minimal realizations

goal: connect the concepts of controllability & observability to minimality of state-space realizations of transfer matrices

ref: Hespanha 17.1, 17.2

· consider LTI-DE $\quad \dot{x}/x^+ = Ax + Bu, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k$

$$y = Cx + Du, \quad y \in \mathbb{R}^m$$

def: (LTI-DE) is a <u>realization</u> of transfer matrix $\hat{G} : \mathbb{C} \to \mathbb{C}^{m \times k}$ $: s \mapsto \hat{G}(s)$ if $\hat{G}(s) = C(sI - A)^{-1}B + D$

note: (A, B, C, D) are not unique:

– given nonsingular $T \in \mathbb{R}^{n \times n}$, $\underbrace{(T^{-1}AT, T^{-1}B, CT, D)}_{\text{i.e. change coordinates: } \bar{x} = T^{-1}x}$ is also a realization of $\hat{G}$

- $\left(\begin{bmatrix} A & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right)$ is <u>also</u> a realization of

but if $A_{22} \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ then state dimension $n + \tilde{n} > n$

Q: how do we know whether (A, B, C, D) is <u>minimal</u>, i.e. has the smallest possible state dimension?

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aside: Markov parameters

- recall that $(sI - A)^{-1} = \mathcal{L}[e^{At}] = \mathcal{L}\left[\sum_{\ell=0}^{\infty} \frac{t^\ell}{\ell!} A^\ell \right]$
- note that $\mathcal{L}\left[\frac{t^\ell}{\ell!}\right] = s^{-(\ell+1)}$ so $= \sum_{\ell=0}^{\infty} s^{-(\ell+1)} A^\ell$
- conclude $\hat{G}(s) = C(sI - A)^{-1}B + D = D + \sum_{\ell=0}^{\infty} s^{-(\ell+1)} \underbrace{CA^\ell B}_{\text{Markov parameters}}$
- so the <u>impulse response</u> $G(t) = \mathcal{L}^{-1}[\hat{G}(s)] = Ce^{At}B + D\delta(t)$
- differentiating both sides of this equation: $\lim_{t \to 0^+} \frac{d^\ell}{dt^\ell} G(t) = CA^\ell B$

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thm: (A,B,C,D) is a minimal realization for

$\hat{G}(s) = C(sI-A)^{-1}B + D$ if and only if (A,B) controllable

(A,C) observable

pf: $(\Leftarrow)$ suppose (A,B) <u>not</u> controllable or (A,C) <u>not</u> observable

then Kalman decomposition yields $(A_{co}, B_{co}, C_{co}, D)$

with $A_{co} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\bar{n} < n$, $\hat{G}(s) = C_{co}(sI - A_{co})^{-1}B_{co} + D$

so (A,B,C,D) not minimal, which is a contradiction

$(\Rightarrow)$ suppose (A,B) controllable, (A,C) observable,

$\hat{G}(s) = C(sI-A)^{-1}B + D$,

<u>and</u> there exists $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, $\bar{A} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $\bar{n} < n$

$\hat{G}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$

- letting $\mathcal{C}$, $\mathcal{O}$ denote controllability & observability matrices for (A,B,C,D):

$$\overbrace{\underset{\in\mathbb{R}^{mn\times n}}{\mathcal{O}}\;\underset{\in\mathbb{R}^{n\times nk}}{\mathcal{C}}}^{\in\mathbb{R}^{mn\times nk}} = \cdots \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\operatorname{rank}\mathcal{O} = n} \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{\operatorname{rank}\mathcal{C} = n}$$

$$= \underbrace{\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix}}_{} \Bigg\} \text{ matrix of Markov parameters}$$

$\longrightarrow$ show that: rank $\mathcal{O}\mathcal{C} = n$

- letting $\bar{\mathcal{C}}$, $\bar{\mathcal{O}}$ denote controllability & observability matrices for $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$:

$$\bar{\mathcal{O}}\bar{\mathcal{C}} = \underbrace{\begin{bmatrix} \bar{C}\bar{B} & \bar{C}\bar{A}\bar{B} & \cdots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \bar{C}\bar{A}\bar{B} & \bar{C}\bar{A}^2\bar{B} & \cdots & \bar{C}\bar{A}^n\bar{B} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{C}\bar{A}^{n-1}\bar{B} & \bar{C}\bar{A}^n\bar{B} & \cdots & \bar{C}\bar{A}^{2n-2}\bar{B} \end{bmatrix}}_{\text{rank } \bar{\mathcal{O}}\bar{\mathcal{C}}}$$

- conclude $\boxed{\text{rank } \bar{\mathcal{O}}\bar{\mathcal{C}} \le \text{rank } \bar{\mathcal{C}} \le \bar{n} < n}$

$\bar{n}$: # rows in $\bar{\mathcal{C}}$

contradiction!

fact: (A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ have same Markov parameters since they define the same impulse response!

$$\Longrightarrow \quad \bar{\mathcal{O}}\bar{\mathcal{C}} = \mathcal{O}\mathcal{C} \Rightarrow \boxed{\text{rank } \bar{\mathcal{O}}\bar{\mathcal{C}} = \text{rank } \mathcal{O}\mathcal{C} = n}$$