An Introduction to Semidefinite Programming for Combinatorial Optimization (Lecture 1)

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IPCO Summer School
May 20-21, 2019

Outline

1. Definitions
2. Optimization
3. Algorithms
4. "Classic" Applications

Definitions

- $\mathbb{R}^p$ = real column vectors of length p
- $\mathbb{R}^{p \times q}$ = real matrices of size $p \times q$
- $\mathbb{S}^p \subseteq \mathbb{R}^{p \times p}$ denotes symmetry, ensures real eigenvalues

- *Trace inner product* defined on $\mathbb{R}^{p \times q}$:

$$M \bullet N := \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} N_{ij} = \operatorname{trace}(M^T N)$$

- Induced *Frobenius norm*:

$$\|M\|_F := \sqrt{M \bullet M}$$

A matrix $X \in \mathbb{S}^p$ is *positive semidefinite* iff:

- $v^T X v \geq 0$ for all $v \in \mathbb{R}^p$
- $\lambda_{\min}[X] \geq 0$
- $X = VV^T$ for some $V \in \mathbb{R}^{p \times r}$ (note: $\mathbf{rank}(X) \leq r$)
- every principal submatrix of X has determinant ≥ 0

We write:

- $X \succeq 0$ or "X is PSD"
- $\mathbb{S}^p_+ =$ set of $p \times p$ PSD matrices

Important properties of $\mathbb{S}^p_+$:

- *Proper*, i.e., closed, convex, pointed, full-dimensional
- *Self-dual*, i.e., $\{S \in \mathbb{S}^p : S \bullet X \geq 0 \; \forall \; X \succeq 0\} = \mathbb{S}^p_+$
- In particular, $X, S \succeq 0 \;\Rightarrow\; X \bullet S \geq 0$

What is the dimension of $\mathbb{S}_{+}^{p}$?

- Ambient dimension is p^2
- But symmetry takes away $\binom{p}{2}$ degrees of freedom
- So dimension is $\binom{p+1}{2}$

When is the following matrix PSD?

$$D := \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix}$$

- D is a diagonal matrix
- Its eigenvalues are D_{11}, D_{22}, D_{33}
- So $D \succeq 0$ iff $\mathrm{diag}(D) \geq 0$

Is this matrix PSD?

$$\begin{pmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{pmatrix}$$

- Yes, because it equals vv^T where $v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Theorem (spectral decomposition). $X \succeq 0$ iff $\exists$ orthogonal $V \in \mathbb{R}^{p \times p}$ and nonnegative, diagonal $D \in \mathbb{S}^p$ s.t. $X = VDV^T$

- Specifically, $\mathrm{diag}(D)$ contains the eigenvalues of X, and the columns of V are its eigenvectors

Optimization

Given:

- Dimensions n and m
- $C, A_1, \ldots, A_m \in \mathbb{S}^n$
- $b \in \mathbb{R}^m$

Let $X \in \mathbb{S}^n$ denote our variable

Primal SDP problem (P):

$$\begin{aligned} p^* := \inf \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i \;\; \forall\, i = 1, \ldots, m \\ & X \succeq 0 \end{aligned}$$

Specify the data for this problem:

$$\begin{aligned} \inf \quad & X_{12} \\ \text{s.t.} \quad & X_{11} + X_{22} = 1 \\ & \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0 \end{aligned}$$

- $n = 2$ and $m = 1$
- $C = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $b_1 = 1$

Now optimize it:

$$\begin{aligned} \inf \quad & X_{12} \\ \text{s.t.} \quad & X_{11} + X_{22} = 1 \\ & \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0 \end{aligned}$$

$$
\begin{aligned}
\inf \quad & X_{12} \\
\text{s.t.} \quad & X_{11} + X_{22} = 1 \\
& X_{11} \geq 0, X_{22} \geq 0 \\
& X_{12}^2 \leq X_{11} X_{22}
\end{aligned}
$$

$$
\begin{aligned}
\inf \quad & X_{12} \\
\text{s.t.} \quad & 0 \leq X_{11} \leq 1 \\
& X_{12}^2 \leq X_{11}(1 - X_{11})
\end{aligned}
$$

$$p^* = -\tfrac{1}{2} \qquad \text{and} \qquad X^* = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

LP is a special case of SDP

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & a_i^T x = b_i \;\; \forall\, i = 1, \ldots, m \\
& x \geq 0
\end{aligned}
$$

$$\downarrow$$

$$
\begin{aligned}
\min \quad & \mathrm{Diag}(c) \bullet X \\
\text{s.t.} \quad & \mathrm{Diag}(a_i) \bullet X = b_i \;\; \forall\, i = 1, \ldots, m \\
& X \succeq 0
\end{aligned}
$$

SOCP is a special case of SDP

$$\|x\| \leq t \quad \iff \quad \begin{pmatrix} t & x^T \\ x & tI \end{pmatrix} \succeq 0$$

Dual SDP problem (D):

$$\begin{aligned} d^* := \sup \quad & b^T y \\ \text{s.t.} \quad & C - \textstyle\sum_{i=1}^m y_i A_i \succeq 0 \end{aligned}$$

or

$$\begin{aligned} d^* := \sup \quad & b^T y \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{aligned}$$

What is the dual?

$$\begin{aligned} \inf \quad & X_{12} \\ \text{s.t.} \quad & X_{11} + X_{22} = 1 \\ & \begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} \succeq 0 \end{aligned}$$

$$
\begin{aligned}
\sup \quad & y_1 \\
\text{s.t.} \quad & \begin{pmatrix} -y_1 & 1/2 \\ 1/2 & -y_1 \end{pmatrix} \succeq 0
\end{aligned}
$$

- $d^* = -\frac{1}{2}$
- $y_1^* = -\frac{1}{2}, \quad S^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Theorem (weak duality). If (X, y, S) is primal-dual feasible, then $C \bullet X - b^T y = X \bullet S \geq 0$

Proof.

$$
\begin{aligned}
C \bullet X - b^T y \quad &= \left(\sum_{i=1}^m y_i A_i + S\right) \bullet X - b^T y \\
&= \sum_{i=1}^m y_i A_i \bullet X + S \bullet X - b^T y \\
&= S \bullet X \geq 0
\end{aligned}
$$

Corollary. If (X, y, S) is primal-dual feasible and $C \bullet X = b^T y$, then (X, y, S) is primal-dual optimal. Moreover, $X \bullet S = 0 \Leftrightarrow XS = 0$

Corollary. $p^* \geq d^*$

In our previous example:

$$X^* = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad S^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$X^* S^* = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Strong Duality? Does $p^* = d^*$?

