Exact Semidefinite Formulations for a Class of (Random and Non-Random) Nonconvex Quadratic Programs

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Abstract

We study a class of quadratically constrained quadratic programs (QCQPs), called *diagonal QCQPs*, which contain no off-diagonal terms $x_j x_k$ for $j \neq k$, and we provide a sufficient condition on the problem data guaranteeing that the basic Shor semidefinite relaxation is exact. Our condition complements and refines those already present in the literature and can be checked in polynomial time. We then extend our analysis from diagonal QCQPs to general QCQPs, i.e., ones with no particular structure. By reformulating a general QCQP into diagonal form, we establish new sufficient conditions for the semidefinite relaxations of general QCQPs to be exact. Finally, these ideas are extended to show that a class of random general QCQPs has exact semidefinite relaxations with high probability as long as the number of variables is significantly larger than the number of constraints. To the best of our knowledge, this is the first result establishing the exactness of the semidefinite relaxation for random general QCQPs.

Keywords: quadratically constrained quadratic programming, semidefinite relaxation, low-rank solutions.

1 Introduction

We study *quadratically constrained quadratic programming* (QCQP), i.e., the minimization of a nonconvex quadratic objective over the intersection of nonconvex quadratic constraints:

$$\begin{aligned} \min \;\; & x^T C x + 2c^T x \\ \text{s.t.} \;\; & x^T A_i x + 2a_i^T x \le b_i \;\; \forall\, i = 1, \ldots, m. \end{aligned} \tag{1}$$

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The variable is $x \in \mathbb{R}^n$ and the data consist of the symmetric matrices $\{C, A_i\}$ and column vectors $\{c, a_i\}$. QCQPs subsume a wide variety of NP-hard optimization problems, and hence a reasonable approach is to approximate them via tractable classes of optimization problems.

Semidefinite programming (SDP) is one of the most frequently used tools for approximating QCQPs in polynomial time. The standard approach constructs an SDP relaxation of (1) by replacing the rank-1 matrix inequality $\binom{1}{x}\binom{1}{x}^T \succeq 0$ by $Y(x, X) \succeq 0$, where

$$Y(x,X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathbb{S}^{n+1}$$

and $\mathbb{S}^{n+1}$ denotes the symmetric matrices of size $(n+1) \times (n+1)$. In this paper, we focus on the simplest SDP relaxation of (1), called the Shor relaxation:

$$\begin{aligned} \min \quad & C \bullet X + 2c^T x \\ \text{s. t.} \quad & A_i \bullet X + 2a_i^T x \le b_i \quad \forall\, i = 1, \dots, m \\ & Y(x, X) \succeq 0. \end{aligned} \tag{2}$$

where $M \bullet N := \operatorname{trace}(M^T N)$ is the trace inner product.

1.1 Rank bounds

Let r^* be the smallest rank among all optimal solutions $Y^* := Y(x^*, X^*)$ of (2). When $r^* = 1$, the relaxation (2) solves (1) exactly, and loosely speaking, r^* is an important measure for understanding the quality of the SDP relaxation, e.g., a low r^* might allow one to develop an approximation algorithm for (1) by solving (2). Furthermore: in many cases the true objective of interest is to find a low-rank feasible solution of (2) [17]; and knowing r^*, or simply preferring a smaller rank, can even help with solving (2) via so-called *low-rank approaches* for solving SDPs [10].

We are interested in *a priori* upper bounds on r^*. Pataki [16] and Barvinok [4] proved that $r^*(r^*+1)/2 \le m+1$, or equivalently $r^* \le \lceil \sqrt{2(m+1)} \rceil$.[1] Note that this result depends neither on n nor on the data of the SDP. In general, to reduce the bound further, one must exploit the particular structure and/or data of the instance, and there are many examples in which this is indeed possible [23, 29, 21, 8]. For example, one classical result establishes

[1] In fact, if the number of inactive linear inequalities at Y^* is known ahead of time, then this bound can be improved. For example, suppose (2) contains the two inequalities $0 \le X_{12} \le 1$. Then the rank bound can be improved to $\lceil \sqrt{2m} \rceil$ since both inequalities cannot be active at the same time.

that, if all C, A_i are positive semidefinite, then $r^* = 1$ is guaranteed.

A recent approach bounds r^* by studying the structure of the simple, undirected graph G associated with the aggregate nonzero structure of the matrices

$$\begin{pmatrix} 1 & c^T \\ c & C \end{pmatrix}, \begin{pmatrix} 1 & a_1^T \\ a_1 & A_m \end{pmatrix}, \ldots, \begin{pmatrix} 1 & a_m^T \\ a_m & A_m \end{pmatrix}.$$

Laurent and Varvitsiotis [12] show in particular that r^* is bounded above by the tree-width of G plus 1. So, for example when G is a tree, $r^* \leq 2$. Similar approaches and extensions can be found in [21, 14, 15]. In fact, [14] proves that any polynomial optimization problem can be reformulated as a QCQP with a corresponding SDP relaxation having $r^* \leq 2$. This demonstrates that, in a certain sense, the difference between approximating and solving (1) is precisely the difference between $r^* = 2$ and $r^* = 1$.

