Tue Mar 29

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- [x] course logistics: flipped lectures, (a)synchronous participation
- [x] Canvas: schedule, Syllabus, Announcements, Assignments
- [ ] first assignment (HW0) due <u>this</u> Fri Apr 1
- [ ] first lectures (week 1) available
- [ ] questions / office hours

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- [x] course logistics, Canvas, etc

Thu Mar 31
- [x] HW0
- [x] week 1 lectures
- [ ] questions / office hours

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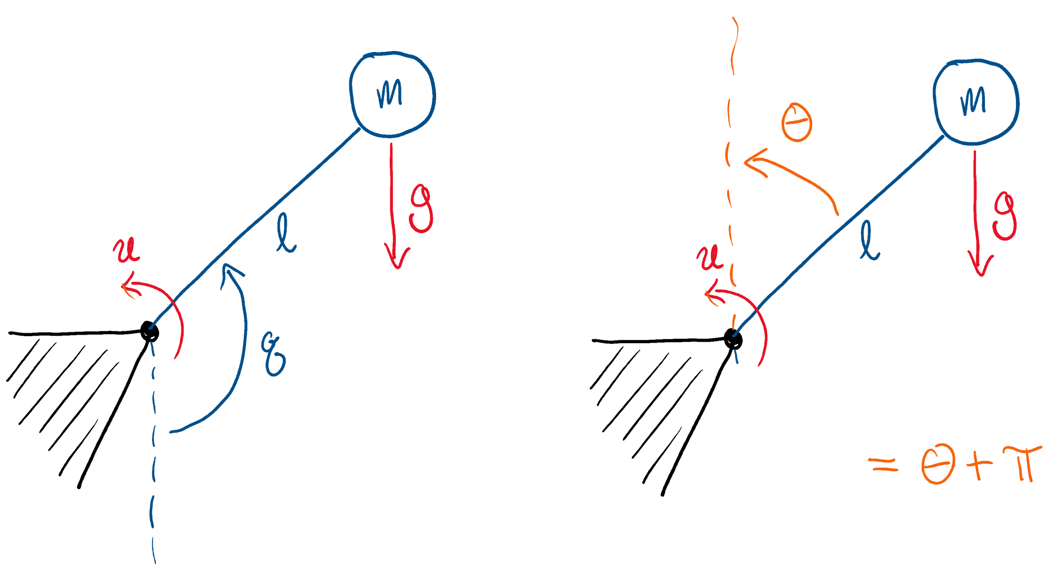
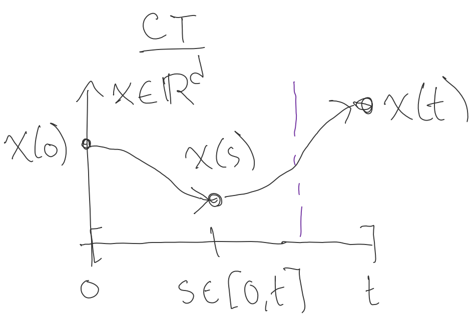
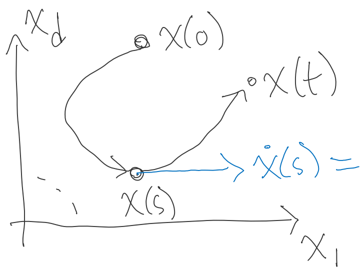
ex: inverted pendulum $\quad m\ell^2 \ddot{q} = -mg\sin q + u - b\dot{q}$



CT

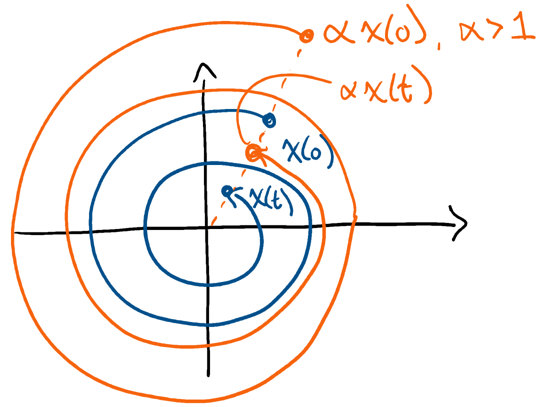




$\dot{x}(s) = \underbrace{f(x(s), u(s))}_{\text{"vector field"}}$

what if f is linear?
(eg LTI?)

$\dot{x} = Ax \iff x(t) = e^{At} x(0)$



Linear systems:

$z(0) = x(0) + y(0)$

$\Downarrow$

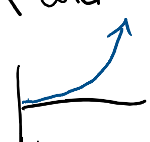
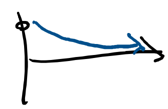
$z(t) = x(t) + y(t)$

Q: what can the flow of an LTI system do?

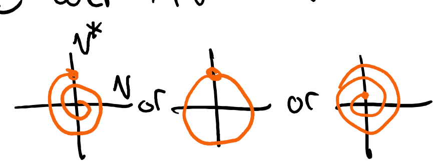
$\dot{x} = Ax$: if $Av = \lambda v$ then $x(0) = \alpha \cdot v \Rightarrow x(t) = \alpha \cdot e^{\lambda t} v$

$\alpha \in \mathbb{R}$

so there are 2 cases: 1°. if $\lambda \in \mathbb{R}$ then $v \in \mathbb{R}^d$ and

$e^{\lambda t}$ looks like  or 

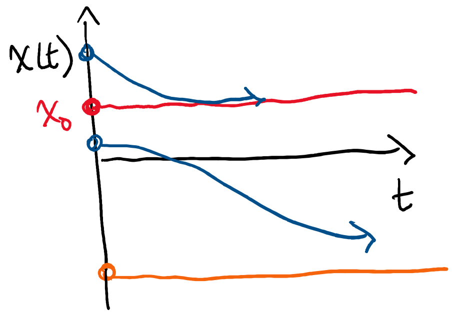
2°. if $\lambda \in \mathbb{C}$ then $v \in \mathbb{C}^d$ and $Av^* = \lambda^* v^*$

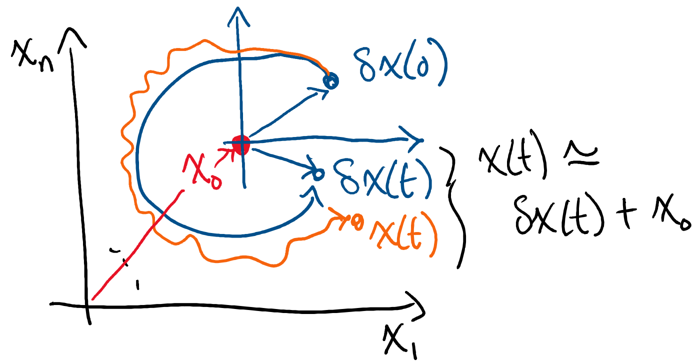
so $e^{\lambda t}$ looks like 

(NL) $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

def: $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an <u>equilibrium</u> for (NL) if $f(x_0, u_0) = 0$

↳ if $x(0) = x_0$, $u(t) = u_0$ then $x(t) = x_0$





* letting $\delta \dot{x} = A \cdot \delta x + B \cdot \delta u$ where $A = \partial_x f(x_0, u_0)$, $B = \partial_u f(x_0, u_0)$.

then $x \simeq x_0 + \delta x$ if $u = u_0 + \delta u$

this approx. gets better as $\|\delta x\|$ gets smaller

Tue Apr 5

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☑ HW0 self-assessment – due next Monday
- ☑ HW 1 – due this Friday
- ☐ week 2 lectures
- ☐ questions / office hours

TODO: ☐ post notes

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consider $\dot{x} = Ax$, $A \in \mathbb{R}^{n \times n}$

fact: if A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$
with corresponding eigenvectors $v_1, \dots, v_n \in \mathbb{C}^n$
then $D = VAV^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Rightarrow A = V^{-1} D V$

where $V = [v_1, \dots, v_n]$

A is diagonalizable

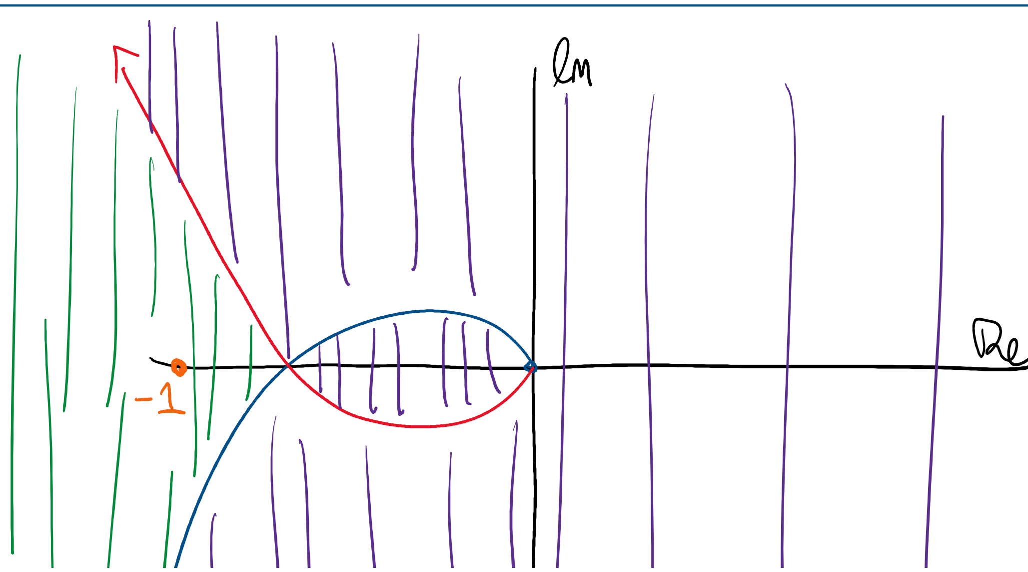
so $e^{At} = e^{V^{-1}DVt} = V^{-1} e^{Dt} V$

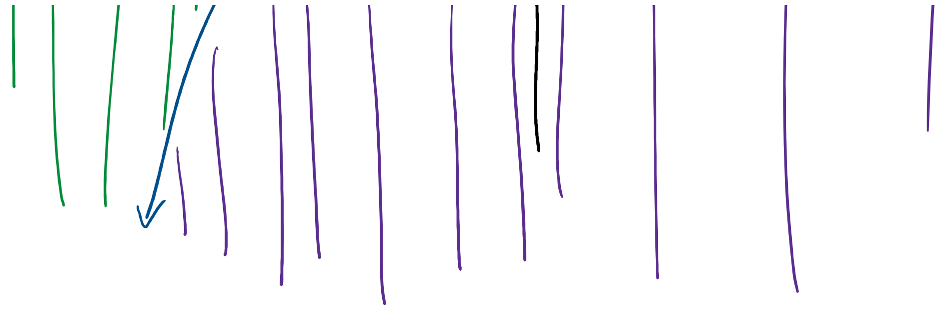
so $e^{At} = e^{V^{-1}DVt} = V^{-1}e^{Dt}V$