Strong duality counterexample:

$$\begin{aligned} 1 = \inf \quad & X_{33} \\ \text{s.t.} \quad & X_{11} = 0 \\ & X_{12} + X_{21} + 2X_{33} = 2 \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} 0 = \sup \quad & 2\,y_2 \\ \text{s.t.} \quad & \begin{pmatrix} -y_1 & -y_2 & 0 \\ -y_2 & 0 & 0 \\ 0 & 0 & 1 - 2\,y_2 \end{pmatrix} \succeq 0 \end{aligned}$$

A matrix $X \in \mathbb{S}^p$ is *positive definite* iff:

- $v^T X v > 0$ for all $v \in \mathbb{R}^p \setminus \{0\}$
- $\lambda_{\min}[X] > 0$
- $X = VV^T$ for some invertible $V \in \mathbb{R}^{p \times p}$
- every principal submatrix of X has determinant > 0

We write:

- $X \succ 0$ or "X is PD"
- $\mathbb{S}^p_{++}$ = set of $p \times p$ PD matrices

In fact, $\mathbb{S}^p_{++} = \mathrm{interior}(\mathbb{S}^p_{+})$

Theorem (strong duality). Suppose (P) and (D) are both feasible.

- If $\exists$ primal feasible X with $X \succ 0$, then $p^* = d^*$ and d^* is attained
- If $\exists$ dual feasible (y, S) with $S \succ 0$, then $p^* = d^*$ and p^* is attained
- If both regularity conditions hold, then $\exists$ primal-dual optimal solution (X^*, y^*, S^*) such that $p^* = C \bullet X^* = b^T y^* = d^*$ and $X^* S^* = 0$

Remark. Algorithmic papers and results often assume both (P) and (D) have interior. But...in any given application, make sure to double check!

Algorithms

Observation. Relying on rational or floating-point arithmetic, we cannot expect to optimize SDPs exactly

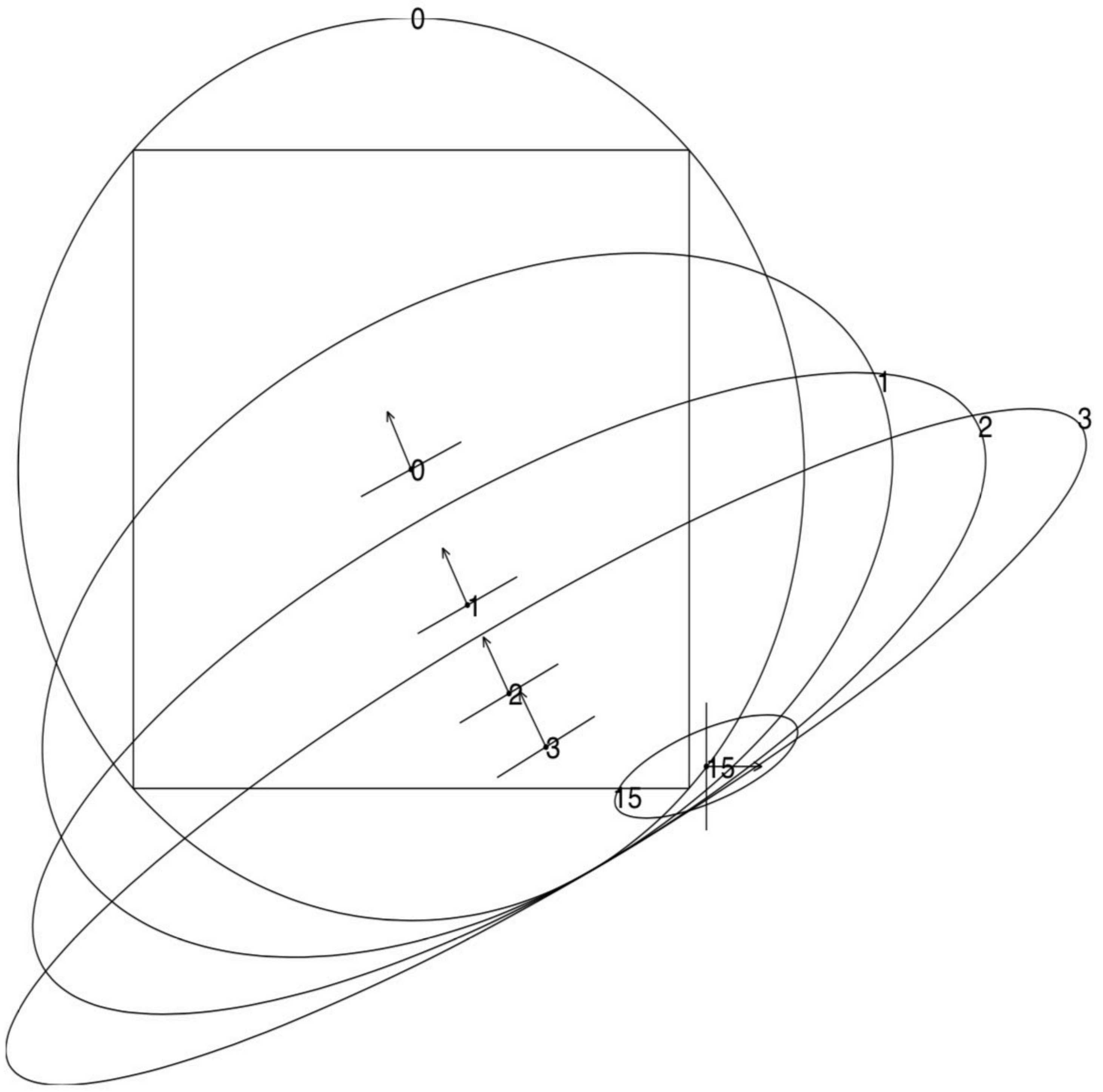
Irrational, despite rational data:

$$
\begin{aligned}
-\sqrt{5} = \min \quad & \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \bullet X \\
\text{s.t.} \quad & \operatorname{trace}(X) = 1 \\
& X \succeq 0
\end{aligned}
$$

Hence, for a user-specified $\epsilon > 0$, a reasonable goal is to find an ϵ-optimal (dual) solution y^ϵ, i.e., one satisfying:

- $C - \sum_{i=1}^{m} y_i^\epsilon A_i + \epsilon I \succeq 0$
- $b^T y^\epsilon \geq d^* - \epsilon$

Ellipsoid Method



Source: *Lectures on Modern Convex Optimzation* by Ben-Tal and Nemirovski

The setup:

- $\epsilon > 0$
- $v =$ vector encoding $(n, m, C, A_1, \ldots, A_m)$
- $\sigma =$ length of v
- $\Sigma := \sigma + \|v\|_1$

$$
\begin{aligned}
d^* := \max \quad & b^T y \\
\text{s.t.} \quad & C - \textstyle\sum_{i=1}^m y_i A_i \succeq 0 \\
& \|y\|_2 \leq \Sigma^\sigma
\end{aligned}
$$

Introduce relaxation (D_ϵ), whose optimal value is within ϵ of d^*:

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & C - \sum_{i=1}^m y_i A_i + \epsilon I \succeq 0 \\ & \|y\|_2 \leq \Sigma^\sigma \end{array}$$

Key parameter:

$$\theta := \frac{\epsilon}{(\sigma + \sqrt{m}\Sigma^\sigma)^\sigma + \epsilon} \in (0,1)$$