In this paper, we study, new sufficient conditions guaranteeing $r^* = 1$ (in fact any optimal solution Y^* has rank equal to 1), and we do so in two stages.

First, in Section 2, we consider a subclass of QCQPs that we call *diagonal QCQPs*: each data matrix $C, A_1, \ldots, A_m$ is diagonal. This means that no cross terms $x_j x_k$ for $j \neq k$ appear in (1), and hence each quadratic function is separable, although the entire problem is not. Under a linear transformation, this is equivalent to the conditions that all C, A_i pairwise commute and that all C, A_i share a common basis of eigenvectors. This subclass is itself NP-hard since it contains, for example, 0-1 binary integer programs. In addition, in this case, the aggregate nonzero structure of G is a star with the first n vertices connected to the $(n+1)$-st vertex, and hence, as discussed above, $r^* \leq 2$ for diagonal QCQPs. A constant approximation algorithm based on the SDP relaxation was given by [28].

With respect to diagonal QCQPs, our main result provides a sufficient condition on the data of (1) guaranteeing $r^* = 1$. Independent of the Laurent-Varvitsiotis bound, which is based only on the graph structure G, our approach shows that r^* is bounded above by $n - f + 1$, where f is a data-dependent integer that can be computed in a prepocessing step by solving n linear programs (LPs). Specifically, before solving the relaxation (2), we construct and solve n auxiliary LPs using the data of (1) to assess the feasibility of n polyhedral systems. The integer f is the number of those systems, which are feasible, and then we can prove $r^* \leq n - f + 1$. Thus the condition $f = n$ implies $r^* = 1$. In particular, the j-th linear system employs the data $C, A_1, \ldots, A_m$ and $c_j, a_{1j}, \ldots, a_{mj}$ and contains 1 equation, m inequalities, $n - 1$ nonnegative variables, and 2 free variables; see (5) below. Note that f does not depend on b. In contrast with the Laurent-Varvitsiotis bound, our bound depends both on the graph structure and the problem data itself. Also, while our bound $r^* \leq n - f + 1$ is not as strong as theirs in general, it can be stronger in specific cases as we will demonstrate. For example, we will reprove a result from [21], which also exploits conditions on the data to guarantee $r^* = 1$ for a sub-class of diagonal QCQPs. We will also provide an example showing that our analysis can be stronger than that of [21] in certain cases.

Second, in Section 3, we study the case of general, non-diagonal QCQPs by first reducing to the case of diagonal QCQPs. This is done by a standard introduction of auxiliarly variables that lifts (1) to a higher dimension in which the QCQP is diagonal.[2] Then, by applying the theory for diagonal QCQPs to this lifted QCQP, we obtain sufficient conditions for the SDP relaxation (2) of the original (1) to be exact with $r^* = 1$. These sufficient conditions involve only the eigenvalues of the matrices $C, A_1, \ldots, A_m$ and do not depend on the vectors $c, a_1, \ldots, a_m, b$.

1.2 Rank bounds under data randomness

Our proof techniques in Sections 2 and 3 reveal an interesting property of the bound $r^* \leq n - f + 1$ mentioned above, namely that it can often be improved by a simple perturbation of the data of (1). In this paper, in addition to examining data perturbation, we also consider how the rank bound $r^* \leq n - f + 1$ behaves under random-data models. Our interest in this subject arises from the fact that optimization algorithms have recently been applied to solve problems for which data are random, often because data themselves contain randomness in a big-data environment or are randomly sampled from large populations. Besides, many optimization software developers also test their solvers on randomly generated data.

It has been shown that data randomness typically makes algorithms run faster in the so-called *average behavior analysis*. The idea is to obtain rigorous probabilistic bounds on the number of iterations required by an algorithm to reach some termination criterion when the algorithm is applied to a random instance of a problem drawn from some probability distribution. In the case of the simplex method for LP, average-case analyses have provided some theoretical justification for the observed practical efficiency of the method, despite its exponential worst case bound; see for example [1, 7, 19, 24] and more recently [22].

In the case of interior-point algorithms for LP, a "high probability" bound of $O(\sqrt{n} \ln n)$ iterations for termination (independent of the data size) has been proved using a variety of algorithms applied to several different probabilistic models. Here, n is the dimension or number of variables in a standard form problem, and "high probability" means the probability of the bound holding goes to 1 as $n \to \infty$; see, e.g., [27, 2]. The paper [25] analyzed a condition number of the constraint matrix A of dimension $m \times n$ for an interior-point LP algorithm

[2] Interestingly, compared to [14], this provides a simple proof that every polynomial optimization problem has a corresponding SDP relaxation in which $r^* = 2$.

and showed that, if A is a standard Gaussian matrix, then the expected condition number equals $O(\min\{m \ln n, n\})$. Consequently, the algorithm terminates in strongly polynomial time in expectation.

