LTI -- frequency domain

goal: derive input/output frequency-domain transformation determined by LTI DE

ref: Hespanha Ch 4 — LTI DE

[textbooks on "signals and systems"] — Laplace / Fourier / Z transform

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from time- to frequency-domain

- consider the LTI DE $\dot{x} / x^+ = Ax + Bu \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k$

  $y = Cx + Du \qquad y \in \mathbb{R}^m$

- applying frequency-domain transformation (Laplace / Fourier in CT; "Z" in DT)

  yields $s\hat{x} = A\hat{x} + B\hat{u}, \quad \hat{y} = C\hat{x} + D\hat{u}$

- rearranging: $\underbrace{(sI - A)\hat{x} = B\hat{u} \implies}$ $\hat{x} = (sI - A)^{-1} B\hat{u}$

assuming $(sI - A)$ invertible, i.e. $\det(sI - A) \neq 0$, i.e. $s \notin \operatorname{spec} A$

assuming $(sI-A)$ invertible, i.e. $\det(sI-A) \neq 0$, $\underbrace{\text{i.e. } s \notin \operatorname{spec} A}_{\text{i.e. } s \text{ isn't an eigenvalue of } A}$

- substituting: $\hat{y} = \underbrace{\left(C(sI-A)^{-1}B + D\right)}_{T_{yu}(s) \in \mathbb{R}^{m \times k}}\hat{u}$

$T_{yu}(s) \in \mathbb{R}^{m\times k}$ is the transfer matrix, i.e. a matrix of transfer functions

* note: $[T_{yu}]_{j,i}$ is transfer function from u_i to y_j

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from frequency- to time-domain

- given $G : \mathbb{C} \to \mathbb{C}^{m \times k}$, i.e. a matrix of transfer functions,
  $: s \mapsto G(s)$

we say $\dot{x}/x^+ = Ax + Bu$, $y = Cx + Du$ is a realization
if $G(s) = C(sI-A)^{-1}B + D$

* note that realizations are not unique!

→ given nonsingular $T \in \mathbb{R}^{n\times n}$, define $\tilde{x} = Tx$ and:

1°. derive LTI $\widetilde{DE}$ for state $\tilde{x}$

2°. show that transfer matrix is same for (DE) & ($\widetilde{DE}$)

→ we'll say (DE) & ($\widetilde{DE}$) are algebraically equivalent

1°. $\tilde{x} = Tx \Rightarrow \dot{\tilde{x}}$ or $\tilde{x}^+ = T\dot{x}$ or Tx^+

$\Updownarrow$

$x = T^{-1}\tilde{x}$

$$
\begin{aligned}
&= T(Ax + Bu) \\
&= \underbrace{TAT^{-1}}_{\tilde{A}}\tilde{x} + \underbrace{TB}_{\tilde{B}}u
\end{aligned}
$$

$$
\begin{aligned}
y &= Cx + Du \\
&= \underbrace{CT^{-1}}_{\tilde{C}}\tilde{x} + \underbrace{D}_{\tilde{D}}u
\end{aligned}
$$

LTI-$\widetilde{DE}$

$$
\begin{aligned}
2°.\ \tilde{G}(s) &= \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D} \\
&= CT^{-1}\underbrace{(sI - TAT^{-1})^{-1}}_{= T(sI-A)^{-1}T^{-1}}TB + D \\
&= C(sI - A)^{-1}B + D = G(s) \quad \boxed{\checkmark}
\end{aligned}
$$

$\rightarrow$ find two DE w/ same transfer matrices but different state dimensions