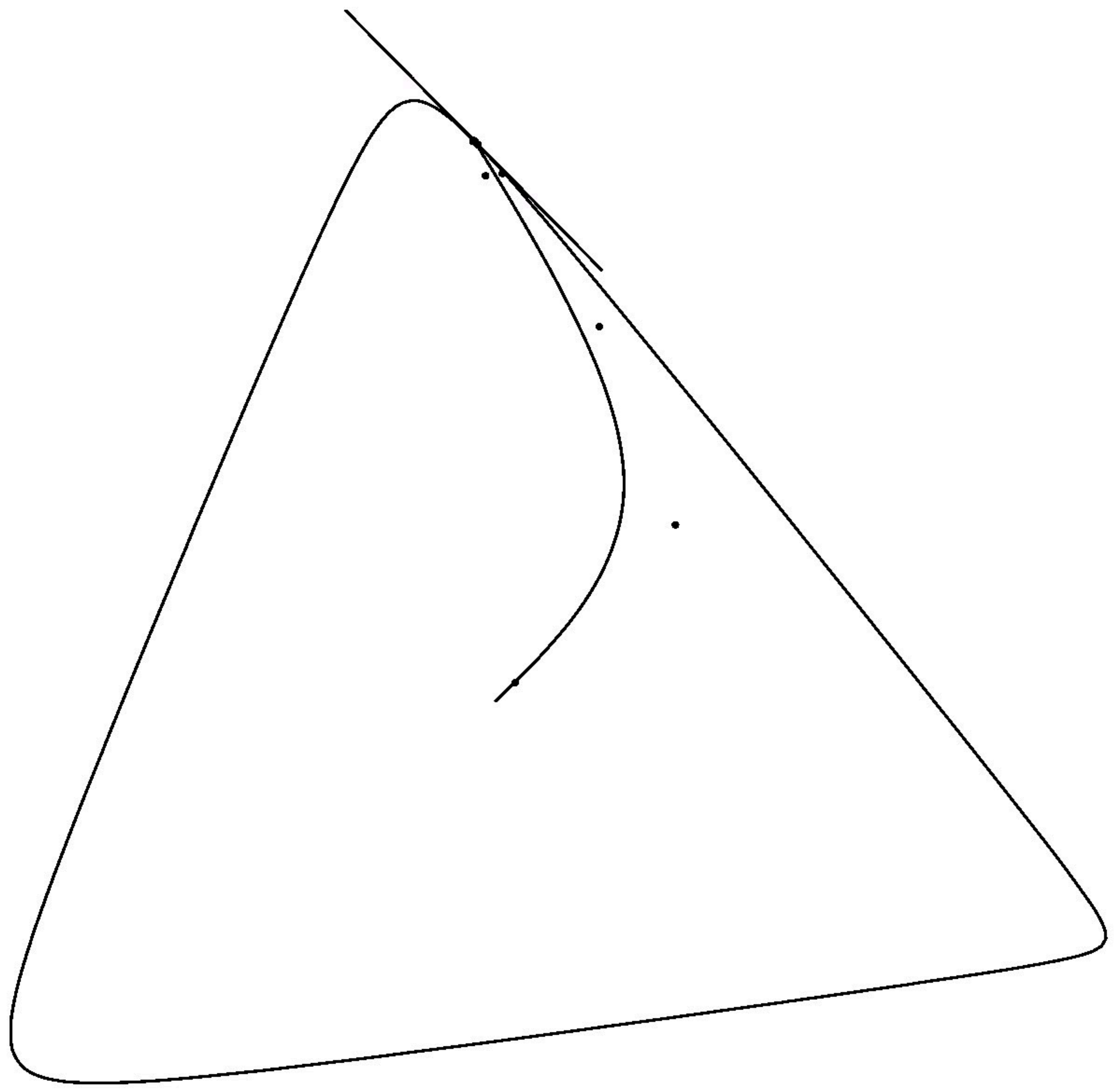
Note $\theta \to 0$ as $\epsilon \to 0$

Lemma. If problem (D) is feasible, then feasible set of (D_ϵ) contains a Euclidean ball of radius $\theta\Sigma^\sigma$

Theorem. The ellipsoid method requires $O(m^2 \log(1/\theta))$ iterations to return an ϵ-optimal solution to (D). In addition, each iteration requires $O(m^2 + mn^2 + n^3)$ floating point operations

Note: $\log(1/\theta) \approx \log(1/\epsilon)$, but θ takes into account σ and Σ

Interior-Point Methods



The setup:

- $\epsilon > 0$
- σ and Σ as before
- Both (P) and (D) interior feasible, which implies:
    - Strong duality holds
    - Central path exists
- Initial (X^0, y^0, S^0) interior primal-dual solution

Key parameter:

$$\rho := \frac{\epsilon}{\sigma + \Sigma + \epsilon^2}$$

Note $\rho \to 0$ as $\epsilon \to 0$

Theorem. The primal-dual short-step path-following method requires $O(\sqrt{m}\log(1/\rho))$ iterations to return an ϵ-optimal primal-dual solution. In addition, each iteration requires $O(m^2 + mn^2 + n^3)$ floating point operations

Note: $\log(1/\rho) \approx \log(1/\epsilon)$, but ρ takes into account σ and Σ

Mosek is one of the leading software packages for SDP:

The interior-point optimizer [in Mosek] is an implementation of the so-called homogeneous and self-dual algorithm... A solution to the homogeneous model can be computed using a primal-dual interior-point algorithm.[1]

[1] https://docs.mosek.com/8.1/capi/solving-conic.html

Other Methods

- Interior-point methods effectively deliver high accuracy for small- to medium-scale problems
- But they slow down for large instances
  - Definition of "large"? When the algorithm slows down 😉
  - Even if the instance is sparse, the linear algebra operations can be dense
- Many methods to address this, often by exploiting problem structure

Theorem (Pataki, Barvinok, Shapiro, Deza-Laurent, 1990's). For problem (P), there exists an optimal solution X^* with $r^* := \mathrm{rank}(X^*)$ satisfying $r^* \leq \lceil \sqrt{2m} \rceil$

Idea (B-Monteiro, 2003). Solve (P) as an NLP by replacing X with VV^T, where the number of columns p of V is at least $\lceil \sqrt{2m} \rceil$

Observation. Despite being "dumb," this idea works well for finding a global (!) optimal X^* in practice. In 2003, we had a limited theory to explain why

Theorem (Boumal-Voroninski-Bandeira, 2018). Under a regularity condition, for almost all cost matrices C, choosing $p > \lceil \sqrt{2m} \rceil$ guarantees that the first- and second-order KKT conditions in V are sufficient for global (!) optimality

