linearization

goal: understand "solution" of differential/difference equations (DE)

$$\dot{x} / x^+ = f(x,u)$$

ref: Strogatz Ch 2 – geometric perspective

3°: linearization

- given DE $\dot{x} / x^+ = f(x,u)$, let $\phi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$ be flow
  and $\psi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$ be sim flow

* we'll study how ϕ, ψ vary wrt. both arguments;
in particular, we'll use linearization to determine first-order variation

→ in CT case $\dot{x} = f(x,u)$, what is $D_1 \phi(s,\xi) = \frac{\partial}{\partial s} \phi(s,\xi)$?

– since $\phi(s,\xi) = x_\xi(s)$, $D_s \phi(s,\xi) = D_s x_s(s)$

$$= f(x_\xi(s), u(s)) = f(\phi(s,\xi), u(s))$$

- now consider how DT flow varies with respect to ξ
  - recall $\phi(s+1,\xi) = f(\phi(s,\xi))$ ⟵ consider autonomous case (no u)

→ compute $D_2 \phi(s+1,\xi)$ in terms of Df & $D_2 \phi(s,\xi)$

– applying the chain rule,

$$D_2 \phi(s+1,\xi) = D_\xi \left[f(\phi(s,\xi)) \right]$$

$$= Df(\phi(s,\xi)) \cdot D_2 \phi(s,\xi)$$

→ compute $D_2\phi(s+1,\xi)$ in terms of Df

- recursively substituting $D_2\phi$: $D_2\phi(s+1,\xi) = \prod_{\ell=0}^{s} Df(\phi(\ell,\xi))$
  (note that $\phi(0,\xi)=\xi$, so $D_2\phi(0,\xi)=I$)

∘ same computation yields $D_2\psi(s+1,\xi) = \prod_{\ell=0}^{s} Df(\psi(\ell,\xi))$

∘ continuous-time case isn't as straightforward;
  start with simulation flow, which satisfies

$$\psi(s+\Delta,\xi) = \psi(s,\xi) + \Delta\cdot f(\psi(s,\xi))$$

→ compute $D_2\psi(s+\Delta,\xi)$ in terms of Df & $D_2\psi(s,\xi)$

- $$\begin{aligned} D_2\psi(s+\Delta,\xi) &= D_\xi\left[\psi(s,\xi) + \Delta\cdot f(\psi(s,\xi))\right] \\ &= D_2\psi(s,\xi) + \Delta\cdot Df(\psi(s,\xi))\cdot D_2\psi(s,\xi) \\ &= \left(I + \Delta\cdot Df(\psi(s,\xi))\right)\cdot D_2\psi(s,\xi) \end{aligned}$$

→ " " " Df

- $$D_2\psi(\overbrace{s+\Delta}^{s=N\cdot\Delta},\xi) = \prod_{k=0}^{N}\left[I + \Delta\cdot Df(\psi(k\cdot\Delta,\xi))\right]$$
  (note $\psi(0,\xi)=\xi$ so $D_2\psi(0,\xi)=I$)

→ apply forward Euler simulation algo to matrix differential eqn:

$$\forall s\in[0,t]:\ \dot{X}(s) = Df(\phi(s,\xi))\cdot X(s),\quad X(0)=I$$

$$- \; X(s+\Delta) = X(s) + \Delta \cdot Df(\phi(s,\xi)) \cdot X(s)$$

$$= \left[I + \Delta \cdot Df(\phi(s,\xi)) \right] \cdot X(s)$$

$$= \prod_{k=0}^{N} \left[I + \Delta \cdot Df(\phi(k\Delta,\xi)) \right]$$

<u>fact</u>: linearization of CT flow is obtained by solving

so that $D_\xi \phi(s,\xi) = X(s)$, where $X : [0,t] \to \mathbb{R}^{d \times d}$ solves

$\to$ how does this simplify at an equilibrium? $(f(x_0) = 0)$

$$- \; f(x_0) = 0 \implies \phi(s, x_0) = x_0$$

$$\implies \dot{X}(s) = Df(\underbrace{\phi(s, x_0)}_{\equiv x_0}) \cdot X(s)$$

define $A = Df(x_0)$

$$= Df(x_0) \cdot X(s) = A \cdot X(s)$$