tests -- LTV

goal: characterize when and how any state of LTV-DE can be reached from the origin

ref: Hespanha Ch 11.4, 12.1, 11.7

- given LTV-DE $\dot{x}/x^+ = A(t)x + B(t)u$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

def: if $t_1 > t_0 \geq 0$ define <u>reachability Grammian</u>

by $$\underbrace{W(t_0,t_1)}_{\in \mathbb{R}^{n\times n}} = \int_{t_0}^{t_1} \underbrace{\Phi(t_1,\tau)}_{n\times n} \underbrace{B(\tau)B(\tau)^T}_{n\times n} \underbrace{\Phi(t_1,\tau)^T}_{n\times n} d\tau$$

note: $W(t_0,t_1)^T = W(t_0,t_1)$ i.e. $W(t_0,t_1)$ is <u>symmetric</u>

<u>thm</u>: $\forall t_1 > t_0 \geq 0$: $\mathcal{R}(W(t_0,t_1)) = \mathcal{R}[t_0,t_1]$

moreover, if $x_1 = W(t_0,t_1)\eta_1 \in \mathcal{R}(W(t_0,t_1))$

then $u:[t_0,t_1]\to\mathbb{R}^k$ defined by $u(t)=B(t)^T\Phi(t_1,t)^T\eta_1$

steers $x(t_0)=0$ to $x_1=x(t_1)=\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)d\tau$

pf: $\left(x_1\in\mathcal{R}(W(t_0,t_1))\Rightarrow x_1\in\mathcal{R}[t_0,t_1]\right)$

$$x(t_1)=\int_{t_0}^{t_1}\Phi(t_1\tau)B(\tau)B(\tau)^T\Phi(t_1,\tau)^T\eta_1\,d\tau$$

$$=W(t_0,t_1)\cdot\eta_1=x_1\Rightarrow x_1\in\mathcal{R}[t_0,t_1]$$

$\rightarrow$ if $(x_1-x_0)\in\mathcal{R}(W(t_0,t_1))$, find u that steers state from $x(t_0)=x_0$ to $x(t_1)=x_1$

$W^T=W$

$\left(x_1\in\mathcal{R}[t_0,t_1]\Rightarrow x_1\in\mathcal{R}(W(t_0,t_1))=\mathcal{N}(W(t_0,t_1)^T)^\perp\right)$

know $\exists\, u:[t_0,t_1]\to\mathbb{R}^k$ s.t. $x_1=\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u(\tau)d\tau$

wts $x_1\in\mathcal{N}(W(t_0,t_1))^\perp$ $\left[=\mathcal{R}(W(t_0,t_1))\right]$

i.e. $\forall\,\eta_1\in\mathcal{N}(W(t_0,t_1)):\ x_1^T\cdot\eta_1=0$

but $x_1^T\cdot\eta_1=\int_{t_0}^{t_1}u(\tau)^TB(\tau)^T\Phi(t_1,\tau)^T\cdot\eta_1\,d\tau$

and $W(t_0,t_1)\cdot\eta_1=0$

so $0=\eta_1^TW(t_0,t_1)\eta_1=\int_{t_0}^{t_1}\eta_1^T\Phi(t_1,\tau)B(\tau)B(\tau)^T\Phi(t_1,\tau)^T\eta_1\,d\tau$

$$=\int^{t_1}\|B(\tau)^T\Phi(t_1,\tau)^T\eta_1\|^2d\tau$$

$$= \int_{t_0}^{t_1} \| B(\tau)^T \Phi(t_1, \tau)^T \eta_1 \|^2 \, d\tau$$

which implies $B(\tau)^T \Phi(t_1, \tau)^T \eta_1 = 0 \quad \forall \tau \in [t_0, t_1]$

i.e. $x_1^T \cdot \eta_1 = 0$, so $x_1 \in \mathcal{N}(W(t_0, t_1)^T)^\perp = \mathcal{R}(W(t_0, t_1))$