"Classic" Applications

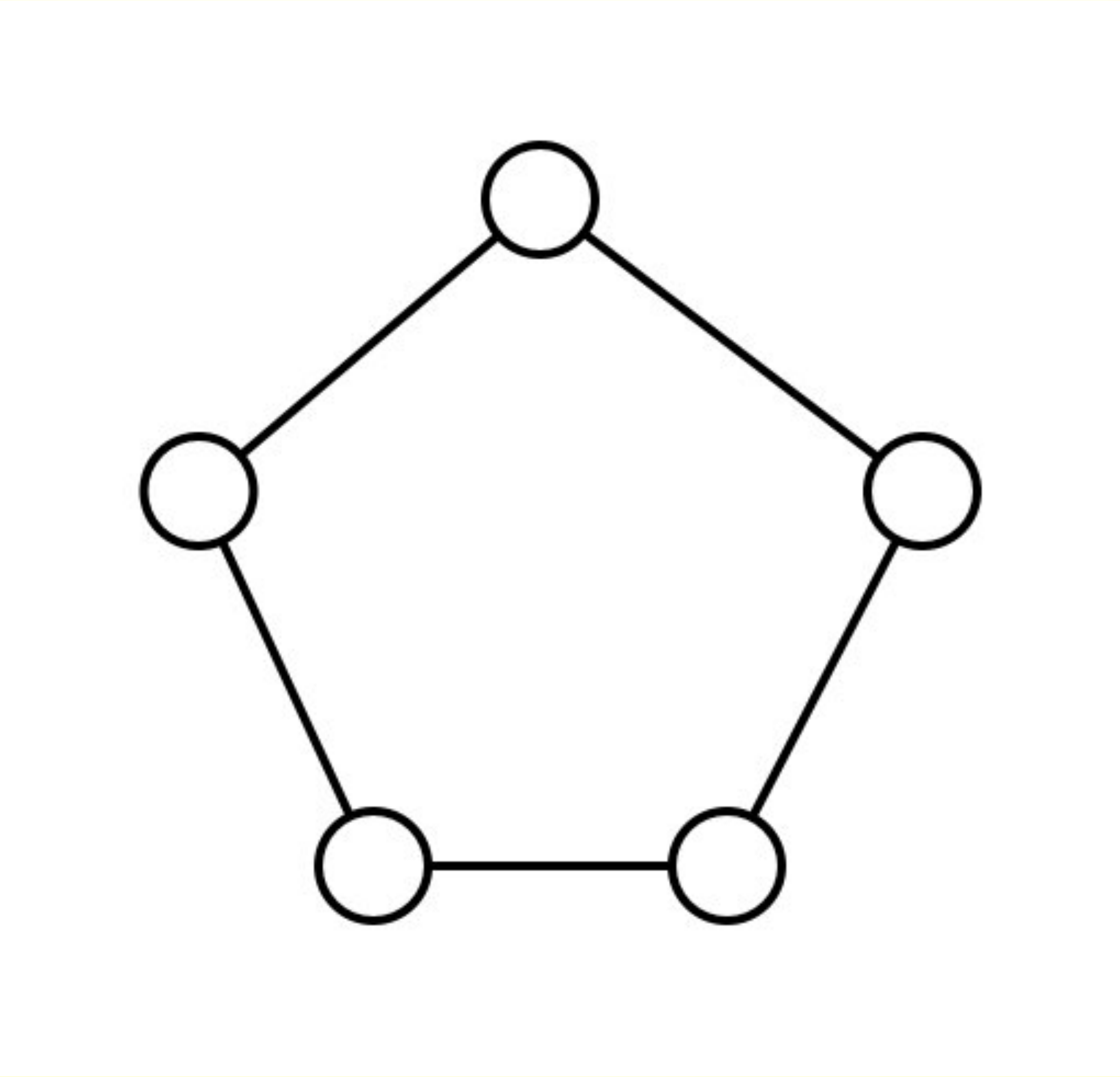
$$\begin{aligned} \min \quad & x^T C x + 2\, c^T x \\ \text{s.t.} \quad & x^T A_i x + 2\, a_i^T x \le b_i \ \ \forall\, i \end{aligned}$$

