How to Convexify the Intersection of a Second Order Cone and a Nonconvex Quadratic

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Abstract

A recent series of papers has examined the extension of disjunctive-programming techniques to mixed-integer second-order-cone programming. For example, it has been shown—by several authors using different techniques—that the convex hull of the intersection of an ellipsoid, $\mathcal{E}$, and a split disjunction, $(l - x_j)(x_j - u) \le 0$ with $l < u$, equals the intersection of $\mathcal{E}$ with an additional second-order-cone representable (SOCr) set. In this paper, we study more general intersections of the form $\mathcal{K} \cap \mathcal{Q}$ and $\mathcal{K} \cap \mathcal{Q} \cap H$, where $\mathcal{K}$ is a SOCr cone, $\mathcal{Q}$ is a nonconvex cone defined by a single homogeneous quadratic, and H is an affine hyperplane. Under several easy-to-verify assumptions, we derive a simple, computable convex relaxations $\mathcal{K} \cap \mathcal{S}$ and $\mathcal{K} \cap \mathcal{S} \cap H$, where $\mathcal{S}$ is a SOCr cone. Under further assumptions, we prove that these two sets capture precisely the corresponding conic/convex hulls. Our approach unifies and extends previous results, and we illustrate its applicability and generality with many examples.

Keywords: convex hull, disjunctive programming, mixed-integer linear programming, mixed-integer nonlinear programming, mixed-integer quadratic programming, nonconvex quadratic programming, second-order-cone programming, trust-region subproblem.

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1 Introduction

In this paper, we study nonconvex intersections of the form $\mathcal{K} \cap \mathcal{Q}$ and $\mathcal{K} \cap \mathcal{Q} \cap H$, where the cone $\mathcal{K}$ is second-order-cone representable (SOCr), $\mathcal{Q}$ is a nonconvex cone defined by a single homogeneous quadratic, and H is an affine hyperplane. Our goal is to develop tight convex

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relaxations of these sets and to characterize the conic/convex hulls whenever possible. We are motivated by recent research on Mixed Integer Conic Programs (MICPs), though our results here enjoy wider applicability.

Prior to the study of MICPs in recent years, cutting plane theory has been fundamental in the development of efficient and powerful solvers for Mixed Integer Linear Programs (MILPs). In this theory, one considers a convex relaxation of the problem, e.g., its continuous relaxation, and then enforces integrality restrictions to eliminate regions containing no integer feasible points—so-called *lattice-free sets*. A valid two-term linear disjunction, say $x_j \leq l \vee x_j \geq u$, is a simple form of a lattice free set. The additional inequalities required to describe the convex hull of such a disjunction are known as *disjunctive cuts*. Such a disjunctive point of view was introduced by Balas [5] in the context of MILPs, and it has since been studied extensively in mixed integer linear and nonlinear optimization [6, 7, 16, 17, 20, 28, 38, 39], complementarity [25, 26, 41, 35] and other non-convex optimization problems [9, 16]. In the case of MILPs, several well-known classes of cuts such as *Chvátal-Gomory*, *lift-and-project*, *mixed-integer rounding (MIR)*, *split*, and *intersection cuts* are known to be special types of disjunctive cuts. Stubbs and Mehrotra [40] extended cutting plane theory from MILP to convex mixed integer linear problems. This work was followed by several papers [14, 22, 23, 28, 43] that investigated linear-outer-approximation based approaches, as well as others that extended specific classes of inequalities, such as Chvatal-Gomory cuts [18] for MICPs, and MIR cuts [4] for SOC-based MICPs.

Recently there has been growing interest in developing closed-form expressions for convex inequalities that fully describe the convex hull of a disjunctive set involving a SOC. This line of work has been initiated by Dadush et al. [21], who derived cuts for ellipsoids based on (parallel) split disjunctions. Modaresi et al. [32] extended this work by studying split disjunctions under the name of *intersection cuts* for SOC and all of its cross-sections (i.e., all conic sections), as well as a number of other sets involving the SOC. A theoretical and computational comparison of intersection cuts from [32] with extended formulations and conic MIR inequalities from [4] is given in [33]. Taking a different approach, Andersen and Jensen [1] derived a SOC constraint describing the convex hull of a split disjunction applied to a SOC. Belotti et al. [10] studied the families of quadratic surfaces having fixed intersections with two given hyperplanes, and in [11], they identified a procedure for constructing two-term disjunctive cuts, when the sets defined by the disjunctions are bounded and disjoint, or when the disjunctions are parallel. Kılınç-Karzan [29] examined minimal valid linear inequalities for general conic sets with a disjunctive structure, showed that these are sufficient to describe the closed convex hull, and analyzed their properties. In the case of two-term disjunctions on regular (closed, convex, pointed with nonempty interior) cones, Kılınç-Karzan and Yıldız

[30] studied the structure of undominated valid linear inequalities by refining the minimal inequality definition of [29], and for the particular case of SOC they derived a class of convex valid inequalities that is sufficient to describe the convex hull by following a conic duality perspective. Conditions under which these inequalities are SOCr, as well as when a single inequality from this class is sufficient, were also established in [30]. Bienstock and Michalka [13] studied the characterization and separation of valid linear inequalities that convexify the epigraph of a convex, differentiable function restricted to a non-convex domain given by a quadratic. Although all of these authors take slightly different approaches, their results are comparable, for example, in the case of analyzing split disjunctions of the SOC. We remark also that these methods convexify in the space of the original variables, i.e., they do not involve lifting. For additional convexification approaches for nonconvex quadratic programming, which convexify in the lifted space of products $x_i x_j$ of variables, we refer the reader to [3, 8, 15, 16, 42], for example.

In this paper, our main contributions can be summarized as follows (see Section 3 and Theorem 1 in particular). First, we derive a simple, computable convex relaxation $\mathcal{K} \cap \mathcal{S}$ of $\mathcal{K} \cap \mathcal{Q}$, where $\mathcal{S}$ is an additional SOCr cone. This also provides the convex relaxation $\mathcal{K} \cap \mathcal{S} \cap H \supseteq \mathcal{K} \cap \mathcal{Q} \cap H$. The derivation relies on several easy-to-verify assumptions. Second, we identify stronger assumptions guaranteeing moreover that $\mathcal{K} \cap \mathcal{S} = \mathrm{cl.\,conic.\,hull}(\mathcal{K} \cap \mathcal{Q})$ and $\mathcal{K} \cap \mathcal{S} \cap H = \mathrm{cl.\,conv.\,hull}(\mathcal{K} \cap \mathcal{Q} \cap H)$, where *cl* indicates the closure, *conic.hull* indicates the conic hull, and *conv.hull* indicates the convex hull. Our approach unifies and extends previous results, and we illustrate its applicability and generality with many examples.

Our approach can be seen as a variation of the following basic, yet general, idea of conic aggregation to generate valid inequalities. Suppose that $f_0 = f_0(x)$ is convex, while $f_1 = f_1(x)$ is nonconvex, and suppose we are interested in the closed convex hull of the set $S := \{x : f_0 \leq 0, f_1 \leq 0\}$. For any $0 \leq t \leq 1$, the inequality $f_t := (1-t)f + tf_1 \leq 0$ is valid for S, but f_t is generally nonconvex. Hence, it is natural to seek values of t such that the function f_t is convex for all x. One might even conjecture that some particular convex f_s with $0 \leq s \leq 1$ guarantees $\mathrm{cl.\,conv.\,hull}(S) = \{x : f_0 \leq 0, f_s \leq 0\}$. However, it is known that this approach cannot generally achieve the convex hull even when f_0, f_1 are quadratic functions; see [32].

In this paper, we follow a similar approach in spirit, but instead of determining $0 \leq t \leq 1$ guaranteeing the convexity of f_t for all x, we only require convexity on $\{x : f_0 \leq 0\}$. This weakened requirement is crucial. In particular, it allows us to obtain convex hulls for many cases where $\{x : f_0 \leq 0\}$ is SOCr and f_1 is a nonconvex quadratic, and we are able to replicate all of the known results cited above in this domain (see Section 6). As a practical and technical matter, instead of working directly with convex functions in this paper, we work in the equivalent realm of convex sets, in particular SOCr cones. Section 2 discusses in detail the features of SOCr cones required for our analysis.

Compared to the earlier literature on MICPs, our work here is broader in that we study a general nonconvex cone $\mathcal{Q}$. In particular, $\mathcal{Q}$ allows much more variety than the cases studied in [1, 11, 13, 30, 32]. For example, beyond just splits, we can handle general two-term linear disjunctions on the SOC, and while [11] also studies more than splits under certain assumptions of disjointness and boundedness, our assumptions here are much weaker. While [30] derives cuts and convex hulls for two-term disjunctions on the SOC in even greater generality than us, their results only apply to the SOC. On the other hand, we handle the SOC, all of its cross-sections, and even more general $\mathcal{Q}$ in a unified framework. Bienstock and Michalka [13] also consider more general $\mathcal{Q}$, but their approach is quite different than ours. Whereas [13] relies on polynomial time procedures for separating and tilting valid linear inequalities, we directly give the convex hull description. Our approach can, for example, characterize: the convex hull of the deletion of an arbitrary ball from another ball; and the convex hull of the deletion of an arbitrary ellipsoid from another ellipsoid sharing the same center. In addition, we can use our results to solve the classical trust region subproblem [19] using SOC optimization, whereas previous algorithms rely on specialized nonlinear algorithms [24, 34] or semidefinite programming [37]. Section 7 discusses these examples.

The paper is structured as follows. Section 2 discusses the details of SOCr cones, and Section 3 states our assumptions and main theorem. Section 4 then provides several low-dimensional examples with figures. A reader desiring just the main ideas of the paper could safely stop after Section 4. In Section 5, we prove the main theorem, and then in Sections 6 and 7 we discuss and prove many interesting general examples covered by our theory. Section 8 concludes the paper with a few final remarks. Our notation is mostly standard. We will define any particular notation upon its first use.

2 Second-Order-Cone Representable Sets

Our analysis in this paper is based on the concept of SOCr (second-order-cone representable) cones. In this section, we define and introduce the basic properties of such sets.

A cone $\mathcal{F}^+ \subseteq \Re^n$ is said to be *second-order-cone representable* (or *SOCr*) if there exists a matrix $B \in \Re^{n\times(n-1)}$ and a vector $b \in \Re^n$ such that the nonzero columns of B are linearly independent, $b \notin \mathrm{Range}(B)$, and

$$\mathcal{F}^+ = \{x : \|B^T x\| \le b^T x\}, \tag{1}$$

where $\|\cdot\|$ denotes the usual Euclidean norm. The negative of $\mathcal{F}^+$ is also SOCr:

$$\mathcal{F}^- := -\mathcal{F}^+ = \{x : \|B^T x\| \leq -b^T x\}. \tag{2}$$

Defining $A := BB^T - bb^T$, the union $\mathcal{F}^+ \cup \mathcal{F}^-$ corresponds to the homogeneous quadratic inequality $x^T A x \leq 0$:

$$\mathcal{F} := \mathcal{F}^+ \cup \mathcal{F}^- = \{x : \|B^T x\|^2 \leq (b^T x)^2\} = \{x : x^T A x \leq 0\}. \tag{3}$$

We also define

$$\begin{aligned} \operatorname{int}(\mathcal{F}^+) &:= \{x : \|B^T x\| < b^T x\} \\ \operatorname{bd}(\mathcal{F}^+) &:= \{x : \|B^T x\| = b^T x\} \\ \operatorname{apex}(\mathcal{F}^+) &:= \{x : B^T x = 0, b^T x = 0\}. \end{aligned}$$

The following proposition establishes some important features of SOCr cones:

Proposition 1. *Let $\mathcal{F}^+$ be SOCr as in (1), and define $A := BB^T - bb^T$. Then* $\operatorname{apex}(\mathcal{F}^+) = \operatorname{null}(A)$*, and A has at most one negative eigenvalue. If A has exactly one negative eigenvalue, then* $\operatorname{int}(\mathcal{F}^+) \neq \emptyset$.

Proof. For any x, we have the equation

$$Ax = (BB^T - bb^T)x = B(B^T x) - b(b^T x) \tag{4}$$

So $x \in \operatorname{apex}(\mathcal{F}^+)$ implies $x \in \operatorname{null}(A)$. The converse also holds by (4) because, by definition, the nonzero columns of B are independent and $b \notin \operatorname{Range}(B)$. Hence, $\operatorname{apex}(\mathcal{F}^+) = \operatorname{null}(A)$.

The equation $A = BB^T - bb^T$, with $BB^T \succeq 0$ and rank-1 $bb^T \succeq 0$, implies that A has at most one negative eigenvalue. If A has exactly one negative eigenvalue with associated negative eigenvector $\bar{x}$, then $\bar{x}^T A \bar{x} < 0$, and so $\operatorname{int}(\mathcal{F}^+)$ contains either $\bar{x}$ or $-\bar{x}$. $\square$

We define analogous sets $\operatorname{int}(\mathcal{F}^-)$, $\operatorname{bd}(\mathcal{F}^-)$, and $\operatorname{apex}(\mathcal{F}^-)$ for $\mathcal{F}^-$. In addition:

$$\begin{aligned} \operatorname{int}(\mathcal{F}) &:= \{x : x^T A x < 0\} = \operatorname{int}(\mathcal{F}^+) \cup \operatorname{int}(\mathcal{F}^-) \\ \operatorname{bd}(\mathcal{F}) &:= \{x : x^T A x = 0\} = \operatorname{bd}(\mathcal{F}^+) \cup \operatorname{bd}(\mathcal{F}^-). \end{aligned}$$

Similarly, we have $\operatorname{apex}(\mathcal{F}^-) = \operatorname{null}(A) = \operatorname{apex}(\mathcal{F}^+)$, and if A has exactly one negative eigenvalue, then $\operatorname{int}(\mathcal{F}^-) \neq \emptyset$ and $\operatorname{int}(\mathcal{F}) \neq \emptyset$.

When considered as a pair of sets $\{\mathcal{F}^+, \mathcal{F}^-\}$, it is possible that another choice $(\bar{B}, \bar{b})$ in place of (B, b) leads to the same pair and hence to the same $\mathcal{F}$. For example, $(\bar{B}, \bar{b}) = (-B, -b)$ simply switches the roles of $\mathcal{F}^+$ and $\mathcal{F}^-$, but $\mathcal{F}$ does not change. However, if A has a single negative eigenvalue, then the next proposition shows that any alternative $(\bar{B}, \bar{b})$ yields $A = \rho(\bar{B}\bar{B} - \bar{b}\bar{b}^T)$ for some $\rho > 0$, i.e., A is essentially invariant with respect to its (B, b) representation.

Proposition 2. *Let $\{\mathcal{F}^+, \mathcal{F}^-\}$ be SOCr sets as in (1) and (2), and define $A := BB^T - bb^T$. Suppose A has a single negative eigenvalue, and let $(\bar{B}, \bar{b})$ be another choice in place of (B, b) leading to the same pair $\{\mathcal{F}^+, \mathcal{F}^-\}$. Then $A = \rho(\bar{B}\bar{B} - \bar{b}\bar{b}^T)$ for some $\rho > 0$.*

Proof. Define $\bar{A} := \bar{B}\bar{B}^T - \bar{b}\bar{b}^T$. We claim that $\bar{A} = \rho A$ for some scalar $\rho > 0$.

Since $\{x : x^T A x \le 0\} = \mathcal{F} = \{x : x^T \bar{A} x \le 0\}$ by assumption, we have $-x^T A x \ge 0 \Rightarrow -x^T \bar{A} x \ge 0$, and because $\mathcal{F}$ is strictly feasible, the S-lemma [12] implies the existence of $\lambda \ge 0$ such that $-\bar{A} \succeq -\lambda A$, i.e., $\lambda A \succeq \bar{A}$. By symmetry, $\beta \bar{A} \succeq A$ for some $\beta \ge 0$; in fact, $\beta > 0$ since otherwise $0 \succeq A$. So $\lambda A \succeq \bar{A} \succeq \beta^{-1} A$. We can also see $\lambda > 0$. Therefore, we conclude that for any x, $\operatorname{sign}(x^T A x) = \operatorname{sign}(x^T \bar{A} x)$, where sign is 1 for positive inputs, 0 for zero inputs, and -1 for negative inputs.