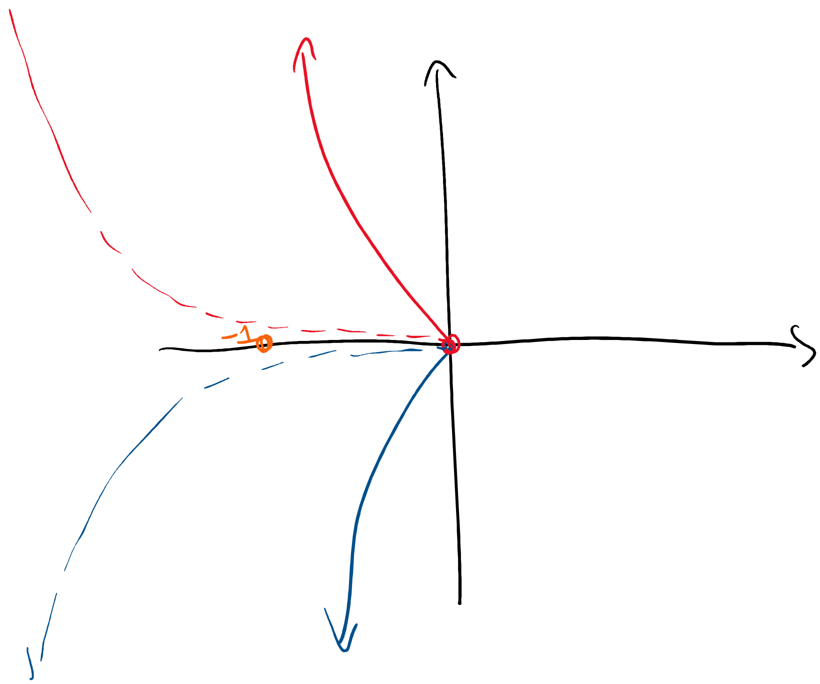
$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

more generally, for any A there exists T^{-1} s.t.

$$J = TAT^{-1} = \begin{bmatrix} \Lambda_1 & & 0 \\ & \ddots & \\ 0 & & \Lambda_n \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{bmatrix}$$







$\xi = $ xi

$\phi = $ phi

$$D_2 \phi(s+1, \xi) = D_\xi \left[f(\phi(s,\xi)) \right]$$
$$= Df(\phi(s,\xi)) \cdot D_2 \phi(s,\xi)$$

$$\begin{array}{ccccccc} \phi(0,x) & & \phi(1,x) & & \phi(2,x) & & \\ \| & & \| & & \| & & \\ x & \mapsto & f(x) & \mapsto & f(f(x)) & \mapsto & \end{array}$$

$$\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \qquad f : \mathbb{R}^d \to \mathbb{R}^d$$
$$: (t, x_0) \mapsto x_t \qquad : x \mapsto x^+$$

$\hookrightarrow Df(x) : \mathbb{R}^d \to \mathbb{R}^d$ linear

$\in \mathbb{R}^{d \times d}$

$$\phi(s+1, x) = f(\phi(s,x))$$

$$D_x \left[\phi(s+1, x) \right] = D_x \left[f(\phi(s,x)) \right]$$

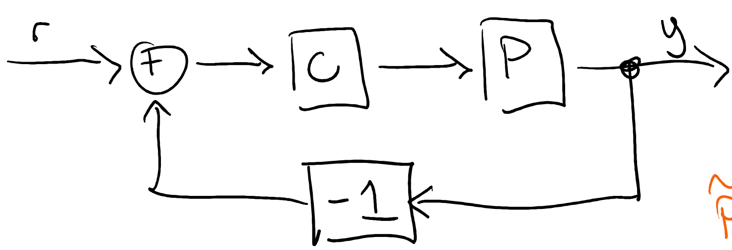
$$D_x[\phi(s+1, x)] = D_x[f(\phi(s,x))]$$
$$= Df \cdot D_2\phi$$
$$= Df(\phi(s,x)) \cdot D_2\phi(s,x)$$

Thu Apr 7

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☑ HW0 self-assessment – due next Monday
- ☑ HW1 – due this Friday
- ☐ week 2 lectures
- ☐ questions / office hours

TODO: ☐ link notebooks / point to Python intro

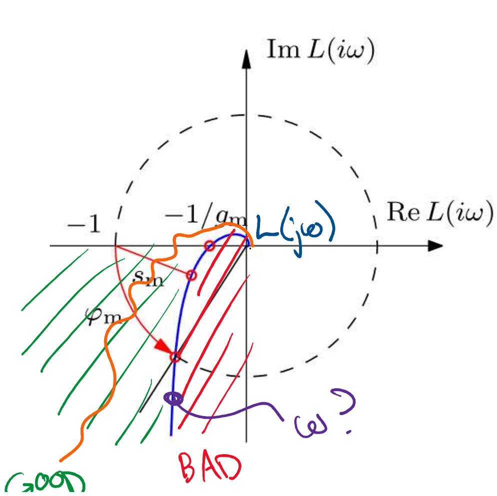


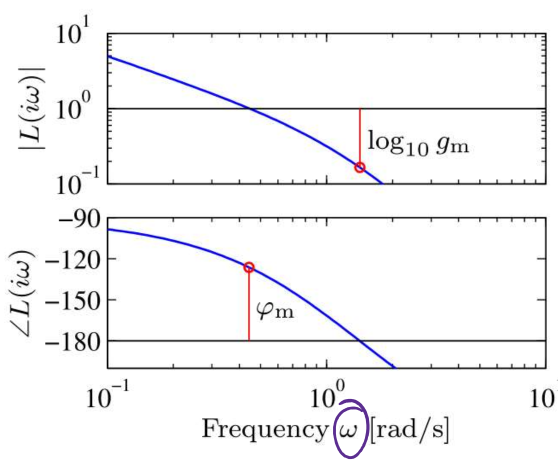
$L = PC$ "open loop"

$\tilde{L} = \tilde{P}\tilde{C}$ xfer func

$\tilde{P}/\tilde{C}$ = perturbed process/controller

* Nyquist tells me to look at $\Omega = \{L(j\omega) : \omega \in \mathbb{R}\} \subset \mathbb{C}$ & its relation to $-1 \in \mathbb{C}$





L strictly proper:

$$L(s) = \frac{b s^m + \cdots}{a s^n + \cdots}$$

$n > m$

$$\lim_{|\omega| \to \infty} |L(j\omega)| = 0$$


GOOD
BAD
ω?
Frequency ω [rad/s]

Tue Apr 12

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- [x] course logistics, Canvas, etc
- [x] HW0 self-assessment – due next Monday
- [x] HW1 – due this Friday
- [ ] week 2 lectures
- [ ] questions / office hours

TODO:
- [ ] link notebooks / point to Python intro

Thu Apr 14

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- [x] course logistics, Canvas, etc
- [x] HW 1 self-assessment – due next ~~Monday~~ thursday
- [x] HW 2 – due this Friday
- [x] week 3 lectures
- [ ] questions / office hours

TODO:
- [ ] HW1 solution ⇝ Canvas
- [ ] 2π in p1(a)
- [ ] link notebooks / point to Python intro
- [ ] ECE Colloq
- [ ] Robotics Colloquium
- [ ] Northwest Robotics Symp.

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* how does continuous-time LQR relate to discrete-time?

$$\dot{x} = Ax + Bu \quad \rightsquigarrow \quad x^+ = e^{\Delta A} x = A_\Delta x, \quad \Delta > 0 \text{ timestep}$$

$$e^{\Delta A} x \approx (I + \Delta A)x$$

$$c = \int_0^{k\cdot\Delta} x^T Q x + u^T R u \quad \rightsquigarrow \quad \sum_{\ell=0}^{k-1} x_\ell^T Q_\Delta x_\ell + u_\ell^T R_\Delta u_\ell = c_\Delta$$

$\downarrow$ LQR $\qquad\qquad\qquad \downarrow$

$$u^* = -Kx \qquad\qquad u^* = -K_\Delta x$$

then: $\lim_{\Delta \to 0} K_\Delta = K, \quad \lim_{\Delta \to 0} c_\Delta = c$

assuming $A_\Delta = A + O(\Delta)$

assuming $A_\Delta = A + O(\Delta)$

$Q_\Delta = Q + O(\Delta)$

$R_\Delta = R + O(\Delta)$

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* compare finite-horizon vs infinite-horizon

finite-horiz: $\dot{x}(t)$ or $x(t+1) = A_t x(t) + B_t u(t)$

cost is $\int_0^T$ or $\sum_{t=0}^{T} x_t^T Q_t x_t + u_t^T R_t u_t$

$\rightsquigarrow$ opt ctrl is $u_t^* = -K_t x_t$

∞-horiz: $\dot{x} / x^+ = Ax + Bu$

cost $\int_0^\infty$ or $\sum_{t=0}^{\infty} x^T Q x + u^T R u$

$\rightsquigarrow$ opt ctrl is $u^* = -Kx$

$$K_s = (B_s^T P_{s+1} B_s + R_s)^{-1} B_s^T P_{s+1} A_s$$

$$P_s = (A_s - B_s K_s)^T P_{s+1} (A_s - B_s K_s) + K_s^T R_s K_s + Q_s$$

$$u_s^* = -K_s x_s, \quad v_s^* = \frac{1}{2} x_s^T P_s x_s$$

$K = (B^T P B + R)^{-1} B P A$

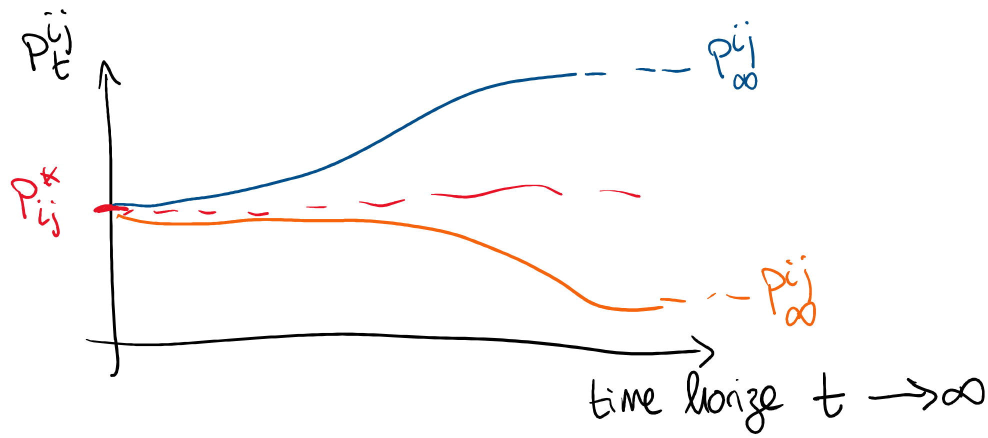
(LTI) $P^- = (A - BK)^T P (A - BK) + KRK + Q$

↳ (LTI) $P^- = (A-BK)^T P (A-BK) + KRK + Q$

* not linear, but: 1°. preserves symmetry & definiteness

2°. has unique GES equilibrium P^*

* P^* is solution of ∞-horiz LTI LQR !



Tue Apr 19

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☑ course logistics, Canvas, etc
☑ exam 1 next week
☐ HW2 self-assessment – due next Monday
☐ HW3 – due this Friday
☐ week 4 lectures
☐ questions / office hours

☑ HW1 solution ⇝ Canvas ☐ 2π in p1(a)

TODO: ☐ link notebooks / point to Python intro
☐ ECE Colloq ☐ Robotics Colloquium ☐ Northwest Robotics Symp.