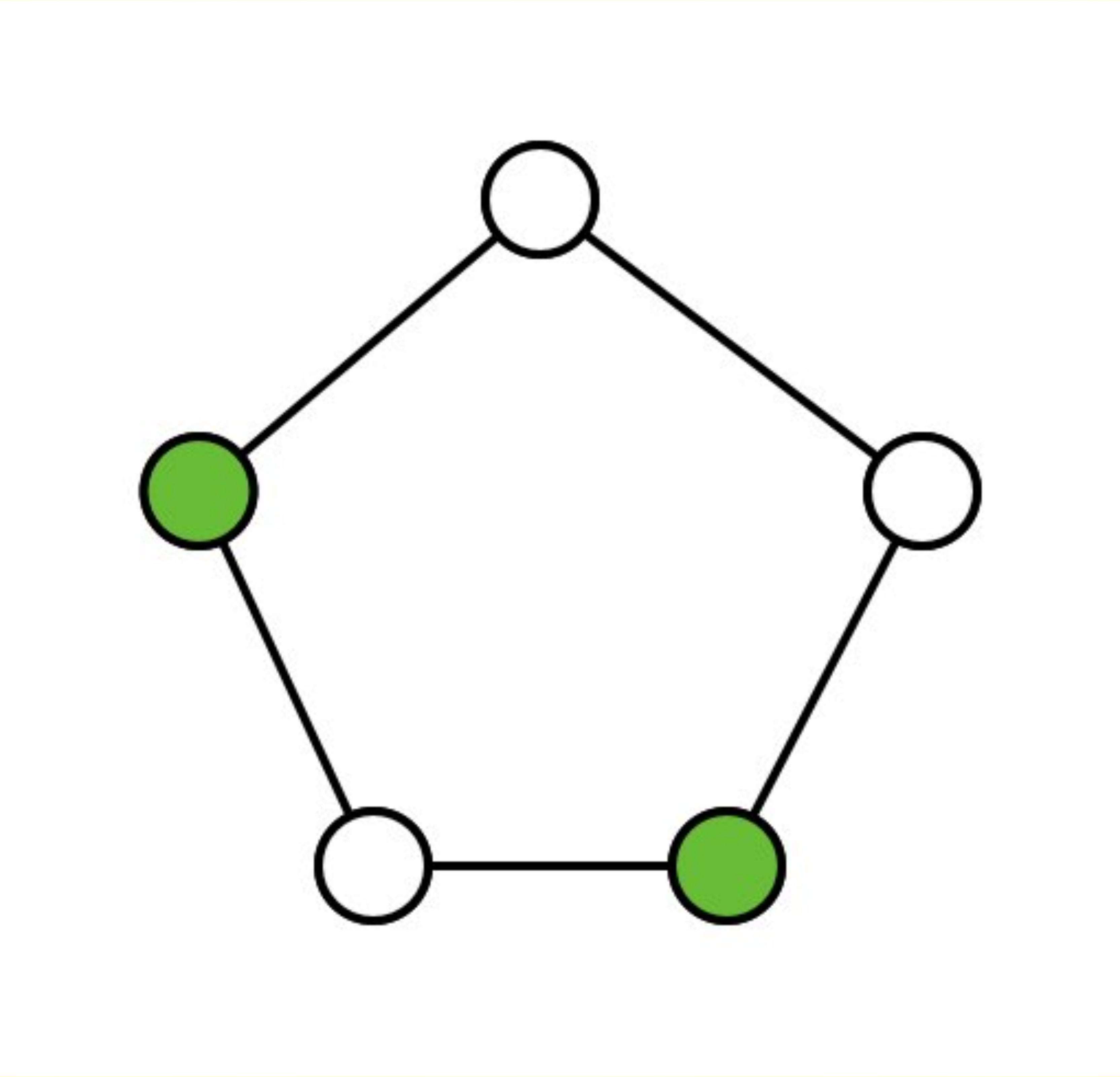
$$\downarrow$$

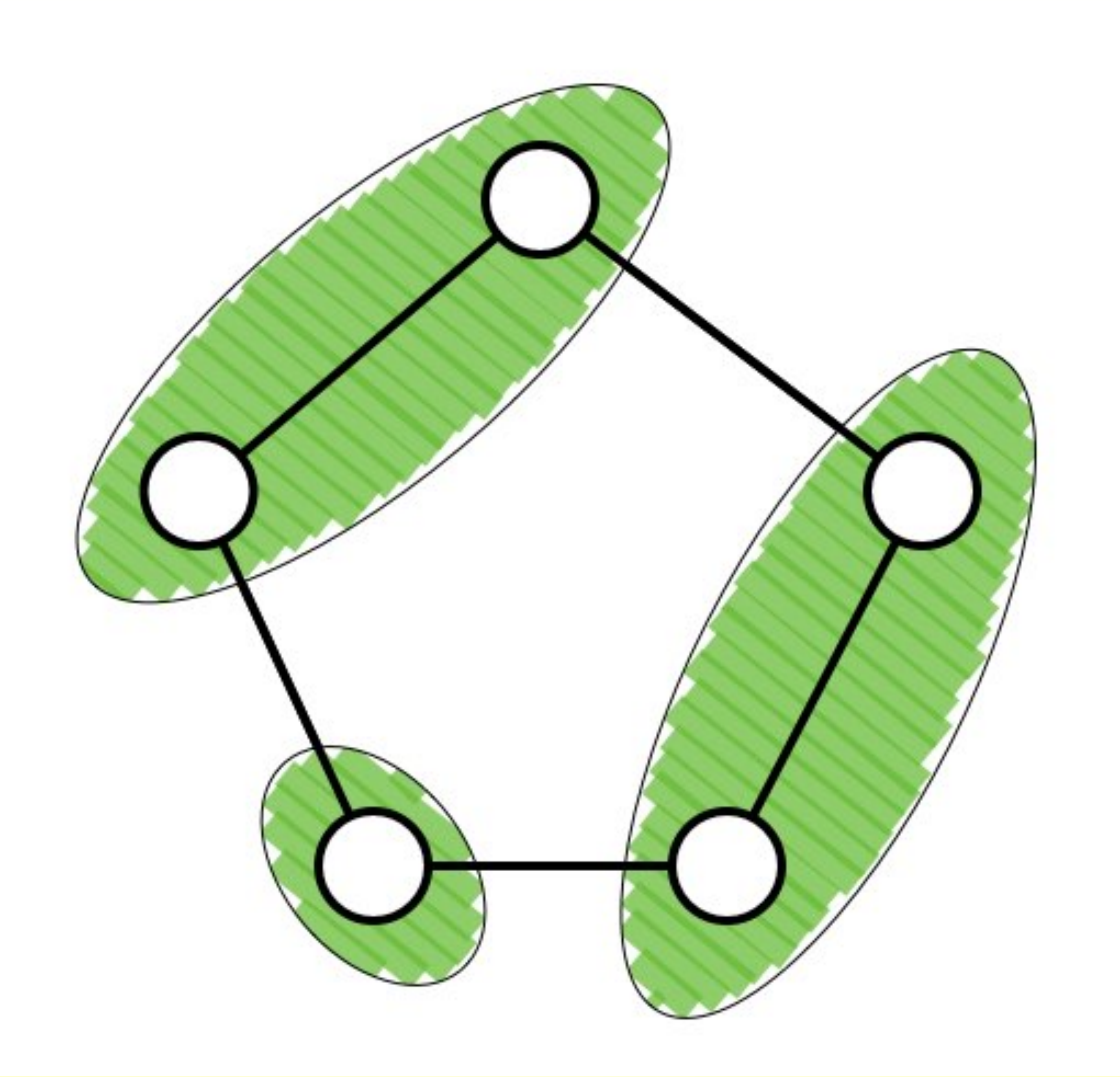
$$\begin{aligned} \min \quad & C \bullet X + 2\, c^T x \\ \text{s.t.} \quad & A_i \bullet X + 2\, a_i^T x \le b_i \ \ \forall\, i \\ & X \succeq x x^T \ \Leftrightarrow \ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \end{aligned}$$

Maximum Stable Set

- $G = (V, E)$ undirected graph
- $\alpha =$ max size of stable set in G (NP-hard)
- $\bar{\chi} =$ min size of clique cover of G (NP-hard)
- $\alpha \leq \bar{\chi}$







$$
\begin{aligned}
\alpha := \max \quad & e^T x \\
\text{s.t.} \quad & x_i + x_j \leq 1 \ \ \forall \, (i,j) \in E \\
& x_j \in \{0,1\} \ \ \forall \, i \in V
\end{aligned}
$$

