

goal: characterize stability of LTI systems using Lyapunov functions

ref: Hespanha ch 8.5

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properties of positive-definite matrices

def:  $Q = Q^T$  is positive-definite if  $\forall x \neq 0 : x^T Q x > 0$

" " "negative-definite" "  $< 0$

" " "semidefinite" "  $\geq 0$   
or  $\leq 0$

equivalently (i.e.  $\Leftrightarrow$ ),  $Q = Q^T$  is positive-definite

$\stackrel{(i)}{\Leftrightarrow}$  all eigenvalues are positive (and real)

fact: if  $Q = Q^T > 0$  then

(ii)  $\forall x : 0 < \lambda_{\min}(Q) \|x\|^2 \leq x^T Q x \leq \lambda_{\max}(Q) \|x\|^2$

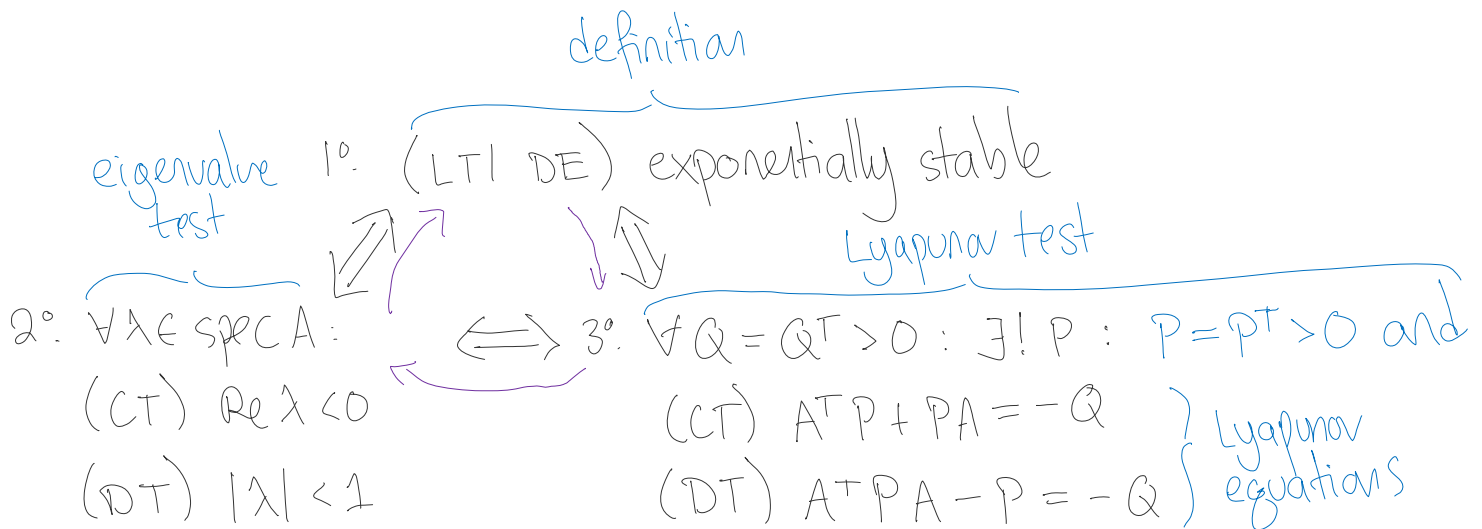
→ derive (i) & (ii)

→ derive analogous facts for negative - (semi) definite

## Lyapunov test for stability

• consider LTI DE  $\dot{x}/x^+ = Ax$

facts: the following are equivalent:



why? let's consider (CT) case

(1°  $\Rightarrow$  3°) assume (LTI DE) exp. stable,  $Q = Q^T > 0$  given,  
and define  $P = \int_0^\infty e^{A^T t} Q e^{A t} dt$

→ verify that  $A^T P + P A = -Q$  (use integration-by-parts)

→ verify that  $P = P^T > 0$  (use def. of pos.-def.)

→ verify that  $P$  unique (assume not, derive contradiction)

→ verify that  $P$  unique

(suppose not, derive contradiction)

(3°  $\Rightarrow$  1°) suppose  $Q = Q^T > 0$ ,  $P = P^T > 0$  are s.t.  $A^T P + P A = -Q$

considers  $v(t) = x(t)^T P x(t) \geq 0$  where  $\dot{x} = A x$

$v: [0, \infty) \rightarrow [0, \infty)$  called a Lyapunov function  
 $: t \mapsto v(t)$

and compute  $\frac{d}{dt} v(t) = \dot{v} = \dot{x}^T P x + x^T P \dot{x}$

$$= x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + P A) x = -x^T Q x \leq 0$$

key step - requires  
a lot of work

(~) can show  $v \rightarrow 0$  at an exponential rate

$\Rightarrow \|x(t)\|^2 \rightarrow 0$  at exp. rate  $\Rightarrow$  (LTI-DE) exp. stable