Without loss of generality by diagonal and symmetric orthogonal scalings, let us assume that $A = \operatorname{Diag}(1, \ldots, 1, 0, \ldots, 0, -1)$, and let e_i denote the i-th standard coordinate vector. By taking $x = e_i$, we see $\operatorname{sign}(A_{ii}) = \operatorname{sign}(\bar{A}_{ii})$ for all i. So the sign pattern of $\operatorname{diag}(A)$ equals that of $\operatorname{diag}(\bar{A})$.

Next let $i < n$ be such that $A_{ii} = 1$. By considering $x = e_i + e_n$, we get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{nn} + 2\bar{A}_{in}$. Likewise, by considering $x = -e_i + e_n$, we get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{nn} - 2\bar{A}_{in}$. These together imply that $\bar{A}_{ii} = -\bar{A}_{nn}$ and $\bar{A}_{in} = 0$. In particular, every nonzero diagonal element of $\bar{A}$ has the same magnitude.

For any distinct $i, j < n$ satsifying $A_{ii} = A_{jj} = 1$, taking $x = e_i + e_j + \sqrt{2}\, e_n$, we get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{jj} + 2\bar{A}_{nn} + 2\bar{A}_{ij} + 2\sqrt{2}\,\bar{A}_{in} + 2\sqrt{2}\,\bar{A}_{jn} = 2\bar{A}_{ij}$, which implies $\bar{A}_{ij} = 0$. Similarly, for any distinct $i, j < n$ with $A_{ii} = A_{jj} = 0$, by considering $x = e_i + e_j$, we get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{jj} + 2\bar{A}_{ij} = 2\bar{A}_{ij}$, which implies $\bar{A}_{ij} = 0$. Finally, for any distinct $i, j < n$ with $A_{ii} = 1$ and $A_{jj} = 0$, we take $x = e_i + e_j + e_n$ and get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{jj} + \bar{A}_{nn} + 2\bar{A}_{ij} + 2\bar{A}_{in} + 2\bar{A}_{jn} = 2\bar{A}_{ij} + 2\bar{A}_{jn}$. Similarly, for $x = -e_i + e_j + e_n$, we get $\operatorname{sign}(x^T A x) = 0 = \bar{A}_{ii} + \bar{A}_{jj} + \bar{A}_{nn} - 2\bar{A}_{ij} - 2\bar{A}_{in} + 2\bar{A}_{jn} = -2\bar{A}_{ij} + 2\bar{A}_{jn}$. Thus, $\bar{A}_{jn} = \bar{A}_{ij} = 0$ as well.

In total, the preceding paragraphs prove that the sign pattern of $\operatorname{diag}(A)$ equals that of $\operatorname{diag}(\bar{A})$, every nonzero diagonal element of $\bar{A}$ has the same magnitude, $\bar{A}_{in} = 0$ for all $i < n$ and $\bar{A}_{ij} = 0$, for all distinct $i, j < n$. It follows that $\bar{A} = \rho A$ for some scalar ρ. Proposition

1 also ensures that there exists $\bar{x} \in \operatorname{int}(\mathcal{F})$ such that $\bar{x}^T A \bar{x} < 0$. Since $\mathcal{F}$ does not change based on $(\bar{B}, \bar{b})$, we know $\bar{x}^T \bar{A} \bar{x} < 0$ also, which ensures $\rho > 0$. □

We can reverse the discussion thus far to start from a symmetric matrix A with a single negative eigenvalue and define associated SOCr cones $\mathcal{F}^+$ and $\mathcal{F}^-$. Indeed, given such an A, let $Q \operatorname{Diag}(\lambda) Q^T$ be a spectral decomposition of A such that $\lambda_1 < 0$. Let q_j be the j-th column of Q, and define

$$B := \begin{pmatrix} \lambda_2^{1/2} q_2 & \cdots & \lambda_n^{1/2} q_n \end{pmatrix} \in \Re^{n \times (n-1)}, \qquad b := (-\lambda_1)^{1/2} q_1 \in \Re^n. \tag{5}$$

Note that the nonzero columns of B are linearly independent and $b \notin \operatorname{Range}(B)$. Then $A = BB^T - bb^T$, and $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$ can be defined as in (1)–(3). An important observation is that, as a collection of sets, $\{\mathcal{F}^+, \mathcal{F}^-\}$ is independent of the choice of spectral decomposition.

Proposition 3. *Let A be a given symmetric matrix with a single negative eigenvalue, and let $A = Q \operatorname{Diag}(\lambda) Q^T$ be a spectral decomposition such that $\lambda_1 < 0$. Define the SOCr sets $\{\mathcal{F}^+, \mathcal{F}^-\}$ according to (1) and (2), where (B, b) is given by (5). Similarly, let $\{\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^-\}$ be defined by an alternative spectral decomposition $A = \bar{Q} \operatorname{Diag}(\bar{\lambda}) \bar{Q}^T$. Then $\{\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^-\} = \{\mathcal{F}^+, \mathcal{F}^-\}$.*

Proof. Let $(\bar{B}, \bar{b})$ be given by the alternative spectral decomposition. Because A has a single negative eigenvalue, $\bar{b} = b$ or $\bar{b} = -b$. In addition, we claim $\|\bar{B}^T x\| = \|B^T x\|$ for all x. This holds because $\bar{B}\bar{B}^T = BB^T$ is the positive semidefinite part of A. This proves the result. □

To resolve the ambiguity inherent in Proposition 3, one could choose a specific $\bar{x} \in \operatorname{int}(\mathcal{F})$, which exists by Proposition 1, and enforce the convention that, for any spectral decomposition, $\mathcal{F}^+$ is chosen to contain $\bar{x}$. This simply amounts to flipping the sign of b so that $b^T \bar{x} > 0$.

3 The Result

In this section, we state our main theorem (Theorem 1) and its assumptions. The proof of Theorem 1 is delayed until Section 5.

To begin, let A_0 be a symmetric matrix satisfying the following assumption:

Assumption 1. *A_0 has exactly one negative eigenvalue.*

As described in Section 2, we may define SOCr cones $\mathcal{F}_0 = \mathcal{F}_0^+ \cup \mathcal{F}_0^-$ based on A_0. We also introduce a symmetric matrix A_1 and define the cone $\mathcal{F}_1 := \{x : x^T A_1 x \le 0\}$ in analogy with $\mathcal{F}_0$. However, we do *not* assume that A_1 has exactly one negative eigenvalue, so $\mathcal{F}_1$ does not necessarily decompose into two SOCr cones.

We investigate the set $\mathcal{F}_0^+ \cap \mathcal{F}_1$, which was expressed as $\mathcal{K} \cap \mathcal{Q}$ in the Introduction. In particular, we would like to develop strong convex relaxations of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ and, whenever possible, characterize its closed conic hull. We focus on the full-dimensional case, and so we assume:

Assumption 2. *There exists* $\bar{x} \in \operatorname{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.

Note that $\operatorname{int}(\mathcal{F}_0^+ \cap \mathcal{F}_1) = \operatorname{int}(\mathcal{F}_0^+) \cap \operatorname{int}(\mathcal{F}_1)$, and so Assumption 2 is equivalent to

$$\bar{x}^T A_0 \bar{x} < 0 \qquad \text{and} \qquad \bar{x}^T A_1 \bar{x} < 0. \tag{6}$$

In particular, this implies A_1 has at least one negative eigenvalue.

Our first result (the first part of Theorem 1 below) establishes that $\operatorname{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ is contained within the convex intersection of $\mathcal{F}_0^+$ with a second set of the same type, i.e., one that is SOCr. In addition to Assumptions 1 and 2, we require the following assumption, which handles the singularity of A_0 carefully:

Assumption 3. *Either (i) A_0 is nonsingular, (ii) A_0 is singular and A_1 is positive definite on* $\operatorname{null}(A_s)$*, or (iii) A_0 is singular and A_1 is negative definite on* $\operatorname{null}(A_0)$.

Verifying Assumption 3(i) is easy. Conditions (ii) and (iii) are also easy to verify by computing the eigenvalues of $Z_0^T A_1 Z_0$, where Z_0 is a matrix whose columns span $\operatorname{null}(A_0)$.

Assumptions 1–3 will ensure (see Section 5.1) the existence of a maximal $s \in [0, 1]$ such that

$$A_t := (1 - t)A_0 + tA_1$$

has a single negative eigenvalue for all $t \in [0, s]$, A_t is invertible for all $t \in (0, s)$, and A_s is singular—that is, $\operatorname{null}(A_s)$ is non-trivial. (Actually, A_s may be nonsingular when s equals 1, but this is a small detail.) Then, for all A_t with $t \in [0, s]$, SOCr sets $\mathcal{F}_t = \mathcal{F}_t^+ \cup \mathcal{F}_t^-$ can be defined as described in Section 2. Furthermore, for $\bar{x}$ of Assumption 2, noting that $\bar{x}^T A_t \bar{x} = (1 - t)\, \bar{x} A_0 \bar{x} + t\, \bar{x}^T A_1 \bar{x} < 0$ by (6), we can choose without loss of generality that $\bar{x} \in \mathcal{F}_t^+$ for all such t. Then Theorem 1 asserts that $\operatorname{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ is contained in $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$.

Our second result (the second part of Theorem 1) provides an additional condition under which $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ actually equals the closed conic hull. The required condition is:

Assumption 4. *When $s < 1$,* $\operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \neq \emptyset$.

While Assumption 4 may appear quite strong, we will actually show (see Lemma 3) that Assumptions 1–3 and the definition of s already ensure $\operatorname{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$. So Assumption 4 is a type of regularity condition guaranteeing that the set $\operatorname{apex}(\mathcal{F}_s^+) = \operatorname{null}(A_s)$ is not restricted to the boundary of $\mathcal{F}_1$. Note also that, given $s < 1$, Assumption 4 can be checked by computing $Z_s^T A_1 Z_s$, where Z_s has columns spanning $\operatorname{null}(A_s)$. We know $Z_s^T A_1 Z_s \preceq 0$ because $\operatorname{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$, and then Assumption 4 holds as long as $Z_s^T A_1 Z_s \neq 0$.

In fact, s has a straightforward definition. Let $\mathcal{T} := \{t \in \Re : A_t \text{ is singular}\}$. We will show in Section 5 (see Lemma 1 in particular) that $\mathcal{T} \subsetneq \Re$. So there exists some $\delta \neq 1$ such that A_δ is invertible. Consequently, $\mathcal{T}$ is easily computable as follows. Define $\bar{A}_0 := A_\delta$, $\bar{A}_1 := A_1$, and $\bar{A}_t := (1-t)\bar{A}_0 + t\bar{A}_1$. Note that $\{\bar{A}_t\}$ is simply an affine reparameterization of $\{A_t\}$. Hence, the set $\overline{\mathcal{T}} := \{t : \bar{A}_t \text{ is singular}\}$ is an affine transformation of $\mathcal{T}$. So we can compute $\overline{\mathcal{T}}$ instead. The following calculation with $t \neq 0$ then shows that the elements of $\overline{\mathcal{T}}$ are in bijective correspondence with the real eigenvalues of $\bar{A}_0^{-1}\bar{A}_1$:

$$
\begin{aligned}
\bar{A}_t \text{ is singular} \quad &\iff \quad \exists\, x \neq 0 \text{ s.t. } \bar{A}_t x = 0 \\
&\iff \quad \exists\, x \neq 0 \text{ s.t. } \bar{A}_0^{-1}\bar{A}_1 x = -\left(\tfrac{1-t}{t}\right) x \\
&\iff \quad -\left(\tfrac{1-t}{t}\right) \text{ is an eigenvalue of } \bar{A}_0^{-1}\bar{A}_1.
\end{aligned}
$$

In particular, $|\overline{\mathcal{T}}| = |\mathcal{T}|$ is finite. Once $\mathcal{T}$ is computed, then s is defined by

$$
s := \begin{cases} \min(\mathcal{T} \cap (0,1]) & \text{under Assumption 3(i) or 3(ii)} \\ 0 & \text{under Assumption 3(iii).} \end{cases} \tag{7}
$$

Additional insight into the definition of s, particularly as it relates to Assumption 3, will be given in Section 5.

We also include in Theorem 1 a specialization for the case when $\mathcal{F}_0^+ \cap \mathcal{F}_1$ is intersected with an affine hyperplane, which was expressed as $\mathcal{K} \cap \mathcal{Q} \cap H$ in the Introduction. For this, let $h \in \Re^n$ be given, and define the hyperplanes

$$
H^0 := \{x : h^T x = 0\}, \tag{8}
$$

$$
H^1 := \{x : h^T x = 1\}. \tag{9}
$$

We introduce an additional condition related to H_0:

Assumption 5. *When $s < 1$, $\operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \cap H^0 \neq \emptyset$ or $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$.*

In general, it seems challenging to verify Assumption 5 for a given s. However, we will show many examples of interest in which it can be verified.

We now state the main theorem of the paper. See Section 5 for its proof.

Theorem 1. *Suppose Assumptions 1–3 are satisfied, and let s be defined by (7). Then* $\mathrm{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$*, and equality holds under Assumption 4. Moreover, Assumptions 1–5 imply* $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 = \mathrm{cl.conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

4 Low-Dimensional Examples

In this section, we illustrate Theorem 1 with several low-dimensional examples. Later, Section 6 will be devoted to the important case, where the dimension n is arbitrary, $\mathcal{F}_0^+$ is the second-order cone, and $\mathcal{F}_1$ represents a two-term linear disjunction $c_1^T x \geq d_1 \vee c_2^T x \geq d_2$. Section 7 investigates cases in which $\mathcal{F}_1$ is given by a (nearly) general quadratic inequality.

4.1 A proper split of the second-order cone

In $\Re^3$, consider the intersection of the canonical second-order cone, defined by $\|(y_1; y_2)\| \leq y_3$, and a specific linear disjunction, defined by $y_1 \leq -1 \vee y_1 \geq 1$, which is a proper split. By homogenizing via $x = \binom{y}{x_4}$ with $x_4 = 1$ and noting that the disjunction is equivalent to $y_1^2 \geq 1 \Leftrightarrow y_1^2 \geq x_4^2$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \mathrm{Diag}(1, 1, -1, 0), \quad A_1 := \mathrm{Diag}(-1, 0, 0, 1), \quad H^1 := \{x : x_4 = 1\}.$$

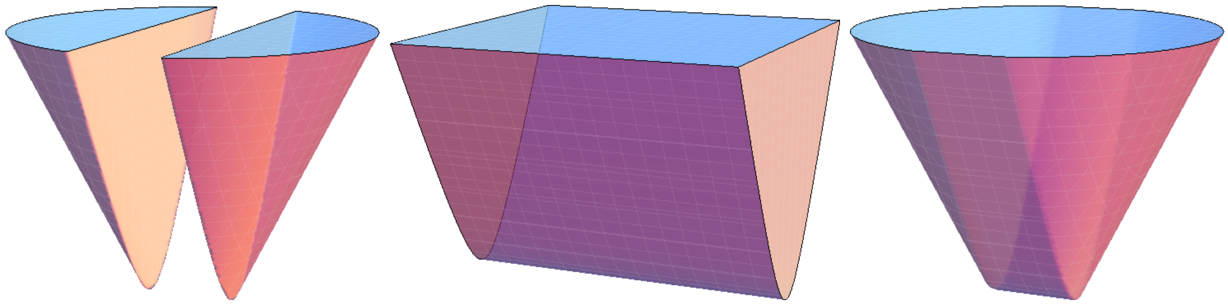
Note that $A_t = \mathrm{Diag}(1-2t, 1-t, -1+t, t)$. Assumptions 1 and 3(ii) are easily verified, and Assumption 2 holds with $\bar{x} := (2; 0; 3; 1)$, for example.

In this case, $s = \frac{1}{2}$, $A_s = \frac{1}{2}\mathrm{Diag}(0, 1, -1, 1)$, $\mathcal{F}_s = \{x : x_2^2 + x_4^2 \leq x_3^2\}$, and $\mathcal{F}_s^+ = \{x : \|(x_2; x_4)\| \leq x_3\}$, which contains $\bar{x}$. Note that $\mathrm{apex}(\mathcal{F}_s^+) = \mathrm{null}(A_s) = \mathrm{span}\{d\}$, where $d := (1; 0; 0; 0)$. It is easy to check that $d \in H^0$ with $d^T A_1 d < 0$, and so Assumptions 4 and 5 are simultaneously verified.