$$\downarrow$$

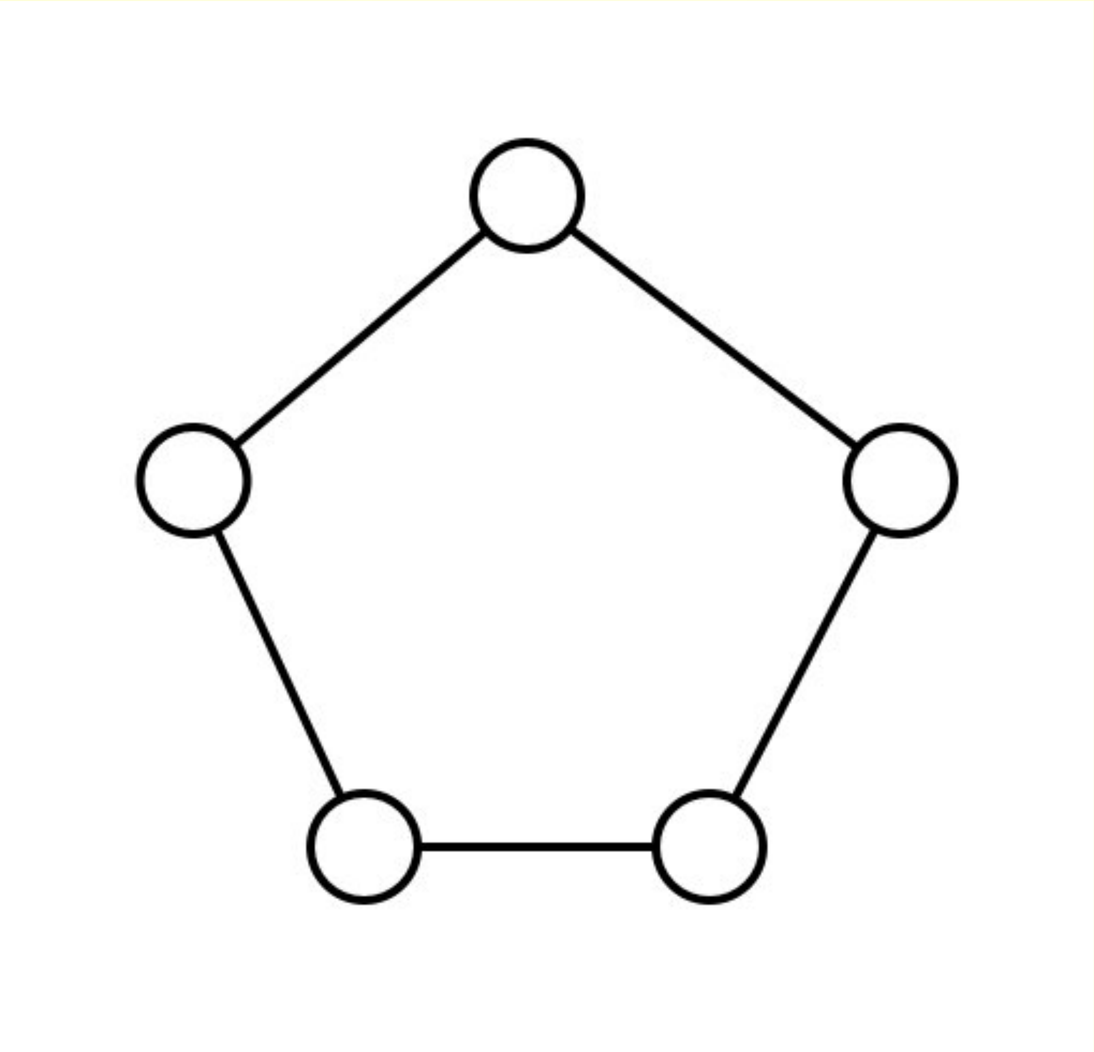
$$
\begin{aligned}
\alpha = \max \quad & x^T x \\
\text{s.t.} \quad & x_i x_j = 0 \ \ \forall \, (i,j) \in E \\
& x_j^2 = x_j \ \ \forall \, i \in V
\end{aligned}
$$

$$
\begin{aligned}
\vartheta := \max \quad & \operatorname{trace}(X) \\
\text{s.t.} \quad & X_{ij} = 0 \;\; \forall\, (i,j) \in E \\
& x = \operatorname{diag}(X) \\
& X \succeq xx^T
\end{aligned}
$$

Theorem (Grötschel-Lovász-Schrijver, 1981). $\alpha \leq \vartheta \leq \bar{\chi}$

Definition. G is *perfect* when $\alpha(G') = \bar{\chi}(G')$ for all induced subgraphs G'

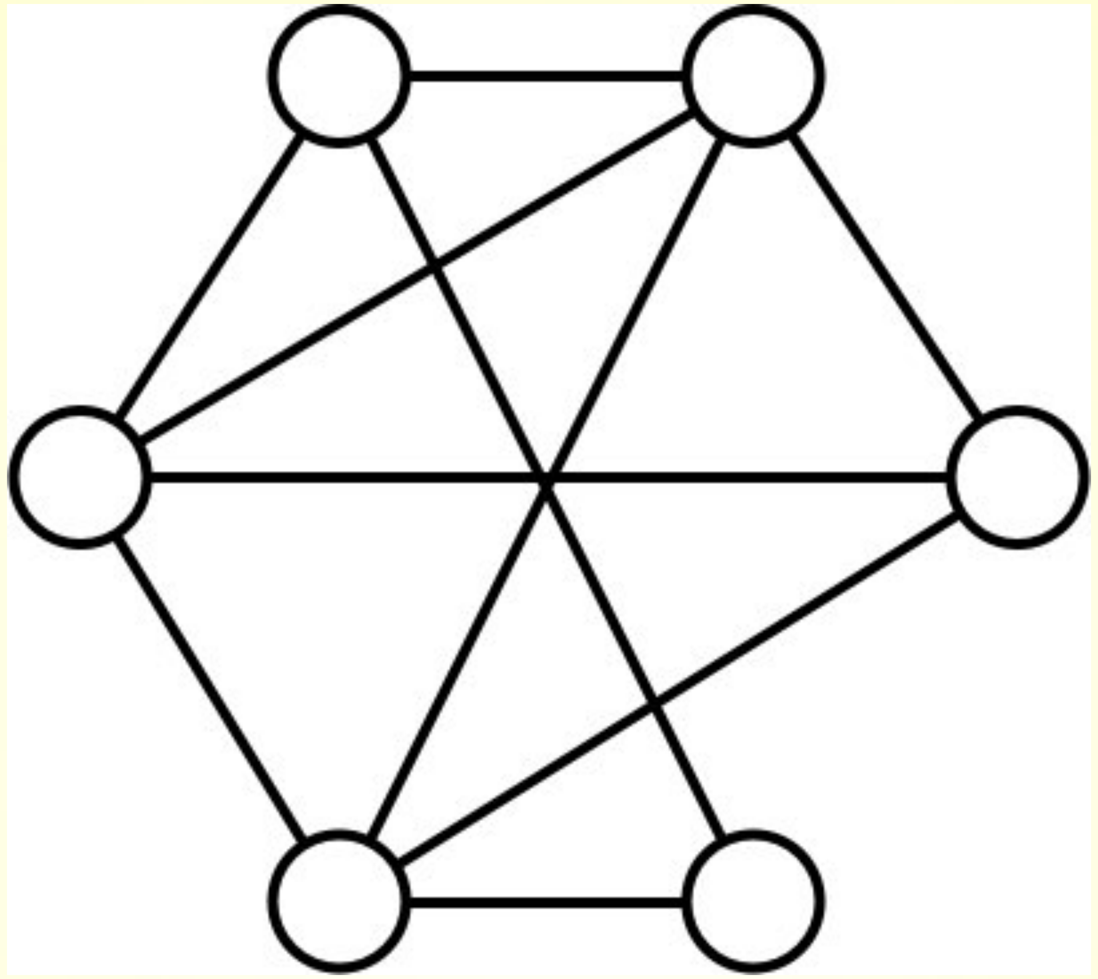
Corollary. α and $\bar{\chi}$ are polynomial-time computable for perfect graphs

