H2 and Hoo system norms -- definition & interpretation

goal: definitions & interpretations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ system norms

refs: definitions – Doyle, Glover, Khargonekar, Francis 1989

**State-Space Solutions to Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control Problems**

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interpretations – Dullerud & Paganini 2013 Ch 6, 7

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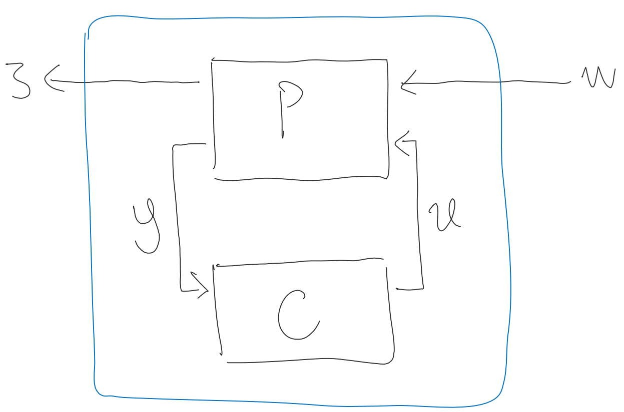
- consider the following feedback block diagram between process P and controller C:




- w contains disturbances (i.e. both input & output disturbances)
- z contains "errors" to be minimized (i.e. tracking error, control effort)

$$T_{zw} = P_{zw} + P_{zu}(I - C P_{yu})^{-1} C P_{yw}$$ → verify this formula

* our goal as control engineers is to minimize $\|T_{zw}\|$

↑ our goal as control engineers is to minimize $\|z_w\|$

↳ we'll focus on $H_2$ and $H_\infty$ norms

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$H_2$ norm

- suppose we know the disturbance a priori (eg a fixed reference)

  $w : (-\infty, \infty) \to \mathbb{R}^n$

  – since system & controller are LTI, focus on scalar Dirac delta:

  $w = \delta$

- by definition:
$$
\begin{aligned}
\|z\|_2^2 &= \int_{-\infty}^{\infty} z^*(t)\, z(t)\, dt && \leftarrow z \in L_2 \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{z}^*(j\omega)\, \hat{z}(j\omega)\, d\omega && \leftarrow \|z\|_2 = \|\hat{z}\|_2 \text{ where } \hat{z} = \Gamma z \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{T}^*(j\omega)\, \hat{T}(j\omega)\, d\omega && \leftarrow z = Tw = T\delta \Rightarrow \hat{z} = \hat{T} \cdot 1 \\
&= \|\hat{T}\|_2^2
\end{aligned}
$$

in other words, $\|z\|_2 = \|\hat{T}\|_2$, i.e. the $H_2$-norm of $T$

↳ "H" is for "Hardy"

- similar calculations apply when statistics of $w$ are known

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$z = (x, u)$ ← i.e. we want to minimize $\|x\|_2$, $\|u\|_2$

the controller $C$ that minimizes $\|T_{zw}\|_2$ is LQG

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$H_\infty$ norm

- what if we have no a priori knowledge of $w$?

→ then it makes sense to consider worst-case scenario:

minimize $\|T\|_{L_2 \to L_2}$ where $z = Tw$

$= \|\hat{T}\|_\infty$, i.e. the $H_\infty$-norm of $\hat{T}$

* $\|\hat{T}\|_\infty$ quantifies how much "energy" in disturbance passes through $T$ to output