flows and simulations

goal: understand "solution" of differential/difference equations (DE)

$$\dot{x} \,/\, x^+ = f(x, u)$$

ref: Strogatz Ch 2 – geometric perspective

2°. flows and simulations

- given general $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, don't expect to be able to "solve" DE $\dot{x} / x^+ = f(x,u)$ } literally a Thm.! except: linear DE

$\rightarrow$ instead, rely on computational tools to approximate traj's

$\rightarrow$ propose simulation algorithms for CT ($\dot{x}$) & DT (x^+), that is, a step-by-step procedure that takes $t, x(0), u, f$ as inputs and returns $\tilde{x} : [0,t] \to \mathbb{R}^d$, which is an approximation of traj $x : [0,t] \to \mathbb{R}^d$ for DE

(DT) $\tilde{x}(s+1) = f(\tilde{x}(s), u(s))$ $\longleftarrow$ note: small errors due to floating-point arithmetic

(CT) using the fact that $\dot{x}(s) = \frac{d}{ds} x(s) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}\left(x(s+\Delta) - x(s)\right)$

$$\simeq \frac{1}{\Delta}\left(x(s+\Delta) - x(s)\right)$$

then rearranging the approximate equation yields

$$x(s+\Delta) \simeq x(s) + \Delta \cdot \underbrace{\dot{x}(s)}_{= f(x(s), u(s))}$$

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so I'll propose

$$\underbrace{\tilde{x}(s+\Delta) = \tilde{x}(s) + \Delta \cdot f(\tilde{x}(s), u(s))}_{\text{errors primarily due to step size } \Delta > 0} \Big\} \text{"forward Euler"}$$

- now think about tri / sim indexed by initial condition $\xi \in \mathbb{R}^d$:

$x_\xi : [0,t] \to \mathbb{R}^d$, $x_\xi(0) = \xi$ $\qquad$ $\tilde{x}_\xi : [0,t] \to \mathbb{R}^d$, $\tilde{x}_\xi(0) = \xi$

- given another initial condition ξ', get a new tri / sim:

$x_{\xi'} : [0,t] \to \mathbb{R}^d$, $x_{\xi'}(0) = \xi'$ $\qquad$ $\tilde{x}_{\xi'} : [0,t] \to \mathbb{R}^d$, $\tilde{x}_{\xi}(0) = \xi'$

* letting initial condition range over all possible vectors in $\mathbb{R}^d$,
we define a function called the <u>flow</u>:

$\phi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$ $\qquad$ $\psi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$

$: (s, \xi) \longmapsto x_\xi(s)$ $\qquad$ $: (s, \xi) \longmapsto \tilde{x}_\xi(s)$

$\to$ key utility of defining flow: we can study how trajectories vary with respect to their initial conditions

<u>fact</u>: when f is continuously differentiable,
flow ϕ, ψ is well-defined and continuously differentiable wrt x

$\to$ for linear DE $\dot{x} / x^+ = Ax$, determine ϕ

(CT) since $x(s) = e^{As} \cdot x(0)$ is tri in continuous time, (CT) since $x(s) = e^{As} \cdot x(0)$ is trj in continuous time,

flow $\phi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$ is $\phi(s,\xi) = e^{As} \cdot \xi$

(DT) since $x(s) = A^s \cdot x(0)$ is trj in discrete time,

flow $\phi : [0,t] \times \mathbb{R}^d \to \mathbb{R}^d$ is $\phi(s,\xi) = A^s \cdot \xi$

→ for linear DE $\dot{x} / x^+ = Ax$, compute $D_2 \phi(t,\xi)$

(CT) since $\phi(s,\xi) = e^{As} \cdot \xi$, $D_2 \phi(s,\xi) = e^{As} \in \mathbb{R}^{d \times d}$ ☑

(DT) since $\phi(s,\xi) = A^s \cdot \xi$, $D_2 \phi(s,\xi) = A^s \in \mathbb{R}^{d \times d}$ ☑