1°: (a) if $L : \mathbb{R}^n \to \mathbb{R}^n$ injective, then a basis $\{v_j\}_{j=1}^n$ for $\mathbb{R}^n$ pushes forward to a basis $\{L(v_j)\}_{j=1}^n$ for $\mathcal{R}(L)$, hence $\dim \mathcal{R}(L) = n$ so $\mathcal{R}(L) = \mathbb{R}^n$, i.e. L surjective

(b) if L surjective, given basis $\{w_j\}_{j=1}^n$ for $\mathbb{R}^n$ there exists $\{v_j\}_{j=1}^n \subset \mathbb{R}^n$ s.t. $\forall j : Lv_j = w_j$; can verify that $\{v_j\}_{j=1}^n$ linearly independent, so $\mathcal{N}(L) = \{0\}$, so L injective

(c) if $\mathcal{R}(L) \subset \mathcal{N}(L)$, then $L^n = 0_{n \times n}$, so $\operatorname{spec} L = \{0\}$

(d) (c) $\Rightarrow \operatorname{spec} A_L = \{0\} \Rightarrow \det(sI - A) = s^n$

2. (a) noting that $L(L^k b) = L^{k+1} b$,
and letting $\det(sI - L) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0$,

$$A_{L,B} = \begin{bmatrix} 0 & \cdots & & 0 & -\alpha_0 \\ 1 & 0 & & & -\alpha_1 \\ & \ddots & \ddots & \vdots & \vdots \\ & & & & \\ 0 & & 1 & 0 & -\alpha_{n-1} \end{bmatrix}$$

(b)

$$A_{L,V} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & 0 \\ & & & & 1 \\ & & & & \lambda \end{bmatrix}$$

(c) given basis $\{a_i\}_{i=1}^n$, let M, W

be matrix rep's for L, Γ, respectively

since $\forall v \in \mathbb{R}^n$: $L\Gamma v = \Gamma L v = v$,

then $MWv = WMv = v$;

since matrix inverses are unique, $W = M^{-1}$

3: (a) if f is linear, then $\exists$ matrix A

s.t. $\forall x \in \mathbb{R}^n$: $f(x) = Ax$

then know $\phi(t,x) = e^{At}x$ so ϕ_t linear

(b) if ϕ_t is linear then

$$\forall \alpha \in \mathbb{R},\ x, \xi \in \mathbb{R}^n:\ \phi_t(x+\alpha\xi) = \phi_t(x) + \alpha\phi_t(\xi)$$

$$\Rightarrow D_t\phi_t(x+\alpha\xi) = D_t\phi_t(x) + \alpha D_t\phi_t(\xi)$$

$$\Rightarrow f(\phi_t(x+\alpha\xi)) = f(\phi_t(x) + \alpha\phi_t(\xi))$$

$$= f(\phi_t(x)) + \alpha f(\phi_t(\xi)) \boxed{\checkmark}$$

4°: (a) $\forall \alpha \in \mathbb{R},\ x, \xi \in \mathbb{R}^{\mathbb{R}}$: $S_t(x + \alpha \xi) = x(t) + \alpha \xi(t)$ $\boxtimes$

(b) consider $\{x_j\}_{j=1}^{\infty}$ defined by
$$x_j(\tau) = \begin{cases} j, & \tau = t \\ 0, & \text{else} \end{cases}$$

then $\|x_j\| = 0$ but $\|S_t x_j\| = j$ so $\|S_t\| = \infty$

5°. (a) $P(a,b)P(b,a) = I \Rightarrow P(a,b)^{-1} = P(a,b)$

(b) $D_b(P(a,b)\,P(b,a)) = D_bP(a,b)\,P(b,a) + P(a,b)\,D_bP(b,a)$

$\Rightarrow D_bP(a,b) = -P(a,b)\;D_bP(b,a)\,P(a,b)$

6: (a) $\phi(s,x) = \Phi(s,0)x$

(b) since $\Phi(s,0)$ invertible, $\Phi(s,0)x = 0 \Leftrightarrow x = 0$

(c) $\Phi = \tilde{\Phi}$

(d) $\tilde{\phi}(s,\tilde{x}) = \Phi(s,0)\tilde{x} + \int_0^s \Phi(s,\tau)\phi(\tau,x)\,d\tau$

(e) $\tilde{x} = -\Phi(0,s)\left[\int_0^s \Phi(s,\tau)\phi(\tau,x)\,d\tau\right]$