definitions

goal: characterize when any state of a linear system can be reached from the origin

ref: Hespanha Ch 11.3, 11.1

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fundamental Theorem of linear equations: rank-nullity Theorem

- given $W \in \mathbb{R}^{m \times n}$, we define the range space

$$\mathcal{R}(W) = \{ y = Wx \in \mathbb{R}^m : x \in \mathbb{R}^n \}$$

and the null space

$$\mathcal{N}(W) = \{ x \in \mathbb{R}^n : Wx = 0 \in \mathbb{R}^m \}$$

* note: both $\mathcal{R}(W)$ & $\mathcal{N}(W)$ are subspaces:

so $x_1, x_2 \in \mathcal{N}(W) \Rightarrow x_1 + \alpha x_2 \in \mathcal{N}(W)$

$y_1, y_2 \in \mathcal{R}(W) \Rightarrow y_1 + \alpha y_2 \in \mathcal{R}(W)$

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$$0 \le \underbrace{\dim \mathcal{R}(W)}_{= \text{rank } W} \le m \,, \quad 0 \le \underbrace{\dim \mathcal{N}(W)}_{= \text{null } W} \le n$$

$= \text{rank } W$, "rank of W" $\quad$ $= \text{null } W$, "nullity of W"

thm: $\dim \mathcal{R}(W) + \dim \mathcal{N}(W) = n$

def: given subspace $V \subset \mathbb{R}^n$ define orthogonal complement

$$V^\perp = \{ x \in \mathbb{R}^n \mid \forall v \in V : v^T x = 0 \}$$

lem: $\mathcal{R}(W) = \mathcal{N}(W^T)^\perp$, $\quad \mathcal{N}(W) = \mathcal{R}(W^T)^\perp$

$\longrightarrow$ prove these facts

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definition of reachable subspace

- recall that linear system $\dot{x} / x^+ = A(t)x + B(t)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

yields $x(t_1) = \Phi(t_1, t_0)\, x(t_0) + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau$

$$\text{i.e.} \quad x_1 = M \cdot x_0 + \mu \,,$$

that is, the final state is a linear function of x_0 & μ

(from the origin)

def: given $t_1 > t_0 \geq 0$ define the $\underbrace{\text{reachable}}_{}$ subspace

also called "controllable-from-the-origin"

$$\mathcal{R}[t_0,t_1] = \left\{ x_1 \in \mathbb{R}^n \;\middle|\; \exists\, u:[t_0,t_1] \to \mathbb{R}^k \text{ such that } x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau \right\}$$

* note that characterizing $\mathcal{R}[t_0,t_1]$ amounts to finding all solutions to linear equation $x_1 = \int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)u(\tau)\,d\tau$