So, in the original variable y, the explicit convex hull is given by

$$\left\{ y : \begin{array}{l} \|(y_1; y_2)\| \leq y_3 \\ \|(y_2; 1)\| \leq y_3 \end{array} \right\} = \mathrm{cl.conv.hull} \left\{ y : \begin{array}{l} \|(y_1; y_2)\| \leq y_3 \\ y_1 \leq -1 \vee y_1 \geq 1 \end{array} \right\}.$$

Figure 1 depicts the original intersection, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Figure 1: A proper split of the second-order cone

4.2 A paraboloid and a second-order-cone disjunction

In $\Re^3$, consider the intersection of the paraboloid defined by $y_1^2+y_2^2 \le y_3$ and the "two-sided" second-order cone disjunction defined by $y_1^2 + y_3^2 \le y_2^2$. One side has $y_2 \ge 0$, while the other has $y_2 \le 0$. By homogenizing via $x = \binom{y}{x_4}$ with $x_4 = 1$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_4 = 1\}.$$

Assumptions 1 and 3(i) are straightforward to verify, and Assumption 2 is satisfied with $\bar{x} = (0; \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}}; 1)$, for example. We can also calculate $s = \frac{1}{2}$ from (7). Then

$$A_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 \end{pmatrix}, \qquad \mathcal{F}_s = \left\{x : x_1^2 + \tfrac{1}{2}\,x_3^2 \le \tfrac{1}{2}\,x_3 x_4\right\}.$$

The negative eigenvalue of A_s is $\lambda_{s1} := (1-\sqrt{2})/4$ with corresponding eigenvector $q_{s1} := (0; 0; \sqrt{2}-1; 1)$, and so, in accordance with the Section 2, we have that $\mathcal{F}_s^+$ equals all $x \in \mathcal{F}_s$ satisfying $b_s^T x \ge 0$, where

$$b_s := (-\lambda_{s1})^{1/2} q_{s1} = \frac{\sqrt{\sqrt{2}-1}}{2} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2}-1 \\ 1 \end{pmatrix}.$$

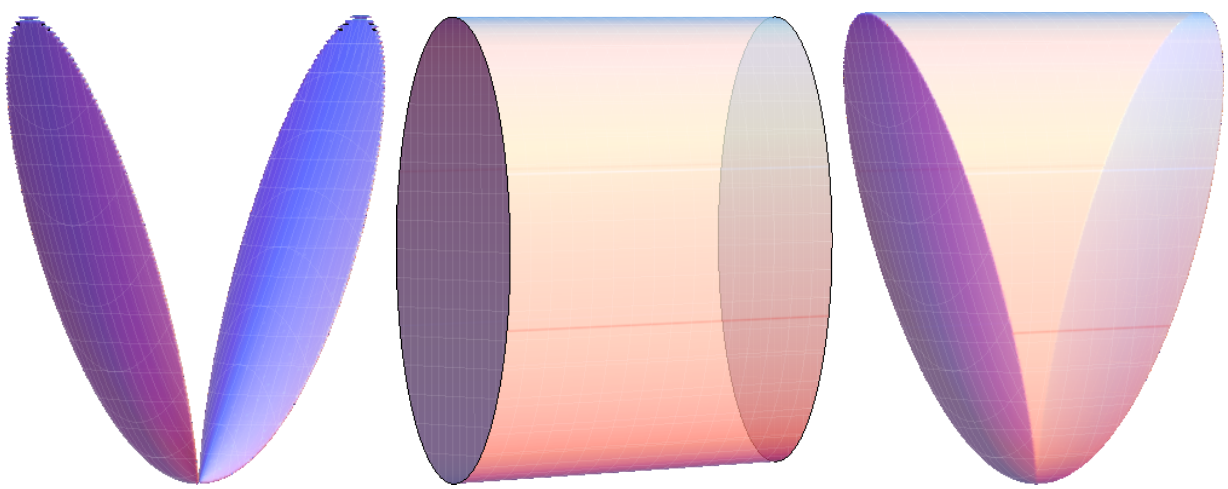
Scaling b_s by a positive constant, we thus have

$$\mathcal{F}_s^+ := \left\{ x : \begin{array}{l} x_1^2 + \frac{1}{2} x_3^2 \le \frac{1}{2} x_3 x_4 \\ (\sqrt{2}-1)x_3 + x_4 \ge 0 \end{array} \right\}.$$

Note that $\bar{x} \in \mathcal{F}_s^+$. In addition, $\mathrm{apex}(\mathcal{F}_s^+) = \mathrm{null}(A_s) = \mathrm{span}\{d\}$, where $d = (0; 1; 0; 0)$. Clearly, $d \in H^0$ and $d^T A_1 d < 0$, which verifies Assumptions 4 and 5 simultaneously. Setting $x_4 = 1$ and returning to the original variable y, we see

$$\left\{ y : \begin{array}{l} y_1^2 + y_2^2 \le y_3 \\ y_1^2 + \frac{1}{2} y_3^2 \le \frac{1}{2} y_3 \end{array} \right\} = \mathrm{cl.\,conv.\,hull} \left\{ y : \begin{array}{l} y_1^2 + y_2^2 \le y_3 \\ y_1^2 + y_3^2 \le y_2^2 \end{array} \right\},$$

where the now redundant constraint $(\sqrt{2}-1)y_3 + 1 \ge 0$ has been dropped. Figure 2 depicts the original intersection, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Figure 2: A paraboloid and a second-order-cone disjunction

4.3 An ellipsoid and a nonconvex quadratic

In $\Re^3$, consider the intersection of the unit ball defined by $y_1^2 + y_2^2 + y_3^2 \le 1$ and the nonconvex set defined by the quadratic $-y_1^2 - y_2^2 + \frac{1}{2} y_3^2 \le y_1 + \frac{1}{2} y_2$. By homogenizing via $x = \binom{y}{x_4}$ with

$x_4 = 1$, we can represent the intersection as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -1 & 0 & 0 & -\frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_4 = 1\}.$$

Assumptions 1 and 3(i) are straightforward to verify, and Assumption 2 is satisfied with $\bar{x} = (\frac{1}{2}; 0; 0; 1)$, for example. We can also calculate $s = \frac{1}{2}$ from (7). Then

$$A_s = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 6 & 0 \\ -2 & -1 & 0 & -4 \end{pmatrix}, \qquad \mathcal{F}_s = \left\{x : 3x_3^2 \leq 2x_1x_4 + x_2x_4 + 2x_4^2\right\}.$$

The negative eigenvalue of A_s is $\lambda_{s1} := -\frac{5}{8}$ with corresponding eigenvector $q_{s1} := (2; 1; 0; 5)$, and so, in accordance with the Section 2, we have that $\mathcal{F}_s^+$ equals all $x \in \mathcal{F}_s$ satisfying $b_s^T x \geq 0$, where

$$b_s := (-\lambda_{s1})^{1/2} q_{s1} = \sqrt{5/8} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 5 \end{pmatrix}.$$

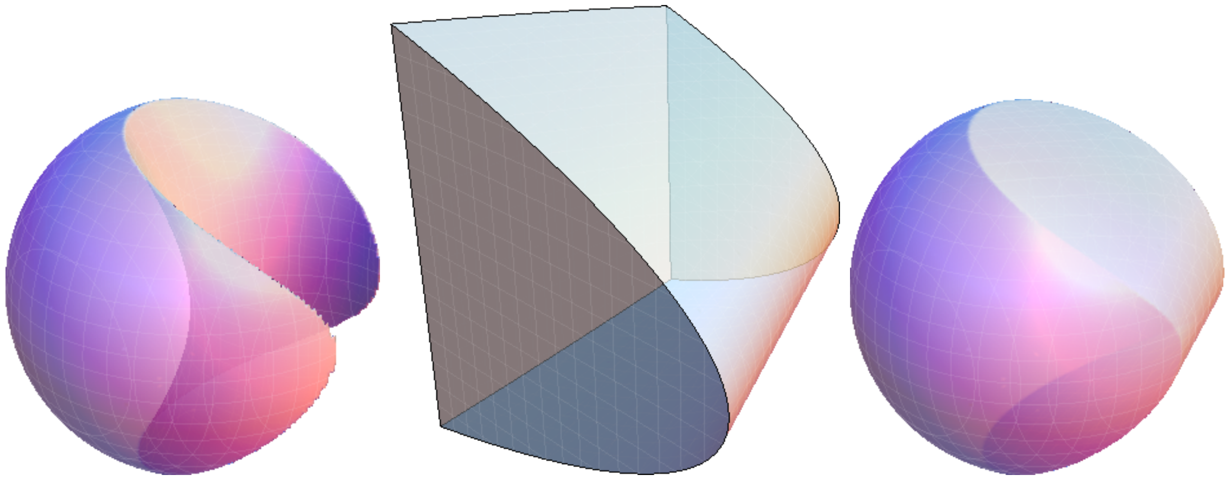
In other words,

$$\mathcal{F}_s^+ := \left\{ x : \begin{array}{l} 3x_3^2 \leq 2x_1x_4 + x_2x_4 + 2x_4^2 \\ 2x_1 + x_2 + 5x_4 \geq 0 \end{array} \right\}.$$

Note that $\bar{x} \in \mathcal{F}_s^+$. In addition, $\text{apex}(\mathcal{F}_s^+) = \text{null}(A_s) = \text{span}\{d\}$, where $d = (1; -2; 0; 0)$. Clearly, $d \in H^0$ and $d^T A_1 d < 0$, which verifies Assumptions 4 and 5 simultaneously. Setting $x_4 = 1$ and returning to the original variables y, we see

$$\left\{ y : \begin{array}{l} y_1^2 + y_2^2 + y_3^2 \leq 1 \\ 3x_3^2 \leq 2x_1 + x_2 + 2 \end{array} \right\} = \text{cl. conv. hull} \left\{ y : \begin{array}{l} y_1^2 + y_2^2 + y_3^2 \leq 1 \\ -y_1^2 - y_2^2 + \frac{1}{2}y_3^2 \leq y_1 + \frac{1}{2}y_2 \end{array} \right\},$$

where the now redundant constraint $2y_1 + y_2 \geq -5$ has been dropped. Figure 3 depicts the original set, $\mathcal{F}_s^+ \cap H^1$, and the closed convex hull.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_s^+ \cap H^1$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

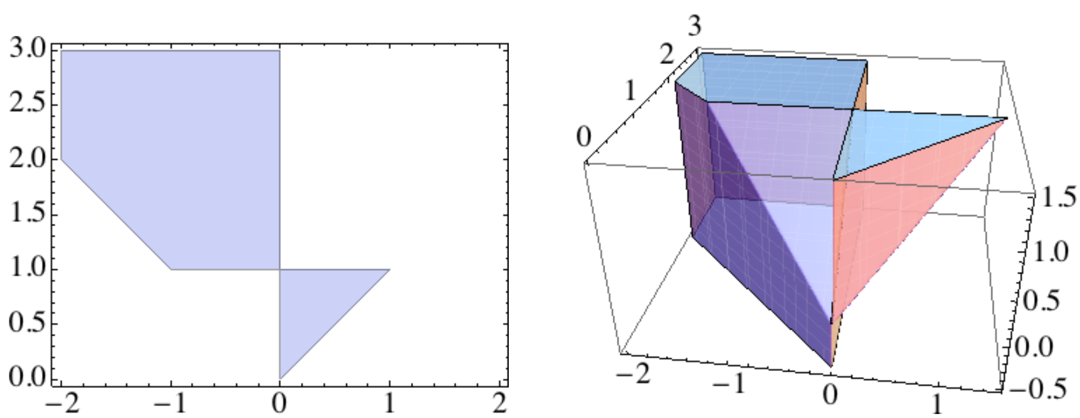
Figure 3: An ellipsoid and a nonconvex quadratic

4.4 An example violating Assumption 3

In $\Re^2$, consider the intersection of the canonical second-order cone defined by $|y_1| \leq y_2$ and the set defined by the quadratic $y_1(y_2 - 1) \leq 0$. By homogenizing via $x = \binom{y}{x_3}$ with $x_3 = 1$, we can represent the set as $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ with

$$A_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad H^1 := \{x : x_3 = 1\}.$$

While Assumptions 1 and 2 hold, Assumption 3 does not hold because A_0 is singular and A_1 is zero on the null space $\operatorname{span}\{(0; 0; 1)\}$ of A_0. Figure 4 depicts $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ and $\mathcal{F}_0^+ \cap \mathcal{F}_1$.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_1$

Figure 4: An example violating Assumption 3

4.5 An example violating Assumption 4

In $\Re^2$, consider the intersection of the second-order cone defined by $|x_1| \le x_2$ and the two-term linear disjunction defined by $x_1 \le 0 \vee x_2 \le x_1$. Note that, in the second-order cone, $x_2 \le x_1$ implies $x_1 = x_2$. So one side of the disjunction is contained in the boundary of the second-order cone. We also note that—in the second-order cone—the disjunction is equivalent to the quadratic $x_1(x_2 - x_1) \le 0$. Thus, to compute the closed conic hull of the intersection of cone and the disjunction, we define

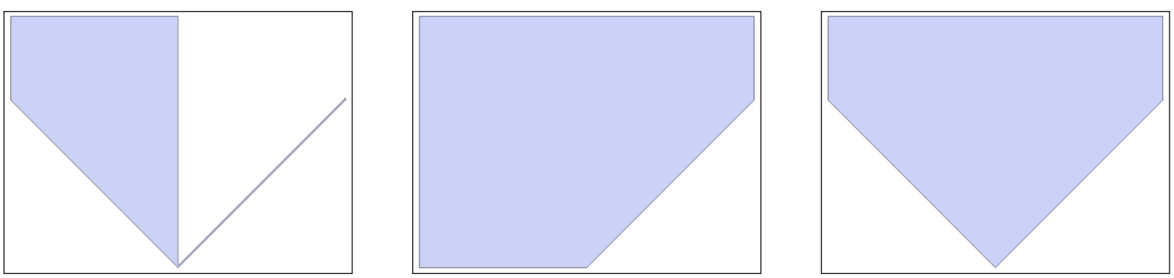
$$A_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix},$$

and we wish to calculate $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.

Assumptions 1, 2, and 3(i) are easily verified, and the eigenvalues of $A_0^{-1} A_1$ are -1 (with multiplicity 2). This implies $s = \frac{1}{2}$ by (7), and so

$$A_s = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$\mathrm{null}(A_s)$ is spanned by $d = (1; 1)$, and yet $d^T A_1 d = 0$, which violates Assumption 4.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1$ (b) $\mathcal{F}_s^+$ (c) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$

Figure 5: An example violating Assumption 4

Note that $A_s = -\frac{1}{2}\binom{1}{-1}\binom{1}{-1}^T$, and so $\mathcal{F}_s^+ = \{x : x_2 \ge x_1\}$. Figure 5 depicts $\mathcal{F}_0^+ \cap \mathcal{F}_1$, $\mathcal{F}_s^+$, and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Since Assumptions 1–3 are satisfied, we know that $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, and it is evident from the figures that—in this particular example—equality holds. This simply indicates that the results of Theorem 1 may still hold even when Assumption 4 is violated.

