duality and separation (LQ + KF = LQG)

goal: compare & combine optimal controller and estimator for linear systems with Gaussian noise

ref: Stengel Ch 5.3

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duality – i.e. comparison of LQ controller & KF estimator

- consider $x_{s+1} = A_s x_s + B_s u_s + \delta_s$, $E[\delta_s] = 0$, $Cov[\delta_s] = \bar{Q}_s$

  $y_s = C_s x_s + \eta_s$, $E[\eta_s] = 0$, $Cov[\eta_s] = \bar{R}_s$

control cost $\sum_{s=0}^{t} x_s^T Q_s x_s + u_s^T R_s u_s$

estimation error $\sum_{s=0}^{t} \delta_s^T \bar{Q}_s^{-1} \delta_s + \eta_s^T \bar{R}_s^{-1} \eta_s$

optimal controller $u_s^* = -K_s x_s$

optimal estimator $\hat{x}_s = \tilde{x}_s + L_s (y_s - C_s \tilde{x}_s)$

where K_s solves Riccati DE:

$$K_s = (B_s^T P_{s+1} B_s + R_s)^{-1} B_s^T P_{s+1} A_s$$

$$P_s = (A_s - B_s K_s)^T P_{s+1} (A_s - B_s K_s)$$

where L_s comes from Kalman filter:

$$\tilde{x}_s = A_{s-1} \hat{x}_{s-1} + B_{s-1} u_{s-1}$$

$$L_s = \tilde{P}_s C_s^T (C_s \tilde{P}_s C_s^T + \bar{R}_s)^{-1}$$

$$P_s = (A_s - B_s K_s)^T P_{s+1} (A_s - B_s K_s) + K_s^T R_s K_s + Q_s$$

$$L_s = \tilde{P}_s C_s^T (C_s \tilde{P}_s C_s^T + R_s)^{-1}$$
$$\tilde{P}_s = A_{s-1} \hat{P}_{s-1} A_{s-1} + \bar{Q}_s$$
$$\hat{P}_s = (I - L_s C_s) \tilde{P}_s (I - L_s C_s)^T + L_s \bar{R}_s L_s^T$$

* solutions are eerily similar ! mathematically: dual

— in continuous-time, on infinite horizon (time-invariant):

$$K = LQR(A, B, Q, R) \quad \Big| \quad L^T = LQR(A^T, C^T, \bar{Q}^{-1}, \bar{R}^{-1})$$

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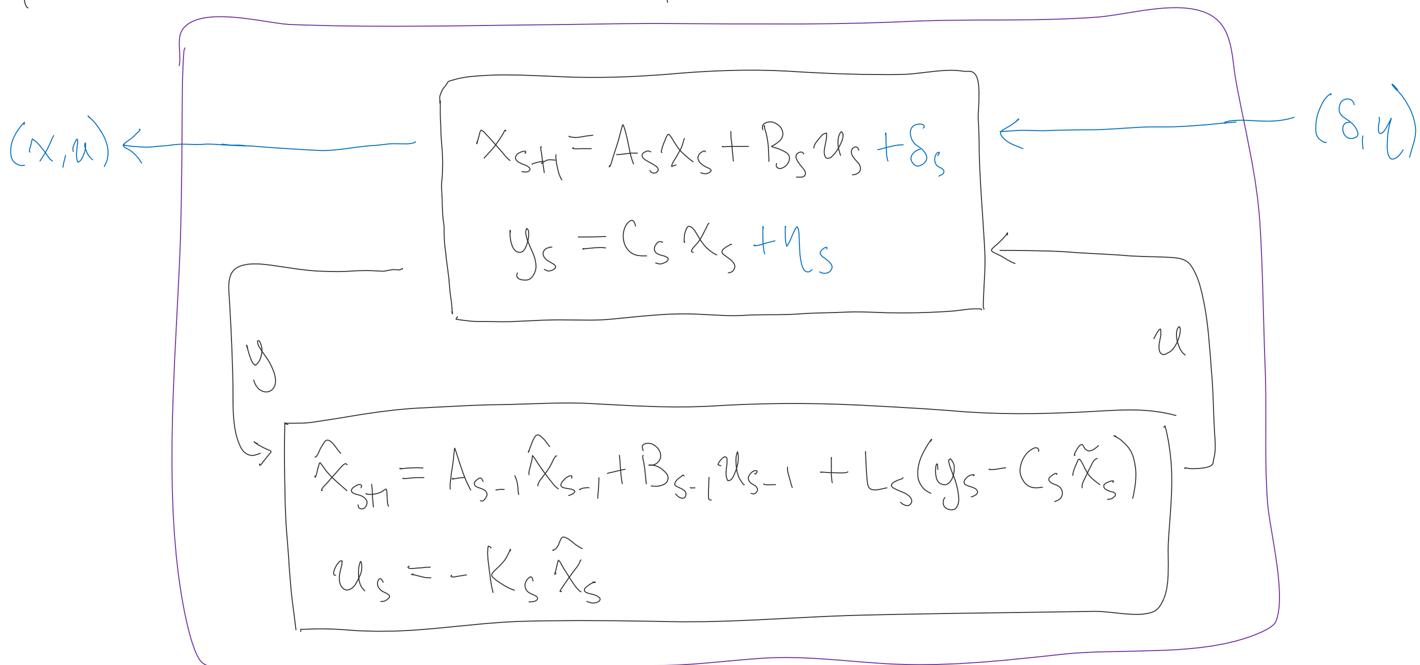
separation — i.e. combination of LQ controller & KF estimator



$\{K_s\}$ minimizes $\sum_{s=0}^{t} x_s^T Q_s x_s + u_s^T R_s u_s$

$\{L_s\}$ minimizes $\sum_{s=0}^{t} \delta_s^T \bar{Q}_s^{-1} \delta_s + \eta_s^T \bar{R}_s^{-1} \eta_s$

even though controller & estimator are designed separately, $e = x - \hat{x} \to 0$, $\hat{x} \to 0$ as $t \to \infty$

* furthermore, this linear-quadratic-Gaussian (LQG) regulator

minimizes $E\left[\sum_{s=0}^{t} x_s^T Q_s x_s + u_s^T R_s u_s\right]$ where expectation is taken minimizes $E\left[\sum_{s=0}^{t} x_s^T Q_s x_s + u_s^T R_s u_s\right]$ where expectation is taken with respect to $\{\delta_s\}, \{\eta_s\}$

— termed "separation principle" in stochastic optimal control (or "certainty equivalence")