C

Convex Hulls in Quadratic Space

Boshi Yang[1] and Sam Burer[2]
[1]School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, USA
[2]Department of Business Analytics, University of Iowa, Iowa, IA, USA

Article Outline

Keywords

Quadratic hull · Convex hull · Convexification · Quadratically constrained quadratic programming · Semidefinite programming · Trust-region subproblem

MSC Codes

90C20, 90C22, 90C26

Introduction

In mathematical optimization, nonconvex problems are generally considered difficult since they harbor many locally optimal solutions. One approach to deal with nonconvexity is to reformulate the problem as a convex one, a process known as *convexification*, and in some cases, convexification yields a polynomial-time variant of the original problem. In such cases, the nonconvex problem is thus considered easy to solve. Convexification can furthermore be applied to substructures within a problem, leading to improved convex relaxations and better bounds on the optimal value of the original problem.

Consider a generic optimization problem with a linear objective function:

$$\inf\{\, c^T x \mid x \in \mathcal{F} \,\},$$

where $c \in \mathbb{R}^n$ and $\mathcal{F} \subseteq \mathbb{R}^n$ is a nonempty set. If $\mathcal{F}$ is nonconvex, the problem can be convexified to

$$\inf\{\, c^T x \mid x \in \operatorname{conv}(\mathcal{F}) \,\},$$

where $\operatorname{conv}(\mathcal{F})$ equals the convex hull of $\mathcal{F}$. For example, in integer linear programming, when $\mathcal{F}$ equals a subset of the integer lattice $\mathbb{Z}^n$, $\operatorname{conv}(\mathcal{F})$ is referred to as the *integer hull* of $\mathcal{F}$, and understanding $\operatorname{conv}(\mathcal{F})$ is crucial for solving integer linear programs.

When the linear objective above is replaced by a quadratic function, convexification involves

P. M. Pardalos, O. A. Prokopyev (eds.), *Encyclopedia of Optimization*,
https://doi.org/10.1007/978-3-030-54621-2_892-1

convex hulls in the space of linear terms x_i and quadratic terms $x_i x_j$. Understanding the structure of this so-called *quadratic hull* plays an important role in solving quadratically constrained quadratic programs (QCQPs), a class of NP-hard problems [9].

Definitions

Let $\mathcal{F}$ be a nonempty set in $\mathbb{R}^n$, and let S^n_+ denote the set of $n \times n$ symmetric positive semidefinite matrices. The *quadratic hull* of $\mathcal{F}$ is the convex set in $\mathbb{R}^n \times S^n_+$ defined as

$$C(\mathcal{F}) := \operatorname{conv}\left\{ (x, xx^T) \;\middle|\; x \in \mathcal{F} \right\}. \quad (1)$$

The closure of $C(\mathcal{F})$, denoted by $\overline{C}(\mathcal{F})$, is called the *closed quadratic hull*. When $\mathcal{F}$ is a compact set, then $C(\mathcal{F})$ is also compact and furthermore $\overline{C}(\mathcal{F}) = C(\mathcal{F})$.

Connection to QCQP

The quadratic hull is closely related to the convexification of QCQP. Suppose that $\mathcal{F}$ is defined by quadratic constraints, i.e.,

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid x^T A_i x + 2a_i^T x + \alpha_i \le 0,\ i = 1, \ldots, m\}, \quad (2)$$

where each $A_i \in S^n$ is a real symmetric matrix, $a_i \in \mathbb{R}^n$, and $\alpha_i \in \mathbb{R}$. Then QCQP minimizes a quadratic function of x over $\mathcal{F}$:

$$v := \inf\left\{ x^T Q x + 2c^T x \;\middle|\; x \in \mathcal{F} \right\}, \quad (3)$$

where $Q \in S^n$ and $c \in \mathbb{R}^n$. Introducing an extra matrix variable $X = xx^T$, the objective function of (3) can be linearized as $Q \bullet X + 2c^T x$, where $Q \bullet X := \sum_{i,j} Q_{ij} X_{ij}$ is the Frobenius inner product of Q and X. Due to the linearity of the objective function in (x, X), one can convexify the feasible region, obtaining the following convex reformulations of (3):

$$\begin{aligned} v &= \inf\left\{ Q \bullet X + 2c^T x \;\middle|\; (x, X) \in C(\mathcal{F}) \right\} \\ &= \inf\left\{ Q \bullet X + 2c^T x \;\middle|\; (x, X) \in \overline{C}(\mathcal{F}) \right\}. \end{aligned} \quad (4)$$

