state estimation

goal: estimate state of an observable LTI system in closed-loop by filtering output through an LTI observer

ref: Hespanha Ch 16.5

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recall: open-loop state estimation — offline, i.e. from batch data

- given (LTV-DE) $\dot{x}/x^+ = A(t)x + B(t)u, \quad x \in \mathbb{R}^n, \; u \in \mathbb{R}^k$
  $y = C(t)x + D(t)u, \quad y \in \mathbb{R}^m$
- let $\tilde{y}(t) = C(t)\Phi(t, t_0)x_0$

$$= y(t) - \int_{t_0}^{t} C(\tau)\Phi(\tau, t_0)B(\tau)u(\tau)d\tau + D(t)u(t)$$

be "augmented output" produced by x_0

and $M(t_0, t_1) = \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau)\Phi(\tau, t_0)d\tau$

... assume CT LTV observable,

be observability Gramian — assume CT-LTV observable
so $M(t_0, t_1)$ nonsingular

- then: $x_0 = M(t_0,t_1)^{-1} \int_{t_0}^{t_1} \Phi(t_1,t_0)^T C(t)^T \tilde{y}(t)\, dt \in \mathbb{R}^n$

  is an "open-loop" estimate of initial state

→ sensitive to errors: – unmodeled dynamics / model inaccuracy
- external disturbances
- measurement noise

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closed-loop state estimation — online, i.e. using data stream

- consider LTI-DE $\dot{x} / x^+ = Ax + Bu,\quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k$
$$y = Cx + Du,\quad y \in \mathbb{R}^m$$

- recall that, if (A,B) controllable, we can synthesize $K \in \mathbb{R}^{k \times n}$ such that full-state feedback $u = -Kx$ stabilizes (LTI-DE)
  - all eigenvalues of "closed-loop" matrix $A - BK$ is stable

* need to know — or estimate — system state x at all times

- consider $\widehat{\text{LTI-DE}}$ $\quad \dot{\hat{x}} / \hat{x}^+ = A\hat{x} + Bu$

(observer system)

and dynamics of error $e = \hat{x} - x$ are LTI:

$$\dot{e} / e^+ = \dot{\hat{x}} / \hat{x}^+ - \dot{x} / x^+ = A\hat{x} + \cancel{Bu} - (Ax + \cancel{Bu})$$

$$= A\hat{x} - Ax = Ae$$

* if A stable then $e(t) \to 0$ as $t \to \infty$!

i.e. observer state $\hat{x}(t) \to x(t)$ as $t \to \infty$!

◦ if A is not stable: augment error dynamics by augmenting observer dynamics: $\dot{\hat{x}} / \hat{x}^+ = A\hat{x} + Bu \underbrace{- L(\hat{y} - y)}_{\text{"output error injection"}}$, $L \in \mathbb{R}^{n \times m}$

$$\hat{y} = C\hat{x} + Du$$

$$\dot{e} / e^+ = \dot{\hat{x}} / \hat{x}^+ - \dot{x} / x^+ = A\hat{x} + \cancel{Bu} - L(\hat{y} - y) - (Ax + \cancel{Bu})$$

$$= A\hat{x} - Ax - L(C\hat{x} + \cancel{Du} - (Cx + \cancel{Du}))$$

$$= Ae - LC(\hat{x} - x)$$

$$= Ae - LCe = (A - LC)e$$

* if $(A - LC)$ stable then $e(t) \to 0$ as $t \to \infty$!

i.e. observer state $\hat{x}(t) \to x(t)$ as $t \to \infty$!

fact: can synthesize L s.t. $A-LC$ stable

$\iff$ " " L^T s.t. $A^T - C^T L^T$ stable

$\iff$ (A^T, C^T) controllable $\iff$ (A, C) observable