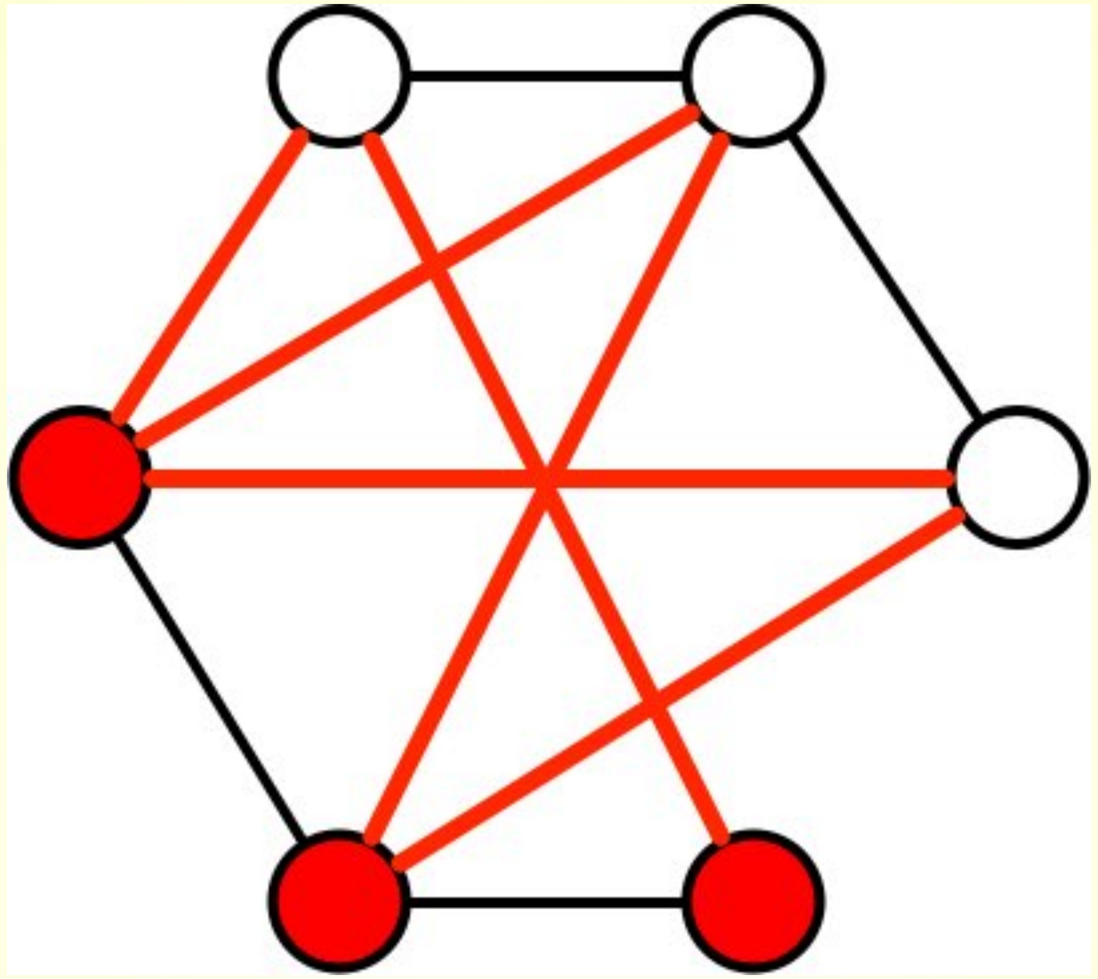


$$\alpha = 2 \qquad \vartheta = \sqrt{5} \qquad \bar{\chi} = 3$$

MaxCut

- $G = (V, E)$ undirected graph
- $U \subseteq V$
- $\mathrm{cut}(U) =$ # of edges from U to $V \setminus U$
- Which U maximizes $\mathrm{cut}(U)$?
- This is NP-hard





$$\text{cut} = 6$$

$$\begin{aligned} \text{MaxCut} := \max \quad & \textstyle\sum_{ij \in E} \frac{1}{2}(1 - x_i x_j) \\ \text{s.t.} \quad & x_j \in \{-1, 1\} \;\; \forall\, i \in V \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \text{MaxCut} = \max \quad & \textstyle\sum_{ij \in E} \frac{1}{2}(1 - x_i x_j) \\ \text{s.t.} \quad & x_j^2 = 1 \;\; \forall\, i \in V \end{aligned}$$

$$
\begin{array}{ll}
\max & \sum_{ij \in E} \frac{1}{2}(1 - X_{ij}) \\
\text{s.t.} & \operatorname{diag}(X) = e \\
& X \succeq xx^T
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{ll}
\max & \sum_{ij \in E} \frac{1}{2}(1 - X_{ij}) \\
\text{s.t.} & \operatorname{diag}(X) = e \\
& X \succeq 0
\end{array}
$$

YALMIP code (in Matlab)

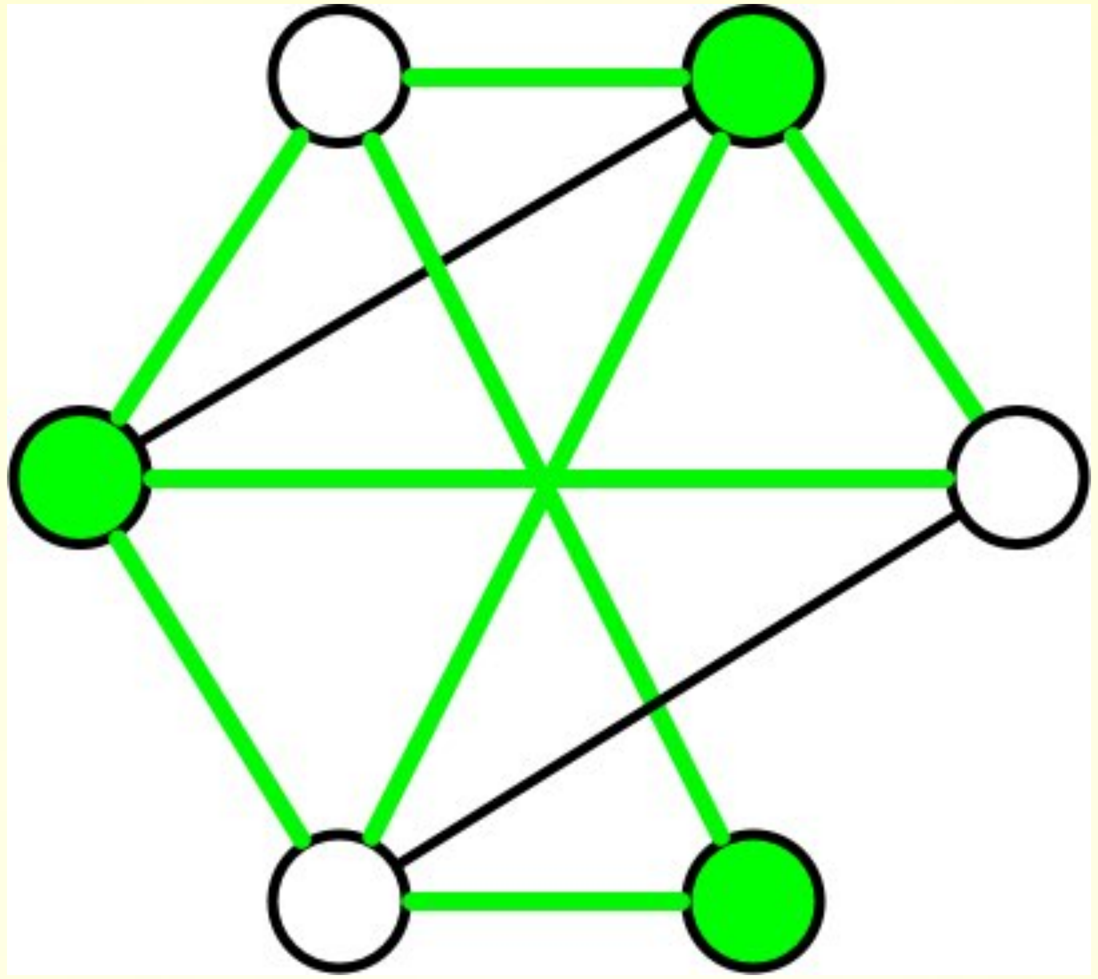
```
n = 6; m = 10;
E = [1, 2; 1, 4; 1, 5; 1, 6; 2, 4; 2, 5; 3, 4; 3, 6; 4, 5; 5, 6];

A = full(sparse(E(:, 1), E(:, 2), ones(m, 1), n, n));
A = A + A';

C = 0.25 * (diag(sum(A)) - A);
X = sdpvar(n);
con = [diag(X) == 1; X >= 0];
obj = C(:)' * X(:);
solvesdp(con, -obj);
```