$w \in \mathbb{R}^2,\ z \in \mathbb{R}^2$

$$\therefore T_{wz} = -\frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} = \begin{bmatrix} T_{w_1 z_1} & T_{w_1 z_2} \\ T_{w_2 z_1} & T_{w_2 z_2} \end{bmatrix}$$

This implies that $T_{w_1,z_1} = T_{w_2,z_2} = -\frac{1}{s+1}$.

$$w = T_{wz} \cdot z$$

$$\Updownarrow$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} T_{w_1 z_1} \cdot z_1 + T_{w_1 z_2} \cdot z_2 \\ T_{w_2 z_1} \cdot z_1 + T_{w_2 z_2} \cdot z_2 \end{bmatrix}$$

multiple notions of stability:

$$\dot{x} = Ax + Bu \xrightarrow[\text{output stability}]{y = Cx + Du}$$

$$\Big\downarrow \text{ state space stability}$$

$$\operatorname{spec} A \subset \mathbb{C}_0^-$$

$$\Downarrow$$

asymptotically $\Leftrightarrow$ exponentially stable $\Leftarrow$ internally stable $\longrightarrow$ BIBO stable

$$\xrightarrow{u} \boxed{\;T\;} \xrightarrow{y = T(u)}$$

BIBO stability

$$* \quad \max_{\substack{u \neq 0 \\ u \in L_2}} \frac{\|T(u)\|}{\|u\|} \overset{?}{<} \infty$$

$$u : \mathbb{R} \to \mathbb{R}^m : \quad \|u\|_2^2 = \int \|u(t)\|_2^2 \, dt$$

Thu Apr 21

Thursday, April 21, 2022 1:07 PM

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☑ course logistics, Canvas, etc
☑ exam 1 next week
☐ HW2 self-assessment – due next Monday
☑ HW3 – due this Friday
☑ week 4 lectures
☐ questions / office hours

probably I need to remove 2π from HW

☑ HW1 solution ⇝ Canvas ☐ 2π in p1(a)

TODO: ☐ link notebooks / point to Python intro
☐ ECE Colloq ☐ Robotics Colloquium ☐ Northwest Robotics Symp.
☐ HW2 1(e) explanation

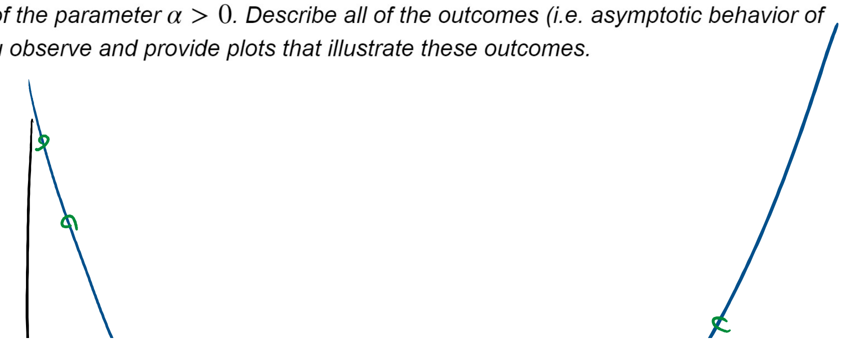
Consider the following cost function of a scalar decision variable $u \in \mathbb{R}$:

$$J(u) = \frac{u^6}{6} - \frac{7u^5}{5} + \frac{17u^4}{4} - \frac{17u^3}{3} + 3u^2 .$$

(a) *Plot $J(u)$, $DJ(u)$, and $D^2J(u)$ versus u; use the subplot(3,1,n) for n = 1, 2, 3 to align the u-axes of the three plots.*

(b) *Determine all local minima of J and the corresponding minimizing u.*

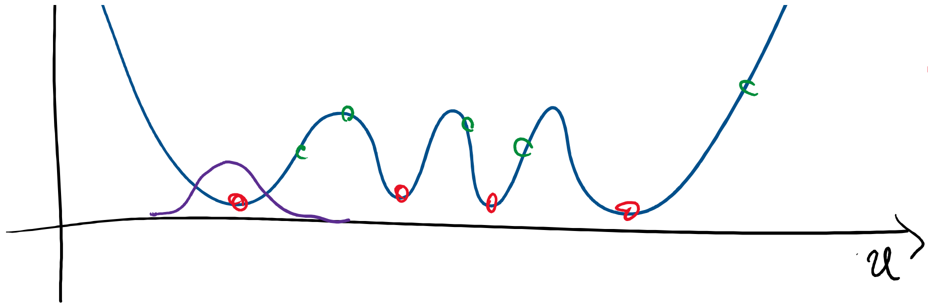
(c) *Run the gradient descent iteration $u^+ = u - \alpha\, DJ(u)$ starting from multiple initial u's and with multiple values of the parameter $\alpha > 0$. Describe all of the outcomes (i.e. asymptotic behavior of the iteration) you observe and provide plots that illustrate these outcomes.*

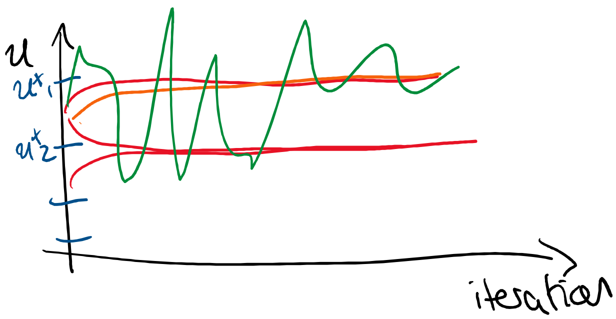


Q: do outcomes change if we consider

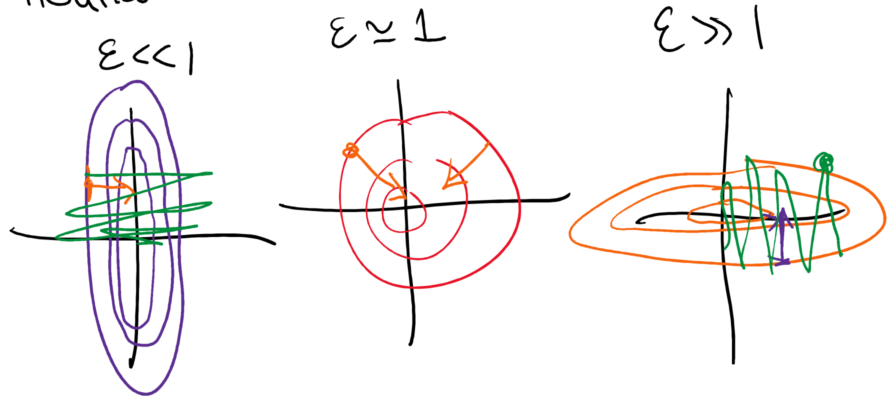
$\min_{v = -u} J(-v)$

$$\min_{v=-u} J(-v)$$





ex: $J(u) = u^T \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix} u$



Tue Apr 26

Tuesday, April 26, 2022 12:57 PM

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☐ course logistics, Canvas, etc
- ☐ exam 1
- ☐ HW3 self-assessment – due next Monday
- ☐ questions / office hours

TODO:
- ☐ link notebooks / point to Python intro
- ☐ ECE Colloq ☐ Robotics Colloquium ☐ Northwest Robotics Symp.
- ☐ HW2 1(e) explanation

Thu Apr 28

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: □ course logistics, Canvas, etc
□ exam 1
□ HW 3 self-assessment – due Monday-after-next
□ questions / office hours

Tue May 3

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☑ course logistics, Canvas, etc

☑ exam 1 — solution done, working on grading

☑ HW 3 self-assessment — due next Monday

☑ HW 4 — due this Friday

☐ week 6 lectures

☐ questions / office hours

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* connection between $\underbrace{\text{stability}}_{\text{i.e. Lyapunov}}$ & $\underbrace{\text{optimal control}}_{\text{i.e. LQR}}$

consider $\min \int_0^\infty x^T Q x + u^T R u$ s.t. $\dot{x} / x^+ = Ax + Bu$

* know that optimal $u^* = -Kx$ where $K = R^{-1} B^T P$ $(\dot{x})$

or $K = (B^T P B + R)^{-1} B^T P A$ (x^+)

where $P = P^T > 0$ solves a <u>Riccati equation</u>

fact: closed-loop dynamics $\dot{x} / x^+ = Ax + Bu = (A - BK)x$

are (exponentially) stable and $\underbrace{v(x) = x^T P x}$ is a Lyapunov function are (exponentially) stable and $\underbrace{V(x) = x' P x}$ is a Lyapunov function

* also the (optimal) cost-to-go

---

ex: $\ddot{y} = u \longleftrightarrow x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$= Ax + Bu$

so $u = -Kx = -\begin{bmatrix} k_P & k_D \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

* if we choose K to $\min \int_0^\infty x^T Q x + u^T R u$

then we've synthesized the "optimal" PD controller

ex: $\ddot{y} = u$, $\dot{z} = y \longleftrightarrow x = \begin{bmatrix} y \\ z \\ \dot{y} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

so $u = -Kx = -\begin{bmatrix} k_P & k_I & k_D \end{bmatrix} \begin{bmatrix} y \\ z \\ \dot{y} \end{bmatrix}$

* if we choose K to $\min \int_0^\infty x^T Q x + u^T R u$

then we've synthesized the "optimal" PID controller

(alternatively cf Zeigler-Nichols)

given nonzero time-varying reference r to track,

might want to choose K to $\min \int_0^\infty q\|r-y\|^2 + r\|u\|^2$

but this problem doesn't have a solution!

$= (r-y)^T \cdot q \cdot (r-y)$

$= \begin{bmatrix} r-y \\ z \\ \dot{y} \end{bmatrix}^T \begin{bmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r-y \\ z \\ \dot{y} \end{bmatrix}$

let $e = r-y$

$\ddot{e} = \ddot{r} - u$

$\dot{z} = e$

$\longleftrightarrow \quad x = \begin{bmatrix} e \\ z \\ \dot{e} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ z \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u$

$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{r}$

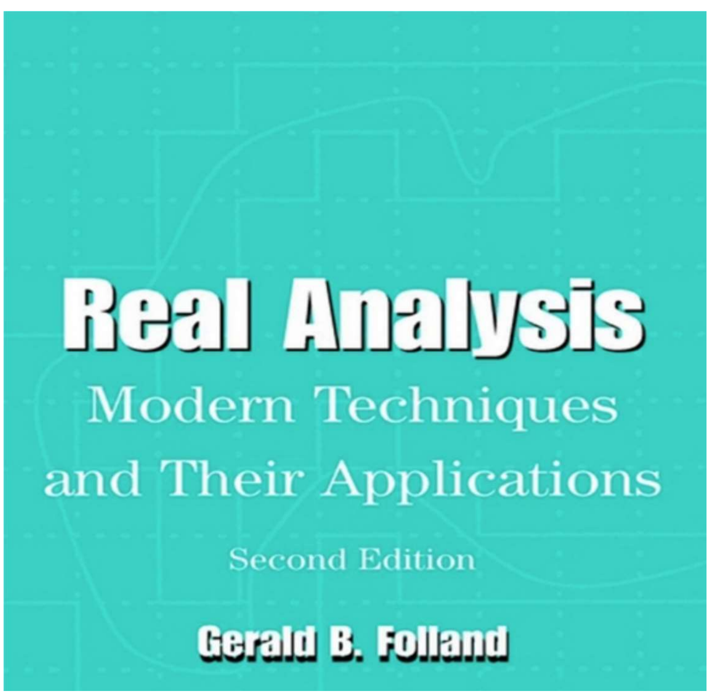
Thu May 5

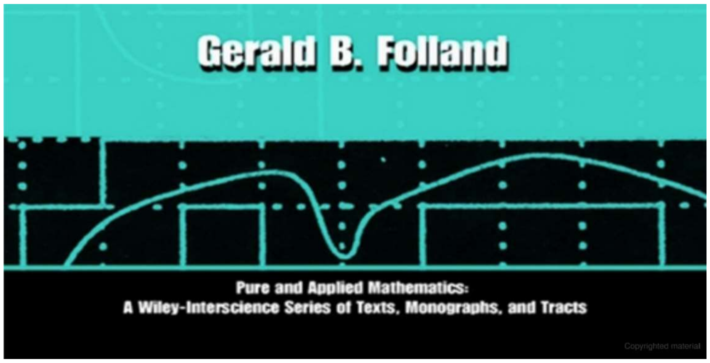
AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☑ exam 1 – solution done, working on grading
- ☑ HW3 self-assessment – due next Monday
- ☑ HW4 – due this Friday
- ☐ week 6 lectures
- ☐ questions / office hours

TODO: ☐ class notes (Tue & today)

* real analysis textbook – if you know measure theory & want to learn probability theory