On the other hand, specific recovery problems with random data/sampling have been proved to be exact via convex optimization approaches, which include digital communication [20], sensor-network localization [18], PhaseLift signal recovery [11], and max-likelihood angular synchronization [3]; see also the survey paper [13] and references therein. In these approaches, the recovery problems are relaxed to semidefinite programs (SDPs), where each randomly sampled measurement becomes a constraint in the relaxation. When the number of random constraints or measurements is sufficiently large—$O(n \ln n)$ relative to the dimension n of the variable matrix—then the relaxation contains the unique solution to be recovered. However, these problems can actually be solved by more efficient, deterministic, targeted sampling using only $O(n)$ measurements.

In Section 4, for general QCQPs, we give further evidence to show that a nonconvex optimization problem, for which the data are random and the variable dimension n is larger than the number of constraints m (in contrast to the recovery problems just mentioned), can be globally solved with high probability via convex optimization, specifically SDP. The proof is based on the ideas developed in Sections 2 and 3.

1.3 Assumptions and basic setup

We make the following assumptions throughout:

Assumption 1. *The feasible set of (1) is nonempty.*

Assumption 2. *There exists $y \in \mathbb{R}^m$ such that $y \le 0$ and $\sum_{i=1}^m y_i A_i \prec 0$.*

Assumption 2 can be equivalently stated with $y \ge 0$ and $\sum_{i=1}^m y_i A_i \succ 0$. However, this form will match the SDP dual (3) below. Assumptions 1–2 together imply that the feasible set of (1) is contained within a full-dimensional ellipsoid and hence (1) has an optimal solution. We also assume:

Assumption 3. *The interior feasible set of (2) is nonempty.*

The dual of (2) is

$$\begin{aligned} \max \quad & b^T y - \lambda \\ \text{s.t.} \quad & y \le 0, \; Z(\lambda, y) \succeq 0 \end{aligned} \tag{3}$$

where

$$Z(\lambda, y) := \begin{pmatrix} \lambda & s(y)^T \\ s(y) & S(y) \end{pmatrix}$$

and

$$s(y) := c - \sum_{i=1}^m y_i a_i, \qquad S(y) := C - \sum_{i=1}^m y_i A_i.$$

Assumption 2 implies that the feasible set of (3) has interior, which together with Assumption 3 ensures that strong duality holds between (2) and (3), i.e., there exist feasible $Y^* := Y(x^*, x^*)$ and $Z^* := Z(\lambda^*, y^*)$ such that $Y^* Z^* = 0$. In particular, we have $\operatorname{rank}(Y^*) + \operatorname{rank}(Z^*) \leq n + 1$.

2 Diagonal QCQPs

In this section, we assume (1) is a diagonal QCQP, i.e., the matrices C, A_i are diagonal. For any fixed index $1 \leq j \leq n$, consider the feasibility system

$$y \leq 0, \quad S(y) \succeq 0, \quad S(y)_{jj} = 0, \quad s(y)_j = 0. \tag{4}$$

Because $S(y)$ is diagonal, this is in fact a polyhedral system, which by Farkas' Lemma is feasible if and only if the polyhedral system

$$\begin{aligned} & C \bullet X + c_j x_j = -1 \\ & A_i \bullet X + a_{ij} x_j \leq 0 \ \ \forall\, i = 1, \ldots, m \\ & X \text{ diagonal}, \ X_{kk} \geq 0 \ \ \forall\, k \neq j \\ & X_{jj} \text{ free}, \ x_j \text{ free} \end{aligned} \tag{5}$$

is infeasible. It turns out that systems (4)–(5) are key to understanding the possible ranks of dual feasible $Z(\lambda, y)$. Define

$$f := |\{j : (4) \text{ is infeasible}\}| = |\{j : (5) \text{ is feasible}\}|.$$

We call f the *feasibility number* for (1), although it is important to note that f does not depend on the right-hand side b.

Lemma 1. *For any feasible $Z := Z(\lambda, y)$, it holds that* $\operatorname{rank}(Z) \geq f$.

Proof. Define $S := S(y)$ and $s := s(y)$. We first note that $\operatorname{rank}(Z) \geq \operatorname{rank}(S)$ since S is a principal submatrix of Z. If $\lambda = 0$, then $Z \succeq 0$ implies $s = 0$, which in turn implies that at least f entries of $\operatorname{diag}(S)$ are positive. Hence, $\operatorname{rank}(Z) \geq \operatorname{rank}(S) \geq f$. If $\lambda > 0$, then the Schur complement $S - \lambda^{-1} s s^T$ is positive semidefinite; in particular, $s_j = 0$ whenever $S_{jj} = 0$. Hence, the number of positive entries of $\operatorname{diag}(S)$ is at least f, and $\operatorname{rank}(Z) \geq \operatorname{rank}(S) \geq f$. □

Using Lemma 1, we prove our main result in this section, which bounds the rank of any optimal Y^* of (2).

