tests -- LTV

goal: characterize when and how state of LTV-DE can be uniquely determined from output

ref: Hespanha 15.5, 15.6 (DT in 15.7)

- consider LTV-DE $\dot{x}/x^+ = A(t)x + B(t)u, \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^k$

$$y = C(t)x + D(t)u, \quad y \in \mathbb{R}^m$$

def: the <u>observability Gramian</u> on time interval $[t_0, t_1]$ is

$$M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0)\, d\tau$$

if $\operatorname{rank} M(t_0, t_1) = n$ we say (LTV-DE) <u>observable</u> on $[t_0, t_1]$

thm: $\mathcal{N}[t_0, t_1] = \mathcal{N}(M(t_0,t_1))$

pf: for every $x_0 \in \mathbb{R}^n$:

$$x_0^T M(t_0,t_1)\, x_0 = \int_{t_0}^{t_1} x_0^T \Phi(\tau,t_0)^T C(\tau)^T C(\tau)\Phi(\tau,t_0) x_0 \, d\tau$$

$$= \int_{t_0}^{t_1} \| C(\tau)\Phi(\tau,t_0)\, x_0\|_2^2 \, d\tau$$

- $x_0 \in \mathcal{N}(W(t_0,t_1)) \Rightarrow x_0 \in \mathcal{N}[t_0,t_1]$
- $x_0 \in \mathcal{N}[t_0,t_1] \Rightarrow x_0 \in \mathcal{N}(W(t_0,t_1))$

- going back to (LTV-DE) $\dot{x}/x^+ = A(t)x + B(t)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k$

$$y = C(t)x + D(t)u, \quad y \in \mathbb{R}^m$$

- given $x_0 \in \mathbb{R}^n$ let $\tilde{y}(t) = C(t)\Phi(t,t_0)x_0$

(note: $\tilde{y}(t) = y(t) - \int_{t_0}^{t} C(\tau)\Phi(\tau,t_0)B(\tau)u(\tau)d\tau + D(t)u(t)$)

- multiply both sides of the equation by $\Phi(t,t_0)^T C(t)^T$ and integrate wrt t between t_0 and t_1:

$$\int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T \tilde{y}(t)\, dt = \int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T C(t)\Phi(t,t_0)x_0 \, dt$$

$$= M(t_0, t_1)\, x_0$$

- if rank $M(t_0, t_1) = n$ (ie (LTV-DE) observable)

then $$x_0 = M(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t)\, dt \in \mathbb{R}^n$$

is the uniquely determined initial state !