tests -- eigenvector

goal: characterize controllability of LTI-DE
using an eigenvector test

ref: Hespanha Ch 12.2

- consider (LTI-DE) $\dot{x} / x^+ = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

and recall controllability matrix $\mathcal{C} = [B \;\; AB \;\; A^2 B \cdots A^{n-1}B] \in \mathbb{R}^{n \times (n \cdot k)}$

characterizes controllability:

thm: $\mathcal{R}[0,t] = \left\{ x(t) = \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau \;\middle|\; u : [0,t] \to \mathbb{R}^k \right\}$
$= \mathcal{R}(\mathcal{C})$

def: we'll say (LTI-DE) is (completely) controllable if $\dim R(\mathcal{C}) = n$ i.e. $R(\mathcal{C}) = \mathbb{R}^n$

thm: (LTI-DE) controllable $\iff$ there is no eigenvector of A^T in the kernel of B^T

pf: $(\Rightarrow)$ suppose $\exists v \neq 0$ s.t. $A^T v = \lambda v$ and $B^T v = 0$

then
$$\mathcal{C}^T v = \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} v = \begin{bmatrix} B^T v \\ \lambda B^T v \\ \lambda^2 B^T v \\ \vdots \\ \lambda^{n-1} B^T v \end{bmatrix} = 0 \in \mathbb{R}^{n \cdot k}$$

so $\dim \mathcal{N}(\mathcal{C}^T) \geq 1$, which means $\dim R(\mathcal{C}^T) = n - \dim \mathcal{N}(\mathcal{C}^T) < n$
$= \dim R(\mathcal{C})$

thm: (Popov-Belevitch-Hautus test - PBH)

(LTI-DE) is controllable $\iff$ $\forall \lambda \in \mathbb{C}$: $\operatorname{rank} \overbrace{[A - \lambda I \; \vdots \; B]}^{\in \mathbb{R}^{n \times (n+k)}} = n$

pf: equivalently, $\dim \mathcal{N} \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = 0$,

i.e. $\mathcal{N} \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = \{ v \in \mathbb{R}^n : A^T v = \lambda v, \; B^T v = 0 \} = \{0\}$

which is clearly equivalent to previous eigenvector test