Theorem 1. *Let $Y^* := Y(x^*, X^*)$ be any optimal solution of (2). It holds that $1 \leq \operatorname{rank}(Y^*) \leq n - f + 1$.*

Proof. As discussed above, $\operatorname{rank}(Y^*) + \operatorname{rank}(Z^*) \leq n + 1$, where $Z^* := Z(\lambda^*, y^*)$ is optimal for (3). Lemma 1 guarantees $\operatorname{rank}(Z^*) \geq f$, which implies $\operatorname{rank}(Y^*) \leq n - f + 1$. Also, since Y^* is nonzero due to its top-left entry, $\operatorname{rank}(Y^*) \geq 1$. □

As mentioned in the Introduction, the Laurent-Varvitsiotis rank bound is $r^* \leq 2$, while Theorem 1 ensures $r^* \leq n - f + 1$. Sections 2.2 and 2.3 below give classes of examples for which $n - f + 1 = 1 < 2$, i.e., $f = n$ and our bound is tighter than the Laurent-Varvitsiotis bound, but here we would briefly like to give an example for which our bound is worse. Consider the standard binary knapsack problem

$$\min \left\{ c^T x : a_1^T x \leq b_1, x \in \{0,1\}^n \right\},$$

where every $c_j > 0$ and every $a_{1j} > 0$. In this case, using the fact that $x_j \in \{0,1\}$ if and only if $x_j = x_j^2$, the j-th system (5) is

$$c_j x_j = -1, \quad a_{1j} x_j = 0, \quad X_{kk} = 0 \ \ \forall\, k \neq j, \quad X_{jj} - x_j = 0$$

which is clearly infeasible since $c_j x_j = -1$ implies $x_j \neq 0$, while $a_{1j} x_j = 0$ implies $x_j = 0$. Hence, in this example, $f = 0$, and our bound is $r^* \leq n + 1$.

An alternative approach to provide a sufficient condition is to check the feasibility system

$$y \leq 0, \quad S(y) \succeq 0, \quad S(y)_{jj} = s(y)_j = 0 \ \forall\, j \in J \tag{6}$$

for a fixed index set $J \subset \{1, ..., n\}$. Again, because $S(y)$ is diagonal, this is in fact a polyhedral system. Then we have

Corollary 1. *Let $Y^* := Y(x^*, X^*)$ be any optimal solution of (2). It holds that $1 \leq \operatorname{rank}(Y^*) \leq j^*$ where j^* is the smallest-cardinality such that all systems (6) with $|J| = j^*$ are infeasible.*

For example, suppose (6) is infeasible for all J of size 2. This means that $S(y)$ must have $n-1$ positive entries, in which case $\operatorname{rank}(Z(y)) \geq \operatorname{rank}(S(y)) \geq n-1$, in which case $\operatorname{rank}(Y^*) \leq n+1-(n-1) = 2$. This condition could be stronger than the bound given by Theorem 1 (since the quantities of multiple indexes need to be 0 at the same time), but it needs to solve a larger collection of linear programs.

2.1 The convex case and a perturbation

As a first application of Theorem 1, we reprove the classical result—for the case of diagonal QCQPs—mentioned in the Introduction that the minimum rank r^* equals 1 when (1) is a convex program. Of course, Proposition 1 holds even when C, A_i are general positive semidefinite matrices, not just diagonal ones, but the theory of this section only applies directly to the diagonal case. (Section 3 will generalize this result further.)

Proposition 1. *Suppose (1) satisfies $C \succeq 0$ and $A_i \succeq 0$ for all $i = 1, \ldots, m$. Then there exists an optimal solution $Y^* := Y(x^*, X^*)$ of (2) with* $\operatorname{rank}(Y^*) = 1$.

Proof. Let us first consider the subcase $C \succ 0$, i.e., $C_{jj} > 0$ for all j. Each of the n linear systems (5) has the form

$$\begin{aligned} C \bullet X + c_j x_j &= -1 \\ A_i \bullet X + a_{ij} x_j &\leq 0 \quad \forall\, i = 1, \ldots, m \end{aligned}$$

where X_{jj} and x_j are free, while the remaining variables in the diagonal X are nonnegative. By setting $x_j = 0$ and $X_{kk} = 0$ for all $k \neq j$, the system reduces to $C_{jj} X_{jj} = -1$ and $[A_i]_{jj} X_{jj} \leq 0$ for all i. Then taking $X_{jj} = -C_{jj}^{-1} < 0$ and using the fact that every $[A_i]_{jj} \geq 0$, we see that each system is feasible. It follows that $f = n$, and so $r^* = 1$ by Theorem 1.

