = c(x, u)$,
$\bar{f}(\bar{x}, \bar{u}) = 0 \iff x(s+1) = F(s, x(s), u(s))$

- so (NLP) gives alternate solution

strategy when we know initial $x(0)$

- for $u : [0,t] \to \mathbb{R}^m$, $x : [0,t] \to \mathbb{R}^n$ to be optimal, they must be stationary:
  - $\bar{F}(\bar{x}, \bar{u}) = 0$ ensured if x is traj generated by input u from $x(0) \in \mathbb{R}^n$
- $D\bar{c}(\bar{x}, \bar{u}) + \bar{\lambda} D\bar{F}(\bar{x}, \bar{u}) = 0$

- start at step t: (stationary wrt x_t) solve for λ_t

$$D_{x_t} \bar{c} = D_x l_t(x_t)$$

$$D x_t \left[F_{t-1}(x_{t-1}, u_{t-1}) - x_t \right] = -I_n$$

so $D_x l_t(x_t) - \lambda_t = 0$,

i.e. $\lambda_t = D_x l_t(x_t)$

- at step s: (stationary wrt x_s) solve for λ_s

$$D_{x_s} \bar{c} = D_x \mathcal{L}_s(x_s, u_s)$$

$$D_{x_s} \left[F_s(x_{s-1}, u_{s-1}) - x_s \right] = -I_n$$

$$D_{x_s} \left[F_s(x_s, u_s) - x_{s+1} \right] = D_x F_s(x_s, u_s)$$

so $D_x \mathcal{L}_s(x_s, u_s) - \lambda_{s-1} + \lambda_s D_x F_s(x_s, u_s) = 0$

i.e. $\lambda_{s-1} = D_x \mathcal{L}_s(x_s, u_s) + \lambda_s D_x F_s(x_s, u_s)$

- also at step s: (stationary wrt u)

$D_{u_s} \bar{C} = D_u \mathcal{L}_s(x_s u_s)$

$D_{u_s}[F_s(x_s, u_s) - x_{s+1}] = D_u F_s(x_s, u_s)$

so $D_u \mathcal{L}_s(x_s, u_s) + \lambda_s D_u F_s(x_s, u_s) = 0$

necessary conditions for optimality in DT-OCP

- to summarize: if $(x^*, u^*) : [0, t] \to \mathbb{R}^n \times \mathbb{R}^m$ is optimal for (OCP), then:

1°. x^* is trj generated by u^* from $x(0) \in \mathbb{R}^n$

2°. $D_u L_s(x_s^*, u_s^*) + \lambda_s^* D_u F_s(x_s^*, u_s^*) = 0$

where $\lambda^* : [0, t] \to \mathbb{R}^m$ satisfies (DLTV-DE)

3°. $\lambda_{s-1}^* = D_x \mathcal{L}_s(x_s^*, u_s^*) + \lambda_s^* D_x F_s(x_s^*, u_s^*)$

from final condition $\lambda_t^* = D_x l_t(x_t^*)$

- if $(x^*, u^*, \lambda^*) : [0, t] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ satisfy 1°. – 3°. and, in addition, $\forall s \in [0, t-1]$:

$D_{u_s}^2 C = D_u^2 \mathcal{L}_s(x_s^*, u_s^*) + D_u[\lambda_s^* D_u F_s(x_s^*, u_s^*)] > 0$

$$D^2_{u_s} C = D^2_u \mathcal{L}_s(x^*_s, u^*_s) + D_u\left[\lambda^*_s D_u F_s(x^*_s, u^*_s)\right] > 0$$

then (x^*, u^*, λ^*) is strict local min for DT-OCP

sufficient conditions for optimality in DT-OCP

→ suppose $(x, u, \lambda) : [0, t] \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ satisfy 1°. & 3°. but not 2°.;
how should u be modified to descent cost?

– we computed

$$D_{u_s} C = D_u \mathcal{L}_s(x_s, u_s) + \lambda_s D_u f_s(x_s, u_s),$$

so u_s should descend using this gradient:

$$u_s^+ = u_s - \alpha_s D_{u_s} C \quad \text{(steepest descent)}$$

$$u_s^+ = u_s - \left[D^2_{u_s} C\right]^{-1} D_{u_s} C \quad \text{(Newton–Raphson)}$$