Complete proofs of Eqs. (4) can be found in, for example, [5, 8].

An Example

Consider

$$\inf\left\{ -\tfrac{1}{2}x^2 + x \;\middle|\; x \in \mathcal{F} \right\}, \quad (5)$$

where $\mathcal{F} = \{ x \in \mathbb{R} \mid 0.5 \le x^2 \le 4 \}$. The graph of the objective function and $\mathcal{F}$ are depicted in Fig. 1a. By inspection, it is observed that $C(\mathcal{F}) = \{ (x, X) \in \mathbb{R} \times \mathbb{R} \mid 0.5 \le X \le 4,\ X \ge x^2 \}$. Problem (5) is then convexified to

$$\inf\left\{ -\tfrac{1}{2}X + x \;\middle|\; (x, X) \in C(\mathcal{F}) \right\},$$

which is depicted in Fig. 1b.

Relaxation of $C(\mathcal{F})$

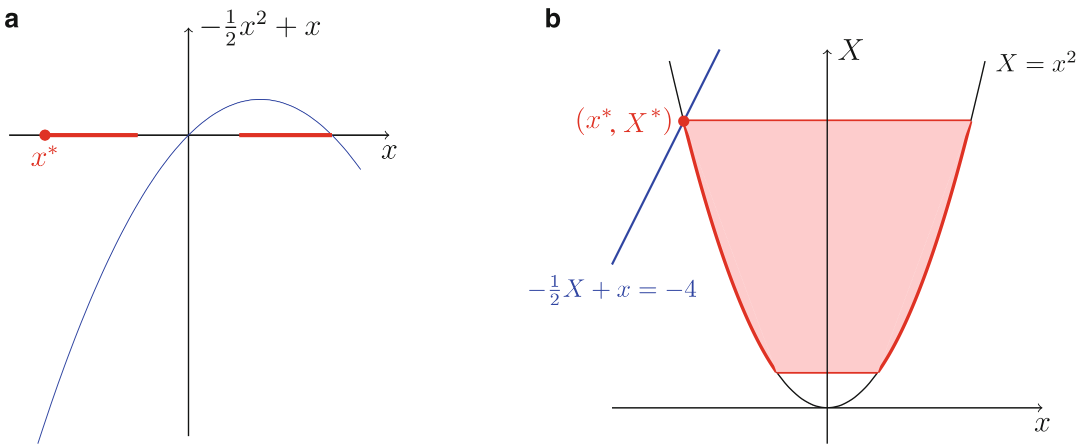
In general, determining an explicit expression for $C(\mathcal{F})$ or $\overline{C}(\mathcal{F})$ is at least as difficult as solving the corresponding QCQP. On the other hand, a polynomial-time solvable semidefinite-programming relaxation of $C(\mathcal{F})$ can be constructed in a standard manner. In particular, the *Shor relaxation* of (3) is

$$\inf\left\{ Q \bullet X + 2c^T x \;\middle|\; (x, X) \in S(\mathcal{F}) \right\}, \quad (6)$$

where

$$S(\mathcal{F}) := \{ (x, X) \mid A_i \bullet X + 2a_i^T x + \alpha_i \le 0,\ i = 1, \ldots, m,\ X \succeq xx^T \}.$$

Here, $X \succeq xx^T$ means that $X - xx^T \in S^n_+$ is a positive semidefinite matrix. Note that (6) is indeed a relaxation of (3) since, for any feasible solution x of (3), (x, xx^T) is an element of $S(\mathcal{F})$ with the same objective value. Clearly, the set is a convex set relaxation of $C(\mathcal{F})$; abusing terminology, $S(\mathcal{F})$ is also referred to as the Shor relaxation of $C(\mathcal{F})$.