Now consider the case when some $C_{jj} = 0$. Perturbing C to $C + D$, where D is a small, positive diagonal matrix, we can apply the previous paragraph to prove that the SDP relaxation of the perturbed problem has $r^* = 1$. Now, to complete the proof, we let $D \to 0$. Note that the perturbation only affects the objective, and hence we obtain a bounded sequence $\{Y^*\}$ of rank-1 matrices, each of which is feasible for (2) and optimal for its corresponding perturbed SDP. Thus, there exists a limit point $\bar{Y}$, which is optimal for (2) and has rank 1. This proves $r^* = 1$ as desired. □

The proof of Proposition 1 relies on a perturbation idea that we will use several times below. The basic insight is that the feasibility number f can increase under slight perturbations of the data of (1), which means that a nearby SDP relaxation might enjoy a smaller rank. By letting the perturbation go to 0, we can ensure that the original SDP contains at least one optimal solution with rank smaller than could otherwise be guaranteed by a direct application of Theorem 1.

2.2 Sign-Definite Linear Terms

We next reprove a result of [21], tailored to our diagonal case, that $r^* = 1$ when, for every j, the coefficients $c_j, a_{1j}, \ldots, a_{mj}$ are all nonnegative or all nonpositive. In such a case, the coefficients are said to be *sign-definite*.

Lemma 2. *Given $1 \le j \le n$, suppose $c_j \neq 0$ and $a_{1j}, \ldots, a_{mj}$ are sign-definite. Then (5) is feasible.*

Proof. Take $X = 0$ and $x_j = -c_j^{-1}$. Then the equation $C \bullet X + c_j x_j = -1$ is satisfied, and the inequalities $A_i \bullet X + a_{ij} x_j \le 0$ are satisfied becaues a_{ij} and x_j have opposite signs. □

Proposition 2 (see also [21])**.** *Suppose (1) has the property that, for all $j = 1, \ldots, n$, c_j and $a_{ij} = 0$ for all $i = 1, \ldots, m$ are sign-definite. Then there exists an optimal solution $Y^* := Y(x^*, X^*)$ of (2) with* $\operatorname{rank}(Y^*) = 1$.

Proof. We consider two subcases. First, when $c_j \neq 0$ for all $j = 1, \ldots, n$, by Lemma 2 and Theorem 1, we have $\operatorname{rank}(Y^*) = 1$. When some $c_j = 0$, choose a fixed $d \in \mathbb{R}^n$ such that $d_j \neq 0$ for all j with $c_j = 0$, $d_j = 0$ otherwise, and the sign-definite property is maintained. Also choose $\epsilon > 0$ and perturb c to $c + \epsilon d$, which in particular does not change the feasible set of (1). By the previous case, the perturbed SDP relaxation has a rank-1 optimal solution. By letting $\epsilon \to 0$ and using an argument similar to the proof of Proposition 1, we conclude that the unperturbed (2) also has a rank-1 optimal solution. □

The diagonal assumption in Proposition 2 is necessary because Burer and Anstreicher [9] provide an example in which $m = 2$, C is non-diagonal, A_1, A_2 are diagonal, $c \neq 0$, $a_1 = a_2 = 0$, and the Shor relaxation is not tight; in particular, it has no optimal solution with rank 1. On the other hand, the diagonal assumption can at least be relaxed when $m = 2$ for the purely homogeneous case: Ye and Zhang [29] showed that, if $m = 2$ with C, A_1, A_2 arbitrary and $c = a_1 = a_2 = 0$, then the corresponding Shor relaxation has a rank-1 optimal solution.

An interesting application of Proposition 2 occurs for the feasible set

$$\{x \ : \ \|x\|_2 \le r_1, \ \|x\|_\infty \le r_2\} = \left\{x \ : \ x^T x \le r_1^2, \ x_j^2 \le r_2^2 \ \forall \ j\right\} \tag{7}$$

which is the intersection of concentric 2-norm and ∞-norm balls. It is well known that, for only the 2-norm ball $\{x : x^T x \le r_1^2\}$, problem (1) is equivalent to the trust-region subproblem, which can be solved in polynomial time. On the other hand, for only the ∞-norm ball $\{x : x_j^2 \le r_2^2 \ \forall\ j\}$, problem (1) is clearly separable and hence solvable in polynomial time. Proposition 2 shows that (1) over the intersection (7) can also be solved in polynomial-time.

According to theorem 2 of [26], the fact that (2) solves (1) when the sign-definiteness property holds also allows us to relate the feasible set of (2) to the closed convex hull

$$\mathcal{K} := \overline{\operatorname{conv}}\,\{(x, x \circ x) : x \text{ feasible for } (1)\}$$

where $\circ$ denotes the Hadamard, i.e., component-wise, product of vectors. Such convex hulls are important for studying QCQPs in general. Specifically, we know that

$$\mathcal{K} = \{(x, \operatorname{diag}(X)) : (x, X) \text{ is feasible for } (2)\}$$

when sign-definiteness holds. In this sense, problem (1) is a "hidden convex" problem in this case.