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Tue May 10

Tuesday, May 10, 2022 1:08 PM

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☑ HW 4 self-assessment – due next Monday
- ☑ HW 6 – due this Friday
- ☑ week 7 lectures
- ☐ questions / office hours
- ☐ exam 1 solution @ 2pm today

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given rv $x : \Omega \to \mathbb{R}^n$

define expectation $E[x] = \begin{bmatrix} E[x_1] \\ \vdots \\ E[x_n] \end{bmatrix} \in \mathbb{R}^n$

covariance $Cov[x] = E\big[\underbrace{(x - E[x])}_{\in \mathbb{R}^{n\times 1}} \underbrace{(x - E[x])^T}_{\in \mathbb{R}^{1\times n}}\big] \in \mathbb{R}^{n\times n}$

$= \big[Cov(x_i, x_j) \big]_{i,j=1}^{n}$

$diag(Cov[x]) = (Var(x_1), Var(x_2), \ldots, Var(x_n))$

$$\mathrm{Cov}(a,b) = E\left[(a - E[a]) \cdot (b - E[b])\right]$$

$x_1 = Ax_0 + Bu_0 + \delta_0$, x_0 & u_0 deterministic

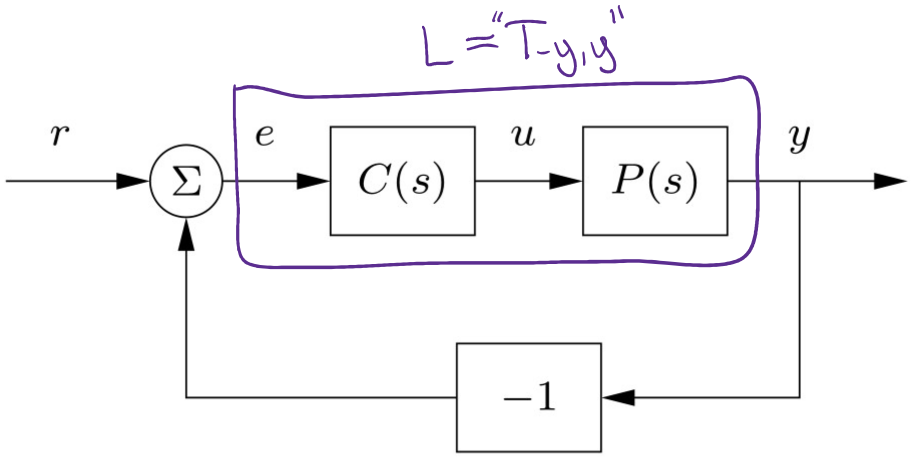
$\delta_0 \sim \mathcal{N}(d, \Delta)$

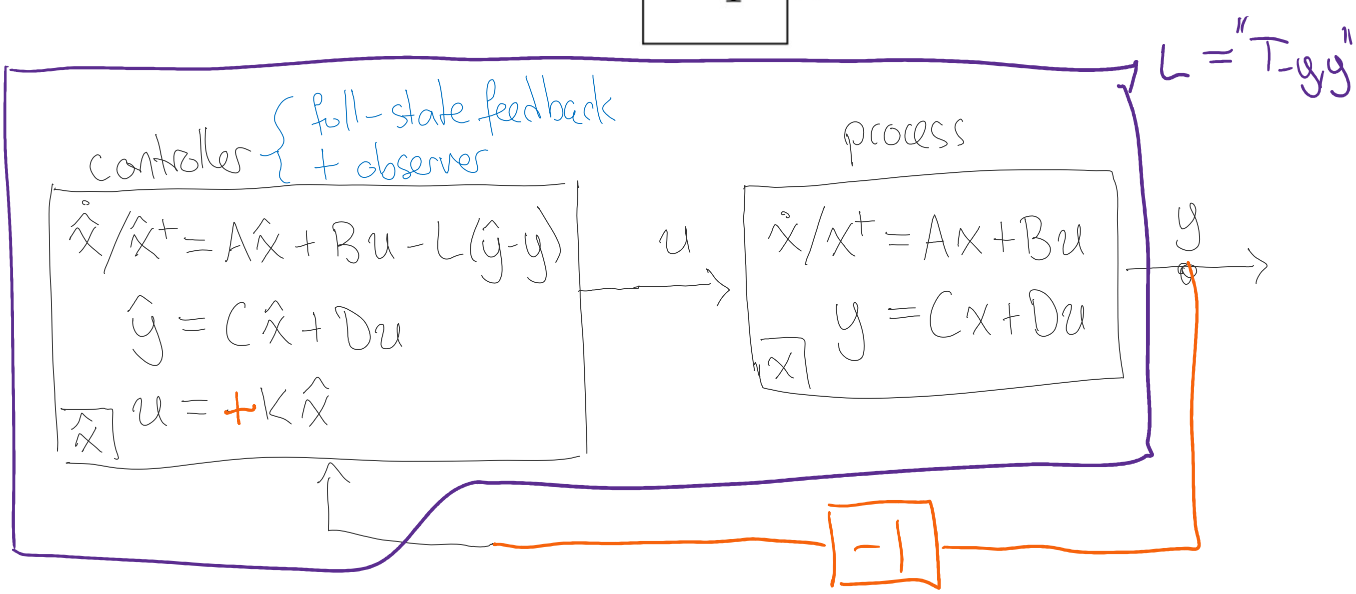
then $x_1 \sim \mathcal{N}(Ax_0 + Bu_0 + d, \Delta)$

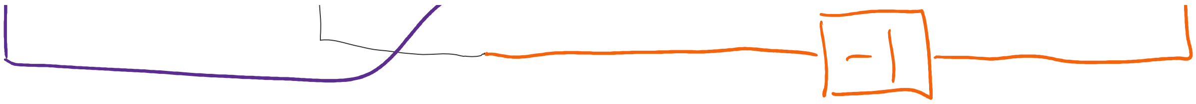
Thu May 12

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☐ exam 1 solution
- ☐ HW 4 self-assessment – due next Monday
- ☐ HW 6 – due this Friday
- ☐ week 7 lectures
- ☐ questions / office hours







$$\begin{bmatrix}\hat{x}\\ e\end{bmatrix}^{\circ/+} = \begin{bmatrix} A\hat{x} + \overset{-BK\hat{x}}{\overset{\parallel}{Bu}} - L\overset{=-Ce}{\overbrace{(\hat{y}-y)}} \\ (A-LC)e \end{bmatrix} = \begin{bmatrix}(A-BK) & LC \\ 0 & (A-LC)\end{bmatrix}\begin{bmatrix}\hat{x}\\ e\end{bmatrix}$$

$$e = x - \hat{x}$$

$$y = Cx = C(e-\hat{x}) = \begin{bmatrix}-C & C\end{bmatrix}\begin{bmatrix}\hat{x}\\ e\end{bmatrix} = \bar{A}\begin{bmatrix}\hat{x}\\ e\end{bmatrix}$$

$L =$ "$T_{-y,y}$"

$$\xrightarrow{-y}$$

$$\begin{bmatrix}\hat{x}\\ e\end{bmatrix}^{\circ/+} = \overbrace{\begin{bmatrix}(A-BK) & LC \\ 0 & (A-LC)\end{bmatrix}}^{\bar{A}}\begin{bmatrix}\hat{x}\\ e\end{bmatrix} + \overbrace{\begin{bmatrix}-L\\ 0\end{bmatrix}}^{\bar{B}}(-y)$$

$$y = \underbrace{\begin{bmatrix}-C & C\end{bmatrix}}_{\bar{C}}\begin{bmatrix}\hat{x}\\ e\end{bmatrix} \qquad \bar{D} = 0$$

? need to check...

$$\xrightarrow{y}$$

$$x_{s+1} = A_s x_s + B_s u_s + \delta_s$$

$$y_s = C_s x_s + \overset{0}{\cancel{D_s}} u_s + \eta_s$$

$$\hat{x}_{s+1} = \text{TBD}$$

$$\hat{y}_s = C_s \hat{x}_s + \overset{0}{\cancel{D_s}} u_s$$

$$\min \sum^{t} (\hat{x}_s - x_s)^T Q_s^{-1} (\hat{x}_s - x_s) + (\hat{y} - y)^T R_s^{-1} (\hat{y} - y)$$

$$\min \sum_{s=0}^{k} \underbrace{(\hat{x}_s - x_s)^T Q_s^{-1} (\hat{x}_s - x_s)}_{\text{consistency w/ process model}} + \underbrace{(\hat{y} - y)^T R_s^{-1} (\hat{y} - y)}_{\text{consistency w/ observations}}$$

* $\hat{x}^*$

Tue May 17

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☑ course logistics, Canvas, etc
- ☑ survey about class experience – anonymous/ungraded
- ☑ HW 6 self-assessment – due next Monday
- ☑ HW 7 – due this Friday
- ☑ week 8 lectures
- ☐ questions / office hours

* won't meet on Thu this week (only) – instead 9:30–11a Fri

$$\mathrm{Cov}[x] = E[\underbrace{(x - E[x])(x - E[x])^T}_{=:X}] \in \mathbb{R}^{n\times n}$$

$=: X$, so $X : \Omega \to \mathbb{R}^{n\times n}$ is a rv

$$\mathbb{R}^{n\times n} \updownarrow \mathbb{R}^{n^2}$$

$$X_{ij} = (x_i - E[x_i])(x_j - E[x_j])$$

$$E[X] = \left[E[X_{ij}]\right]_{i,j=1}^{n}$$

Fri May 20

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

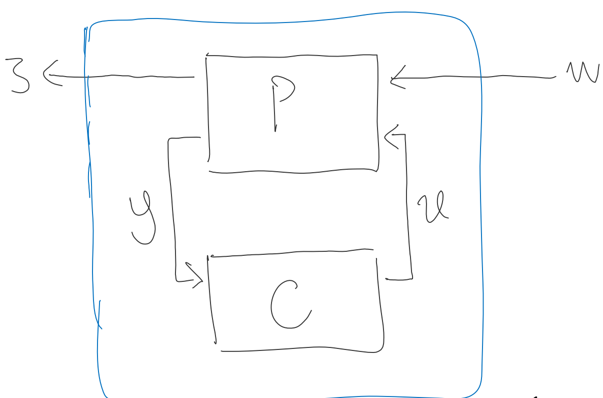
today: ☐ course logistics, Canvas, etc

☐ HW 6 self-assessment – due next Monday

☐ HW 7 – due this Friday

☐ week 8 lectures

☐ questions / office hours



Q: what is T_{zw}?

$$u = Cy$$

$$C = T_{uy} \quad \text{i.e.} \quad \cancel{y = C \cdot u}$$

$$P = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|c} P_{zw} & P_{zu} \\ \hline P_{yw} & P_{yu} \end{array}\right]$$

i.e. $$\begin{aligned} z &= P_{11} \cdot w + P_{12} \cdot u \\ y &= P_{21} \cdot w + P_{22} \cdot u \end{aligned} \iff \begin{bmatrix} z \\ y \end{bmatrix} = P \cdot \begin{bmatrix} w \\ u \end{bmatrix}$$

$$y = P_{21} w + P_{22} u = P_{21} w + P_{22} C y \iff (I - P_{22} C) y = P_{21} w$$

$$\iff y = (I - P_{22} C)^{-1} P_{21} w$$

$$z = P_{11} w + P_{12} u = P_{11} w + P_{12} C y = P_{11} w + P_{12} C (I - P_{22} C)^{-1} P_{21} w$$

$$z = P_{11} w + P_{12} u = P_{11} w + P_{12} C y = P_{11} w + P_{12} C (I - P_{22} C)^{-1} P_{21} w$$

$$\Longleftrightarrow T_{zw} = P_{zw} \cdot w + P_{zu} C (I - P_{yu} C)^{-1} P_{yw}$$