Convex Hulls in Quadratic Space, Fig. 1 An example of the quadratic hull convexifying QCQP. (**a**) The set $\mathcal{F} = \{x \in \mathbb{R} \mid 0.5 \le x^2 \le 4\}$ consists of the two intervals in red. The blue curve depicts the quadratic objective function. The optimal solution is $x^* = -2$. (**b**) The quadratic hull $C(\mathcal{F})$ is depicted in red. The blue line represents a level curve of the linearized objective function. The optimal solution is $(x^*, X^*) = (-2, 4)$

Special Cases

Since QCQP is NP-hard in general, it is unrealistic to expect a tractable expression for $C(\mathcal{F})$ in general. On the other hand, $C(\mathcal{F})$ has been proved to be semidefinite representable in special cases. For all the cases below, $\mathcal{F}$ is assumed to be nonempty.

One Quadratic Constraint

When $\mathcal{F} = \{x \in \mathbb{R}^n \mid \|x\| \le \Delta\} = \{x \in \mathbb{R}^n | x^T x - \Delta^2 \le 0\}$, problem (3) is referred to as the *trust-region subproblem* (TRS), where $\Delta > 0$ is the radius of the ball constraint. TRS, which can be solved efficiently despite being nonconvex, arises as a subroutine in the trust-region method of nonlinear optimization. More generally, problem (3) is called a *generalized TRS* when

$$\mathcal{F} = \{x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \le 0\}$$

for $A \in S^n$, $a \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. In this case, the quadratic hull of $\mathcal{F}$ is a closed set and can be expressed [12] as

$$C(\mathcal{F}) = S(\mathcal{F}) = \left\{ (x, X) \;\middle|\; A \bullet X + 2a^T x + \alpha \le 0,\ X \succeq xx^T \right\}.$$

When $\mathcal{F} = \{x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha = 0\}$ is defined by a single quadratic equality, it is known that

$$C(\mathcal{F}) = S(\mathcal{F}) = \left\{ (x, X) \;\middle|\; A \bullet X + 2a^T x + \alpha = 0,\ X \succeq xx^T \right\}.$$

Sturm and Zhang [12] first prove this equation when A is definite. Xia et al. [14] show the result for general A as long as the following two-sided Slater's condition holds: there exist $\hat{x}$, $\tilde{x} \in \mathbb{R}^n$ such that $\hat{x}^T A\hat{x} + 2a^T \hat{x} + \alpha < 0 < \tilde{x}^T A\tilde{x} + 2a^T \tilde{x} + \alpha$. Furthermore, Joyce and Yang [7] show that the equation holds without qualification as long as $A \ne 0$.

When $\mathcal{F} = \{x \in \mathbb{R}^n | \ \ell \le x^T Ax + 2a^T x + \alpha \le u\}$, where $-\infty < \ell < u < \infty$, the corresponding QCQP (3) is often referred to as the *interval bounded generalized TRS*. Pong and Wolkowicz [10] show that

$$C(\mathcal{F}) = S(\mathcal{F}) = \left\{ (x, X) \;\middle|\; \ell \le A \bullet X + 2a^T x + \alpha \le u,\ X \succeq xx^T \right\}$$

under five assumptions. The assumptions are reduced in [13] and then [7] to the single assumption that $A \neq 0$. When $A = 0$, $\mathcal{F}$ has an alternative quadratic expression:

$$\mathcal{F} = \{ x \in \mathbb{R}^n \mid \ell \le 2a^T x + \alpha \le u \} =$$
$$\{ x \in \mathbb{R}^n \mid (2a^T x + \alpha - \ell)(2a^T x + \alpha - u) \le 0 \}.$$