2.3 Arbitrary C and each $A_i \in \{\pm I, 0\}$

As mentioned above, Proposition 2 of the previous subsection was first proved in [21], and it involves only conditions on the data c_j and a_{ij} of the linear terms in (1). In fact, the authors of [21] provide a broader theory, one that studies more general nonzero structures—not just diagonal—but one that only considers data corresponding to off-diagonal terms X_{ij} in the SDP relaxations. In particular, they do not consider data such as C_{jj} and $[A_i]_{jj}$. This is indeed a key difference of our theory compared to theirs, i.e., our feasibility number f takes into account the data matrices C, A_i. We now give an example to illustrate this point further—an example in which the sign-definiteness assumption in Proposition 2 can be relaxed when C, A_i are taken into account.

As discussed in the Introduction, the assumption that the matrices $C, A_1, \ldots, A_m$ are diagonal is equivalent (after a linear transformation) to the matrices pairwise commuting. When C is arbitrary and $A_i \in \{\pm I, 0\}$ for all i, this assumption is clearly satisfied. Geometrically, the feasible set is then an intersection of balls, complements of balls, and half-spaces. Although this problem is strongly NP-hard in general, Bienstock and Michalka [6] show that it can be solved in polynomial-time, for example, when the number of ball constraints is fixed. More recently, Beck and Pan [5] study precisely this special case of (1) and develop a branch-and-bound algorithm for its global optimization; [5] also contains a detailed literature review of this problem.

Assume that the data has already been transformed so that C is diagonal and, without loss of generality, $C_{11} \geq \cdots \geq C_{nn}$. In particular, the diagonal of C contains the eigenvalues of the original C. In addition, let us consider the sub-case in which c_n and a_{in} for all i are sign-definite. (This is a weaker condition than the sign-definiteness of Proposition 2.) By Theorem 1, the rank of an optimal solution Y^* of the corresponding Shor relaxation (2) is bounded above by $n - f + 1$, where f is the feasibility number associated with the systems

$$\begin{aligned} C \bullet X + c_j x_j &= -1 \\ \pm I \bullet X + a_{ij} x_j &\leq 0 \quad \forall\, i \text{ with } A_i = \pm I \\ a_{ij} x_j &\leq 0 \quad \forall\, i \text{ with } A_i = 0 \end{aligned}$$

where X_{jj} and x_j are free, while the remaining variables X_{kk} are nonnegative. Our next proposition shows that, in essence, f equals n, so that $r^* = 1$.

Proposition 3. *If C is diagonal with $C_{11} \geq \cdots \geq C_{nn}$, $A_i \in \{\pm I, 0\}$ for all $i = 1, \ldots, m$, and $c_n, a_{1n}, \ldots, a_{mn}$ sign-definite, then $r^* = 1$.*

Proof. We first examine the sub-case when $C_{(n-1)(n-1)} > C_{nn}$ and $c_n \neq 0$. For $j = 1, \ldots, n-1$, consider the system described above the statement of the proposition. Fixing $x_j = 0$, it reduces to the system $C \bullet X = -1$, $\pm I \bullet X = 0$, $\pm I \bullet X \leq 0$. Next substituting $X_{jj} = -\sum_{k \neq j} X_{kk}$, the system further simplifies to

$$\sum_{k \neq j} (C_{kk} - C_{jj}) X_{kk} = -1.$$

We may then take $X_{nn} = (C_{jj} - C_{nn})^{-1}$ and all other $X_{kk} = 0$, showing that the system is feasible. Now consider the system for $j = n$ above. Substituting $X_{nn} = -\sum_{k=1}^{n-1} X_{kk}$, the system reduces to $c_n x_n = -1$ and $a_{in} x_n \leq 0$ for all i. Because c_n and a_{in} are sign-definite, this system is solvable. It follows that $f = n$ when $C_{(n-1)(n-1)} > C_{nn}$ and $c_n \neq 0$.

Finally, if $C_{(n-1)(n-1)} = C_{nn}$ or $c_n = 0$, then we may make an arbitrarily small perturbation of the objective such that the previous paragraph applies. As in the proof of Proposition 1, the perturbation can be removed, thus establishing $r^* = 1$. □

3 General QCQPs

We now turn our attention to the case of general, non-diagonal QCQPs, keeping in mind that Assumptions 1–3 still apply. In particular, the feasible set of (1) is bounded, i.e., it exists in a ball defined by $x^T x \le r^2$ for some radius r. Note that for the following development, r does not need to be known explicitly.