Tue May 24

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today:
- ☐ course logistics, Canvas, etc
- ☑ results from "check-in" survey Sat Jun 4 (for +2 bonus)
- ☑ exam 2 next week – due ~~Fri Jun 3 (?)~~
- ☑ HW 7 self-assessment – due ~~next Monday~~ Mon Jun 6
- ☑ HW 8 – due this Friday
- ☐ week 9 lectures
- ☐ questions / office hours

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ex: $L_p(-\infty,\infty) = \{ u : \underbrace{(-\infty,\infty)}_{= \mathbb{R} \text{ or } \mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}} \to \mathbb{C}^n \mid \|u\|_p < \infty \}$

given $u, v \in L_p(-\infty,\infty)$, $\alpha \in \mathbb{C}$

define $w = u + \alpha \cdot v$ where $\forall t \in (-\infty,\infty) : w(t) = u(t) + \alpha \cdot v(t) \in \mathbb{C}^n$

ex: $W = \{ f : S \to V \}$ where V is vec space, S is set
eg $V = \mathbb{C}^n$ eg $S = (-\infty,\infty)$

$u, v \in W$, $\alpha \in \mathbb{C}$: $w = u + \alpha \cdot v$ defined $\forall a \in S : w(a) = u(a) + \alpha \cdot v(a)$

$$\|u\|_p = \left(\int_{-\infty}^{\infty} \|u(t)\|_p^p \, dt\right)^{1/p} \qquad \|u\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \|u(t)\|_\infty = \max_{t\in\mathbb{R}} \|u(t)\|_\infty$$

ex: $V = \{ u : \underbrace{\{0,1,2,\cdots,t-1\}}_{=: S} \to \mathbb{R}^n \}$

given $u \in V$ define $\bar{u} \in \mathbb{R}^{t\cdot n}$ by $\bar{u} = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}$

then $\|\bar{u}\|_2^2 = \bar{u}^T \cdot \bar{u} = \sum_{s=0}^{t-1} u(s)^T \cdot u(s)$

so I might propose $\|w\|_2^{\cancel{2}} := \sqrt{\int_{-\infty}^{\infty} w^T(s) \cdot \underbrace{w(s)}_{\in\mathbb{R}^n} ds} = \sqrt{\int_{-\infty}^{\infty} \|w(s)\|_2^2 \, ds}$

for any $w : (-\infty, \infty) \to \mathbb{R}^n$

more generally, $\|w\|_p = \left(\int_{-\infty}^{\infty} \|w(s)\|_p^p \, ds\right)^{1/p}$

$p \in [1, \infty)$

$\|T\|_{L_2 \to L_2}$ is the induced norm of LTI system T

$\|T\|_{L_2 \to L_2}$ is the induced norm of LTI system T

$$\text{"disturbances"} \quad w \longrightarrow \boxed{\;T\;} \longrightarrow z \quad \text{"errors"}$$

$w \in L_2$ $\qquad$ $z \in L_2$

$w(t) \in \mathbb{R}^m$ $\qquad$ $z(t) \in \mathbb{R}^q$

→ quantifies the maximum amount of "power" (quantified as $\|w\|_2$) that T transfers from w to z (quantified as $\|z\|_2$)

* in contrast, $\|\hat{T}\|_2$ quantifies power in output ($\|z\|_2$) produced by <u>one</u> <u>specific</u> <u>input</u> ($\|w^*\|_2$)

Thu May 26

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☐ course logistics, Canvas, etc

☐ exam 2 next week – due Sat Jun 4

☐ HW 7 self-assessment – due ~~next Monday~~ Mon Jun 6

☐ HW 8 – due this Friday – self-assessment due

☐ week 9 lectures

☐ questions / office hours

TODO: ☐ mixed H_2 / H_∞ control papers

☐ post today's notes & recording

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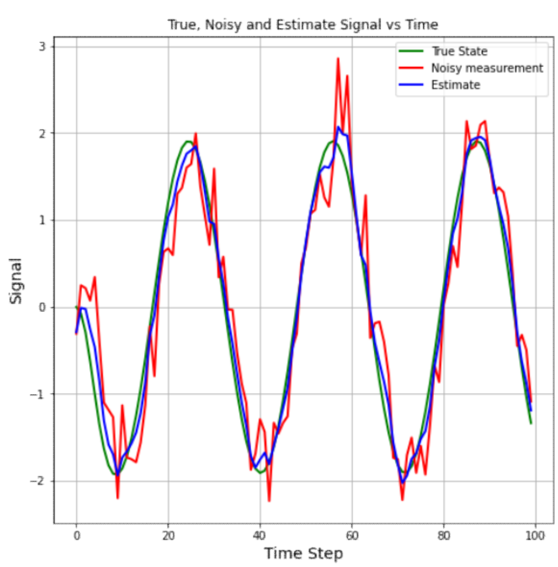
example results from applying Kalman Filter (HW7):



* looks like a low-pass filter

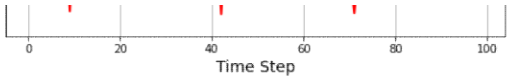
→ is it? if so, why? if not, why not?

◦ our Kalman filter is defined by:

$$\hat{x}_s = \tilde{x}_s + L_s(y_s - C_s\tilde{x}_s)$$

$$\tilde{x}_s = A_{s-1}\hat{x}_{s-1} + B_{s-1}u_{s-1}$$

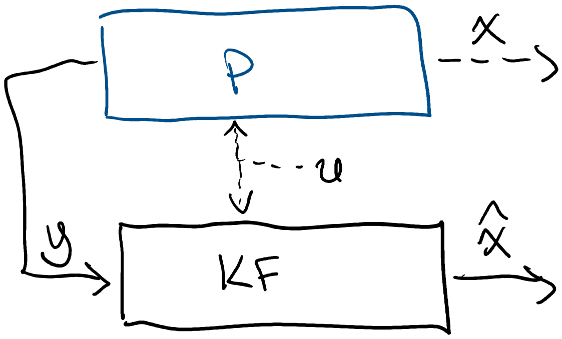
i.e. $\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u + Ly$



i.e. $\hat{x}^+ = \hat{A}\hat{x} + \hat{B}u + Ly$

$$\Updownarrow$$

$$\hat{X} = \underbrace{\left[(sI-\hat{A})^{-1}L\right]}_{KF} \cdot Y$$

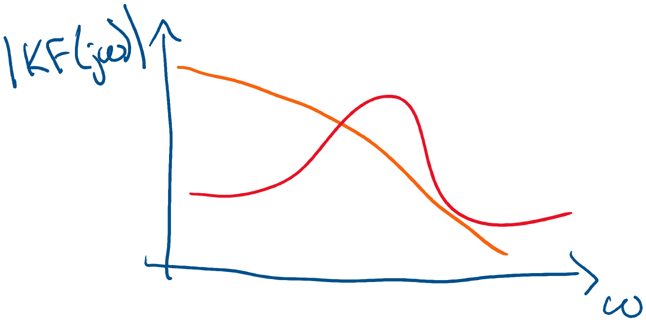


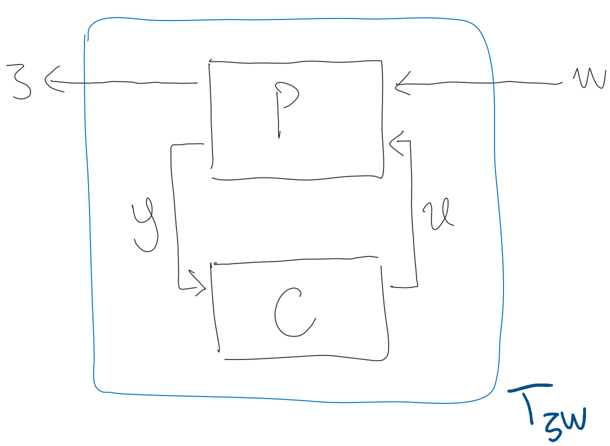
$$x^+ = Ax + Bu + \delta$$

$$y = Cx + Du + \eta$$

$$\Updownarrow$$

$$X = \underbrace{\left[C(sI-A)^{-1}B + D\right]}_{P} Y$$





$\boxed{\text{LQG problem:}}$

$$P: \begin{array}{c|c|c|c} & & w = [\delta \;\; \eta] & u \\ \hline & A & [E \;\; 0] & B \\ z & \begin{bmatrix} Q^{1/2} \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ R^{1/2} \end{bmatrix} \\ y & C & [0 \;\; F] & \cancel{D}\, 0 \end{array}$$

ex: LQG $w = (\delta, \eta)$, $\delta, \eta \sim \mathcal{N}(0, I)$, $z = (Q^{1/2}x, R^{1/2}u)$

$x^+ = Ax + Bu + E\delta$ $\Rightarrow$ $\mathrm{Cov}[E\delta] = E^T E$

$$y = Cx + F\eta$$

WANT: $\sum_{s=0}^{\infty} x_s^T Q_s x_s + u_s^T R_s u_s$ small $\iff$ $\|T_{zw}\|_2$ small

Tue May 31

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☐ course logistics, Canvas, etc

☐ exam 2

☐ questions / office hours

$$\min_{x_0, u_0} \| f(x_0, u_0) \|_2^2$$

$\dot{x} / x^+ = f(x, u)$ $\quad y = h(x, u)$ $\quad$ suppose $f(x_0, u_0) = 0 \ (\dot{x})$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m$ $\quad f(x_0, u_0) = x_0 \ (x^+)$

$\delta\dot{x} / \delta x^+ = A \delta x + B \delta u$ $\quad A = D_x f(x_0, u_0)$ $\quad B = D_u f(x_0, u_0)$

$\delta y = C \delta y + D \delta u$ $\quad C = D_x h(x_0, u_0)$ $\quad D = D_u h(x_0, y_0)$

$$\Rightarrow C = \left.\begin{bmatrix} \partial_{x_1} h_1 & \partial_{x_2} h_1 & \cdots & \partial_{x_n} h_1 \\ \vdots & & & \\ \partial_{x_1} h_o & \partial_{x_2} h_o & \cdots & \partial_{x_n} h_o \end{bmatrix}\right|_{\substack{x = x_0 \\ u = u_0}}$$

$$\left| \partial_{x_1} h_\theta \quad \partial_{x_2} h_\theta \quad \cdots \quad \partial_{x_n} h_\theta \right| \Big|_{\substack{x = x_0 \\ u = u_0}}$$

