linear quadratic (LQ) regulation

goal: derive optimal controller for $\underbrace{\text{linear}}_{L}$ DE with $\underbrace{\text{quadratic}}_{Q}$ cost

(i.e. L Q regulator)

refs: Stengel 1994 Ch 3.7

Lewis, Vrabie, Syrmos 2012 Ch 6

- DT linear-quadratic regulation (DT-LQR)

$$\min_u C_\tau(x,u) \quad \text{s.t.} \quad x_{\tau+1} = A_\tau x_\tau + B_\tau u_\tau \leftarrow \text{linear DE}$$

where $C_\tau(x,u) = \underbrace{\frac{1}{2} x_t^T P_t x_t}_{\ell(t,x_t)} + \frac{1}{2}\sum_{s=\tau}^{t-1} \underbrace{x_s^T Q_s x_s + u_s^T R_s u_s}_{\mathcal{L}(s,x_s,u_s)}$

entire control signal $u:[0,t] \to \mathbb{R}^m$

$\ell(t,x_t) \leftarrow$ quadratic costs $\to \mathcal{L}(s,x_s,u_s)$

* assume $P_t^T = P_t > 0$, $Q_s^T = Q_s > 0$, $R_s^T = R_s > 0$

→ we'll use Bellman's principle to determine optimal control:

letting $v_\tau^* : \mathbb{R}^d \to \mathbb{R} : x_\tau \mapsto v_\tau^*(x_\tau)$ be value function,

$$v_\tau^*(x_\tau) = \min_{u_\tau \in \mathbb{R}^m} \left[\mathcal{L}(s,x_s,u_s) + v_{\tau+1}^*(x_{\tau+1}) \right]$$

→ determine optimal control & value at $\tau = t$

– since $C_t = \frac{1}{2} x_t^T P_t x_t$, $u_t^* = 0$ ← not uniquely determined

and $v_t^*(x_t) = \frac{1}{2} x_t^T P_t x_t$

→ determine optimal control & value at $\tau = t-1$

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- $V^*_{t-1}(x_{t-1}) = \min_{u_{t-1}} \left[\frac{1}{2} x_{t-1}^T Q_{t-1} x_{t-1} + \frac{1}{2} u_{t-1}^T R_{t-1} u_{t-1} + \frac{1}{2} x_t^T P_t x_t \right]$

- substituting $x_t = A_{t-1} x_{t-1} + B_{t-1} u_{t-1}$ & differentiating $D_{u_{t-1}}$

  yields $u_{t-1}^T R_{t-1} + (A_{t-1} x_{t-1} + B_{t-1} u_{t-1})^T P_t B_{t-1}$

  $$= u_{t-1}^T (R_{t-1} + B_{t-1}^T P_t B_{t-1}) + x_{t-1}^T A_{t-1}^T P_t B_{t-1}$$

  $$= 0 \iff u^*_{t-1} = -\Big(\underbrace{\underbrace{B_{t-1}^T P_t B_{t-1}}_{*\text{ verify: } \geq 0} + \underbrace{R_{t-1}}_{>0}}_{*\text{ verify } > 0 \implies \text{all eigvals} > 0}\Big)^{-1} B_{t-1}^T P_t A_{t-1} x_{t-1}$$

- differentiating wrt u_{t-1} yields $(R_{t-1} + B_{t-1}^T P_t B_{t-1}) > 0$

  $\implies u^*_{t-1}$ is the minimum, i.e. optimal control

- for simplicity, define $K_{t-1} = (B_{t-1}^T P_t B_{t-1} + R_{t-1})^{-1} B_{t-1}^T P_t A_{t-1}$

  so $u^*_{t-1} = -K_{t-1} x_{t-1}$ i.e. optimal control is <u>linear</u>!

- optimal value $V^*_{t-1} = \frac{1}{2} x_{t-1}^T P_{t-1} x_{t-1}$ where

  $$P_{t-1} = (A_{t-1} - B_{t-1} K_{t-1})^T P_t (A_{t-1} - B_{t-1} K_{t-1}) + K_{t-1}^T R_{t-1} K_{t-1} + Q_{t-1}$$

- at time $\tau = t-2$, we'll perform analogous calculations, so conclude:

  - define
  $$\boxed{\begin{aligned} K_s &= (B_s^T P_{s+1} B_s + R_s)^{-1} B_s^T P_{s+1} A_s \\ P_s &= (A_s - B_s K_s)^T P_{s+1} (A_s - B_s K_s) + K_s^T R_s K_s + Q_s \end{aligned}}$$

$$- \text{then:} \quad \boxed{\underbrace{\begin{cases} P_s = (A_s - B_s K_s)' P_{s+1} (A_s - B_s K_s) + K_s' R_s K_s + Q_s \\ u_s^* = -K_s x_s, \quad V_s^* = \frac{1}{2} x_s^T P_s x_s \end{cases}}_{\text{solution to DT-LQR optimal control problem}}}$$

↳ called a Riccati DE (Jacopo Riccati, 1676–1754)