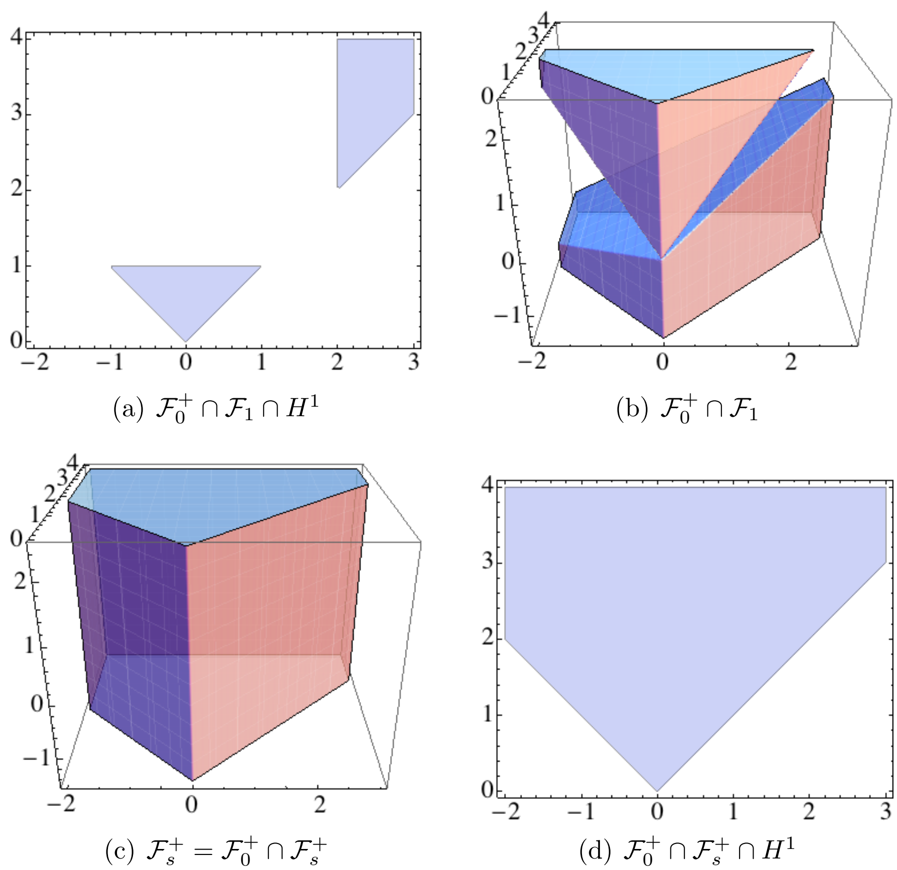
4.6 An example violating Assumption 5

In $\Re^2$, consider the intersection of the second-order cone defined by $|y_1| \le y_2$ and the two-term linear disjunction defined by $y_1 \ge 2 \vee y_2 \le 1$. Note that, in the second-order cone, the disjunction is equivalent to the quadratic $(y_1 - 2)(1 - y_2) \le 0$. Thus, to compute the closed conic hull of the intersection of cone and the disjunction, we define $x = \binom{y}{x_3}$ and

$$A_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_1 := \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & -4 \end{pmatrix}, \quad H^1 := \{x : x_3 = 1\}$$

and we wish to calculate $\text{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Assumptions 1, 2, and 3(iii) are easily verified, and so $s = 0$ with $\text{null}(A_s)$ spanned by $d = (0; 0; 1)$. Then Assumption 4 is clearly satisfied. However, $d_3 \neq 0$, and so the first option for Assumption 5 is not satisfied. The second option is the containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$, which simplifies to $\mathcal{F}_0^+ \cap H^0 \subseteq \mathcal{F}_1$ in this case. This is also *not* true because the point $x = (1; 2; 0) \in \mathcal{F}_0^+ \cap H^0$ but $x \notin \mathcal{F}_1$.



(a) $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ (b) $\mathcal{F}_0^+ \cap \mathcal{F}_1$

(c) $\mathcal{F}_s^+ = \mathcal{F}_0^+ \cap \mathcal{F}_s^+$ (d) $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

Figure 6: An example violating Assumption 5. Note that $s = 0$ in this case.

Figure 6 depicts this example. Note that the inequality $y_1 \geq -1$ is valid for the convex hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$. In addition, $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ = \text{cl. conic. hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ because Assumptions 1–4 are satisfied. However, the projection $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ is not the desired convex hull since, for example, it violates $y_1 \geq -1$.

5 The Proof

In this section, we build the proof of Theorem 1, and we provide important insights along the way. The key results are Propositions 5–7, which state

$$
\begin{aligned}
\mathcal{F}_0^+ \cap \mathcal{F}_1 &\subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \text{conic. hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \\
\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1 &\subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 \subseteq \text{conv. hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1).
\end{aligned}
$$

Recall that the definition of s is given by (7). In each line, the first containment depends only on Assumptions 1–3, which proves the first part of Theorem 1. On the other hand, the second containments require Assumption 4 and Assumptions 4–5, respectively. Then the second part of Theorem 1 follows by simply taking the closed conic hull and the closed convex hull, respectively, and noting that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ are already closed and convex.

5.1 The interval $[0, s]$

Our next result, Lemma 1, is quite technical but critically important. For example, it establishes that the line $\{A_t\}$ contains at least one invertible matrix not equal to A_1. As discussed in Section 3, this proves that the set $\mathcal{T}$ used in the definition (7) of s is finite and easily computable. The lemma also provides additional insight into the definition of s. Specifically, the lemma clarifies the role of Assumption 3 in (7). Since the proof of Lemma 1 is involved, we delay it until the end of this subsection.

Lemma 1. *Let $\epsilon > 0$ be small and consider A_ϵ and $A_{-\epsilon}$. Relative to Assumption 3:*

- *if (i) holds, then A_ϵ and $A_{-\epsilon}$ are each invertible with one negative eigenvalue;*
- *if (ii) holds, then only A_ϵ is invertible with one negative eigenvalue;*
- *if (iii) holds, then only $A_{-\epsilon}$ is invertible with one negative eigenvalue.*

If Assumption 3(i) or 3(ii) holds, then Lemma 1 shows that the interval $(0, \epsilon)$ contains invertible A_t, each with exactly one negative eigenvalue, and (7) takes s to be the largest ϵ

with this property. By continuity, A_s is singular (when $s < 1$) but still retains exactly one negative eigenvalue, a necessary condition for defining $\mathcal{F}_s^+$ in Theorem 1. On the other hand, if Assumption 3(iii) holds, then A_0 is singular and no $\epsilon > 0$ has the property just mentioned. Yet, $s = 0$ is still the natural "right-hand limit" of invertible $A_{-\epsilon}$, each with exactly one negative eigenvalue. This will be all that is required for Theorem 1.

With Lemma 1 in hand, we can prove the following key result, which sets up the remainder of this section. The proof of Lemma 1 follows afterward.

Proposition 4. *For all $t \in [0, s]$, A_t has exactly one negative eigenvalue. In addition, A_t is nonsingular for all $t \in (0, s)$, and if $s < 1$, then A_s is singular.*

Proof. Assumption 2 implies (6), and so $\bar{x}^T A_t \bar{x} = (1 - t)\, \bar{x}^T A_0 \bar{x} + t\, \bar{x}^T A_1 \bar{x} < 0$ for every t. So each A_t has at least one negative eigenvalue. Also, the definition of s ensures that all A_t for $t \in (0, s)$ are nonsingular and that A_s is singular when $s < 1$.

Suppose that some A_t with $t \in [0, s]$ has two negative eigenvalues. Then by Assumption 1 and the continuity of eigenvalues, there exists some $0 \le r < t \le s$ with at least one zero eigenvalue, i.e., with A_r singular. From the definition of s, it must be the case that $r = 0$, and A_ϵ has two negative eigenvalues for $\epsilon > 0$ small. Then Assumption 3(ii) holds since $s > 0$. However, we then encounter a contradiction with Lemma 1, which states that A_ϵ has exactly one negative eigenvalue. $\square$

Proof of Lemma 1. The lemma is clearly true under (i) since A_0 is invertible with exactly one negative eigenvalue and since the eigenvalues are continuous in ϵ.

Suppose (ii) holds. We first construct general bounds for $x^T A_0 x$ and $x^T A_1 x$ in terms of the eigenvalues and eigenvectors of A_0. Let P denote a matrix with columns consisting of the positive eigenvectors of A_0, and let Z and N consist of the zero and negative eigenvectors, respectively. Note that N has only one column. Also let the diagonal matrix Π and scalar ν correspond to the positive and negative eigenvalues such that

$$A_0 = P \Pi P^T - \nu N N^T.$$

In particular, $\nu = |\lambda_{\min}[A_0]|$. Any vector x of unit length may be expressed as

$$x = Pp + Zz + Nn,$$

with vectors p, z and scalar n such that $\|p\|^2 + \|z\|^2 + n^2 = \|x\|^2 = 1$. (Note that, in this proof, n is not a dimension, but rather just a scalar. We do this for a mnemonic to remember the association with the "negative" eigenvalue and eigenvector.) Then

$$\begin{aligned} x^T A_0 x &= (Pp + Zz + Nn)^T (P\Pi P^T - \nu NN^T)(Pp + Zz + Nn) \\ &= (Pp + Zz + Nn)^T (P\Pi p - \nu Nn) \\ &= p^T \Pi p - \nu\, n^2 \end{aligned}$$

and

$$\begin{aligned} x^T A_1 x &= (Pp + Zz + Nn)^T A_1 (Pp + Zz + Nn) \\ &= p^T (P^T A_1 P) p + z^T (Z^T A_1 Z) z + (N^T A_1 N) n^2 + \\ &\qquad 2p^T (P^T A_1 Z) z + 2p^T (P^T A_1 N) n + 2z^T (Z^T A_1 N) n. \end{aligned}$$

Defining

$$\pi_{\max} := \lambda_{\max}[\Pi] > 0, \qquad \text{and} \qquad \pi_{\min} := \lambda_{\min}[\Pi] > 0,$$

$$\begin{aligned} \alpha_p &:= \|P^T A_1 P\|_2 \geq 0 \\ \alpha_z &:= \lambda_{\min}[Z^T A_1 Z] > 0 \text{ (since (ii) holds)} \\ \alpha_n &:= |N^T A_1 N| \geq 0, \end{aligned}$$

where $\|\cdot\|_2$ indicates the matrix 2-norm, and

$$\begin{aligned} \beta_{pz} &:= \|P^T A_1 Z\|_2 \geq 0, \\ \beta_{pn} &:= \|P^T A_1 N\|_2 \geq 0, \\ \beta_{zn} &:= \|Z^T A_1 N\|_2 \geq 0, \end{aligned}$$

we can bound $x^T A_0 x$ from above and below and $x^T A_1 x$ from below:

$$\begin{array}{rcl} \pi_{\min} \|p\|^2 - \nu\, n^2 \;\leq\; x^T A_0 x & \leq & \pi_{\max} \|p\|^2 - \nu\, n^2, \\ x^T A_1 x & \geq & -\alpha_p \|p\|^2 + \alpha_z \|z\|^2 - \alpha_n n^2 - 2\beta_{pz} \|p\| \|z\| - 2\beta_{pn} \|p\| |n| - 2\beta_{zn} \|z\| |n|. \end{array}$$

Our next step is to use these inequalities to prove facts about A_ϵ and $A_{-\epsilon}$.

Consider all x with $x = Pp + Zz$ and $\|x\|^2 = \|p\|^2 + \|z\|^2 = 1$, i.e., with the scalar $n = 0$. Such x are orthogonal to the subspace generated by the negatic eigenvalue of A_0, and thus span a subspace of dimension $n-1$. We have

$$x^T A_0 x \geq \pi_{\min} \|p\|^2,$$
$$x^T A_1 x \geq -\alpha_p \|p\|^2 + \alpha_z \|z\|^2 - 2\beta_{pz} \|p\| \|z\|,$$

and so

$$\begin{aligned} x^T A_\epsilon x &\geq (1-\epsilon)\pi_{\min} \|p\|^2 + \epsilon \left(-\alpha_p \|p\|^2 + \alpha_z \|z\|^2 - 2\beta_{pz} \|p\| \|z\|\right) \\ &= \begin{pmatrix} \|p\| \\ \|z\| \end{pmatrix}^T \begin{pmatrix} (1-\epsilon)\pi_{\min} - \epsilon\, \alpha_p & -\epsilon\, \beta_{pz} \\ -\epsilon\, \beta_{pz} & \epsilon\, \alpha_z \end{pmatrix} \begin{pmatrix} \|p\| \\ \|z\| \end{pmatrix}. \end{aligned}$$

We claim this 2×2 matrix is positive definite. Indeed, the diagonal entries $(1-\epsilon)\pi_{\min} - \epsilon\, \alpha_p$ and $\epsilon\, \alpha_z$ are positive for $\epsilon > 0$ small. Also, the determinant $\pi_{\min}\, \alpha_z\, \epsilon - (\beta_{pz}^2 + \alpha_p\, \alpha_z + \pi_{\min}\, \alpha_z)\epsilon^2$ is positive. So $x^T A_\epsilon x$ is positive definite on a subspace of dimension $n-1$, which implies that A_ϵ has at least $n-1$ positive eigenvalues. In addition, we know that A_ϵ has at least one negative eigenvalue because $\bar{x}^T A_\epsilon \bar{x} < 0$ according to Assumption 2 and (6). Hence, A_ϵ is invertible with exactly one negative eigenvalue, as claimed.

Now consider all x with $x = Zz + Nn$ and $\|x\|^2 = \|z\|^2 + n^2 = 1$, i.e., with the vector $p = 0$. Such x span a subspace of dimension of at least 2. We have

$$x^T A_0 x \leq -\nu\, n^2$$
$$x^T A_1 x \geq \alpha_z \|z\|^2 - \alpha_n n^2 - 2\beta_{zn} \|z\| |n|$$

and so

$$\begin{aligned} x^T A_{-\epsilon} x &\leq (1+\epsilon)(-\nu n^2) - \epsilon \left(\alpha_z \|z\|^2 - \alpha_n n^2 - 2\beta_{zn} \|z\| |n|\right) \\ &= \begin{pmatrix} \|z\| \\ |n| \end{pmatrix}^T \begin{pmatrix} -\epsilon\, \alpha_z & \epsilon\, \beta_{zn} \\ \epsilon\, \beta_{zn} & -(1+\epsilon)\nu + \epsilon\, \alpha_n \end{pmatrix} \begin{pmatrix} \|z\| \\ |n| \end{pmatrix}. \end{aligned}$$

Using an argument similar to the previous 2×2 matrix, it can be shown that this 2×2 matrix is negative definite. So $x^T A_{-\epsilon} x$ is negative definite on a subspace of dimension at least 2, which implies that $A_{-\epsilon}$ has at least 2 negative eigenvalues, as claimed.

Finally, suppose (iii) holds and define

$$\bar{A}_\epsilon := \left(\tfrac{1}{1+2\epsilon}\right) A_{-\epsilon} = \left(\tfrac{1}{1+2\epsilon}\right) ((1+\epsilon)A_0 - \epsilon A_1) = \left(\tfrac{1+\epsilon}{1+2\epsilon}\right) A_0 + \left(\tfrac{\epsilon}{1+2\epsilon}\right) (-A_1)$$
$$\bar{A}_{-\epsilon} := \left(\tfrac{1}{1-2\epsilon}\right) A_\epsilon = \left(\tfrac{1}{1-2\epsilon}\right) ((1-\epsilon)A_0 + \epsilon A_1) = \left(\tfrac{1-\epsilon}{1-2\epsilon}\right) A_0 + \left(\tfrac{-\epsilon}{1-2\epsilon}\right) (-A_1).$$

Then $\bar{A}_\epsilon$ and $\bar{A}_{-\epsilon}$ are on the line generated by A_0 and $-A_1$ such that $-A_1$ is positive definite on the null space of A_0. Applying the previous case for assumption (ii), we see that only $\bar{A}_\epsilon$ is invertible with a single negative eigenvalue. This proves the result. $\square$

5.2 The containment $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$

For each $t \in [0, s]$, Proposition 4 allows us to define analogs $\mathcal{F}_t = \mathcal{F}_t^+ \cup \mathcal{F}_t^-$ as described in Section 2 based on any spectral decomposition $A_t = Q_t \operatorname{Diag}(\lambda_t) Q_t^T$.

It is an important technical point, however, that in this paper we require λ_t and Q_t to be defined continuously in t. While it is well known that the vector of eigenvalues λ_t can be defined continuously, it is also known that—if the eigenvalues are ordered, say, such that $[\lambda_t]_1 \le \cdots \le [\lambda_t]_n$ for all t—then the corresponding eigenvectors, i.e., the ordered columns of Q_t, cannot be defined continuously in general. On the other hand, if one drops the requirement that the eigenvalues in λ_t stay ordered, then the following result of Rellich [36] (see also [27]) guarantees that λ_t and Q_t can be constructed continuously—in fact, analytically—in t:

Theorem 2 (Rellich [36])**.** *Because A_t is analytic in the single parameter t, there exist spectral decompositions $A_t = Q_t \operatorname{Diag}(\lambda_t) Q_t^T$ such that λ_t and Q_t are analytic in t.*

So we define $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$ using continuous spectral decompositions provided by Theorem 2:

$$\mathcal{F}_t^+ := \{x : \|B_t^T x\| \le b_t^T x\}$$
$$\mathcal{F}_t^- := \{x : \|B_t^T x\| \le -b_t^T x\},$$

where B_t and b_t such that $A_t = B_t B_t^T - b_t b_t^T$ are derived from the spectral decomposition as described in Section 2. Recall from Proposition 3 that, for each t, a different spectral decomposition could flip the roles of $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$, but we observe that Theorem 2 and Assumption 2 together guarantee that each $\mathcal{F}_t^+$ contains $\bar{x}$. In this sense, every $\mathcal{F}_t^+$ has the same "orientation." Our observation is enabled by a lemma that will be independently helpful in subsequent analysis.

