state estimation

goal: estimate state of a linear system from observations obtained over a sequence of samples

refs: Hespanha 2009 Ch 15
Stengel 1994 Ch 4.1

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- consider linear system $\dot{x}(t) / x(t+1) = A(t)x(t) + B(t)u(t)$
  - suppose we don't directly measure $x(t) \in \mathbb{R}^d$ but instead have observations $y(t) = C(t)x(t) + D(t)u(t) \in \mathbb{R}^n$

  i.e. $y(t)$ is the vector of sensor measurements at time $t$

  ex: mechanical system: $y(t)$ contains positions (encoder, GPS) but not velocities

  electrical circuit: $y(t)$ contains voltages but not currents

* key observation: observation $y$ is a linear function of unknown initial state $x(\tau)$ and known input $u$

- recall: if $\Phi(t,\tau)$ is the state transition matrix associated with the linear system:

with the linear system:

$$x(t) = \Phi(t,\tau)x(\tau) + \int_\tau^t \Phi(t,s)B(s)u(s)\,ds$$

$$\Rightarrow \quad y(t) = C(t)x(t) + D(t)u(t)$$

$$= C(t)\Phi(t,\tau)x(\tau) + \int_\tau^t C(t)\Phi(t,s)B(s)u(s)\,ds + D(t)u(t)$$

* state estimate can be obtained by solving linear equations!

→ focus on discrete-time linear time-invariant case (DT-LTI)

set up a system of equations involving known $\{(y(\tau), u(\tau))\}_{\tau=0}^{t}$ and unknown $x(0)$ that can be solved for $x(0)$

(how many samples $t$ are needed? what condition must be imposed on $(A,B,C,D)$?)

– for each $\tau \in \{0,\dots,t\}$:

$$C\,x(0) = y(0) - D\,u(0)$$

→ can I solve for $x(0)$? $\Leftrightarrow$ $\operatorname{rank} C = d$

but $C \in \mathbb{R}^{n\times d}$, so $\operatorname{rank} C \le \min\{n,d\}$

$$C\,x(1) = y(1) - D\,u(1)$$

$$\Leftrightarrow CA\,x(0) + CB\,u(0) = y(1) - D\,u(1)$$

$$\Leftrightarrow CA\,x(0) = y(1) - D\,u(1) - CB\,u(0)$$

→ can I solve for $x(0)$? $\Leftrightarrow$ $\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = d$

$\vdots$

$$CA^t x(0) = y(t) - D\,u(t) - \sum^{t-1} CA^{(t-1)-\tau} B\cdot u(\tau)$$

$$CA^t x(0) = y(t) - Du(t) - \sum_{\tau=0}^{t-1} CA^{(t-1)-\tau} B \cdot u(\tau)$$

$$\Updownarrow$$

$$\underbrace{\mathcal{O}_t\, x(0) = y}, \quad \mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^t \end{bmatrix} \in \mathbb{R}^{(t \cdot n) \times d}$$

$\longrightarrow$ can I solve for $x(0)$? $\Longleftrightarrow$ rank $\mathcal{O}_t = d$

takeaway: linear dynamics + linear observation
$\rightsquigarrow$ state can be estimated by solving system of linear equations