$(x, u) \leftarrow$ LTV/LTI process $\leftarrow (\delta, \eta)$

$$x_{s+1} = A_s x_s + B_s u_s + \delta_s$$

$$y_s = C_s x_s + \eta_s$$

$y \qquad\qquad u$

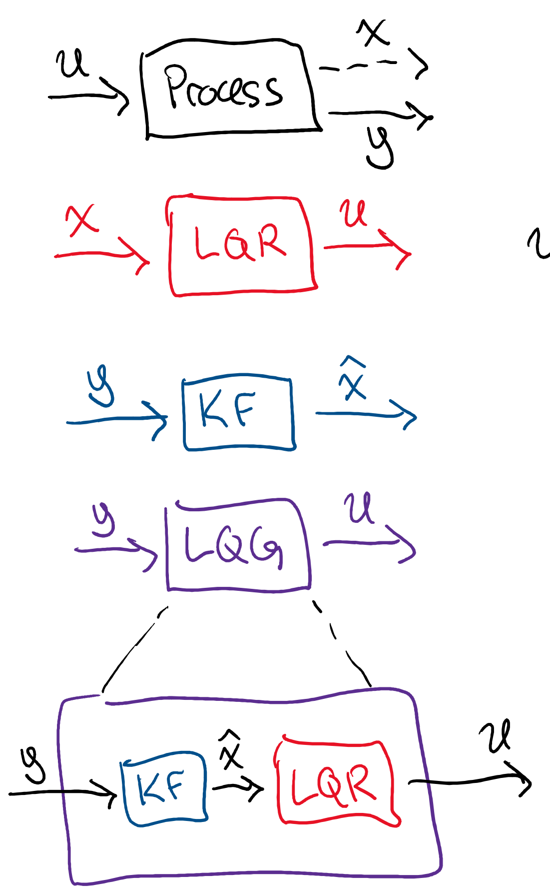
LTV/LTI LQG regulator

$$\hat{x}_{s+1} = A_s \hat{x}_s + B_s u_s + L_s (y_s - C_s \hat{x}_s)$$

Kalman Filter state estimator

$$u_s = -K_s \hat{x}_s$$

LQR state feedback



$$u = -Kx$$

Thu Jun 2

AA/ECE/ME 548 Linear Multivariable Control Sp22 Prof Burden

today: ☐ course logistics, Canvas, etc

☐ exam 2 – let me know if you want more time

☐ course evaluation

☐ questions / office hours

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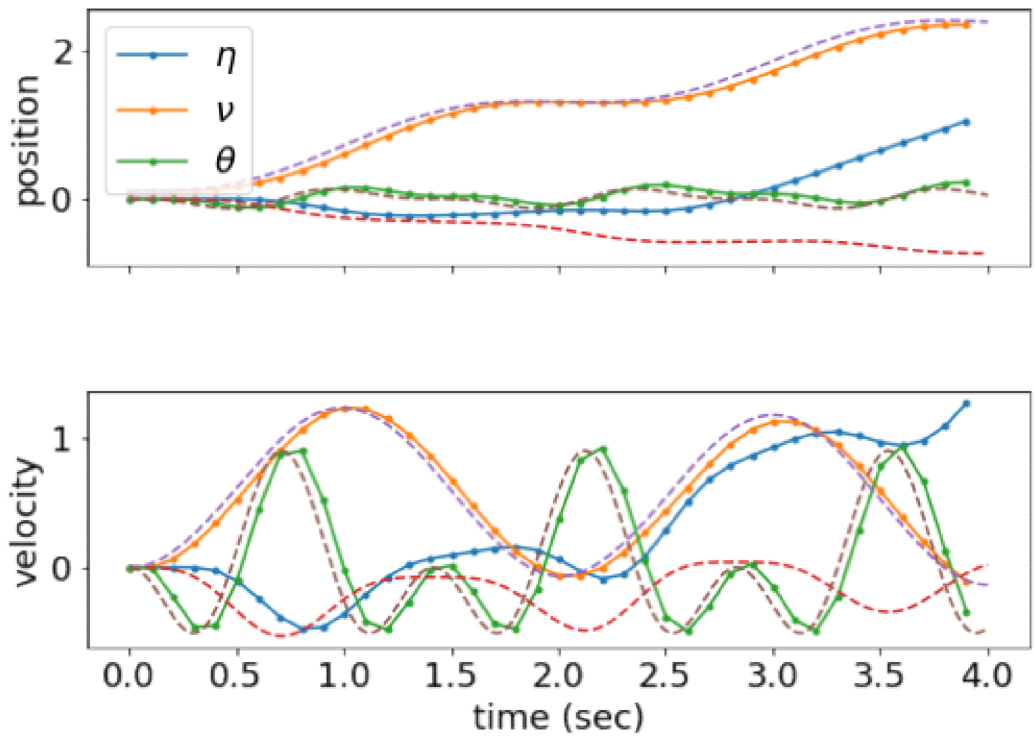
subproblem (a)

Discretize the nonlinear system: choose a step size $\Delta > 0$ and time horizon $T \gg \Delta$ so that the *forward Euler* approximation $x(t+\Delta) \approx x(t) + \Delta f(x(t), u(t))$ yields reasonably good approximation of trajectories on the time interval $t \in [0, T]$. Provide plots that show the approximation is reasonable by simulating the nonlinear system twice -- once with your chosen step size Δ, and once with step size $\nabla \ll \Delta$; the second simulation will be used as the "ground truth". Use non-constant input signals F and τ in your simulation.

Note: there is no one right choice for Δ, but it will be helpful in what follows if you choose a large step size.

ex:



$--- \nabla \sim 10^{-4}$

$-o- \Delta \sim 10^{-1}$

subproblem (b)

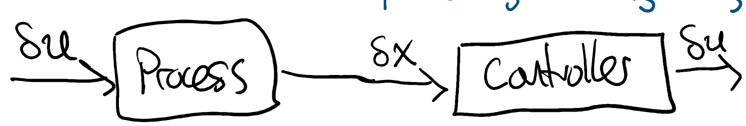
Find an equilibrium (x_0, u_0) for the discretized nonlinear system and linearize the dynamics around this equilibrium to obtain a discrete-time linear time-invariant (DT-LTI) control system model: determine A, B, C, D such that, with $\delta x = x - x_0$, $\delta u = u - u_0$, and $\delta y = y - y_0$ where $y_0 = h(x_0, u_0)$, we have $x^+ \approx x_0 + \delta x^+$ and $y \approx y_0 + \delta y$ where

$$\text{(DT-LTI)} \quad \delta x^+ = A\,\delta x + B\,\delta u, \;\; \delta y = C\,\delta x + D\,\delta u.$$

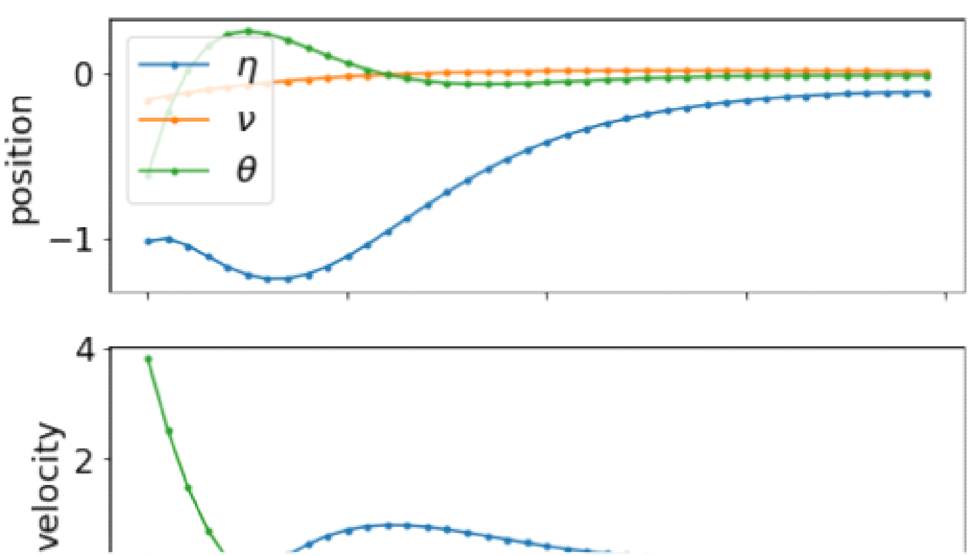
subproblem (c)

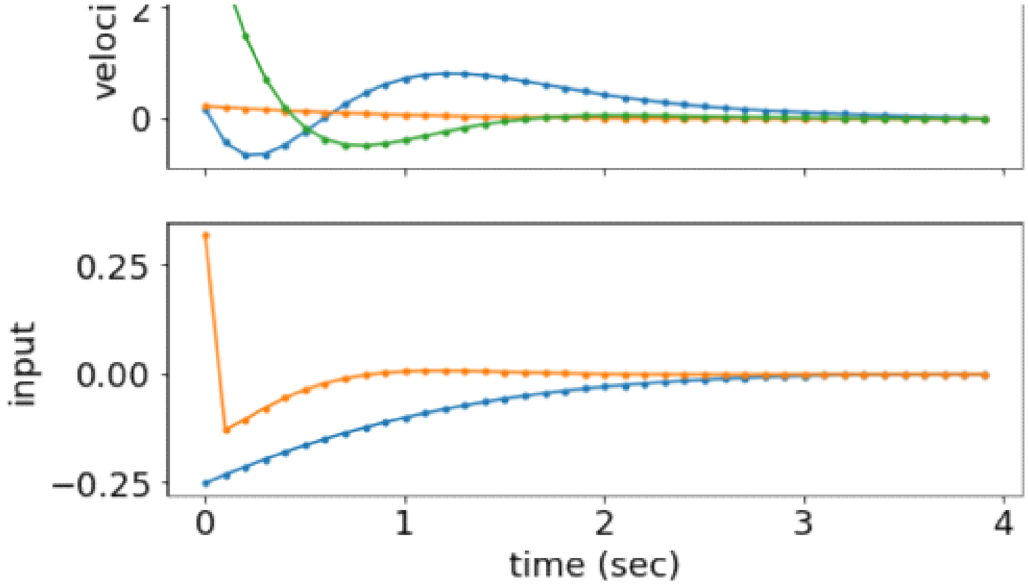
Choose Q and R matrices and solve the finite-horizon LQR problem to synthesize an LTV state-feedback controller for the LTI control system on the time horizon T you chose above. Run a simulation of the controlled system and provide plots of the state δx, control δu, and output δy time series.

$\delta x^+ = A\,\delta x + B\,\delta u, \quad \delta u_s = -K_s\,\delta x_s$



ex:





subproblem (d)

Choose square nonsingular matrices V_1 and V_2 that multiply standard normal random vectors $w_1 \in \mathbb{R}^3$ and $w_2 \in \mathbb{R}^2$ to model process and measurement noise,

$$\delta x^+ = A\,\delta x + B\,\delta u + \begin{bmatrix} 0 \\ V_1 \end{bmatrix} w_1,\ \ \delta y = C\,\delta x + D\,\delta u + V_2 w_2.$$

Solve the finite-horizon Kalman Filtering problem to synthesize an LTV observer for the LTI control system. Run a simulation of the LTI system including samples from the process and measurement noise distributions you chose and non-constant input signals, use the resulting data to run a simulation of your LTV Kalman Filter, and provide plots of: the state δx and your estimate $\delta\hat{x}$; the noisy output δy, noise-free output, and observer output $\delta\hat{y}$.

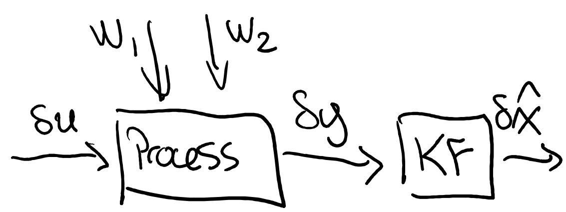
$$\delta x^+ = A\delta x + B\delta u + \begin{bmatrix} 0 \\ V_1 \end{bmatrix} w_1 \qquad w_1 \sim \mathcal{N}(0, I)$$

$$\delta y = C\delta x + D\delta u + V_2 w_2 \qquad w_2 \sim \mathcal{N}(0, I)$$

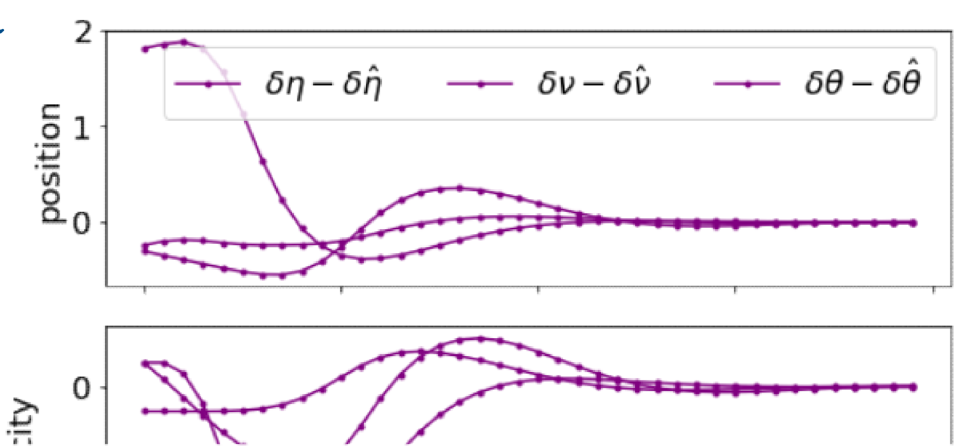
$$\delta\hat{x}_{s+1} = A\delta\hat{x}_s + B\delta u_s + L_s(\delta y_s - \cancel{C}\,\delta\hat{y}_s)$$

typo!