$$\text{relaxation opt val} = 8$$

$$X^* = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$



$$\text{cut} = 8$$

Theorem (Goemans-Williams, 1995). The following (randomized) algorithm is an 0.87856-approximation algorithm for MaxCut:

1. Solve the SDP relaxation to obtain X^*
2. Compute a factorization $X^* = R^*(R^*)^T$
3. Randomly generate a vector $u \in \mathbb{R}^{|V|}$ uniform on the unit sphere
4. Define $U := \{i \in V : [(R^*)^T u]_i \geq 0\}$

Lift and Project

$$P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0,1]^n$$

$$P^{01} := \mathrm{conv}(P \cap \{0,1\}^n)$$

What we can determine about P^{01}?

Homogenization of P:

$$H := \left\{ \begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{n+1} : \begin{array}{rl} x_0 & \geq 0 \\ Ax \leq & x_0 b \end{array} \right\}$$

Observation. $x \in P$ implies $\begin{pmatrix} x_j \\ x_j x \end{pmatrix}, \begin{pmatrix} 1 - x_j \\ x - x_j x \end{pmatrix} \in H$

Idea. Linearize $x_j x$ by introducing a new variable y

Lift

$$\widehat{\mathrm{BCC}}(P,j) := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \begin{array}{c} x_j = y_j \\ \binom{x_j}{y}, \binom{1-x_j}{x-y} \in H \end{array} \right\}$$

Project

$$\mathrm{BCC}(P, j) := \mathbf{proj}_x(\widehat{\mathrm{BCC}}(P, j))$$

Proposition. $\mathrm{BCC}(P, j)$ is a polytope and $P^{01} \subseteq \mathrm{BCC}(P, j) \subseteq P$.

Proof.

1. Projection of a polytope is a polytope
2. As mentioned $x \in P$ implies $(x_j; y), (1 - x_j; x - y) \in H$ when $y = x_j x$. Also, $x_j = y_j$ is valid when $y = x_j x$ and x is binary
3. Summing $(x_j; y)$ and $(1 - x_j; x - y)$, we see $(1; x) \in H$, as desired

"Rinse and Repeat"

- $\mathrm{BCC}^0(P) := \mathrm{BCC}(P, 0) := P$
- $\mathrm{BCC}^k(P) := \mathrm{BCC}(\mathrm{BCC}^{k-1}(P), k)$

Get $P^{01} \subseteq \mathrm{BCC}^n(P) \subseteq \cdots \subseteq \mathrm{BCC}^1(P) \subseteq P$

Theorem (Balas-Ceria-Cornuejols, 1993). $\mathrm{BCC}^n(P) = P^{01}$

Observation. In fact, the iterative BCC procedure depends on the order of variables in the liftings

Idea. Define $N_0(P) := \cap_{j=1}^{n} \mathrm{BCC}(P, j)$ and apply iteratively

This idea works, but unfortunately, still need n steps in general. Can it be strengthened further?

$$\widehat{N_0}(P) = \left\{ \left(x, y^{(1)}, \ldots, y^{(n)}\right) : \begin{array}{r} x_j = y_j^{(j)} \quad \forall\, j \\ \binom{x_j}{y^{(j)}}, \binom{1-x_j}{x-y^{(j)}} \in H \quad \forall\, j \end{array} \right\}$$

$$X := \left(y^{(1)}, \ldots, y^{(n)} \right)$$

$$\widehat{N_0}(P) = \left\{ (x, X) : \begin{array}{c} x = \operatorname{diag}(X) \\ \binom{x_j}{X_{\cdot j}}, \binom{1-x_j}{x-X_{\cdot j}} \in H \ \ \forall\, j \end{array} \right\}$$

Observation (Lovász-Schrijver, 1991). With respect to $\widehat{N_0}(P)$, the constraints $X = X^T$ and $X \succeq xx^T$ are valid for P^{01}

$$N(P) := \mathrm{proj}_x \left(\widehat{N_0}(P) \cap \{X = X^T\} \right)$$

$$N_+(P) := \mathrm{proj}_x \left(\widehat{N_0}(P) \cap \{X = X^T, X \succeq xx^T\} \right)$$

Observation. $N_+(P)$ can also be gotten by constructing the SDP relaxation of the following (redundant) description of $P \cap \{0,1\}^n$:

$$\begin{aligned} &x_j(b - Ax) \geq 0 \;\; \forall\, j \\ &(1 - x_j)(b - Ax) \geq 0 \;\; \forall\, j \\ &x_j^2 = x_j \;\; \forall\, j \end{aligned}$$

$$N^1(P) := N(P), \quad N^k(P) := N(N^{k-1}(P))$$

$$N_+^1(P) := N_+(P), \quad N_+^k(P) := N_+(N_+^{k-1}(P))$$

Corollary. As $k \to n$, $N^k(P)$ and $N_+^k(P)$ converge monotonically to P^{01}. Moreover, $N_+^k(P)$ converges at least as fast as $N^k(P)$

Theorem. For fixed k, can optimize over $N^k(P)$ and $N_+^k(P)$ in polynomial-time

Theorem (Lovász-Schrijver, 1991). For a given graph G, let P be the stable-set LP relaxation. Then $N_+(P)$ implies the

- all inequalities implied by the SDP relaxation defining ϑ
- odd-cycle, odd-anti-hole, odd-wheel, and clique inequalities

For any $\epsilon \in (0,1)$, define

$$P := \{x \in [0,1]^n : x_1 + \cdots + x_n \leq n - \epsilon\}$$

Theorem (see Tuncel, 2010). $P^{01} \subsetneq N_+^{n-1}(P)$

Until Next Time...