tests -- LTI

goal: characterize when and how any state of LTI-DE can be reached from the origin

ref: Hespanha Ch 11.6, 12.2, 12.3

- given $\dot{x}/x^+ = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$

the reachability Grammian is $W(t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \in \mathbb{R}^{n \times n}$

$\longrightarrow$ verify this formula by applying change-of-variables to $W(t_0, t_1)$ with $t = t_1 - t_0$

the controllability matrix is $\mathcal{C} = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \cdots \,\vdots\, A^{n-1}B] \in \mathbb{R}^{n \times nk}$

<u>thm:</u> $\mathcal{R}[t_0, t_1] = \mathcal{R}(W(t_1 - t_0)) = \mathcal{R}(\mathcal{C})$

pf: (DT: $x_1 \in \mathcal{R}(\mathcal{C}) \Rightarrow x_1 \in \mathcal{R}[t_0, t_1]$, $t_1 - t_0 \geq n$)

$x_1 \in \mathcal{R}(\mathcal{C}) \Rightarrow \exists w \in \mathbb{R}^{nk}$ s.t. $x_1 = \mathcal{C} \cdot w$

$$\mathcal{C} \cdot w = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_{n-1} \end{bmatrix}, \quad w_k \in \mathbb{R}^k$$

$$= Bw_0 + ABw_1 + \cdots + A^{n-1}Bw_{n-1}$$

$$= \sum_{k=0}^{n-1} A^k B w_k$$

* since $t_1 - t_0 \geq n$ can define $u : [t_0, t_1] \to \mathbb{R}^k$

by $u(\tau) = \begin{cases} 0, & t_0 \leq \tau < t_1 - n \\ w_{t_1 - \tau - 1} & t_1 - n \leq \tau \leq t_1 - 1 \end{cases}$

which yields $x(t_1) = \sum_{\tau = t_0}^{t_1 - 1} A^{t_1 - \tau - 1} B u(\tau) = \sum_{k=0}^{n-1} A^k B w_k = x_1$

$(x(t_0) = 0)$

so $x_1 \in \mathcal{R}[t_0, t_1]$