Lemma 2. *Given $t \in [0, s]$, suppose some $x \in \mathcal{F}_t^+$ satisfies $b_t^T x = 0$. Then $t = 0$ or $t = s$.*

Proof. Since $x^T A_t x \le 0$ with $b_t^T x = 0$, we have

$$0 = (b_t^T x)^2 \ge \|B_t^T x\|^2 \quad \Longrightarrow \quad A_t x = (B_t B_t^T - b_t b_t^T) x = B_t (B_t^T x) - b_t (b_t^T x) = 0.$$

So A_t is singular. By Proposition 4, this implies $t = 0$ or $t = s$. $\square$

Observation 1. *For all $t \in [0, s]$, $\bar{x} \in \mathcal{F}_t^+$.*

Proof. Assumption 2 implies $b_0^T \bar{x} > 0$. Let $t \in (0, s]$ be fixed. Since $\bar{x}^T A_t \bar{x} < 0$ by (6), either $\bar{x} \in \mathcal{F}_t^+$ or $\bar{x} \in \mathcal{F}_t^-$. Suppose for contradiction that $\bar{x} \in \mathcal{F}_t^-$, i.e., $b_t^T \bar{x} < 0$. Then the continuity of b_t by Theorem 2 implies the existence of $r \in (0, t)$ such that $b_r^T \bar{x} = 0$. Because $\bar{x}^T A_r \bar{x} < 0$ as well, $\bar{x} \in \mathcal{F}_r^+$. By Lemma 2, this implies $r = 0$ or $r = s$, a contradiction. □

In particular, Observation 1 implies that our discussion in Section 3, where we chose $\bar{x} \in \mathcal{F}_t^+$ to facilitate the statement of Theorem 1, is indeed consistent with the discussion here.

We now state the primary result of this subsection, i.e., that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$. This result relies only on Assumptions 1–3.

Proposition 5. $\mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+$.

Proof. If $s = 0$, the result is trivial. So assume $s > 0$. In particular, Assumption 3(i) or 3(ii) holds. Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_1$, that is, $x^T A_0 x \le 0$, $b_0^T x \ge 0$, and $x^T A_1 x \le 0$. We would like to show $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+$. So we need $x^T A_s x \le 0$ and $b_s^T x \ge 0$. The first inequality holds because $x^T A_s x = (1 - s)\, x^T A_0 x + s\, x^T A_1 x \le 0$. Now suppose for contradiction that $b_s^T x < 0$. In particular, $x \ne 0$. Then by the continuity of b_t via Theorem 2, there exists $0 \le r < s$ such that $b_r^T x = 0$. Since $x^T A_r x \le 0$ also, $x \in \mathcal{F}_r^+$, and Lemma 2 implies $r = 0$. So Assumption 3(ii) holds. However, $x \in \mathcal{F}_1$ also, contradicting that A_1 is positive definite on $\operatorname{null}(A_0)$. □

5.3 The containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$

Proposition 5 in the preceding subsection establishes that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ under Assumptions 1–3. We now show that, in essence, the reverse inclusion holds under Assumption 4. Indeed, when $s = 1$, we clearly have $\mathcal{F}_0^+ \cap \mathcal{F}_1^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$. So the true case of interest is $s < 1$, for which Assumption 4 is the key ingredient. (However, results are stated to cover the cases $s < 1$ and $s = 1$ simultaneously.)

As mentioned in Section 3, Assumption 4 is a type of regularity condition in light of Lemma 3 next. The proof of Proposition 6 also relies on Lemma 3.

Lemma 3. $\operatorname{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_1$.

Proof. By Proposition 1, the claimed result is equivalent to $\operatorname{null}(A_s) \subseteq \mathcal{F}_1$. Let $d \in \operatorname{null}(A_s)$. If $s = 1$, then the equation $0 = d^T A_s d = (1 - s)\, d^T A_0 d + s\, d^T A_1 d$ implies $d^T A_1 d = 0$, i.e., $d \in \operatorname{bd}(\mathcal{F}_1) \subseteq \mathcal{F}_1$, as desired. If $s = 0$, then Assumption 3(iii) holds, that is, A_0 is singular and A_1 is negative definite on $\operatorname{null}(A_0)$. Then $d \in \operatorname{null}(A_0)$ implies $d^T A_1 d \le 0$, as desired.

So assume $s \in (0,1)$. Note that it is sufficient to consider $d \in \text{int}(\mathcal{F}_0)$, that is, $d^T A_0 d < 0$. Otherwise, i.e., when $d^T A_0 d \geq 0$, then the equation $0 = (1-s)\, d^T A_0 d + s\, d^T A_1 d$ implies $d^T A_1 d \leq 0$, as desired.

To prove the claim, suppose for contradiction that $d \in \text{int}(\mathcal{F}_0)$ and without loss of generality that $d \in \text{int}(\mathcal{F}_0^+)$ and $-d \in \text{int}(\mathcal{F}_0^-)$. We know $-d \in \text{null}(A_s) = \text{apex}(\mathcal{F}_s^+) \subseteq \mathcal{F}_s^+$. Also, $\mathcal{F}_t^+$ is a full-dimensional set because $\bar{x}^T A_t \bar{x} < 0$ by (6), and because it is defined by a SOC inequality, $\mathcal{F}_t^+$ converges as a set to $\mathcal{F}_s^+$ as $t \to s$. So there exists a sequence $y_t \in \mathcal{F}_t^+$ converging to $-d$. In particular, $\mathcal{F}_t^+ \cap \text{int}(\mathcal{F}_0^-) \neq \emptyset$ for $t \to s$.

We can now achieve the desired contradiction (hence proving the claim) by focusing on the statement $\mathcal{F}_t^+ \cap \text{int}(\mathcal{F}_0^-) \neq \emptyset$. Let x be a member of this set with $t < s$. Then $x^T A_0 x \leq 0, b_0^T x < 0$ and $x^T A_t x \leq 0, b_t^T x \geq 0$. It follows that $x^T A_r x \leq 0, b_r^T x = 0$ for some $0 < r \leq t < s$. Hence, Lemma 2 implies $r = 0$ or $r = s$, a contradiction. □

Proposition 6. $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \subseteq \text{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$.

Proof. First, suppose $s = 1$. Then the result follows because $\mathcal{F}_0^+ \cap \mathcal{F}_1^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_1 \subseteq \text{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$. So assume $s \in [0,1)$.

Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+$, that is, $x^T A_0 x \leq 0$, $b_0^T x \geq 0$ and $x^T A_s x \leq 0$, $b_s^T x \geq 0$. If $x^T A_1 x \leq 0$, we are done. So assume $x^T A_1 x > 0$.

By Assumption 4, there exists $d \in \text{null}(A_s)$ such that $d^T A_1 d < 0$. In addition, d is necessarily perpendicular to the negative eigenvector b_s. For all $\epsilon \in \Re$, consider the affine line of points given by $x_\varepsilon := x + \epsilon\, d$. We have

$$\left.\begin{aligned} x_\varepsilon^T A_s x_\varepsilon = (x + \epsilon\, d)^T A_s (x + \epsilon\, d) = x^T A_s x \leq 0 \\ b_s^T x_\varepsilon = b_s^T (x + \epsilon\, d) = b_s^T x \geq 0 \end{aligned}\right\} \quad \Longrightarrow \quad x_\epsilon \in \mathcal{F}_s^+.$$

Note that $x_\varepsilon^T A_1 x_\varepsilon = x^T A_1 x + 2\,\epsilon\, d^T A_1 x + \epsilon^2\, d^T A_1 d$. Since $d^T A_1 d < 0$, there exist $l < 0 < u$ such that $x_l^T A_1 x_l^T = x_u^T A_1 x_u = 0$, i.e., $x_l, x_u \in \mathcal{F}_1$. Then $s < 1$ and $x_l^T A_s x_l \leq 0$ imply $x_l^T A_0 x_l \leq 0$, and hence $x_l \in \mathcal{F}_0$. Similarly, $x_u^T A_0 x_u \leq 0$ leading to $x_u \in \mathcal{F}_0$. We will prove in the next paragraph that both x_l and x_u are in $\mathcal{F}_0^+$, which will establish the result because then $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}^1$ and x is a convex combination of x_l and x_u.

Suppose that at least one of the two points x_l or x_u is not a member of $\mathcal{F}_0^+$. Without loss of generality, say $x_l \notin \mathcal{F}_0^+$. Then $x_l \in \mathcal{F}_0^-$ with $-b_0^T x_l > 0$. Similar to Proposition 5, we can prove $\mathcal{F}_0^- \cap \mathcal{F}_1 \subseteq \mathcal{F}_0^- \cap \mathcal{F}_s^-$, and so $x_l \in \mathcal{F}_0^- \cap \mathcal{F}_s^-$. Then $x_l \in \mathcal{F}_s^+ \cap \mathcal{F}_s^-$, which implies $b_s^T x_l = 0$ and $B_s^T x_l = 0$, which in turn implies $A_s x_l = 0$, i.e., $x_l \in \text{null}(A_s)$. Then $x + l\, d = x_l \in \text{null}(A_s)$ implies $x \in \text{null}(A_s)$ also. Then $x \in \mathcal{F}_1$ by Lemma 3, but this contradicts the earlier assumption that $x^T A_1 x > 0$. □

5.4 Intersection with an affine hyperplane

With Propositions 5 and 6, we can prove the first two statements of Theorem 1 as discussed at the beginning of this section. To prove the last statement of Theorem 1, recall that H^0 and H^1 are defined according to (8) and (9), where $h \in \Re^n$. Also define

$$H^+ := \{x : h^T x \geq 0\}.$$

The next lemma is the analog of Propositions 5–6 under intersection with H^+, and the proof uses Assumptions 1–3 to prove the first containment, and Assumption 5 to prove the second. Note that Assumption 5 only applies when $s < 1$. When $s = 1$, the second containment is clear (although results are stated covering both $s < 1$ and $s = 1$ simultaneously).

Lemma 4. $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+ \subseteq \operatorname{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+)$.

Proof. Proposition 5 implies that $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+ \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+$. Moreover, we can repeat the proof of Proposition 6, intersecting with H^+ along the way. However, we require one key modification in the proof of Proposition 6.

Let $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^+$ with $x^T A_1 x > 0$. Then, mimicking the proof of Proposition 6 for $s \in [0, 1)$, $x \in \{x_\epsilon := x + \epsilon\, d : \epsilon \in \Re\} \subseteq \mathcal{F}_s^+$ with $d \in \operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1)$. Moreover, x is a strict convex combination of points $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}_1$. Hence, the entire closed interval from x_l to x_u is contained in $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$.

Under Assumption 5, if there exists $d \in \operatorname{apex}(\mathcal{F}_s^+) \cap \operatorname{int}(\mathcal{F}_1) \cap H^0$, then this particular d can be used to show that x_l, x_u also satisfy $h^T x_l = h^T x_u = h^T x \geq 0$, i.e., $x_l, x_u \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+$, which implies $x \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+$, as desired.

So suppose $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$ under Assumption 5. Since $x \in H^+$, either $h^T x = 0$ or $h^T x > 0$. If $h^T x = 0$, then $x \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$, as desired. Finally, consider the case when $h^T x > 0$. At least one of x_l, x_u is in H^+; say, x_l is without loss of generality. If $x_u \in H^+$ also, then we are done. So suppose $h^T x_u < 0$. Now let y be a strict convex combination of x and x_u such that $y \in H^0$. Then $y \in \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$, and so x is a convex combination of $x_l, y \in \mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^+$, as desired. □

Our next main result, Proposition 7, requires the following simple lemma:

Lemma 5. *Let K be a closed cone (not necessarily convex), and let* $\operatorname{rec.cone}(K)$ *be its recession cone. Then* $\operatorname{conv.hull}(K) + \operatorname{conic.hull}(\operatorname{rec.cone}(K)) = \operatorname{conv.hull}(K)$.

Proof. The containment $\supseteq$ is clear. Now let $x + y$ be in the left-hand side such that

$$x = \sum_k \lambda_k x_k, \quad x_k \in K, \quad \lambda_k > 0, \quad \sum_k \lambda_k = 1$$

and

$$y = \sum_j \rho_j y_j, \quad y_j \in \operatorname{rec.cone}(K), \quad \rho_j > 0.$$

Without loss of generality, we may assume the number of x_k's equals the number of y_j's by splitting some $\lambda_k x_k$ or some $\rho_j y_j$ as necessary. Then

$$x + y = \sum_k (\lambda_k x_k + \rho_k y_k) = \sum_k \lambda_k (x_k + \lambda_k^{-1} \rho_k y_k) \in \operatorname{conv.hull}(K).$$

$\square$

Proposition 7. $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1 \subseteq \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1 \subseteq \operatorname{conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Proof. For notational convenience, define $\mathcal{G}_1 := \mathcal{F}_0^+ \cap \mathcal{F}_1$ and $\mathcal{G}_s^+ := \mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Lemma 4 shows $\mathcal{G}_1 \cap H^1 \subseteq \mathcal{G}_s^+ \cap H^1 \subseteq \operatorname{conic.hull}(\mathcal{G}_1 \cap H^+) \cap H^1$. To prove the proposition, we show that the last set is contained in $\operatorname{conv.hull}(\mathcal{G}_1 \cap H^1) + \operatorname{conic.hull}(\mathcal{G}_1 \cap H^0)$, which equals $\operatorname{conv.hull}(\mathcal{G}_1 \cap H^1)$ by Lemma 5. Indeed, let $x \in \operatorname{conic.hull}(\mathcal{G}_1 \cap H^+) \cap H^1$. We may write

$$x = \sum_k \lambda_k x_k, \quad x_k \in \mathcal{G}_1 \cap H^+, \quad \lambda_k > 0,$$

which may further be separated as

$$x = \underbrace{\sum_{k\,:\,h^T x_k > 0} \lambda_k x_k}_{:=y} + \underbrace{\sum_{k\,:\,h^T x_k = 0} \lambda_k x_k}_{:=r} = y + r.$$

Note that $r \in \operatorname{conic.hull}(\mathcal{G}_1 \cap H^0)$, and so it remains to show $y \in \operatorname{conv.hull}(\mathcal{G}_1 \cap H^1)$. We rewrite y as

$$y = \sum_{k\,:\,h^T x_k > 0} \lambda_k x_k = \sum_{k\,:\,h^T x_k > 0} \underbrace{(\lambda_k \cdot h^T x_k)}_{:=\tilde{\lambda}_k} \underbrace{(x_k / h^T x_k)}_{:=\tilde{x}_k} =: \sum_{k\,:\,h^T x_k > 0} \tilde{\lambda}_k \tilde{x}_k.$$

By construction, each $\tilde{x}_k \in \mathcal{G}_1 \cap H^1$. Moreover, each $\tilde{\lambda}_k$ is positive and

$$\sum_{k\,:\,h^T x_k > 0} \tilde{\lambda}_k = \sum_{k\,:\,h^T x_k > 0} \lambda_k \cdot h^T x_k = h^T y = h^T (x - r) = 1 - 0 = 1$$

since $x \in H^1$. So $y \in \operatorname{conv.hull}(\mathcal{G}_1 \cap H^1)$ as needed. $\square$

6 Two-term disjunctions on the second-order cone

In this section (specifically Sections 6.1–6.4), we consider the intersection of the canonical second-order cone

$$\mathcal{K} := \{x : \|\tilde{x}\| \leq x_n\}, \quad \text{where } \tilde{x} = (x_1; \ldots ; x_{n-1}),$$

and a two-term linear disjunction defined by $c_1^T x \geq d_1 \vee c_2^T x \geq d_2$. Without loss of generality, we take $d_1, d_2 \in \{0, \pm 1\}$ with $d_1 \geq d_2$, and we make the following assumption:

Assumption 6. *The disjunctive sets* $\mathcal{K}_1 := \mathcal{K} \cap \{x : c_1^T x \geq d_1\}$ *and* $\mathcal{K}_2 := \mathcal{K} \cap \{x : c_2^T x \geq d_2\}$ *are non-intersecting except possibly on their boundaries, e.g.,*

$$\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \left\{ x \in \mathcal{K} : \begin{array}{c} c_1^T x = d_1 \\ c_2^T x = d_2 \end{array} \right\}.$$

This assumption ensures that, on $\mathcal{K}$, the disjunction $c_1^T x \geq d_1 \ \vee \ c_2^T x \geq d_2$ is equivalent to the quadratic inequality $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$. Assumption 6 is satisfied, for example, when the disjunction is a proper split, i.e., $c_1 \parallel c_2$ with $c_1^T c_2 < 0$, $\mathcal{K}_1 \cup \mathcal{K}_2 \neq \mathcal{K}$, and $d_1 = d_2$. (If $d_1 \neq d_2$, then it can be shown that the closed conic hull is just $\mathcal{K}$.)