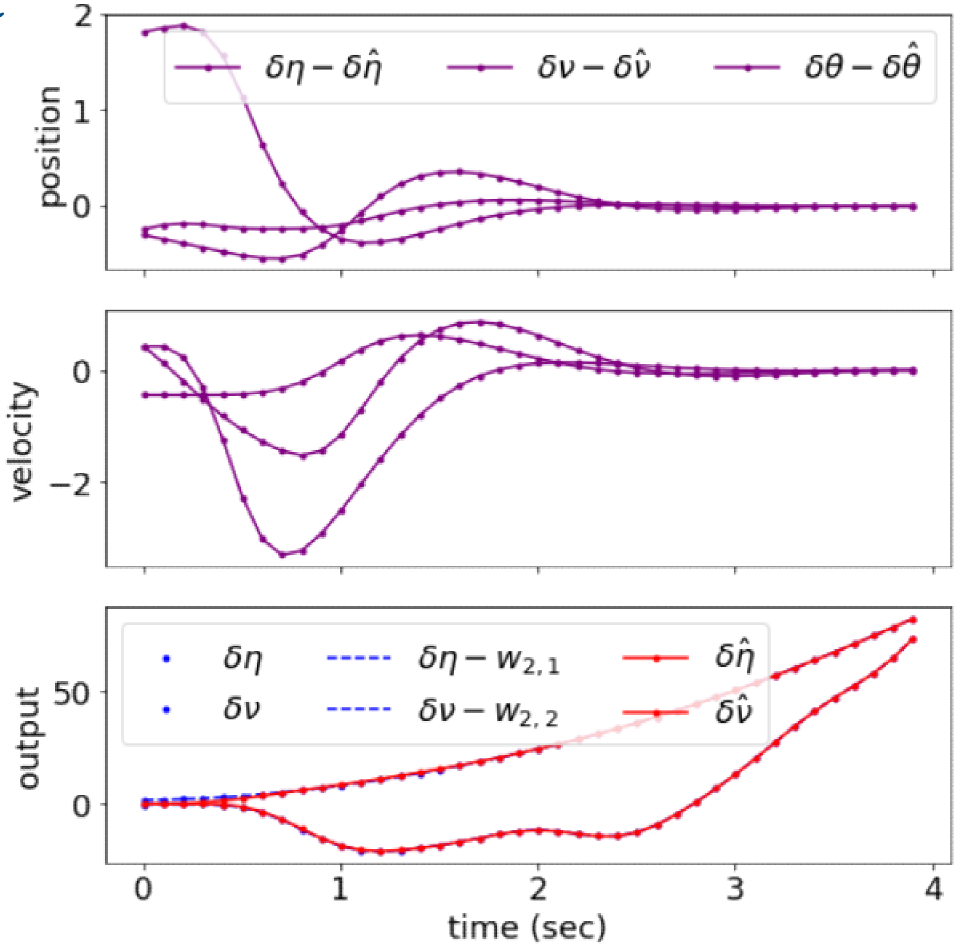
$$\delta\hat{y} = C\delta\hat{x} + D\delta u$$



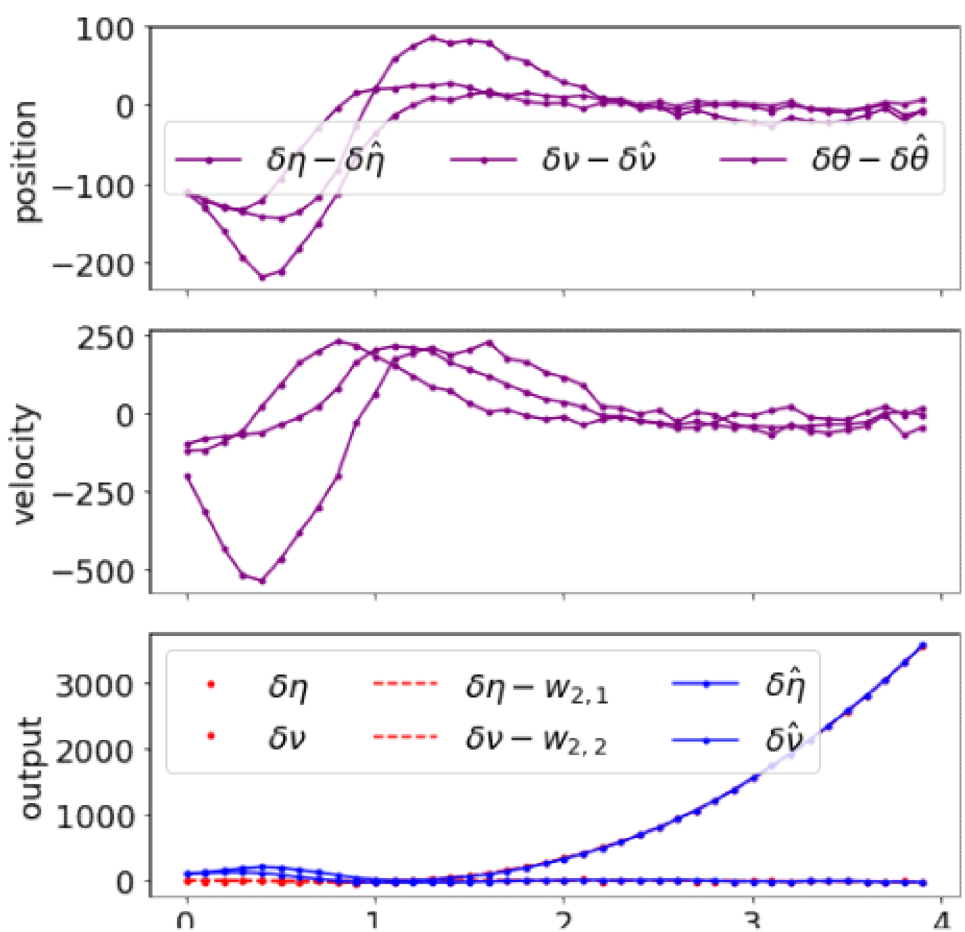
ex: low noise

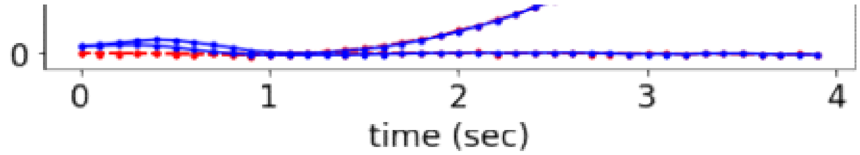


ex: low noise



ex: high noise





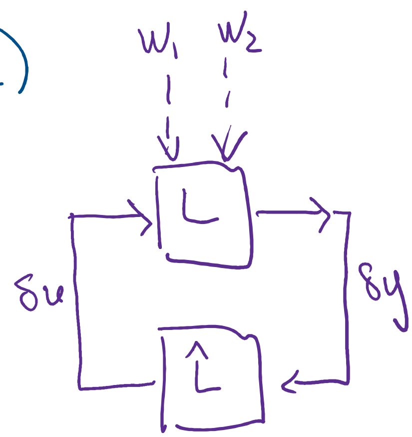
subproblem (e)

Combine your LQR state-feedback law and Kalman Filter observer to obtain a time-varying linear-quadratic Gaussian (LQG) regulator. Run a simulation of the closed-loop system obtained from the feedback interconnection of the LTI system and your LQG regulator. Provide plots of: the state δx and your estimate $\delta\hat{x}$; the noisy output δy, noise-free output, and observer output $\delta\hat{y}$.

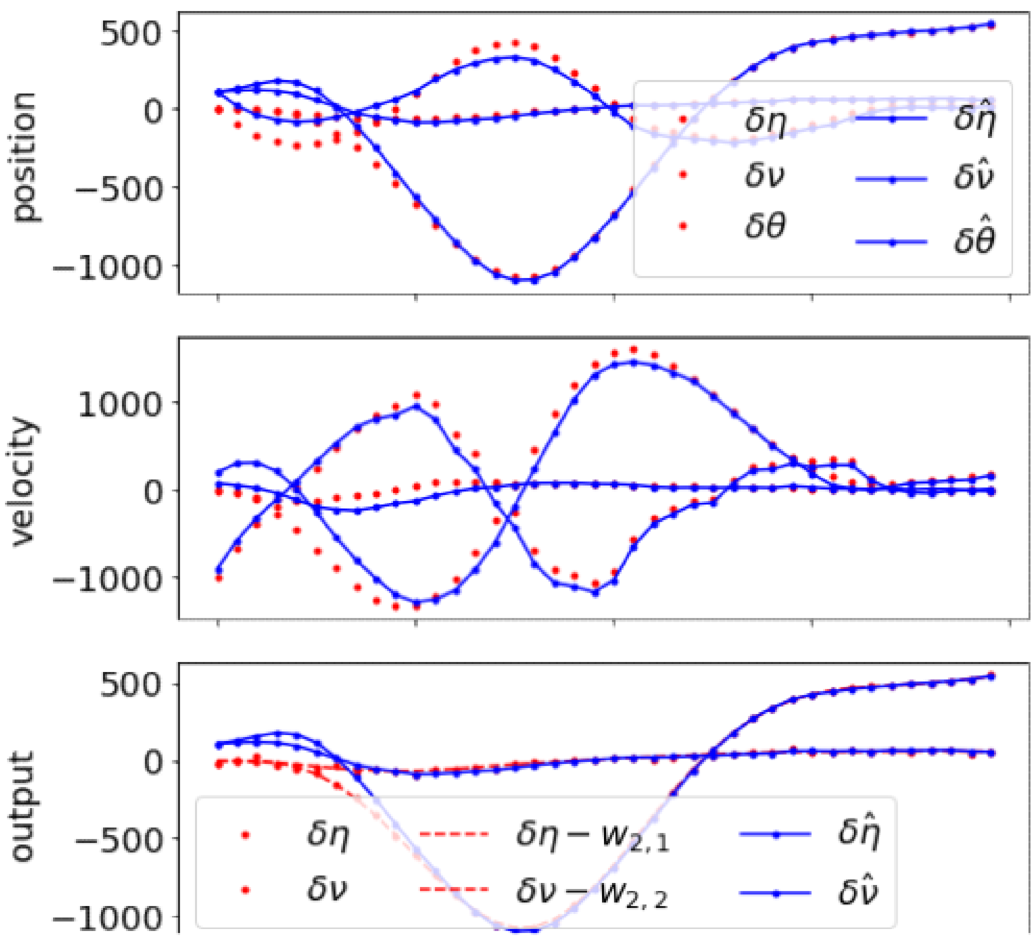
$$L \begin{cases} \delta x^+ = A\delta x + B\delta u + \begin{bmatrix} 0 \\ V_1 \end{bmatrix} w_1 & w_1 \sim \mathcal{N}(0, I) \\ \delta y = C\delta x + D\delta u + V_2 w_2 & w_2 \sim \mathcal{N}(0, I) \end{cases}$$

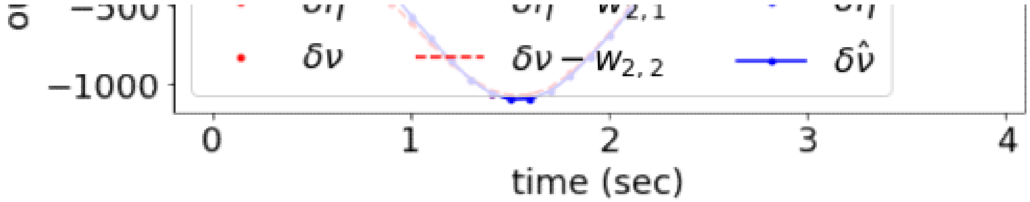
$$\hat{L} \begin{cases} \delta\hat{x}_{s+1} = A\delta\hat{x}_s + B\delta u_s + L_s(\delta y_s - \cancel{C}\delta\hat{y}_s) \\ \delta\hat{y} = C\delta\hat{x} + D\delta u \\ \delta u_s = -K_s\delta\hat{x}_s \end{cases}$$

typo!



ex:





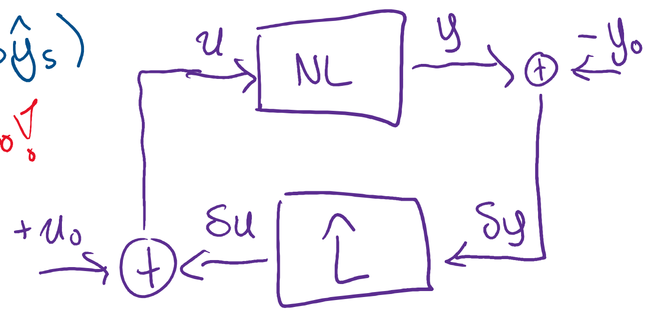
subproblem (f)

Now apply your LQG regulator to the discretized nonlinear system. Run a simulation of the closed-loop system obtained from the feedback interconnection of the DT-NL system and your LQG regulator. Provide plots comparing: the state x and your estimate $\delta\hat{x}$; the noisy output y, noise-free output, and observer output $\delta\hat{y}$.

