goal: understand "solution" of differential/difference equations (DE)
$$\dot{x} / x^+ = f(x, u)$$

ref: Strogatz Ch 2 — geometric perspective

3°: linearization

• given DE $\dot{x} / x^+ = f(x, u)$, let $\phi: [0, t] \times \mathbb{R}^d \to \mathbb{R}^d$ be flow
and $\psi: [0, t] \times \mathbb{R}^d \to \mathbb{R}^d$ be sim flow

* we'll study how ϕ, ψ vary wrt. both arguments;
 in particular, we'll use linearization to determine first-order variation

$\longrightarrow$ in CT case $\dot{x} = f(x, u)$, what is $D_1 \phi(s, \xi) = \frac{\partial}{\partial s} \phi(s, \xi)$?

— since $\phi(s, \xi) = x_\xi(s)$, $D_s \phi(s, \xi) = D_s x_\xi(s)$
$$= f(x_\xi(s), u(s)) = f(\phi(s, \xi), u(s))$$

• now consider how DT flow varies with respect to ξ
 — recall $\phi(s+1, \xi) = f(\phi(s, \xi))$ $\longleftarrow$ consider autonomous case (no u)

$\longrightarrow$ compute $D_2 \phi(s+1, \xi)$ in terms of Df & $D_2 \phi(s, \xi)$

— applying the chain rule,
$$D_2 \phi(s+1, \xi) = D_\xi \left[f(\phi(s, \xi)) \right]$$
$$= Df(\phi(s, \xi)) \cdot D_2 \phi(s, \xi)$$

$\rightarrow$ compute $D_2\phi(s+1, \xi)$ in terms of Df

— recursively substituting $D_2\phi$: $D_2\phi(s+1, \xi) = \prod\limits_{\ell=0}^{s} Df(\phi(\ell, \xi))$

(note that $\phi(0, \xi) = \xi$, so $D_2\phi(0, \xi) = I$)

○ same computation yields $D_2\psi(s+1, \xi) = \prod\limits_{\ell=0}^{s} Df(\psi(\ell, \xi))$

○ continuous-time case isn't as straightforward;
start with simulation flow, which satisfies

$$\psi(s+\Delta, \xi) = \psi(s, \xi) + \Delta \cdot f(\psi(s, \xi))$$

$\rightarrow$ compute $D_2\psi(s+\Delta, \xi)$ in terms of Df & $D_2\psi(s, \xi)$

— $D_2\psi(s+\Delta, \xi) = D_\xi\left[\psi(s, \xi) + \Delta \cdot f(\psi(s, \xi))\right]$

$$= D_2\psi(s, \xi) + \Delta \cdot Df(\psi(s, \xi)) \cdot D_2\psi(s, \xi)$$

$$= \left(I + \Delta \cdot Df(\psi(s, \xi))\right) \cdot D_2\psi(s, \xi)$$

$\rightarrow$ " $\quad s = N \cdot \Delta \quad$ " $\quad\quad$ " Df

— $D_2\psi(s+\Delta, \xi) = \prod\limits_{k=0}^{N}\left[I + \Delta \cdot Df(\psi(k \cdot \Delta, \xi))\right]$

(note $\psi(0, \xi) = \xi$ so $D_2\psi(0, \xi) = I$)

$\rightarrow$ apply forward Euler simulation algo to matrix differential eqn:

$$\forall s \in [0, t]: \dot{X}(s) = Df(\phi(s, \xi)) \cdot X(s), \quad X(0) = I$$

$$- \ X(s+\Delta) = X(s) + \Delta \cdot Df(\phi(s,\xi)) \cdot X(s)$$

$$= \left[I + \Delta \cdot Df(\phi(s,\xi)) \right] \cdot X(s)$$

$$= \prod_{k=0}^{N} \left[I + \Delta \cdot Df(\phi(s,\xi)) \right]$$

<u>fact</u>: linearization of CT flow is obtained by solving

so that $D_\xi \phi(s,\xi) = X(s)$, where $X : [0,t] \to \mathbb{R}^{d \times d}$ solves

$\longrightarrow$ how does this simplify at an equilibrium? ($f(x_0) = 0$)

$- \ f(x_0) = 0 \implies \phi(s,x_0) = x_0$

$$\implies \dot{X}(s) = Df(\underbrace{\phi(s,x_0)}_{\equiv x_0}) \cdot X(s)$$

define $A = Df(x_0)$

$$= Df(x_0) \cdot X(s) = A \cdot X(s)$$