tests -- Lyapunov

goal: characterize stability of LTI systems using Lyapunov functions

ref: Hespanha Ch 8.5

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properties of positive-definite matrices

def: $Q = Q^T$ is positive-definite if $\forall x \neq 0 : x^T Q x > 0$

" " negative-definite " " < 0

" " -semidefinite " " ≥ 0 or ≤ 0

equivalently (i.e. $\iff$), $Q = Q^T$ is positive-definite

$\overset{(i)}{\iff}$ all eigenvalues are positive (and <u>real</u>)

fact: if $Q = Q^T > 0$ then

(ii) $\forall x : 0 < \lambda_{min}(Q) \|x\|^2 \leq x^T Q x \leq \lambda_{max}(Q) \|x\|^2$

→ derive (i) & (ii) → derive analogous facts for negative-(semi)definite

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Lyapunov test for stability

- consider LTI DE $\dot{x} / x^+ = Ax$

facts: the following are equivalent:

definition

eigenvalue test 1°. $\underbrace{\text{(LTI DE) exponentially stable}}$

$\Updownarrow$ Lyapunov test

2°. $\underbrace{\forall \lambda \in \operatorname{spec} A}$: $\iff$ 3°. $\forall Q = Q^T > 0 : \exists!\, P : \underbrace{P = P^T > 0 \text{ and}}$

(CT) $\operatorname{Re} \lambda < 0$ (CT) $A^T P + PA = -Q$

(DT) $|\lambda| < 1$ (DT) $A^T P A - P = -Q$

$\Big\}$ Lyapunov equations

why? let's consider (CT) case

(1° ⇒ 3°) assume (LTI DE) exp. stable, $Q = Q^T > 0$ given, and define $P = \int_0^\infty e^{A^T t} Q e^{At} \, dt$

→ verify that $A^T P + PA = -Q$ (use integration-by-parts)

→ verify that $P = P^T > 0$ (use def. of pos.-def.)

→ verify that P unique (suppose not, derive contradiction)

$\rightarrow$ verify that P unique (suppose not, derive contradiction)

($3^\circ \Rightarrow 1^\circ$) suppose $Q = Q^T > 0$, $P = P^T > 0$ are s.t. $A^T P + PA = -Q$

consider $\underbrace{v(t) = x(t)^T P x(t)}\geq 0$ where $\dot{x} = Ax$

$v : [0,\infty) \rightarrow [0,\infty)$ called a Lyapunov function

$: t \longmapsto v(t)$

and compute $\frac{d}{dt} v(t) = \dot{v} = \dot{x}^T P x + x^T P \dot{x}$

$$= x^T A^T P x + x^T P A x$$

key step - requires a lot of work

$$= x^T (A^T P + PA) x = -x^T Q x \leq 0$$

$\rightsquigarrow$ can show $v \rightarrow 0$ at an exponential rate

$\Rightarrow \|x(t)\|^2 \rightarrow 0$ at exp. rate $\Rightarrow$ (LTI-DE) exp. stable