We do assume, for simplicity and without loss of generality, that C is diagonal, and we let $A_i = Q_i D_i Q_i^T$ denote the spectral decomposition of A_i, where Q_i is an orthogonal matrix. Next, we introduce auxiliary variables $y_i = Q_i^T x \in \mathbb{R}^n$, rewriting (1) as

$$\begin{aligned} \min \quad & x^T C x + 2c^T x \\ \text{s. t.} \quad & y_i^T D_i y_i + 2a_i^T x \le b_i, \; y_i = Q_i^T x \\ & x^T x + \sum_i y_i^T y_i \le (m+1) r^2 \end{aligned} \tag{8}$$

where the last constraint is technically redundant but has been added so that (8) satisfies Assumptions 1–3 on its own. In particular, the feasible set of (8) is bounded.

The lifted problem (8) is a diagonal QCQP, and so we can apply the theory of Section 2. In particular, we would like to determine conditions under which the feasibility number for (8) equals its total number of variables, which is $n(m+1)$. We provide just such a condition in the following theorem, and for this, we introduce the following linear system:

$$\begin{aligned} & C \bullet X = -1 \\ & D_i \bullet Y_i \le 0 \;\forall\; i = 1, \dots, m \\ & I \bullet X + \textstyle\sum_{i=1}^m I \bullet Y_i \le 0 \\ & X, Y_i \text{ diagonal.} \end{aligned} \tag{9}$$

Theorem 2. *Let z represent any single variable X_{jj} or $[Y_i]_{jj}$ in (9). Constrain (9) further by forcing all variables other than z to be nonnegative, while keeping z free. If all such $n(m+1)$ systems corresponding to every possible choice of z are feasible, then $r^* = 1$ for both (8) and (1).*

Proof. The $n(m+1)$ systems described constitute the systems (5) tailored to (8), reduced further by setting the "linear part" x_j in (5) to 0. So we conclude that $r^* = 1$ for (8) by applying Theorem 1.

The result $r^* = 1$ also holds for (1) because the SDP relaxation (2) for (1) is at least as strong as the corresponding relaxation for (8), which we have just proven is exact. □

As we have seen in Section 2, perturbation can be a useful tool for broadening the application of the ideas of Theorem 2. For example, a reasonable perturbation might be to replace the objective of $x^T Cx + 2c^T x$ of (8) with $x^T Cx + 2c^T x + \epsilon \sum_{i=1}^m y_i^T y_i$, where $\epsilon > 0$ is small, resulting in the analog

$$C \bullet X + \epsilon \sum_{i=1}^{m} I \bullet Y = -1, \quad D_i \bullet Y_i \leq 0, \quad I \bullet X + \sum_{i=1}^{m} I \bullet Y_i \leq 0 \tag{10}$$

of (9). Note that the perturbation is consistent with the need to satisfy the inequality $I \bullet X + \sum_{i=1}^m I \bullet Y_i \leq 0$ The following proposition, which proves the general convex case of of (1), is an example of this perturbation.

Proposition 4. *Suppose (1) satisfies $C \succeq 0$ and $A_i \succeq 0$ for all $i = 1, \ldots, m$. Then there exists an optimal solution $Y^* := Y(x^*, X^*)$ of (2) with* $\operatorname{rank}(Y^*) = 1$.

Proof. First assume $C \succ 0$. Using the suggested perturbation, we need to show all such systems (10) are feasible. For the system with X_{jj} free, set all other variables to 0 so that (10) reduces to $C_{jj} X_{jj} = -1$ and $X_{jj} \leq 0$, which is solvable because $C_{jj} > 0$. On the other hand, for the systems with $[Y_i]_{jj}$ free, set all other variables to 0. Then (10) becomes $\epsilon [Y_i]_{jj} = -1$, $[D_i]_{jj}[Y_i]_{jj} \leq 0$, and $[Y_i]_{jj} \leq 0$, which is also solvable because $[D_i]_{jj} \geq 0$. Hence the feasibility number is $n(m+1)$ as desired. The case $C \succeq 0$ is just a limiting case of $C \succ 0$. □

4 Random General QCQPs

In this section, we study the behavior of r^* for (1) under the assumption that C is positive semidefinite and the A_i are generated randomly. The analysis does not depend on c, a_i, or b, although these data are required for satisfying Assumptions 1–3. Our result is as follows:

Theorem 3. *Regarding the general QCQP (1), suppose that C is positive semidefinite, and for each $i = 1, \ldots, m$, A_i is generated randomly with eigenvalues independently following the standard Gaussian distribution. In particular, each A_i is indefinite with probablity 1. Suppose also that $c, a_1, \ldots, a_m$, and b are chosen independently such that Assumptions 1 and 3 are satisfied. Finally, for any finite $M > 0$, add the constraint $x^T x \leq M$ to ensure that Assumption 2 is satisfied, while not violating Assumptions 1 and 3. Then, for fixed m,* Prob$(r^* = 1) \to 1$ *as $n \to \infty$.*

