tests -- Lyapunov

goal: characterize controllability of stable LTI-DE using a Lyapunov test

ref: Hespanha Ch 12.3

- consider (LTI-DE) $\dot{x} / x^+ = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

<u>thm:</u> if (LTI-DE) is stable, then:

(LTI-DE) is controllable $\iff$ there is a unique solution $W = W^T > 0$ to Lyapunov equation: (CT) $AW + WA^T = -BB^T$

(DT) $AWA^T - W = -BB^T$

in this case,

$$\left.\begin{array}{ll} \text{(CT)} & W = \int_0^\infty e^{A\tau} B B^T e^{A^T \tau} d\tau \\ \text{(DT)} & W = \sum_{\tau=0}^{\infty} A^\tau B B^T (A^T)^\tau \end{array}\right\} = \lim_{t \to \infty} W(0,t)$$

pf: (sketch)

($\Leftarrow$) assume $\exists!\ W = W^T > 0$ solution to Lyapunov equation and let $v \neq 0$ satisfy $A^T v = \lambda v \Rightarrow v^* A = \lambda^* v^*$

- then $v^*(AW + WA^T)v = -v^* BB^T v = -\|B^T v\|^2$

- but
$$
\begin{aligned}
v^*(AW + WA^T)v &= v^* A W v + v^* W A^T v \\
&= \lambda^* v^* W v + \lambda v^* W v \\
&= 2 \underbrace{\mathrm{Re}(\lambda)}_{<0} \underbrace{v^* W v}_{>0} < 0
\end{aligned}
$$

- conclude $B^T v \neq 0$, i.e. there is no eigenvector of A^T in the nullspace of B^T, so (LTI-DE) is controllable

($\Rightarrow$) assuming (LTI-DE) controllable (and stable) we know $\exists!\ W = W^T > 0$ to Lyapunov equation with $Q = BB^T \geq 0 \leftarrow$ some subtlety involved in handling positive semi-definite Q, but it can be handled