observers

goal: estimate state of a linear system using an "observer", that is, a second linear system that processes observations from the first

refs: Hespanha 2009 Ch 16
Stengel 1994 Ch 4.3

- consider linear time-invariant system $\dot{x} / x^+ = Ax + Bu$, $x \in \mathbb{R}^d$
  with observations $y = Cx + Du$, $y \in \mathbb{R}^n$

ex: in mechanical system, y contains positions, not velocities
in electrical circuits, y contains voltages, not currents

idea: construct a second linear system — the "observer":

$$\dot{\hat{x}} / \hat{x}^+ = A\hat{x} + Bu$$ ← $\hat{x} \neq x$, but we use the same A, B & u

→ show that error $e = x - \hat{x} \to 0 \Longleftrightarrow$ A is stable

– $\dot{e} / e^+ = \dot{x} / x^+ - \dot{\hat{x}} / \hat{x}^+$ ⇑

$$-\quad \dot{e}/e^+ = \dot{x}/x^+ - \dot{\hat{x}}/\hat{x}^+$$
$$= Ax + \cancel{Bu}^{\,0} - (A\hat{x} + \cancel{Bu}^{\,0})$$
$$= Ax - A\hat{x} = A(x - \hat{x}) = Ae$$

<u>idea:</u> when A is not stable / to tune performance independently:

$$\dot{\hat{x}}/\hat{x}^+ = A\hat{x} + Bu - L(\hat{y} - y)$$
$$\hat{y} = C\hat{x} + Du$$

$\rightarrow$ show that $\dot{e}/e^+ = (A - LC)e$

$$-\quad \dot{e}/e^+ = \dot{x}/x^+ - \dot{\hat{x}}/\hat{x}^+$$
$$= Ax + \cancel{Bu} - (A\hat{x} + \cancel{Bu}^{\,0} - L(\overbrace{\hat{y} - y}^{C\hat{x} + \cancel{Du}^{\,0} - (Cx + \cancel{Du}^{\,0})}))$$
$$= Ax - A\hat{x} + LC\hat{x} - LCx$$
$$= Ae - LCe = (A - LC)e \quad \boxed{\checkmark}$$

<u>takeaway:</u> if $A - LC$ is <u>stable</u>

i.e. $\forall \lambda \in \operatorname{spec}(A - LC)$: $\begin{cases} \operatorname{Re}\lambda < 0, & \dot{x} \\ |\lambda| < 1, & x^+ \end{cases}$

then $e \rightarrow 0$, so $\hat{x} \rightarrow x$

<u>recall:</u> if (A, C) observable then eigenvalues of $A - LC$

<u>recall</u>: if (A, C) observable then eigenvalues of $A - LC$ can be placed anywhere by appropriate choice of L

$\rightarrow$ $L^T = \text{place}(A^T, C^T, \text{eigvals})$ yields L

s.t. $\text{spec}(A - LC) = \text{eigvals}$