Because $d_1, d_2 \in \{0, \pm 1\}$ with $d_1 \geq d_2$, we can break our analysis into the following three cases with a total of six subcases:

(a) $d_1 = d_2 = 0$, covering subcase $(d_1, d_2) = (0, 0)$;

(b) $d_1 = d_2$ nonzero, covering subcases $(d_1, d_2) \in \{(-1, -1), (1, 1)\}$;

(c) $d_1 > d_2$, covering subcases $(d_1, d_2) \in \{(0, -1), (1, -1), (1, 0)\}$.

Case (a) is the homogeneous case, in which we take $A_0 = J := \mathrm{Diag}(1, \ldots, 1, -1)$ and $A_1 = c_1 c_2^T + c_2 c_1^T$ to match our set of interest $\mathcal{K} \cap \mathcal{F}_1$. Note that $\mathcal{K} = \mathcal{F}_0^+$ in this case. For the non-homogeneous cases (b) and (c), we can homogenize via $y = \binom{x}{x_{n+1}}$ with $h^T y = x_{n+1} = 1$. Defining

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -d_2 c_1 - d_1 c_2 \\ -d_2 c_1^T - d_1 c_2^T & 2 d_1 d_2 \end{pmatrix},$$

we then wish to examine $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$.

In fact, by the results in [30] (see that paper's second example, in particular), case (c) implies that $\mathrm{cl.\,conic.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ cannot in general be captured by two conic inequalities, making it unlikely that our desired equality $\mathrm{cl.\,conv.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1) = \mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$

will hold in general. So we will focus on cases (a) and (b). Nevertheless, we include some comments on case (c) in Section 6.4.

Later on, in Section 6.3, we will also revisit Assumption 6 to show that it is unnecessary in some sense. Precisely, even when Assumption 6 does not hold, we can derive a related convex valid inequality, which, together with $\mathcal{F}_0^+$, gives the complete convex hull description. This inequality precisely matches the one already described in [30], but it does not have an SOC form.

In contrast to Sections 6.1–6.4, Section 6.5 examines two-term disjunctions on conic sections of $\mathcal{K}$, i.e., intersections of $\mathcal{K}$ with a hyperplane.

6.1 The case (a) of $d_1 = d_2 = 0$

As discussed above, we have $A_0 := J$ and $A_1 := c_1 c_2^T + c_2 c_1^T$. If either $c_i \in \mathcal{K}$, then the corresponding side of the disjunction $\mathcal{K}_i$ simply equals $\mathcal{K}$, so the conic hull is $\mathcal{K}$. In addition, if either $c_i \in \text{int}(-\mathcal{K})$, then $\mathcal{K}_i = \{0\}$, so the conic hull equals the other $\mathcal{K}_j$. Hence, we assume both $c_i \notin \mathcal{K} \cup \text{int}(-\mathcal{K})$, i.e., $\|\tilde{c}_i\| \geq |c_{i,n}|$, where $c_i = \binom{\tilde{c}_i}{c_{i,n}}$. Since the example in Section 4.5 violates Assumption 4 with $\|\tilde{c}_2\| = |c_{2,n}|$, we further assume that both $\|\tilde{c}_i\| > |c_{i,n}|$.

Assumptions 1 and 3(i) are easily verified. In particular, $s > 0$. Assumption 2 describes the full-dimensional case of interest. It remains to verify Assumption 4. (Note that Assumption 4 is only relevant when $s < 1$ and that Assumption 5 is not of interest in this homogeneous case.) So suppose $s < 1$, and given nonzero $z \in \text{null}(A_s)$, we will show

$$z^T A_1 z = 2(c_1^T z)(c_2^T z) < 0,$$

verifying Assumption 4. We already know from Lemma 3 that $z^T A_1 z \leq 0$. So it remains to show that both $c_1^T z$ and $c_2^T z$ are nonzero.

Since $z \in \text{null}(A_s)$, we know $\left(\frac{1-s}{s}\right) A_0 z = -A_1 z$, i.e.,

$$\left(\tfrac{1-s}{s}\right) \begin{pmatrix} \tilde{z} \\ -z_n \end{pmatrix} = -c_1(c_2^T z) - c_2(c_1^T z). \tag{10}$$

Note that $c_1^T z = \binom{\tilde{c}_1}{-c_{1,n}}^T \binom{\tilde{z}}{-z_n}$, so multiplying both sides of equation (10) with $\binom{\tilde{c}_1}{-c_{1,n}}^T$ and rearranging terms, we obtain

$$\left[\tfrac{1-s}{s} + \tilde{c}_1^T \tilde{c}_2 - c_{1,n} c_{2,n}\right] (c_1^T z) = \left(c_{1,n}^2 - \|\tilde{c}_1\|_2^2\right) (c_2^T z).$$

Similarly, using $\binom{\tilde{c}_2}{-c_{2,n}}^T$, we obtain:

$$\left[\tfrac{1-s}{s} + \tilde{c}_1^T \tilde{c}_2 - c_{1,n} c_{2,n}\right] (c_2^T z) = \left(c_{2,n}^2 - \|\tilde{c}_2\|_2^2\right) (c_1^T z).$$

The inequalities $\|\tilde{c}_1\| > |c_{1,n}|$ and $\|\tilde{c}_2\| > |c_{2,n}|$ thus imply $c_1^T z \neq 0 \Leftrightarrow c_2^T z \neq 0$. Moreover, $c_1^T z$ and $c_2^T z$ cannot both be 0; otherwise, z would be 0 by (10).

6.2 The case (b) of nonzero $d_1 = d_2$

In [30], it was shown that $c_1 - c_2 \in \pm\mathcal{K}$ implies one of the sets $\mathcal{K}_i$ defining the disjunction is contained in the other $\mathcal{K}_j$. So then the desired closed convex hull trivially equals $\mathcal{K}_j$. So we assume $c_1 - c_2 \notin \pm\mathcal{K}$, i.e., $\|\tilde{c}_1 - \tilde{c}_2\|^2 > (c_{1,n} - c_{2,n})^2$, where $c_i = \binom{\tilde{c}_i}{c_{i,n}}$.

Defining $\sigma = d_1 = d_2$, we have

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -\sigma(c_1 + c_2) \\ -\sigma(c_1 + c_2)^T & 2 \end{pmatrix}.$$

Assumptions 1 and 3(ii) are easily verified, and Assumption 2 describes the full-dimensional case of interest. It remains to verify Assumptions 4 and 5. So assume $s < 1$, and note $s > 0$ due to Assumption 3(ii).

For any $z^+ \in \Re^{n+1}$, write $z^+ = \binom{z}{z_{n+1}}$ and $z = \binom{\tilde{z}}{z_n} \in \Re^n$. Suppose $z^+ \neq 0$. Then

$$\begin{aligned} z^+ \in \text{null}(A_s) \quad &\Longleftrightarrow \quad \left(\tfrac{1-s}{s}\right) A_0 z^+ = -A_1 z^+ \\ &\Longleftrightarrow \quad \left(\tfrac{1-s}{s}\right) A_0 z^+ = -\binom{c_1}{-\sigma}\binom{c_2}{-\sigma}^T z^+ - \binom{c_2}{-\sigma}\binom{c_1}{-\sigma}^T z^+ \\ &\qquad\qquad\qquad\quad =: \alpha\binom{c_1}{-\sigma} + \beta\binom{c_2}{-\sigma}. \end{aligned}$$

Since the last component of $A_0 z^+$ is zero, we must have $\beta = -\alpha$, and we claim $\alpha \neq 0$. Assume for contradiction that $\alpha = 0$. Then $z = 0$, but $z_{n+1} \neq 0$ as z^+ is nonzero. On the other hand, Lemma 3 implies $0 \geq (z^+)^T A_1 z^+ = 2z_{n+1}^2$, a contradiction. So indeed $\alpha \neq 0$.

Because $z^+ \in \text{null}(A_s)$ and $s \in (0, 1)$, the equation

$$0 = (z^+)^T A_s z^+ = (1 - s)(z^+)^T A_0 z^+ + s(z^+)^T A_1 z^+$$

implies Assumption 4 holds if and only $(z^+)^T A_0 z^+ > 0$. From the previous paragraph, we have $\left(\frac{1-s}{s}\right) A_0 z^+ = \alpha\binom{c1-c2}{0}$ with $\alpha \neq 0$. Then

$$\begin{aligned}\left(\tfrac{1-s}{s}\right)(z^+)^T A_0 z^+ &= \begin{pmatrix} \alpha(\tilde{c}_1 - \tilde{c}_2) \\ -\alpha(c_{1,n} - c_{2,n}) \\ z_{n+1} \end{pmatrix}^T \begin{pmatrix} \alpha(\tilde{c}_1 - \tilde{c}_2) \\ \alpha(c_{1,n} - c_{2,n}) \\ 0 \end{pmatrix} \\ &= \alpha^2 \left(\|\tilde{c}_1 - \tilde{c}_2\|^2 - (c_{1,n} - c_{2,n})^2\right) > 0,\end{aligned}$$

as desired.

However, it seems difficult to verify Assumption 5 generally. For example, consider its second condition $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$. In the current context, we have $\mathcal{F}_0^+ \cap H^0 = \mathcal{K} \times \{0\}$, and it is unclear if its intersection with $\mathcal{F}_s^+$ would be contained in $\mathcal{F}_1$. Letting $\binom{\hat{h}}{0} \in \mathcal{F}_s^+$ with $\hat{h} \in \mathcal{K}$, we would have to check the following:

$$\begin{pmatrix} \hat{h} \\ 0 \end{pmatrix} \in \mathcal{F}_1 \quad \Longleftrightarrow \quad 0 \geq \begin{pmatrix} \hat{h} \\ 0 \end{pmatrix}^T A_s \begin{pmatrix} \hat{h} \\ 0 \end{pmatrix} = (1-s)\,\hat{h}^T J \hat{h} + 2s\,(c_1^T \hat{h})(c_2^T \hat{h}).$$

If $\hat{h}$ were in the interior of $\mathcal{K}$, then $\hat{h}^T J \hat{h} < 0$ could still allow $(c_1^T \hat{h})(c_2^T \hat{h}) > 0$, so that $\binom{\hat{h}}{0} \in \mathcal{F}_1$ would not be achieved. So it seems Assumption 5 will hold under additional assumptions only.

One such set of assumptions is as follows: there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 c_1 + c_2 \in -\mathcal{K}$ and $\beta_2 c_1 + c_2 \in \mathcal{K}$. These hold, for example, for split disjunctions, i.e., when c_2 is a negative multiple of c_1. To prove Assumption 5, take $\hat{h} \in \mathcal{K}$. Then $c_1^T \hat{h} \geq 0$ implies

$$c_2^T \hat{h} = -\beta_1 c_1^T \hat{h} + (\beta_1 c_1 + c_2)^T \hat{h} \leq 0 + 0 = 0,$$

and similarly $c_1^T \hat{h} \leq 0$ implies $c_2^T \hat{h} \geq 0$. Then overall $\hat{h} \in \mathcal{K}$ implies $(c_1^T \hat{h})(c_2^T \hat{h}) \leq 0$. In the context of the previous paragraph, this ensures $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_0^+ \cap H^0 \subseteq \mathcal{F}_1$, thus verifying Assumption 5.

6.3 Revisiting Assumption 6

For the cases $d_1 = d_2$ of Sections 6.1 and 6.2, we know that $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ is a valid convex relaxation of $\mathcal{F}_0^+ \cap \mathcal{F}_1$ under Assumptions 1–3 and 6. The same holds for the cross-sections $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$ of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$. In particular, $s > 0$. However, when Assumption 6 is violated, it may be possible that $\mathcal{F}_s^+$ is invalid for points simultaneously satisfying both sides of the disjunction, i.e., points x with $c_1^T x \geq d_1$ and $c_2^T x \geq d_2$. This is because such points can violate the quadratic $(c_1^T x - d_1)(c_2^T x - d_2) \leq 0$ from which $\mathcal{F}_s^+$ is derived. In such cases, the set $\mathcal{F}_s^+$ should be relaxed somehow.

Recall that, by definition, $\mathcal{F}_s^+ = \{x : x^T A_s x \le 0, b_s^T x \ge 0\}$. Let us examine the inequality $x^T A_s x \le 0$, which can be rewritten as

$$\begin{aligned} & 0 \ge (1-s)\, x^T J x + 2s\, (c_1^T x - d_1)(c_2^T x - d_2) \\ \iff\ & 0 \ge 2(1-s)\, x^T J x + s\, \left([(c_1^T x - d_1) + (c_2^T x - d_2)]^2 - [(c_1^T x - d_1) - (c_2^T x - d_2)]^2 \right) \\ \iff\ & s\, [(c_1 - c_2)^T x - (d_1 - d_2)]^2 - 2(1-s)\, x^T J x \ \ge\ s\, [(c_1 + c_2)^T x - (d_1 + d_2)]^2. \end{aligned}$$

Note that the left hand-side of the third inequality is nonnegative for any $x \in \mathcal{K}$ since $x^T J x \le 0$. Therefore, $x \in \mathcal{K}$ implies $x^T A_s x \le 0$ is equivalent to

$$\sqrt{[(c_1 - c_2)^T x - (d_1 - d_2)]^2 - 2\left(\tfrac{1-s}{s}\right) x^T J x} \ \ge\ |(c_1 + c_2)^T x - (d_1 + d_2)|. \tag{11}$$

An immediate relaxation of (11) is

$$\sqrt{[(c_1 - c_2)^T x - (d_1 - d_2)]^2 - 2\left(\tfrac{1-s}{s}\right) x^T J x} \ \ge\ (d_1 + d_2) - (c_1 + c_2)^T x \tag{12}$$

since $|(c_1 + c_2)^T x - (d_1 + d_2)| \ge (d_1 + d_2) - (c_1 + c_2)^T x$. Note also that (12) is clearly valid for any x satisfying $c_1^T x \ge d_1$ and $c_2^T x \ge d_2$ since the two sides of the inequality have different signs in this case. In total, the set

$$\mathcal{G}_s^+ := \{x : (12) \text{ holds},\ b_s^T x \ge 0\}$$

is a valid relaxation when Assumption 6 does not hold. Although not obvious, it follows from [30] that (12) is a convex inequality. In that paper, (12) was encountered from a different viewpoint, and its convexity was established directly, even though it does not admit an SOC representation. So in fact $\mathcal{G}_s^+$ is convex.

Now let us assume that Assumption 4 holds as well so that $\mathcal{F}_s^+$ captures the conic hull of the intersection of $\mathcal{F}_0^+$ and $(c_1^T x - d_1)(c_2^T x - d_2) \le 0$. We claim that $\mathcal{F}_0^+ \cap \mathcal{G}_s^+$ captures the conic hull when Assumption 6 does not hold. (A similar claim will also hold when Assumption 5 holds for the further intersection with H^1.) So let $\hat{x} \in \mathcal{F}_0^+ \cap \mathcal{G}_s^+$ be given. If (11) happens to hold also, then $\hat{x}^T A_s \hat{x} \le 0 \Rightarrow \hat{x} \in \mathcal{F}_s^+$. Then $\hat{x}$ is already in the closed convex hull given by $(c_1^T x - d_1)(c_2^T x - d_2) \le 0$ by assumption. On the other hand, if (11) does not hold, then it must be that $(c_1 + c_2)^T \hat{x} > d_1 + d_2$. So either $c_1^T \hat{x} > d_1$ or $c_2^T \hat{x} > d_2$. Whichever the case, $\hat{x}$ satisfies the disjunction. Therefore $\hat{x}$ is in the closed convex hull, which gives the desired conclusion.

We remark that, despite their different forms, (12) and the inequality defining $\mathcal{F}_s^+$ both originate from $x^T A_s x \leq 0$ and match precisely on the boundary of $\text{conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1) \setminus (\mathcal{F}_0^+ \cap \mathcal{F}_1)$, e.g., the points added due to convexification process. Moreover, (12) can be interpreted as adding all of the recessive directions $\{d \in \mathcal{K} : c_1^T d \geq 0, c_2^T d \geq 0\}$ of the disjunction to the set $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$. Finally, [30] shows in addition that the linear inequality $b_s^T x \geq 0$ is in fact redundant for $\mathcal{G}_s^+$.

