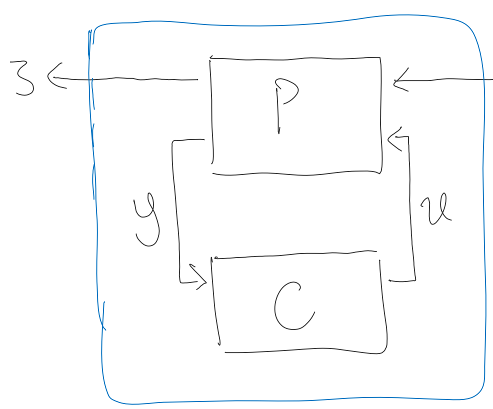
linear analysis

goal: concepts & definitions we'll use to quantify robustness

refs: Zhou, Doyle, Glover – Robust & Optimal Control, 1996

Dullerud & Paganini – A course in robust control theory, 2013

- consider the following feedback block diagram between process P and controller C:



- w contains disturbances (i.e. both input & output disturbances)
- z contains "errors" to be minimized (i.e. tracking error, control effort)

$T_{zw} = P_{zw} + P_{zu}(I - C P_{yu})^{-1} C P_{yw}$ → verify this formula

* our goal as control engineers is to minimize $\|T_{zw}\|$

→ remainder of this lecture will develop math to define

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norms

- a norm $\|\cdot\| : V \to [0,\infty)$ on vector space V satisfies:

  1°. $\|v\| = 0 \Leftrightarrow v = 0$ — definiteness

  2°. $\|\alpha \cdot v\| = |\alpha| \cdot \|v\|, \quad \alpha \in \mathbb{C}$ — positive-homogeneity

  3°. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ — triangle inequality

ex: $\mathbb{C}^n$, $\mathbb{R}^n$ $\quad \|v\|_p = \left(|v_1|^p + \cdots + |v_n|^p\right)^{1/p}$, eg $\|v\|_2 = \sqrt{\sum_{i=1}^n |v_i|^2}$

$$\|v\|_\infty = \max_{1 \leq i \leq n} |v_i|$$

ex: $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}$ $\quad |M|_F = (\mathrm{tr}\, M^* M)^{1/2} = \sqrt{\sum_{i,j} |M_{ij}|^2}$ — Frobenius

$$\sigma_{max}(M) = \left(\lambda_{max}(M^* M)\right)^{1/2}$$ — max singular value

ex: $L_p(-\infty,\infty) = \{ u : (-\infty,\infty) \to \mathbb{C}^n \mid \|u\|_p < \infty \}$

$$\|u\|_p = \left(\int_{-\infty}^{\infty} \|u(t)\|_p^p \, dt\right)^{1/p} \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|u(t)\|_\infty = \max_{t\in\mathbb{R}} \|u(t)\|_\infty$$

inner products

- an <u>inner product</u> $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$ on vector space V:

  1°. $\langle v, v \rangle \geq 0$

  2°. $\langle v, v \rangle = 0 \iff v = 0$

  } definiteness

  3°. $w \mapsto \langle v, w \rangle$ is linear for all v

$\to \|v\| = \sqrt{\langle v, v \rangle}$ is a norm; v, w <u>orthogonal</u> if $\langle v, w \rangle = 0$

<u>ex:</u> $\mathbb{C}^n$ $\langle x, y \rangle = x^* y$ $\mathbb{R}^n$ $\langle x, y \rangle = x^T y =$ "$x \cdot y$"

<u>ex:</u> $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m \times n}$ $\langle A, B \rangle = \operatorname{tr} A^* B$, $\operatorname{tr} M = \sum_{i=1}^{n} M_{i,i}$

<u>ex:</u> L_2 $\langle u_1, u_2 \rangle = \int_{-\infty}^{\infty} u_1^*(t) u_2(t)\, dt = \int_{-\infty}^{\infty} \langle u_1(t), u_2(t) \rangle dt$

$\hookrightarrow$ induces L_2 norm above

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linear functions (linear transformations)

- a linear function $L : V \to W$ between normed vector spaces has an <u>induced norm</u> $\|L\|_{V \to W} = \max_{v \neq 0} \dfrac{\|Lv\|_W}{\|v\|_V}$

<u>ex:</u> $x \mapsto Mx$, $M \in \mathbb{C}^{m\times n}$ $\|M\|_{\mathbb{C}^m \to \mathbb{C}^n} = \sigma_{max}(M)$

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↳ assuming 2-norms on $\mathbb{C}^m$, $\mathbb{C}^n$

Fourier transform – relate time- and frequency-domain

- let $\hat{L}_2(j\mathbb{R})$ be inner product space of signals $\hat{u} : j\mathbb{R} \to \mathbb{C}^n$ with $\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \hat{u}(j\omega), \hat{v}(j\omega) \rangle \, d\omega$, $\|\hat{u}\|_2 = \sqrt{\langle \hat{u}, \hat{u} \rangle} < \infty$

- the <u>Fourier transform</u> of $u : \mathbb{R} \to \mathbb{C}^n$ is

$$\hat{u}(j\omega) = (\Gamma u)(j\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt$$

<u>facts</u> (Plancherel): this defines linear $\Gamma : L_2 \to \hat{L}_2$ with inverse $\Gamma^{-1} : \hat{L}_2 \to L_2$, $u(t) = (\Gamma^{-1}\hat{u})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(j\omega) e^{j\omega t} d\omega$

and, moreover, Γ is an isometry, i.e. preserves lengths $(\|\cdot\|)$ and angles $(\langle \cdot, \cdot \rangle)$

→ so $\|u\|_{L_2} = \|\hat{u}\|_{\hat{L}_2}$, $\quad \hat{u} = \Gamma u \iff u = \Gamma^{-1}\hat{u}$

$\langle u, v \rangle_{L_2} = \langle \hat{u}, \hat{v} \rangle_{\hat{L}_2}$

so we'll drop subscripts $L_2, \hat{L}_2$ when clear from context