Finite-dimensional linear transformations

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear.

a. If L is injective, prove that L is surjective.

b. If L is surjective, prove that L is injective.

Now suppose the rangespace of L is a subset of its nullspace.

c. Compute $\operatorname{spec} L$.

d. Given any matrix representation $A_L \in \mathbb{R}^{n\times n}$ of L, compute $\det(sI - A_L)$.

Matrix representations

Let $L : U \to U$ be linear and $\dim U = n$.

Suppose there exists $b \in U$ such that the vectors in $B = \{b, Lb, L^2 b, \dots, L^{n-1} b\}$ are linearly independent.

a. Obtain the matrix representation of L with respect to the basis B.

Suppose that $\lambda \in \mathbb{C}$ and $V = \{v_j\}_{j=1}^n$ is a basis for U such that $Lv_1 = \lambda v_1$ and $Lv_k = \lambda v_k + v_{k-1}$ for all $k \in \{2, \dots, n\}$.

b. Obtain the matrix representation of L with respect to the basis V.

Suppose that $\operatorname{rank} L = n$, so that there exists $\Gamma : U \to U$ such that $L \circ \Gamma = \Gamma \circ L = \mathrm{id}$ where $\mathrm{id} : U \to U$ is the *identity operator* defined for all $u \in U$ by $\mathrm{id}(u) = u$.

c. Show that, in any basis for U, the matrix representation of L is the inverse of the matrix representation of Γ.

Vector fields and their flows

Let $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow for $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$. Given $t \in \mathbb{R}$, let $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be defined for all $x \in \mathbb{R}^n$ by $\phi_t(x) = \phi(t, x)$.

a. If f is linear, prove that ϕ_t is linear.

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Induced norm

Given $t \in \mathbb{R}$, let $S_t : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ be defined for all signals $x \in \mathbb{R}^{\mathbb{R}}$ by $S_t(x) = x(t)$.

Prove or provide a counterexample.

a. S_t is linear.

b. The induced norm of $\|S_t\|$ is finite. (Use $\|x\| = \int_{-\infty}^{\infty} |x(s)| ds$ for signals $x \in \mathbb{R}^{\mathbb{R}}$ and $|a|$ for scalars $a \in \mathbb{R}$.)

State transition matrix

Suppose the differentiable function $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ is such that
$\forall a, b, c \in \mathbb{R} : P(c, b)P(b, a) = P(c, a)$and $P(a, a) = I$ (the $n \times n$ identity matrix).

a. Prove that $P(a, b)$ is invertible and $P(a, b)^{-1} = P(b, a)$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$.

b. Write $D_b P(a, b)$ in terms of $P(a, b)$ and $D_a P(a, b)$.

Linear time-varying systems

Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ be the state transition matrix and $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow for the (CLTV) system

$$\dot{x} = A(t)x.$$

a. Given $(s, x) \in \mathbb{R} \times \mathbb{R}^n$, write $\phi(s, x)$ in terms of x and Φ.

b. Is it possible to have $\phi(s, x) = 0$ for $x \neq 0$?

Now let $\tilde{\Phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$ be the state transition matrix and $\tilde{\phi} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be the flow for the (CLTV) system

$$\dot{\tilde{x}} = A(t)\tilde{x} + \phi(t, x).$$

c. What is the relationship between Φ and $\tilde{\Phi}$?

d. Given $(s, x, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, write $\tilde{\phi}(s, \tilde{x})$ in terms of s, x, $\tilde{x}$, and Φ.

e. Given $x \neq 0$, determine $\tilde{x} \neq 0$ such that $\tilde{\phi}(s, \tilde{x}) = 0$.