6.4 The case (c) of $d_1 > d_2$

As mentioned above, the results of [30] ensure that $\text{cl.conic.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1)$ requires more than two conic inequalities, making it highly likely that the closed convex hull of $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ requires more than two also. In other words, our theory would not apply in this case in general. So we ask: which assumptions are violated in this case?

Let us first consider when $d_1 d_2 = 0$, which covers two subcases. Then

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & -d_2 c_1 - d_1 c_2 \\ -d_2 c_1^T - d_1 c_2^T & 0 \end{pmatrix},$$

and it is clear that Assumption 3 is not satisfied.

Now consider the remaining subcase when $(d_1, d_2) = (1, -1)$. Then

$$A_0 := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} c_1 c_2^T + c_2 c_1^T & c_1 - c_2 \\ c_1^T - c_2^T & -2 \end{pmatrix}.$$

Assumption 1 holds, and Assumption 2 is the full-dimensional case of interest. Assumption 3(iii) holds as well, so $s = 0$. Then Assumption 4 requires $v^T A_1 v < 0$, where $v = (0; \ldots; 0; 1)$, which is true. On the other hand, Assumption 5 might fail. In fact, the example in Section 4.6 provides just such an instance. This being said, the same stronger assumption discussed in Section 6.2 can be seen to imply Assumption 5, that is, when there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 c_1 + c_2 \in -\mathcal{K}$ and $\beta_2 c_1 + c_2 \in \mathcal{K}$. This covers the case of split disjunctions, for example.

Of course, even when all assumptions do not hold, just Assumptions 1-3, which hold when $d_1 d_2 = -1$, are enough to ensure the valid relaxations $\mathcal{F}_0^+ \cap \mathcal{F}_s^+$ and $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^1$. However, these relaxations may not be sufficient to describe the conic and convex hulls.

If necessary, another way to generate valid conic inequalities when $d_1 > d_2$ is as follows. Instead of the original disjunction, consider the weakened disjunction $c_1^T x \geq d_2 \vee c_2^T x \geq d_2$, where d_2 replaces d_1 in the first term. Clearly any point satisfying the original disjunction will also satisfy the new disjunction. Therefore any valid inequality for the new disjunction will also be valid for the original one. In Sections 6.1 and 6.2, we have discussed the conditions under which Assumptions 1-5 are satisfied when $d_1 = d_2$. Even if the new disjunction violates Assumption 6, as long as the original disjunction satisfies Assumption 6, the resulting inequalities from this approach will be valid.

6.5 Conic sections

Let $\rho_1^T x \geq d_1 \vee \rho_2^T x \geq d_2$ be a disjunction on a cross-section $\mathcal{K} \cap H^1$ of the second-order cone, where $H^1 = \{x : h^T x = 1\}$. We make an assumption in analogy with Assumption 6:

Assumption 7. *The disjunctive sets $\mathcal{K}_1 := \mathcal{K} \cap H^1 \cap \{x : \rho_1^T x \geq d_1\}$ and $\mathcal{K}_2 := \mathcal{K} \cap H^1 \cap \{x : \rho_2^T x \geq d_2\}$ are non-intersecting except possibly on their boundaries, e.g.,*

$$\mathcal{K}_1 \cap \mathcal{K}_2 \subseteq \left\{ x \in \mathcal{K} \cap H^1 : \begin{array}{c} \rho_1^T x = d_1 \\ \rho_2^T x = d_2 \end{array} \right\}.$$

We would like to characterize the convex hull of the disjunction, which is the same as the convex hull of the disjunction $(\rho_1 - d_1 h)^T x \geq 0 \vee (\rho_2 - d_2 h)^T x \geq 0$ on $\mathcal{K} \cap H^1$. Defining $c_1 := \rho_1 - d_1 h$, $c_2 := \rho_2 - d_2 h$, $A_0 := J$, and $A_1 := c_1 c_2^T + c_2 c_1^T$, our goal is to characterize $\operatorname{cl.conv.hull}(\mathcal{K} \cap \mathcal{F}_1 \cap H^1)$. This is quite similar to the analysis in Section 6.1 except that here we also must verify Assumption 5.

Assumptions 1 and 3(i) are easily verified, and Assumption 2 describes the full-dimensional case of interest. Following the development in Section 6.1, we can verify Assumption 4 when $\|\tilde{\rho}_1 - d_1 \tilde{h}\|_2 > |\rho_{1,n} - d_1 h_n|$ and $\|\tilde{\rho}_2 - d_2 \tilde{h}\|_2 > |\rho_{2,n} - d_2 h_n|$, and otherwise the convex hull is easy to determine. For Assumption 5, we consider the cases of ellipsoids, paraboloids, and hyperboloids separately.

Ellipsoids are characterized by $h \in \operatorname{int}(\mathcal{K})$, and so $\mathcal{K} \cap H^0 = \{0\}$. Thus $\mathcal{K} \cap \mathcal{F}_s^+ \cap H^0 = \{0\} \subseteq \mathcal{F}_1$ easily verifying Assumption 5. On the other hand, paraboloids are characterized by $0 \neq h \in \operatorname{bd}(\mathcal{K})$, and in this case, $\mathcal{K} \cap H^0 = \operatorname{cone}\{\hat{h}\}$, where $\hat{h} := -Jh = \binom{-\tilde{h}}{h_n}$. Thus, to verify Assumption 5, it suffices to show $\hat{h} \in \mathcal{F}_s^+ \Rightarrow \hat{h} \in \mathcal{F}_1$. Indeed $\hat{h} \in \mathcal{F}_s^+$ implies

$$0 \geq \hat{h}^T A_s \hat{h} = (1-s)\, \hat{h}^T J \hat{h} + s\, \hat{h}^T A_1 \hat{h} = s\, \hat{h}^T A_1 \hat{h}$$

because $h \in \operatorname{bd}(\mathcal{K})$ ensures $\hat{h}^T J \hat{h} = 0$. So $\hat{h} \in \mathcal{F}_1$.

It remains only to verify Assumption 5 for hyperboloids, which are characterized by $h \notin \pm\mathcal{K}$, i.e., $h = \binom{\tilde{h}}{h_n}$ satisfies $\|\tilde{h}\| > |h_n|$. However, it seems difficult to verify Assumption 5 generally. Still, we note that $\hat{h} \in H^0$ implies

$$\hat{h}^T A_1 \hat{h} = 2(c_1^T \hat{h})(c_2^T \hat{h}) = 2(\rho_1^T \hat{h} - d_1 h^T \hat{h})(\rho_2^T \hat{h} - d_2 h^T \hat{h}) = 2(\rho_1^T \hat{h})(\rho_2^T \hat{h}).$$

Then Assumption 5 would hold, for example, when ρ_1 and ρ_2 satisfy the following, which is identical to conditions discussed in Sections 6.2 and 6.4: there exists $\beta_1, \beta_2 \geq 0$ such that $\beta_1 \rho_1 + \rho_2 \in -\mathcal{K}$ and $\beta_2 \rho_1 + \rho_2 \in \mathcal{K}$. This covers the case of split disjunctions, for example.

We remark that our analysis in this subsection covers all of the various cases of split disjunctions found in [32].

7 General Quadratics with Conic Sections

In this section, we examine the case of (nearly) general quadratics intersected with conic sections of the SOC. For simplicity of presentation, we will employ affine transformations of the sets $\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1$ of interest. It is clear that our theory is not affected by affine transformations.

7.1 Ellipsoids

Consider the set

$$\left\{ y \in \Re^n \ : \ \begin{array}{c} y^T y \leq 1 \\ y^T Q y + 2\, g^T y + f \leq 0 \end{array} \right\},$$

where $\lambda_{\min}[Q] < 0$. Note that if $\lambda_{\min}[Q] \geq 0$, then the set is already convex. Allowing an affine transformation, this set models the intersection of any ellipsoid with a general quadratic inequality. We can model this set in our framework by homogenizing $x = \binom{y}{x_{n+1}}$ and taking

$$A_0 := \begin{pmatrix} I & 0 \\ 0^T & -1 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g \\ g^T & f \end{pmatrix}, \quad H^1 := \{x : x_{n+1} = 1\}.$$

We would like to compute $\mathrm{cl.\,conv.\,hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$.

Assumptions 1 and 3(i) are clear, and Assumption 2 describes the full-dimensional case of interest. When $s < 1$, Assumption 5 is satisfied because, in this case, $\mathcal{F}_0^+ \cap H^0 = \{0\}$ making the containment $\mathcal{F}_0^+ \cap \mathcal{F}_s^+ \cap H^0 \subseteq \mathcal{F}_1$ trivial. So it remains to prove Assumption 4. In Sections 7.1.1 and 7.1.2 below, we break the analysis into two subcases that we are able to handle: (i) when $\lambda_{\min}[Q]$ has multiplicity $k \geq 2$; and (ii) when $\lambda_{\min}[Q] \leq f$ and $g = 0$.

Subcase (i) covers, for example, the situation of deleting the interior of an arbitrary ball from the unit ball. Indeed, consider

$$\left\{ x \in \Re^n \ : \ \begin{array}{c} x^T x \leq 1 \\ (x - c)^T (x - c) \geq r^2 \end{array} \right\},$$

where $c \in \Re^n$ and $r > 0$ are the center and radius of the ball to be deleted. Then case (i) holds with $(Q, g, f) = (-I, c, r^2 - c^T c)$. On the other hand, subcase (ii) can handle, for example, the deletion of the interior of an arbitrary ellipsoid from the unit ball—as long as that ellipsoid shares the origin as its center. In other words, the portion to delete is defined by $x^T E x < r^2$, for some $E \succ 0$ and $r > 0$, and we take $(Q, g, f) = (-E, 0, r^2)$. Note that $\lambda_{\min}[Q] \le -f \Leftrightarrow \lambda_{\max}[E] \ge r^2$, which occurs if and only if the deleted ellipsoid contains a point on the boundary of the unit ball. This is the most interesting case because, if the deleted ellipsoid were either completely inside or outside the unit ball, then the convex hull would simply be the unit ball itself. This case was also studied in Corollary 4.2 of [32].

7.1.1 When $\lambda_{\min}[Q]$ has multiplicity $k \ge 2$

Define $B_t := (1-t)I + tQ$ to be the top-left $n \times n$ corner of A_t. Since $\lambda_{\min}[B_1] < 0$ with multiplicity $k \ge 2$, there exists $r \in (0,1)$ such that: (i) $B_r \succeq 0$; (ii) $\lambda_{\min}[B_r] = 0$ with multiplicity k; (ii) $B_t \succ 0$ for all $t < r$. We claim that $s = r$ as a consequence of the interlacing of eigenvalues with respect to A_t and B_t. Indeed, let $\lambda_{n+1}^t := \lambda_{\min}[A_t]$ and λ_n^t denote the two smallest eigenvalues of A_t, and let ρ_n^t and ρ_{n-1}^t denote the analogous eigenvalues of B_t. It is well known that

$$\lambda_{n+1}^t \le \rho_n^t \le \lambda_n^t \le \rho_{n-1}^t.$$

When $t < r$, we have $\lambda_{n+1}^t < 0 < \rho_n^t \le \lambda_n^t$, and when $t = r$, we have $\lambda_{n+1}^r < 0 \le \lambda_n^r \le 0$, which proves $s = r$.

Since $\dim(\text{null}(B_s)) = k \ge 2$ and $\dim(\text{span}\{g\}^\perp) = n - 1$, there exists $0 \ne z \in \text{null}(B_s)$ such that $g^T z = 0$. We can show that $\binom{z}{0} \in \text{null}(A_s)$:

$$A_s \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s & s\,g \\ s\,g^T & (1-s)(-1) + sf \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s z \\ s\,g^T z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = z^T Q z < 0$ because $z \in \text{null}(B_s)$ if and only if z is a eigenvector of $B_1 = Q$ corresponding to $\lambda_{\min}[Q]$. This verifies Assumption 4.

7.1.2 When $\lambda_{\min}[Q] \le -f$ and $g = 0$

The argument is similar to the preceding subcase in Section 7.1.1. We first observe that

$$A_t = \begin{pmatrix} (1-t)I + tQ & 0 \\ 0 & (1-t)(-1) + tf \end{pmatrix} =: \begin{pmatrix} B_t & 0 \\ 0 & \beta_t \end{pmatrix}$$

is block diagonal, so that the singularity of A_t is determined by the singularity of B_t and β_t. B_t is first singular when $t = 1/(1 - \lambda_{\min}[Q])$, while β_t is first singular when $t = 1/(1+f)$ (assuming $f > 0$; if not, then β_t is never singular). Note that

$$\frac{1}{1-\lambda_{\min}[Q]} \leq \frac{1}{1+f} \quad \Longleftrightarrow \quad \lambda_{\min}[Q] \leq -f,$$

which holds by assumption. So B_t is singular before β_t, leading to $s = 1/(1 - \lambda_{\min}[Q])$. Let $0 \neq z \in \text{null}(B_s)$. Then, we have $Qz = -\frac{1-s}{s}z$, and thus, $\binom{z}{0} \in \text{null}(A_s)$ with $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = z^T Q z < 0$. Assumption 4 is hence verified.

7.2 The trust-region subproblem

We show in this subsection that our methodology can be used to solve the trust-region subproblem

$$\min_{\tilde{y} \in \Re^{n-1}} \left\{ \tilde{y}^T \tilde{Q} \tilde{y} + 2\, \tilde{g}^T \tilde{y} : \ \tilde{y}^T \tilde{y} \leq 1 \right\}, \tag{13}$$

where $\lambda_{\min}[\tilde{Q}] < 0$. Without loss of generality, we assume that $\tilde{Q}$ is diagonal with $\tilde{Q}_{(n-1)(n-1)} = \lambda_{\min}[\tilde{Q}]$ after applying an orthogonal transformation that does not change the feasible set.

We first argue that (13) is equivalent to a trust-region subproblem

$$\min_{y \in \Re^n} \left\{ y^T Q y + 2\, g^T y : \ y^T y \leq 1 \right\} \tag{14}$$

in the n-dimensional variable $y := \binom{\tilde{y}}{y_n}$. Indeed, define

$$Q := \begin{pmatrix} \tilde{Q} & 0 \\ 0^T & \lambda_{\min}[\tilde{Q}] \end{pmatrix}, \quad g := \begin{pmatrix} \tilde{g} \\ -\frac{1}{2}\tilde{g}_{n-1} \end{pmatrix},$$

and note that $\lambda_{\min}[Q]$ has multiplicity at least 2. The following proposition shows that (14) is equivalent to (13).

Proposition 8. *There exists an optimal solution of (14) with $y_n = 0$. In particular, the optimal values of (13) and (14) are equal.*

Proof. Let $\bar{y}$ be an optimal solution of (14). Then $(\bar{y}_{n-1}; \bar{y}_n)$ is an optimal solution of the two-dimensional trust-region subproblem

$$\min_{y_{n-1}, y_n} \left\{ |\lambda_{\min}[\tilde{Q}]|(-y_{n-1}^2 - y_n^2) + 2\tilde{g}_{n-1}(y_{n-1} - \tfrac{1}{2}y_n) : \quad y_{n-1}^2 + y_n^2 \leq \epsilon \right\}.$$

where $\epsilon := 1 - (\bar{y}_1^2 + \cdots \bar{y}_{n-2}^2)$. Since we are minimizing a concave function over the ellipsoid, at least one optimal solution will be on the boundary of this set. In particular, whenever $\tilde{g}_{n-1} > 0$, the solution $\binom{y_{n-1}}{y_n} = \binom{-\sqrt{\epsilon}}{0}$ is optimal, and when $\tilde{g}_{n-1} \le 0$, the solution $\binom{y_{n-1}}{y_n} = \binom{\sqrt{\epsilon}}{0}$ is optimal. Thus, this problem has at least one optimal solution with $y_n = 0$. Hence, $\bar{y}_n$ can equal 0. □

With the proposition in hand, we now focus on the solution of (14).