Proof. We analyze the situation when $C \succ 0$, as $C \succeq 0$ is just a limiting case. Without loss of generality, after a change of variables, we may assume that C is diagonal. We apply the techniques of Section 3 with one adjustment. Let $J_1, \ldots, J_m \in \mathbb{S}^n$ be random diagonal matrices with all diagonal entries independently and uniformly distributed in $[0,1]$. Then, in analogy with (8), our randomly generated problem (1) with the added constraint $x^T x \le M$ is equivalent to the lifted QCQP

$$\begin{aligned} \min \quad & x^T C x + 2c^T x \\ \text{s.t.} \quad & y_i^T D_i y_i + 2a_i^T x \le b_i, \;\; y_i = Q_i^T x \\ & x^T x \le M \\ & \sum_i y_i^T J_i y_i \le M' \end{aligned} \tag{11}$$

where M' is an arbitrarily large positive number. We claim that $r^* = 1$ for (11) with high probability, and since the SDP relaxation for (11) is at least as tight as (2) for (1), this will prove $r^* = 1$ for (1) with high probability as desired.

To prove the claim, we analyze a perturbation of (11). For each i, let $B_i \in \mathbb{S}^n$ be a diagonal matrix with diagonal entries independently following the standard Gaussian distribution, and for $\epsilon > 0$ small, consider the perturbed problem

$$\begin{aligned} \min \quad & x^T C x + 2c^T x + \epsilon \sum_i y_i^T B_i y_i \\ \text{s.t.} \quad & y_i^T D_i y_i + 2a_i^T x \le b_i, \;\; y_i = Q_i^T x \\ & x^T x \le M \\ & \sum_i y_i^T J_i y_i \le M'. \end{aligned} \tag{12}$$

In analogy with systems (9) and (10) in Section 3, we need to analyze the feasibility of systems of the form

$$\begin{aligned} & C \bullet X + \epsilon \textstyle\sum_i B_i \bullet Y_i = -1 \\ & D_i \bullet Y_i \le 0 \;\forall\, i \\ & I \bullet X \le 0 \\ & \textstyle\sum_i J_i \bullet Y_i \le 0 \end{aligned} \tag{13}$$

where all matrices X, Y_i are diagonal and all variables are nonnegative except for one, which is free.

First consider the case of (13) when X_{jj} is free. Set all $Y_i = 0$ so that the system reduces to $C \bullet X = -1$ and $I \bullet X \le 0$, which is feasible since $C_{jj} > 0$ by assumption.

Second, consider the case of (13) when $[Y_k]_{jj}$ is free. Set $X = 0$ and all other $Y_i = 0$ so that the system reduces to $\epsilon B_k \bullet Y_k = -1$, $D_k \bullet Y_k \leq 0$, and $J_k \bullet Y_k \leq 0$, which is certainly feasible if the following equality system is feasible:

$$\begin{aligned} B_k \bullet Y_k &= -1 \\ D_k \bullet Y_k &= 0 \\ J_k \bullet Y_k &= 0. \end{aligned} \tag{14}$$

Note that (14) does not depend on ϵ. The basis size for (14) is 3, and due to the random nature of the data, every 3×3 basis matrix is invertible. Also, because $[Y_k]_{jj}$ is free while all other variables in Y_k are nonnegative, the system (14) has $\binom{n-1}{2}$ bases, and hence, $\binom{n-1}{2}$ basic solutions. One can verify—using Monte Carlo simulation, for example—that the probability of a given basic solution being feasible is 1/6. Hence, due to independence, the probability that (14) is feasible, i.e., there exists at least one basic feasible solution, is

$$\theta := 1 - \left(\frac{5}{6}\right)^{\binom{n-1}{2}}.$$

Thus, θ is also a lower bound on the probability of the feasibility of system (13) when $[Y_k]_{jj}$ is free. To ensure $r^* = 1$ for (12), we need that all such systems (13) are feasible. Again exploiting indepence, this occurs with probability at least θ^{nm}, which tends to 1 as $n \to \infty$ with m fixed.

Finally, since the above probability analysis does not depend on ϵ, we may take $\epsilon \to 0$ so that the probability analysis applies as well to problem (11), which proves the claim, i.e., that $r^* = 1$ for (11) with high probability. $\square$

Although Theorem 3 assumes that $C \succeq 0$, we conjecture that is true even when C is generated randomly in the same manner as the A_i matrices. The proof seems to break down for analyzing the relevant feasibility system for X_{jj}, which corresponds to the smallest eigenvalue C_{jj} of C, when that C_{jj} is negative. A possible work-around could be to put the objective $x^T C x + 2c^T x$ into the constraints using an auxiliary variable t via the constraint $x^T C x + 2c^T x \leq t$ and then to minimize t. However, t would need to be bounded in accordance with Assumptions 1–3 before applying the theory we have developed.

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