Then, by the above discussion of the generalized TRS,

$$C(\mathcal{F}) = \Big\{ (x, X) \mid 4aa^T \bullet X + 2(2\alpha - \ell - u)a^T x$$
$$+ (\alpha - \ell)(\alpha - u) \le 0,\ X \succeq xx^T \Big\}.$$

Note that this alternative expression when $A = 0$ is necessary. For example, when $\mathcal{F} = \{ x \in \mathbb{R} \mid -2 \le x \le 1 \}$,

$$C(\mathcal{F}) = \{ (x, X) \mid x + X \le 2,\ X \ge x^2 \}$$
$$\neq \{ (x, X) \mid -2 \le x \le 1,\ X \ge x^2 \} = S(\mathcal{F}).$$

A Ball Constraint with Linear Constraints

Another class of TRS variants adds linear constraints to the ball constraint $\|x\| \le 1$. Let $\mathcal{F} = \{ x \in \mathbb{R}^n \mid \|x\| \le 1,\ b^T x \le \beta \}$, where $b \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. It is known [3, 12] that

$$C(\mathcal{F}) = \Big\{ (x, X) \mid \|\beta x - Xb\| \le \beta - b^T x,$$
$$\mathrm{trace}(X) \le 1,\ X \succeq xx^T \Big\}.$$

Burer and Anstreicher [3] call the first constraint an *SOC-RLT constraint*, where *SOC* stands for the second-order cone and *RLT* stands for the reformulation-linearization technique [11]. An SOC-RLT constraint is constructed by multiplying the nonnegative expression $\beta - b^T x$ on both sides of the second-order cone constraint $\|x\| \le 1$ before relaxing the quadratic term xx^T to X. (The result in [12] is slightly more general, where the ball constraint in $\mathcal{F}$ can be replaced with any convex quadratic constraint.)

Jin et al. [6] consider the slight generalization

$$\mathcal{F} = \{ (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid \max\{\|x_1\|, \delta_2\}$$
$$\le d_1^T x_1 + d_2^T x_2 + \delta_1 \le \delta_3 \},$$

where $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ with $0 \le \delta_2 \le \delta_3$. Let $d = \binom{d_1}{d_2}$, $x = \binom{x_1}{x_2}$, and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix},$$

where $X_{11} \in \mathcal{S}^{n_1}$, $X_{22} \in \mathcal{S}^{n_2}$, and $X_{12} \in \mathbb{R}^{n_1 \times n_2}$. The authors show

$$\overline{C}(\mathcal{F}) = C(\mathcal{F})$$
$$= \left\{ (x, X) \;\middle|\; \begin{array}{c} \|x_1\| \le d^T x + \delta_1,\ \delta_2 \le d^T x + \delta_1 \le \delta_3, ,\ X \succeq xx^T, \\ \mathrm{trace}(X_{11}) \le \delta_1^2 + 2\delta_1 d^T x + d^T X d, \\ \|(\delta_1 - \delta_2)x_1 + X_{11}d_1 + X_{12}d_2\| \le \delta_1(\delta_1 - \delta_2) + (2\delta_1 - \delta_2)d^T x + d^T X d, \\ \|(\delta_1 - \delta_3)x_1 + X_{11}d_1 + X_{12}d_2\| \le \delta_1(\delta_3 - \delta_1) + (\delta_3 - 2\delta_1)d^T x - d^T X d, \\ (\delta_3 - \delta_1)(\delta_1 - \delta_2) + (\delta_2 + \delta_3 - 2\delta_1)d^T x + d^T X d \ge 0 \end{array} \right\}.$$

The SOC representable constraints are constructed using the same SOC-RLT approach. The last inequality is a result of RLT, where the two nonnegative quantities $d^T x + \delta_1 - \delta_2$ and $\delta_3 - \delta_1 - d^T x$ are multiplied together before relaxing the quadratic term xx^T to X. It is also shown in [6] that if $\delta_3 = +\infty$, then $\overline{C}(\mathcal{F})$ is the same as above without any inequalities involving δ_3.