A typical approach to solve (14) is to introduce an auxiliary variable x_{n+2} (where we reserve the variable x_{n+1} for later homogenization) and to recast the problem as

$$\min \left\{ x_{n+2} : \begin{array}{l} y^T y \le 1 \\ y^T Q y + 2\, g^T y \le x_{n+2} \end{array} \right\}.$$

If one can compute the closed convex hull of this feasible set, then (14) is solvable by simply minimizing x_{n+2} over the convex hull. We can represent this approach in our framework by taking $x = (y; x_{n+1}; x_{n+2})$, homogenizing via $x_{n+1} = 1$, and defining

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & -1 & 0 \\ 0^T & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g & 0 \\ g^T & 0 & -\frac{1}{2} \\ 0^T & -\frac{1}{2} & 0 \end{pmatrix}, \quad H^1 := \{x \in \Re^{n+2} : x_{n+1} = 1\}.$$

Clearly, Assumptions 1 and 2 are satisfied. However, no part of Assumption 3 is satisfied. So we require a different approach.

Since $x = 0$ is feasible for (14), its optimal value is nonpositive. (In fact, it is negative since Q has a negative eigenvector, so that $x = 0$ is not a local minimizer). Hence, (14) is equivalent to

$$v := \min \left\{ x_{n+2}^2 : \begin{array}{l} y^T y \le 1 \\ y^T Q y + 2\, g^T y \le -x_{n+2}^2 \end{array} \right\}, \tag{15}$$

which can be solved in stages: first, minimize x_{n+2} over the feasible set of (15) (let l be the minimal value); second, separately maximize x_{n+2} over the same (let u be the maximal value); and finally take $v = \min\{-l^2, -u^2\}$. If one can compute the closed convex hull of (15), then l and u can be computed easily.

To represent the feasible set of (15) in our framework, we define $x = (y; x_{n+1}; x_{n+2})$ and take

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & -1 & 0 \\ 0^T & 0 & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} Q & g & 0 \\ g^T & 0 & 0 \\ 0^T & 0 & 1 \end{pmatrix}, \quad H^1 := \{x \in \Re^{n+2} : x_{n+1} = 1\}.$$

Clearly, Assumptions 1 and 2 are satisfied, and Assumption 3(ii) is now satisfied. For Assumptions 4 and 5, we note that A_t has a block structure such that s equals the smallest positive t such that

$$B_t := (1-t)\begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} + t\begin{pmatrix} Q & g \\ g^T & 0 \end{pmatrix}$$

is singular. Using an argument similar to Section 7.1.1 and exploiting the fact that $\lambda_{\min}[Q]$ has multiplicity at least 2, we can compute s such that there exists $0 \neq z \in \text{null}(B_s) \subseteq \Re^{n+1}$ with $z^T B_1 z < 0$ and $z_{n+1} = 0$. By appending an extra 0 entry, this z can be easily extended to $z \in \Re^{n+2}$ with $z^T A_1 z < 0$ and $z \in H^0$. This simultaneously verifies Assumptions 4 and 5.

7.3 Paraboloids

Consider the set

$$\left\{ y = \begin{pmatrix} \tilde{y} \\ y_n \end{pmatrix} \in \Re^n \ : \ \begin{array}{c} \tilde{y}^T \tilde{y} \leq y_n \\ \tilde{y}^T \tilde{Q} \tilde{y} + 2\, g^T y + f \leq 0 \end{array} \right\},$$

where $\lambda := \lambda_{\min}[\tilde{Q}] < 0$ and $2g_n \leq -\lambda$. After an affine transformation, this models the intersection of a paraboloid with any quadratic inequality that is strictly linear in x_n, i.e., no quadratic terms involve x_n. Note that if $\lambda_{\min}[Q] \geq 0$, then the set is already convex. The reason for the upper bound on $2g_n$ will become evident shortly.

Writing $g := \binom{\tilde{g}}{g_n}$, we can model this situation with $x = \binom{y}{x_{n+1}}$ and

$$A_0 := \begin{pmatrix} I & 0 & 0 \\ 0^T & 0 & -\frac{1}{2} \\ 0^T & -\frac{1}{2} & 0 \end{pmatrix}, \quad A_1 := \begin{pmatrix} \tilde{Q} & 0 & \tilde{g} \\ 0^T & 0 & g_n \\ \tilde{g}^T & g_n & f \end{pmatrix}, \quad H^1 := \{x : x_{n+1} = 1\},$$

and we would like to compute $\text{cl.conv.hull}(\mathcal{F}_0^+ \cap \mathcal{F}_1 \cap H^1)$. Assumptions 1 and 3(i) are clear, and Assumption 2 describes the full-dimensional case of interest. So it remains to verify Assumptions 4 and 5.

Define

$$B_t := \begin{pmatrix} (1-t)I + t\tilde{Q} & 0 \\ 0 & 0 \end{pmatrix}$$

to be the top-left $n \times n$ corner of A_t, and define $r := 1/(1-\lambda)$. Due to its structure, B_t is positive semidefinite for all $t \leq r$. Moreover, B_t has exactly one zero eigenvalue for $t < r$, and B_r has at least two zero eigenvalues. Those two zero eigenvalues ensure that A_r is singular by the interlacing of eigenvalues of A_t and B_t (similar to Section 7.1.1). So $s \leq r$.

We claim that in fact $s = r$. Let $t < r$, and consider the following system for $\text{null}(A_t)$:

$$\begin{pmatrix} (1-t)I + t\tilde{Q} & 0 & t\,\tilde{g} \\ 0^T & 0 & (1-t)(-\frac{1}{2}) + t\,g_n \\ t\,\tilde{g}^T & (1-t)(-\frac{1}{2}) + t\,g_n & tf \end{pmatrix} \begin{pmatrix} \tilde{z} \\ z_n \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that $2g_n \le -\lambda$ and $0 \le t < r$ imply

$$2\left[(1-t)(-\tfrac{1}{2}) + t\,g_n\right] = t(1 + 2\,g_n) - 1 \le t(1-\lambda) - 1 < r(1-\lambda) - 1 = 0, \tag{16}$$

which implies $z_{n+1} = 0$. This in turn implies $\tilde{z} = 0$ because $(1-t)I + t\tilde{Q} \succ 0$ when $t < r$. Finally, $z_n = 0$ again due to (16). So we conclude that $t < r$ implies $\text{null}(A_t) = \{0\}$. Hence, $s = r$.

We next write

$$A_s = \begin{pmatrix} B_s & g_s \\ g_s & sf \end{pmatrix}.$$

Since $\dim(\text{null}(B_s)) \ge 2$ and $\dim(\text{span}\{g_s\}^\perp) = n - 1$, there exists $0 \ne z \in \text{null}(B_s)$ such that $g_s^T z = 0$. From the structure of B_s, we have $z = \binom{\tilde{z}}{z_n}$, where $\tilde{z}$ is a negative eigenvector of $\tilde{Q}$. We claim that $\binom{z}{0} \in \text{null}(A_s)$. Indeed:

$$A_s \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s & g_s \\ g_s^T & sf \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} = \begin{pmatrix} B_s z \\ g_s^T z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Moreover, $\binom{z}{0}^T A_1 \binom{z}{0} = z^T B_1 z = \tilde{z}^T \tilde{Q} \tilde{z} < 0$. This verifies Assumptions 4 and 5.

We remark that corollary 4.1 in [32] studies the closed convex hull of the set

$$\left\{ y = \begin{pmatrix} \tilde{y} \\ y_n \end{pmatrix} \in \Re^n : \|\tilde{B}(\tilde{y} - \tilde{c})\|^2 \le y_n,\ \|\tilde{A}(\tilde{y} - \tilde{d})\|^2 \ge -\gamma\, y_n + q \right\},$$

where $\tilde{B} \in \Re^{(n-1)\times(n-1)}$ is an invertible matrix, $\tilde{c}, \tilde{d} \in \Re^{n-1}$ and $\gamma, q \ge 0$. This situation is covered by our theory here.

8 Conclusion

This paper provides basic convexity results regarding the intersection of a second-order-cone representable set and a nonconvex quadratic. Although several results have appeared in the prior literature, we unify and extend these by introducing a simple, computable technique for aggregating (with nonnegative weights) the inequalities defining the two intersected sets.

The underlying assumptions of our theory can be checked in many cases of interest.

Beyond the examples detailed in this paper, our technique can be used in other ways. Consider for example, a general quadratically constrained quadratic program, whose objective has been linearized without loss of generality. If the constraints include an ellipsoid constraint, then our techniques can be used to generate valid SOC inequalities for the convex hull of the feasible region by pairing each nonconvex quadratic constraint with the ellipsoid constraint one by one. The theoretical and practical strength of this technique is of interest for future research, and the techniques in [2, 31] could provide a good point of comparison.

In addition, it would be interesting to investigate whether our techniques could be extended to produce valid inequalities or explicit convex hull descriptions for intersections involving multiple second-order cones or multiple nonconvex quadratics.

References

[1] K. Andersen and A. N. Jensen. Intersection cuts for mixed integer conic quadratic sets. In *Proceedings of IPCO 2013*, volume 7801 of *Lecture Notes in Computer Science*, pages 37–48, Valparaiso, Chile, March 2013.

[2] I. P. Androulakis, C. D. Maranas, and C. A. Floudas. αBB: a global optimization method for general constrained nonconvex problems. *Journal of Global Optimization*, 7(4):337–363, 1995. State of the art in global optimization: computational methods and applications (Princeton, NJ, 1995).

[3] K. M. Anstreicher and S. Burer. Computable representations for convex hulls of low-dimensional quadratic forms. *Mathematical programming*, 124(1-2):33–43, 2010.

[4] A. Atamtürk and V. Narayanan. Conic mixed-integer rounding cuts. *Mathematical Programming*, 122(1):1–20, 2010.

[5] E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.

[6] E. Balas. Disjunctive programming. *Annals of Discrete Mathematics*, 5:3–51, 1979.

[7] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58:295–324, 1993.

[8] X. Bao, N. V. Sahinidis, and M. Tawarmalani. Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons. *Mathematical Programming*, 129(1):129–157, 2011.

[9] P. Belotti. Disjunctive cuts for nonconvex MINLP. In J. Lee and S. Leyffer, editors, *Mixed Integer Nonlinear Programming*, volume 154 of *The IMA Volumes in Mathematics and its Applications*, pages 117–144. Springer, New York, NY, 2012.

[10] P. Belotti, J. Góez, I. Pólik, T. Ralphs, and T. Terlaky. On families of quadratic surfaces having fixed intersections with two hyperplanes. *Discrete Applied Mathematics*, 161(16):2778–2793, 2013.

[11] P. Belotti, J. C. Goez, I. Polik, T. K. Ralphs, and T. Terlaky. A conic representation of the convex hull of disjunctive sets and conic cuts for integer second order cone optimization. Technical report, June 2012. Department of Industrial and Systems Engineering, Lehigh University, Bethlehem, PA, `http://www.optimization-online.org/DB_FILE/2012/06/3494.pdf`.

[12] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization*. MPS-SIAM Series on Optimization. SIAM, Philadehia, PA, 2001.

[13] D. Bienstock and A. Michalka. Cutting-planes for optimization of convex functions over nonconvex sets. *SIAM Journal on Optimization*, 24(2):643–677, 2014.

[14] P. Bonami. Lift-and-project cuts for mixed integer convex programs. In O. Gunluk and G. J. Woeginger, editors, *Proceedings of the 15th IPCO Conference*, volume 6655 of *Lecture Notes in Computer Science*, pages 52–64, New York, NY, 2011. Springer.

[15] S. Burer and K. M. Anstreicher. Second-order-cone constraints for extended trust-region subproblems. *SIAM Journal on Optimization*, 23(1):432–451, 2013.

[16] S. Burer and A. Saxena. The MILP road to MIQCP. In *Mixed Integer Nonlinear Programming*, pages 373–405. Springer, 2012.

[17] F. Cadoux. Computing deep facet-defining disjunctive cuts for mixed-integer programming. *Mathematical Programming*, 122(2):197–223, 2010.

[18] M. Çezik and G. Iyengar. Cuts for mixed 0-1 conic programming. *Mathematical Programming*, 104(1):179–202, 2005.

[19] A. R. Conn, N. I. M. Gould, and P. L. Toint. *Trust-Region Methods*. MPS/SIAM Series on Optimization. SIAM, Philadelphia, PA, 2000.

[20] G. Cornuéjols and C. Lemaréchal. A convex-analysis perspective on disjunctive cuts. *Mathematical Programming*, 106(3):567–586, 2006.

[21] D. Dadush, S. Dey, and J. Vielma. The split closure of a strictly convex body. *Operations Research Letters*, 39:121–126, 2011.

[22] S. Drewes. *Mixed Integer Second Order Cone Programming*. PhD thesis, Technische Universität Darmstadt, 2009.

[23] S. Drewes and S. Pokutta. Cutting-planes for weakly-coupled 0/1 second order cone programs. *Electronic Notes in Discrete Mathematics*, 36:735–742, 2010.

[24] N. I. M. Gould, S. Lucidi, M. Roma, and P. L. Toint. Solving the trust-region subproblem using the Lanczos method. *SIAM Journal on Optimization*, 9(2):504–525 (electronic), 1999.

[25] J. Hu, J. E. Mitchell, J.-S. Pang, K. P. Bennett, and G. Kunapuli. On the global solution of linear programs with linear complementarity constraints. *SIAM J. Optim.*, 19(1):445–471, 2008.

[26] J. J. Júdice, H. Sherali, I. M. Ribeiro, and A. M. Faustino. A complementarity-based partitioning and disjunctive cut algorithm for mathematical programming problems with equilibrium constraints. *Journal of Global Optimization*, 136:89–114, 2006.

[27] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin-New York, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[28] M. Kılınç, J. Linderoth, and J. Luedtke. Effective separation of disjunctive cuts for convex mixed integer nonlinear programs. Technical report, 2010. `http://www.optimization-online.org/DB_FILE/2010/11/2808.pdf`.

[29] F. Kılınç-Karzan. On minimal valid inequalities for mixed integer conic programs. Technical report, June 2013. GSIA Working Paper Number: 2013-E20, GSIA, Carnegie Mellon University, Pittsburgh, PA, `http://www.optimization-online.org/DB_FILE/2013/06/3936.pdf`.

[30] F. Kılınç-Karzan and S. Yıldız. Two term disjunctions on the second-order cone. Technical report, April 2014. `http://arxiv.org/pdf/1404.7813v1.pdf`.

[31] S. Kim and M. Kojima. Second order cone programming relaxation of nonconvex quadratic optimization problems. *Optimization Methods and Software*, 15(3-4):201–224, 2001.

[32] S. Modaresi, M. Kılınç, and J. Vielma. Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets. Technical report, March 2013. `http://www.optimization-online.org/DB_FILE/2013/02/3767.pdf`.

[33] S. Modaresi, M. Kılınç, and J. Vielma. Split cuts and extended formulations for mixed integer conic quadratic programming. Technical report, February 2014. `http://web.mit.edu/jvielma/www/publications/Split-Cuts-and-Extended-Formulations.pdf`.

[34] J. J. Moré and D. C. Sorensen. Computing a trust region step. *SIAM Journal on Scientific and Statistical Computing*, 4(3):553–572, 1983.

[35] T. T. Nguyen, M. Tawarmalani, and J.-P. P. Richard. Convexification techniques for linear complementarity constraints. In O. Günlük and G. J. Woeginger, editors, *IPCO*, volume 6655 of *Lecture Notes in Computer Science*, pages 336–348. Springer, 2011.

[36] F. Rellich. *Perturbation theory of eigenvalue problems.* Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[37] F. Rendl and H. Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Mathematical Programming*, 77(2, Ser. B):273–299, 1997.

[38] A. Saxena, P. Bonami, and J. Lee. Disjunctive cuts for non-convex mixed integer quadratically constrained programs. In A. Lodi, A. Panconesi, and G. Rinaldi, editors, *IPCO*, volume 5035 of *Lecture Notes in Computer Science*, pages 17–33. Springer, 2008.

[39] H. Sherali and C. Shetti. Optimization with disjunctive constraints. *Lectures on Econ. Math. Systems*, 181, 1980.

[40] R. A. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming. *Mathematical Programming*, 86(3):515–532, 1999.

[41] M. Tawarmalani, J. Richard, and K. Chung. Strong valid inequalities for orthogonal disjunctions and bilinear covering sets. *Mathematical Programming*, 124(1-2):481–512, 2010.

[42] M. Tawarmalani, J.-P. P. Richard, and C. Xiong. Explicit convex and concave envelopes through polyhedral subdivisions. *Mathematical Programming*, 138(1-2):531–577, 2013.

[43] J. Vielma, S. Ahmed, and G. Nemhauser. A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs. *INFORMS Journal on Computing*, 20(3):438–450, 2008.