randomness

goal: mathematical representation of "random" variables & their statistics

read: Stengel Ch 2.4

---

* regardless of whether the universe is "random",
we can use mathematical models of "randomness" in systems
to represent: unmodeled phenomena or uncertainty

eg vibrations
deformation
friction
backlash

eg parameters
estimated quantities

def: a <u>random variable</u> is a function $x : \Omega \to \mathbb{R}$
over a <u>sample space</u> Ω with associated probabilities:

discrete probability   continuous probability discrete probability

$|\Omega| < \infty$

- each $\omega \in \Omega$ has probability $P(\omega) \in [0,1]$ and $\sum_{\omega \in \Omega} P(\omega) = 1$

- each event $W \subset \Omega$ has probability $P(W) = \sum_{\omega \in \Omega} P(\omega)$

continuous probability

$|\Omega| = \infty$, e.g. $\Omega = \mathbb{R}^n$

- each $\omega \in \Omega$ has density $p(\omega) \in [0,\infty)$ and $\int_\Omega p(\omega)\, d\omega = 1$

- each event $W \subset \Omega$ has probability $P(W) = \int_W p(\omega)\, d\omega$

* if we're not careful, can generate paradoxes, eg Banach-Tarski

---

- random variable $X : \Omega \to \mathbb{R}$ & random vector $X : \Omega \to \mathbb{R}^d$ (both referred to as "rv") quantifying "payout" of each $\omega \in \Omega$

$|\Omega| < \infty$

ex: flipping a coin: $\Omega = \{H, T\}$

$P(H) = P(T) = \frac{1}{2}$

$X : \Omega \to \mathbb{R}$: $X(H) = \$1$

$X(T) = -\$5$

$\Omega = \mathbb{R}^n \quad p : \mathbb{R}^n \to [0,\infty)$

ex: Gaussian or normal: $\mathcal{N}(\mu, \Sigma)$ $\in \mathbb{R}^n$, $\in \mathbb{R}^{n\times n}$

$\forall W \subset \Omega = \mathbb{R}^n$:

$$P(W) = \int_W \frac{\exp\left(-\frac{1}{2}(\omega-\mu)^T \Sigma^{-1}(\omega-\mu)\right)}{\sqrt{(2\pi)^n \det \Sigma}}\, d\omega$$

$X : \Omega \to \mathbb{R}$: $X(\omega) = a^T \cdot \omega$

$X' : \Omega \to \mathbb{R}^d$: $X'(\omega) = \omega$

$x' : \Omega \to \mathbb{R}^d : \quad x'(\omega) = \omega$

$y : \Omega \to \mathbb{R}^m : \quad y(\omega) = M \cdot \omega, \; M \in \mathbb{R}^{m \times n}$

now let's define statistics – function of a rv $x : \Omega \to \mathbb{R}^n$

- <u>expectation</u> or <u>mean</u>

$|\Omega| < \infty$

$$E[x] = \sum_{\omega \in \Omega} x(\omega) \cdot P(\omega)$$

$|\Omega| = \mathbb{R}^n, \; \mathcal{N}(\mu, \Sigma)$

$$E[x] = \int_{\Omega} x(\omega) \cdot p(\omega) d\omega$$

ex: $x(\omega) = \omega \quad E[x] = \mu \in \mathbb{R}^n$

→ "middle" of distribution

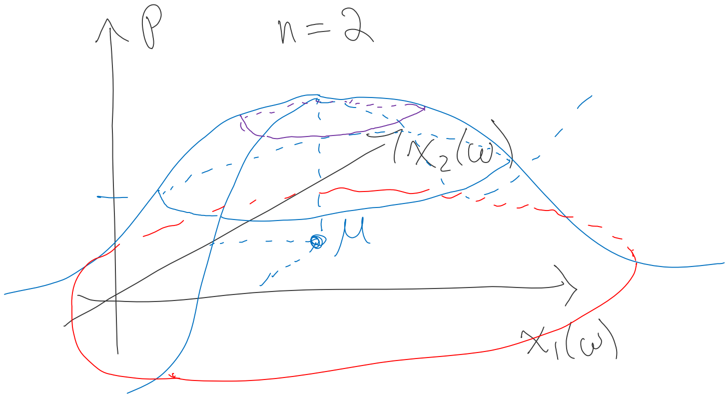
<u>(co-)variance</u>

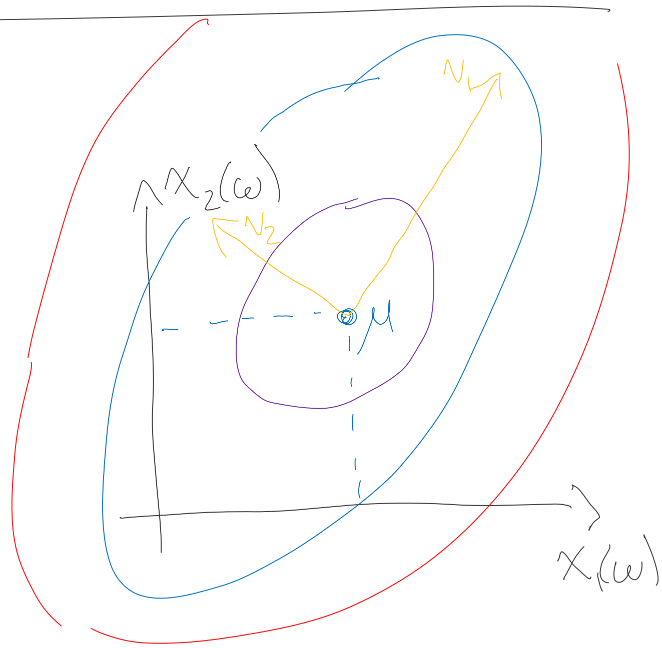
$$Cov[x] = E\left[(x - E[x])(x - E[x])^T \right] \in \mathbb{R}^{n \times n}$$

ex: $x(\omega) = \omega \quad Cov[x] = \Sigma$

→ "spread" of distribution

<u>ex</u>: Gaussian rv $x \sim \mathcal{N}(\mu, \Sigma)$





Σ has eigenvectors $\{v_1, v_2\}$
* corresponding eigenvalues σ_1^2, σ_2^2

- define $\nu_1 = v_1^T \cdot x$, $\nu_2 = v_2^T \cdot x$ so $\nu_1, \nu_2 : \mathbb{R}^n \to \mathbb{R}$

  then $E[\nu_i] = v_i^T \cdot \mu$, $Cov[\nu_i] = \sigma_i^2$

fact: if $x \sim \mathcal{N}(\mu, \Sigma)$ and $y = Mx + b$,

then $y \sim \mathcal{N}(M\mu + b, M \Sigma M^T)$, i.e. y is a Gaussian rv

and $E[y] = M \cdot \mu + b$, $Cov[y] = M \Sigma M^T$

* this is a <u>very</u> special property of Gaussian r.v.s, <u>not</u> satisfied by most r.v.s !