1°. (a) ∘ A injective $\Rightarrow \forall v \neq 0 : Av \neq 0$

∘ suppose $\exists v \neq 0 : A^T A v = 0$

∘ then $0 = v^T A^T A v = \|Av\|^2 \Rightarrow Av = 0$, contradiction

∘ so $\ker(A^T A) = \{0\}$, thus $A^T A$ invertible

(b) $P^2 = A(A^T A)^{-1} \underbrace{A^T A (A^T A)^{-1}}_{= I} A^T = P$

(c) ∘ if $x \in \operatorname{Im} A$ then $\exists v \in \mathbb{R}^n$ s.t. $Av = x$

then $Px = PAv = A\underbrace{(A^T A)^{-1} A^T A}_{= I} v = Av = x$

∘ if $x \in \ker A^T$ then $A^T x = 0$ so $Px = A(A^T A)^{-1} A^T x = 0$

(d) if $y \in \operatorname{Im} A$ then $\exists v \in \mathbb{R}^n$ s.t. $y = Av$

then $y^T(x - P(x)) = v^T A^T (I - A(A^T A)^{-1} A^T) x$

$= v^T (A^T - A^T) x = 0$

* common mistakes:

- A is nonsquare, so can't be invertible (same for A^T)
- $P \neq I$
- not given that $x \in \operatorname{Im} A$ in (d)

2. (a) $\left.\begin{array}{l}\dot{x} = (A - BK)x + Bv \\ y = (C - DK)x + Dv\end{array}\right\}$ (CLTI-CL)

(b). if (CLTI) ctrl'able, then by eigvec test no eigvec of A^T lies in $\ker B^T$

- suppose (CLTI-CL) not ctrlable, so

  $\exists\ v \neq 0,\ \lambda \in \mathbb{C}$ s.t. $(A^T - K^T B^T)v = \lambda v$ and $B^T v = 0$

- then $A^T v = \lambda v$, so v is eigvec of A^T in $\ker B^T$, contradicting (CLTI)'s controllability
- this is a contradiction, so (CLTI-CL) ctrl'able

(c) consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $K = [0\ 1]$

$C = [1\ 0]$

then (A, C) obs'able: C and $CA = [0\ 1]$ are linearly indep

but $(A - BK, C)$ not obs'able: $A - BK = 0$

* common mistakes:

- proving something in (b) that's not sufficient for what's asked, e.g. (CLTI uncontrollable $\Rightarrow$ CLTI-CL uncontrollable)

  $\not\Rightarrow$ (CLTI controllable $\Rightarrow$ CLTI-CL controllable)

3°

(a) $D_x f(\underbrace{1}_{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \overset{\begin{bmatrix}1\\1\end{bmatrix}}{1}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ $\quad D_u f(1,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

$D_x h(1,0) = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$ $\quad D_u h \equiv 0$

(b) ctrl'able: $[B \,\vdots\, AB] = \left[\begin{array}{cc:cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right]$.

lin. indep.

(c) not obs'able: $\begin{bmatrix} C \\ CA \\ CAA \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -4 & 0 \\ 0 & 4 & 0 \end{bmatrix}$

this was trickier than intended;
for $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$ to be an equilibrium, need $u = \begin{bmatrix}1\\1\end{bmatrix}$
* I tried not to dock (a) if you linearized w/ different u;
I did dock (a) if you used different x, but
tried not to penalize (b) or (c) if the
calculations seemed correct given matrices from (a)

4: yes: though (CLTI) is not controllable,
(9, 10, 0) lies in the reachable subspace
the input that takes the ctrl'able subsystem
$\left(\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right)$ from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 9 \\ 10 \end{bmatrix}$
will take whole sys from $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 9 \\ 10 \\ 0 \end{bmatrix}$

* for full credit, needed to observe both
  - x_1 & x_2 ctrl'able
  - $x_3(0) = 0 \Rightarrow x_3(t) = 0$ (regardless of input)

5°. (a) false: if (A,B) not stabilizable, this is not possible

counterex: $A=I$, $B=0$, $\eta=0$, $\xi=0$, $\mu=0$

(b) ∘ we know $(A-BK)$ is stable, i.e. $\mathrm{spec}(A-BK)\subset\mathbb{C}_0^-$

∘ let $\tilde{x}=x-\xi$, $\tilde{u}=u-\mu$

so $\dot{\tilde{x}}=\dot{x}=Ax+Bu=A(\tilde{x}+\xi)+B(\tilde{u}+\mu)$

$=A\tilde{x}+B\tilde{u}$ since $A\xi+B\mu=0$

thus $\tilde{u}=-K\tilde{x}$ yields $\dot{\tilde{x}}=(A-BK)\tilde{x}$

so $\tilde{x}\to 0$, $\tilde{u}\to 0$

∘ in original coords, $x\to\xi$, $u\to\mu$, so $y\to\eta$

∘ thus $u=-K(x-\xi)+\mu$

* common mistakes:
  - in (a), controllability is sufficient but stabilizability is necessary
  - in (b), $\dim x=\dim u$ or $\dim y$, $\dim u\neq\dim y$;

  if you added x (or ξ) to u (or μ) or y (or η) or added u (or μ) to y (or η), I gave zero pts.

6. (a) $\bar{A} = A - BD^{-1}C$, $\bar{B} = BD^{-1}$, $\bar{C} = -D^{-1}C$, $\bar{D} = D^{-1}$

(b) since both systems are stable,
influence of initial conditions is asymptotically forgotten,
so $y \to \bar{u}$ hence $\|y - \bar{u}\| \to 0$

in (b), needed to specify that $x, \bar{x} \to 0$
but not claim that $y, \bar{u} \to 0$
(they need not, yet $\|y - \bar{u}\| \to 0$)