state feedback

goal: synthesize stabilizing state-feedback controller for controllable LTI-DE

ref: Hespanha 12.4

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- consider LTI-DE $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$
- assume (LTI-DE) is controllable - i.e. $(A, B)$ controllable

$\Longrightarrow$ $(-\mu I - A, B)$ contollable for every $\mu \in \mathbb{R}$

$\rightarrow$ verify using eigenvector test

- if $v \neq 0$ is such that $Av = \lambda v$, some $\lambda \in \mathbb{C}$

then $(-\mu I - A)v = (-\mu - \lambda)v$, so $(-\mu - \lambda) \in \operatorname{spec}(-\mu I - A)$

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- note that, if $\mu$ is sufficiently large, $-\mu I - A$ is stable (all eigenvalues have negative real part)
  - suppose we choose $\bar{\mu}$ to make $-\bar{\mu} I - A$ stable
- Lyapunov test for controllability $\Rightarrow \exists!\, W = W^T > 0$ that solves Lyapunov equation

$$(-\bar{\mu} I - A) W + W(-\bar{\mu} I - A^T) = -BB^T$$

$$\iff \quad AW + WA^T - BB^T = -2\bar{\mu} W$$

- multiplying both sides of the equation by $P = W^{-1} > 0$ yields $PA + A^T P - PBB^T P = -2\bar{\mu} P$
  with $K = \frac{1}{2} B^T P$: $\quad P(A-BK) + (A-BK)^T P = \underbrace{-2\bar{\mu} P}_{= -Q}$
- by Lyapunov test for stability: $A - BK$ is stable, i.e. all eigenvalues have negative real part !
  so state feedback $\underline{u = -Kx \text{ stabilizes (LTI-DE)}}$ !

fact: real part of all eigenvalues of $A - BK$ are $\le -\bar{\mu}$