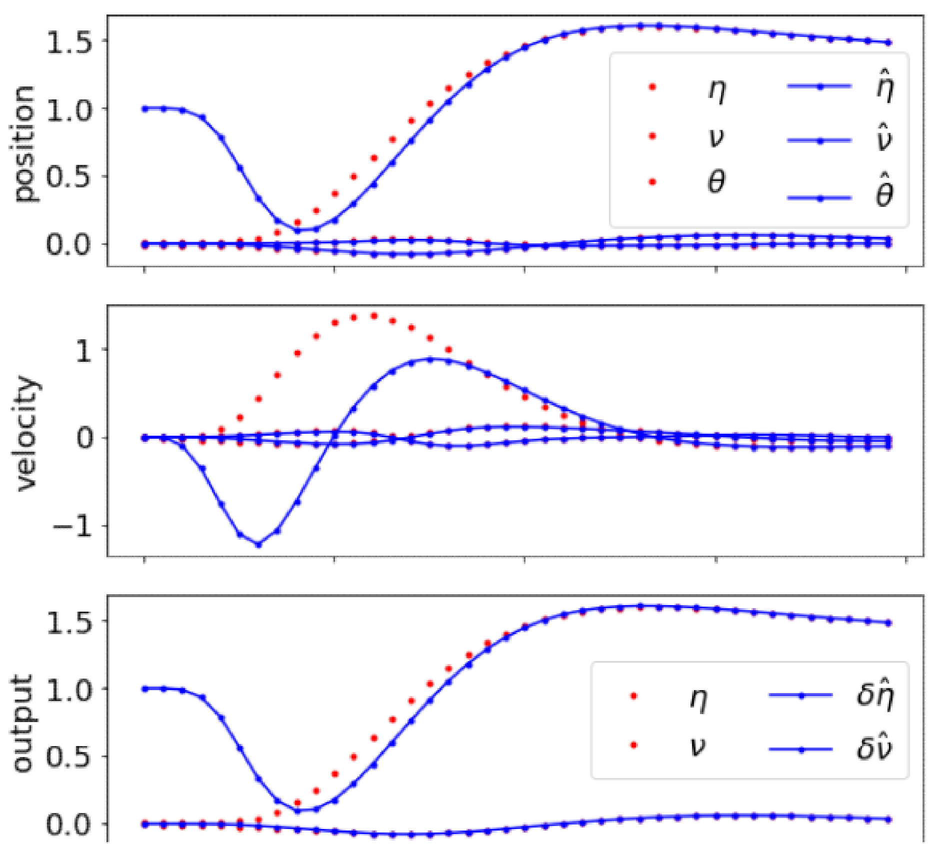
$$\text{NL}\begin{cases} x^+ = f(x,u) + \begin{bmatrix} 0 \\ V_1 \end{bmatrix} w_1 & w_1 \sim \mathcal{N}(0, I) \\ y = h(x,u) + V_2 w_2 & w_2 \sim \mathcal{N}(0, I) \end{cases}$$

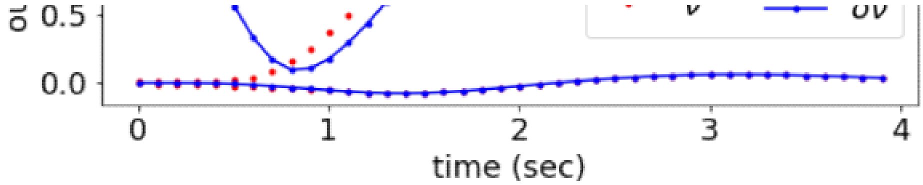
$$\hat{L}\begin{cases} \delta\hat{x}_{s+1} = A\delta\hat{x}_s + B\delta u_s + L_s(\delta y_s - \cancel{C}\delta\hat{y}_s) \\ \delta\hat{y} = C\delta\hat{x} + D\delta u \\ \delta u_s = -K_s\delta\hat{x}_s \end{cases}$$

typo!

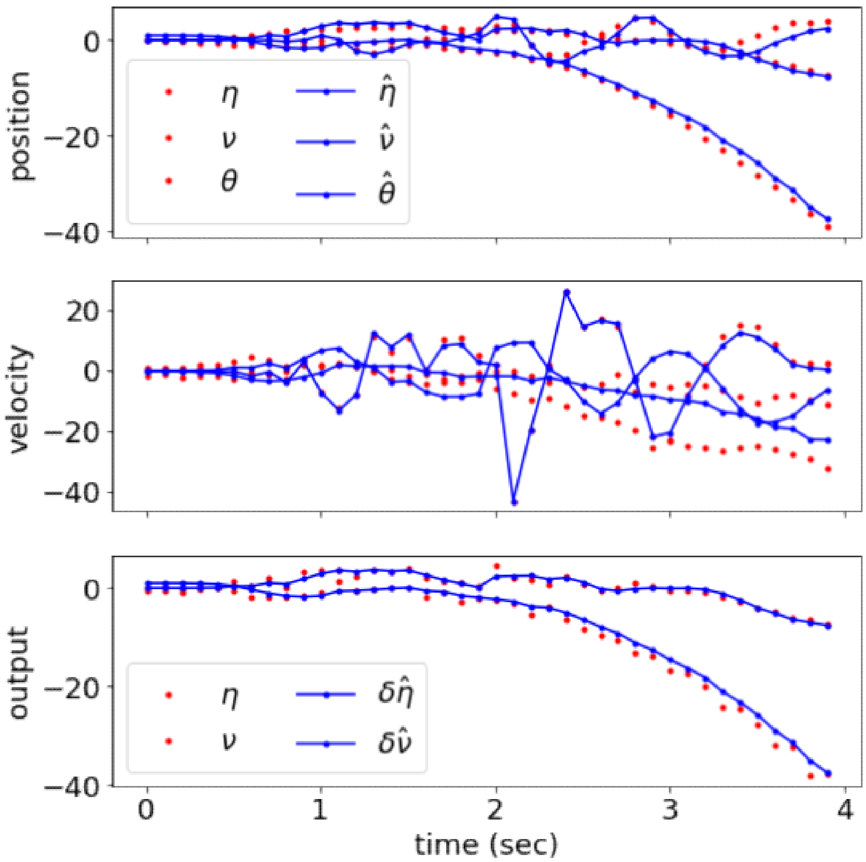


ex: low noise





ex: high(er) noise



interlude

Reinterpreting the DT-LTI system above as a continuous-time process P in the form expected for robust controller synthesis with inputs $w, \delta u$ and outputs $z, \delta y$ yields

$$\begin{aligned} \delta\dot{x} &= \bar{A}\,\delta x + B_1 w + B_2\,\delta u, \\ z &= C_1\,\delta x + D_{11} w + D_{12}\,\delta u, \\ \delta y &= C_2\,\delta x + D_{21} w + D_{22}\,\delta u \end{aligned}$$

TODO: missing $\frac{1}{\Delta}$?

where $\bar{A} = \frac{1}{\Delta}(A - I)$ and

$Q \in \mathbb{R}^{6\times 6}$ from (c)
$R \in \mathbb{R}^{2\times 2}$ from (c)
$V_1 \in \mathbb{R}^{3\times 3}$ from (d)
$V_2 \in \mathbb{R}^{2\times 2}$ from (d)

$$B_1 = \begin{array}{c|c} & \begin{array}{cc} 3 & 2 \end{array} \\ \begin{array}{c} 3 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 \\ \frac{1}{\Delta} V_1 & 0 \end{bmatrix} \end{array}, \quad B_2 = B \in \mathbb{R}^{6\times 2},$$

$$\in \mathbb{R}^{2\times 6}\; C_1 = \begin{bmatrix} \sqrt{\frac{1}{\Delta} Q} \in \mathbb{R}^{6\times 6} \\ 0 \end{bmatrix}, \quad D_{11} \overset{8|\,5}{=} 0, \quad D_{12} = \begin{bmatrix} 0 \in \mathbb{R}^{6\times 2} \\ \sqrt{\frac{1}{\Delta} R} \in \mathbb{R}^{2\times 2} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} C & \cancel{0} \end{bmatrix}, \quad D_{21} = \begin{array}{c|c} & \begin{array}{cc} 3 & 2 \end{array} \\ 2 & \begin{bmatrix} 0 & \frac{1}{\Delta} V_2 \end{bmatrix} \end{array}, \quad D_{22} = D.$$

$\delta x \in \mathbb{R}^6$ $w_1 \in \mathbb{R}^3$ $w_2 \in \mathbb{R}^2$ $w = \begin{bmatrix} w_1 \\ \end{bmatrix} \in \mathbb{R}^5$ $\delta u \in \mathbb{R}^2$ $\delta y \in \mathbb{R}^2$

$V_2 \in \mathbb{R}$ from (a)

$\delta x \in \mathbb{R}^6, \quad w_1 \in \mathbb{R}^3, \quad w_2 \in \mathbb{R}^2, \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^5, \quad \delta u \in \mathbb{R}^2, \quad \delta y \in \mathbb{R}^2$

$$z = \begin{bmatrix} \sqrt{\frac{1}{\Delta} Q}\, \delta x \\ \sqrt{\frac{1}{\Delta} R}\, \delta u \end{bmatrix} \in \mathbb{R}^8 \implies \|z\|_2^2 = \delta x^T \frac{Q}{\Delta} \delta x + \delta u^T \frac{R}{\Delta} \delta u$$

C_{H_2} = control.h2syn(P, nmeas = 2, ncont = 2)

↳ $.A^{H_2}, .B^{H_2}, .C^{H_2}, .D^{H_2}$ define

$$\delta u \leftarrow \boxed{\hat{L} \begin{cases} \delta \dot{\hat{x}} = A^{H_2} \delta \hat{x} + B^{H_2} \delta y \\ \delta u = C^{H_2} \delta \hat{x} + D^{H_2} \delta y \end{cases}} \leftarrow \delta y$$

$\Big\downarrow$ discretize

$$\delta \hat{x}^+ = (I + \Delta A^{H_2}) \delta x + \Delta B^{H_2} \delta y$$
$$\delta u = C^{H_2} \delta \hat{x} + D^{H_2} \delta y$$