1. Orthogonal projection (5pts)

Let $A \in \mathbb{C}^{m \times n}$ and suppose that A is injective.

a. (2pts) Prove that $A^T A$ is invertible.

For (b.), (c.), and (d.), let $P = A(A^T A)^{-1} A^T$.

b. (1pt) Prove that $P^2 = P$.

c. (1pt) Prove that $P(x) = x$ for all $x \in \operatorname{Im} A$ and $P(x) = 0$ for all $x \in \ker A^T$.

d. (1pt) Prove that $x - P(x)$ is orthogonal to $\operatorname{Im} A$, i.e. that $y^T(x - P(x)) = 0$ for all $y \in \operatorname{Im} A$.

2. Effects of feedback (5pts)

Consider the (CLTI) system

$$\dot{x} = Ax + Bu,\ y = Cx + Du,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m$$

subject to the state--feedback control input

$$u(x) = -Kx + v$$

where $v \in \mathbb{R}^k$ denotes a new input.

a. (1pt) Determine the state--space representation of the closed--loop system, treating v as the input and y as the output.

For (b.) and (c.), refer to the state space model from (a.) as (CLTI-CL).

b. (2pts) Prove or provide a counterexample: if (CLTI) is controllable, then (CLTI-CL) is controllable.

c. (2pts) Prove or provide a counterexample: if (CLTI) is observable, then (CLTI-CL) is observable.

3. Linearization (5pts)

Consider the nonlinear system (CNL)

$$\dot{x}_1 = -x_1 + u_1,\ \dot{x}_2 = -x_2 + u_2,\ \dot{x}_3 = x_2 u_1 - x_1 u_2,\ y = x_1^2 + x_2^2 + x_3^2.$$

a. (1pts) Linearize (CNL) around the equilibrium $x_1 = x_2 = x_3 = 1$ to obtain a (CLTI) system. (*Note: this is not the same equilibrium as the practice problem.*)

b. (2pts) Is (CLTI) from (a.) controllable?

c. (2pts) Is (CLTI) from (a.) observable?

4. Reachability (2pts)

Consider the (CLTI) system

$$\dot{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} x + \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} u.$$

(2pts) Does there exist an input $u : [0, t] \to \mathbb{R}$ that takes $x(0) = (0, 0, 0)$ to $x(t) = (9, 10, 0)$?

5. Set-point control (4pts)

We seek a state feedback control law that causes the output of the (CLTI) system

$$\dot{x} = Ax + Bu,\ y = Cx + Du,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m$$

to tend asymptotically to $\eta \in \mathbb{R}^m$; note that η may be nonzero. Note that the linear equations $A\xi + B\mu = 0,\ C\xi + D\mu = \eta$ must necessarily be satisfied if the control objective $y \to \eta$ is to be achieved.

For (a.) and (b.), suppose these equations are satisfied by $\xi \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$.

a. (2pts) Prove or provide a counterexample: it is always possible to determine a state feedback control law $u : \mathbb{R}^n \to \mathbb{R}^k$ that ensures $y \to \eta$ regardless of the initial condition.

b. (2pts) Given a linear state feedback law $u_0 = -Kx$ that ensures $y \to 0$ regardless of the initial condition, design a state feedback control law $u : \mathbb{R}^n \to \mathbb{R}^k$ that ensures $y \to \eta$ regardless of the initial condition.

6. Inverse model (3pts)

Consider the (CLTI) system $\dot{x} = Ax + Bu,\ y = Cx + Du$

and the $(\overline{\text{CLTI}})$ system $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u},\ \bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u}$.

We say that $(\overline{\text{CLTI}})$ *is an inverse of* (CLTI) if, when the output $\bar{y}$ of $(\overline{\text{CLTI}})$ is applied as the input u to (CLTI), then the output y of (CLTI) is the input $\bar{u}$ that was applied to $(\overline{\text{CLTI}})$.

For (a.), suppose D is nonsingular and both systems are initialized at the origin.

a. (2pts) Determine the system matrices $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ such that $(\overline{\text{CLTI}})$ is an inverse of (CLTI).

For (b.), suppose $(\overline{\text{CLTI}})$ is an inverse of (CLTI), and the output $\bar{y}$ of $(\overline{\text{CLTI}})$ is applied as the input u to (CLTI).

b. (1pts) If both $(\overline{\text{CLTI}})$ and (CLTI) are asymptotically stable, but are initialized at different initial conditions, compute $\lim_{t\to\infty} \|y(t) - \bar{u}(t)\|$.