nonlinear programming

goal: derive necessary & sufficient conditions for local optimality in unconstrained nonlinear programs (NLP)

refs: Stengel 1994 pg 29–41
Lewis, Vrabie, Syrmos 2012 Ch 1

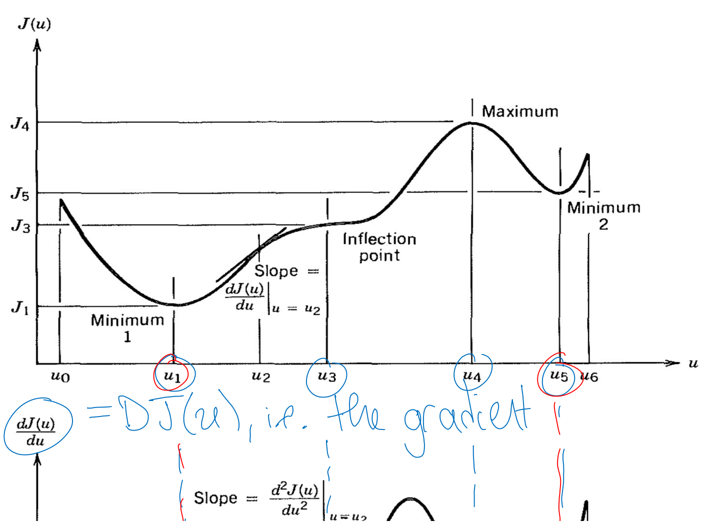
- we want to minimize a given <u>objective</u> function $J : \mathbb{R}^m \to \mathbb{R}$
$: u \mapsto J(u)$

i.e. find $u^* \in \mathbb{R}^m$ s.t. $\forall u \neq u^* : J(u^*) < J(u)$

– write $\min_{u \in \mathbb{R}^m} J(u) \leftarrow$ "nonlinear program" (NLP)

<u>idea</u>: starting from $u \in \mathbb{R}^m$ where $DJ(u) \neq 0$ then
$u^+ = u - \gamma \cdot DJ(u)$ yields $J(u^+) < J(u)$ for all $\alpha > 0$ small

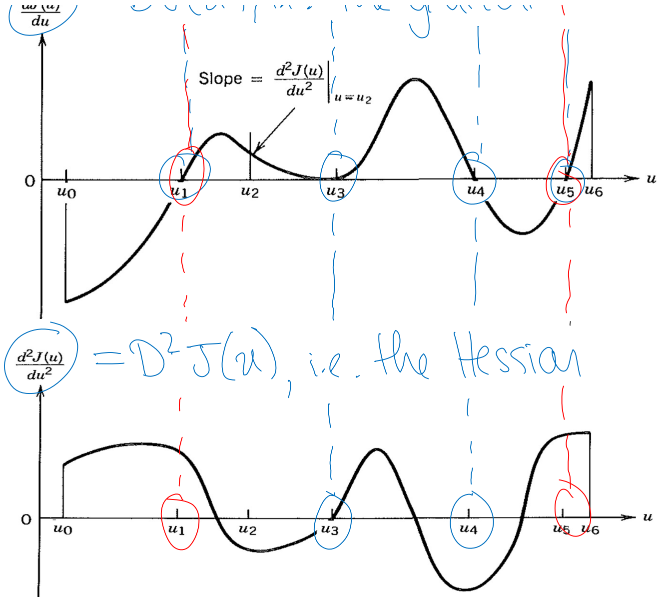
$\longrightarrow$ so we can use gradients to:
1°. detect that we haven't found the minimum
2°. update / iterate to get closer to minimum



$\frac{dJ(u)}{du} = DJ(u)$, i.e. the gradient

def: $u_0 \in \mathbb{R}^m$ is <u>stationary</u> if $DJ(u_0) = 0$
ex: u_1, u_3, u_4, u_5

<u>def</u>: $u^* \in \mathbb{R}^m$ is a local minimum if there is an open neighborhood $\mathcal{U} \subset \mathbb{R}^m$ s.t. $\forall u \in \mathcal{U} : J(u^*) \leq J(u)$



s.t. $\forall u \in \mathcal{U} : J(u^*) \le J(u)$

i.e. a "ball": $\{u : \|u - u^*\| < r\}$

u^* is a strict local min if $<$ strict for $u \neq u^*$

ex: u_1, u_5

thm: (sufficient conditions for optimality)

a $\underbrace{\text{stationary point } u_0 \in \mathbb{R}^m}_{\Rightarrow DJ(u_0) = 0 \in \mathbb{R}^{1\times m}}$ is a strict local min if $\underbrace{\overbrace{D^2J(u_0)}^{\in \mathbb{R}^{m\times m}} > 0}_{\text{i.e. positive-definite}}$

aside: $D^2J(u_0)$ is symmetric if J is twice continuously differentiable

because $D^2J(u_0) = \left[\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} J\Big|_{u=u_0}\right]_{i,j} = \left[\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_i} J\Big|_{u=u_0}\right]_{i,j}$

$\Rightarrow D^2J(u_0)^T = D^2J(u_0)$

aside: if $S = S^T$ then all eigenvalues of S are real

so $S > 0 \iff$ all eigenvalues of S are positive

$\rightarrow$ determine J s.t. $u_0 = 0$ is a local min but $D^2J(u_0) \not> 0$

– $J(u) = 0$ – $J(u) = u^4$ – $J(u) = |u|$ $\leftarrow$ not differentiable

thm: (necessary conditions for optimality):

if $u_0 \in \mathbb{R}^m$ is a local min, then:

– if J is continuously differentiable ($J \in C^1$) then $DJ(u_0) = 0$

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- if J is twice continuously diff'able ($J \in C^2$) then $D^2J(u_0) \geq 0$

→ determine necessary conditions on $b^T \in \mathbb{R}^{1 \times m}$, $C^T = C$
for $u_0 \in \mathbb{R}^m$ to be local min of

$$J(u) = J(u_0) + b^T(u - u_0) + \frac{1}{2}(u - u_0)^T C (u - u_0)$$

- $DJ(u) = b^T + (u - u_0)^T C$ so necessary that $DJ(u_0) = \boxed{b^T = 0}$
- $D^2J(u) = C$, so necessary $DJ(u_0) = \boxed{C \geq 0}$

→ if u_0 is a strict local min, solve for u_0

- if $C \not> 0$ then $\exists\, v_0 \neq 0$ s.t. $C v_0 = 0$

  so $J(u) = J(u_0)$ for all $u - u_0 = \alpha \cdot v_0$, $\alpha \in \mathbb{R}$
- conclude $C > 0$ for u_0 to be a strict local min
- verify that $u_0 = \underbrace{u - (D^2J(u))^{-1} DJ(u)^T}_{\text{Newton-Raphson iteration}}$