LTI -- time domain

goal: derive "solution" (i.e. trajectories and flow)
for linear time-invariant (LTI) systems

ref: Hespanha Ch 6

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linear time-invariant systems

- consider the linear DE (i.e. differential / difference equation)

$$\dot{x} / x^+ = Ax + Bu \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k$$

$$y = Cx + Du \qquad y \in \mathbb{R}^m$$

- say DE is single-input (SI) if $k=1$, multi-input (MI) if $k>1$
  " "-output (SO) if $m=1$, " " output (MO) if $m>1$

→ what does it mean to "solve" this DE?
(i.e. what kind of math object is the "solution"?)

* a "solution" is a collection of signals (i.e. func. from time to state)

$u : [0,t) \to \mathbb{R}^k$, $x : [0,t) \to \mathbb{R}^n$, $y : [0,t) \to \mathbb{R}^m$

if $\dot{x}$, i.e. continuous time (CT), $[0,t) \subset \mathbb{R}$,
if x^+, i.e. discrete time (DT), $[0,t) \subset \mathbb{N}$,
$\Big\}$ $t = \infty$ is OK

that satisfy (DE) at every time:

$$\forall s \in [0,t) : \dot{x}(s) \,/\, x(s+1) = A\,x(s) + B\,u(s)$$

$$\text{and} \quad y(s) = C\,x(s) + D\,u(s)$$

→ i.e. x is a trajectory for DE w/ input u, output y

- for LTI DE:

if $u = 0$ then

$$x(t) = e^{At} x(0) \quad \text{"solves" CT-LTI}$$

$$x(t) = A^t x(0) \quad \text{"solves" DT-LTI}$$

→ verify these expressions

if $u \neq 0$ then

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau \quad \text{"solves" CT-LTI}$$

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-\tau-1} B\,u(\tau) \quad \text{"solves" DT-LTI}$$

→ verify these expressions

- if we define the <u>state transition matrix</u> $\Phi : [0,t) \to \mathbb{R}^{n \times n}$
by $\Phi(t) = e^{At}$ for CT-LTI and $\Phi(t) = A^t$ for DT-LTI

then $x(t) = \Phi(t)x(0) + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)d\tau}$

interpret as $\sum_{\tau=0}^{t-1} \Phi(t-\tau-1)Bu(\tau)$ in DT case

- in all cases: $y(t) = Cx(t) + Du(t)$

$$= C\Phi(t)x(0) + \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

- "bundling" all trajectories into a single function yields the <u>flow</u>:
$\phi : [0,t) \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$ defined by

$$: (s, x_0, u) \longmapsto \Phi(s)x_0 + \int_0^s \Phi(s-\tau)Bu(\tau)d\tau$$

$\longrightarrow$ compute $D_{x_0}\phi$ $\quad$ * $D_{x_0}\phi(s, x_0, u) = \Phi(s)$