Multiple linear constraints are also considered. Burer and Anstreicher [3] study

$$\mathcal{F} = \{ x \in \mathbb{R}^n \mid \|x\| \le 1,\ b_1^T x \le \beta_1,\ b_2^T x \le \beta_2 \},$$

C

when the two linear constraints are parallel, i.e., when b_1 is a negative multiple of b_2. They show

$$C(\mathcal{F}) = \left\{ (x, X) \;\middle|\; \begin{array}{c} \|\beta_1 x - X b_1\| \le \beta_1 - b_1^T x,\ \|\beta_2 x - X b_2\| \le \beta_2 - b_2^T x,\ \operatorname{trace}(X) \le 1, \\ \beta_1\beta_2 - (\beta_2 b_1 + \beta_1 b_2)^T x + b_1^T X b_2 \ge 0,\ X \succeq xx^T \end{array} \right\}.$$

Burer and Yang [4] extend the result to

$$\mathcal{F} = \{x \in \mathbb{R}^n |\ \|x\| \le 1,$$
$$b_i^T x \le \beta_i,\ i = 1, \ldots, m\},$$

where the m linear constraints are non-intersecting inside the unit ball, i.e., for all pairs $i \ne j$, there exists no $x \in \mathcal{F}$ satisfying $b_i^T x = \beta_i$ and $b_j^T x = \beta_j$. They show

$$C(\mathcal{F}) = \left\{ (x, X) \;\middle|\; \begin{array}{c} \|\beta_i x - X b_i\| \le \beta_i - b_i^T x,\ i = 1, \ldots, m, \\ \beta_i\beta_j - (\beta_j b_i + \beta_i b_j)^T x + b_i^T X b_j \ge 0,\ 1 \le i < j \le m, \\ \operatorname{trace}(X) \le 1,\ X \succeq xx^T \end{array} \right\}.$$

A TRS with two complementary linear constraints is also considered in [16], where

$$\mathcal{F} = \{ x \in \mathbb{R}^n \mid \|x\| \le 1,\ b_1^T x \le \beta_1,\ b_2^T x \le \beta_2,\ (\beta_1 - b_1^T x)(\beta_2 - b_2^T x) = 0 \}.$$

In this case, no additional assumptions on b_1 and b_2 are made. As above, the quadratic hull can be expressed using RLT and SOC-RLT as

$$C(\mathcal{F}) = \left\{ (x, X) \;\middle|\; \begin{array}{c} \|\beta_1 x - X b_1\| \le \beta_1 - b_1^T x,\ \|\beta_2 x - X b_2\| \le \beta_2 - b_2^T x,\ \operatorname{trace}(X) \le 1, \\ b_1^T X b_2 - (\beta_1 b_2 + \beta_2 b_1)^T x + \beta_1\beta_2 = 0,\ X \succeq xx^T \end{array} \right\}.$$

Only Linear Constraints

When $\mathcal{F}$ is a polytope in dimension n defined by m linear inequalities, $C(\mathcal{F})$ can be represented by linear constraints plus a matrix constraint, which linearly embeds (x, X) into a cone that is closely related to the $m \times m$ completely positive cone [2]. Then, because the completely positive cone is semidefinite representable for sizes 4×4 or smaller, explicit expressions for $C(\mathcal{F})$ can be obtained for polytopes with four or fewer facets. In particular, let

$$\mathcal{F} = \{ x \in \mathbb{R}^n \mid Ax \le b \},$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. In [1, 2], it is shown that

$$C(\mathcal{F}) = \{(x, X) \mid bb^T - bx^T A^T - Axb^T$$
$$+ AXA^T \ge 0,\ X \succeq xx^T\}$$

in three different cases: $(n, m) = (2, 3)$ and $\mathcal{F}$ is a proper triangle, $(n, m) = (3, 4)$ and $\mathcal{F}$ is a proper tetrahedron, and $(n, m) = (2, 4)$ and $\mathcal{F}$ is a proper quadrilateral. Here, *proper* means that the polytope has nonempty interior. The linear constraints in $C(\mathcal{F})$, which constrain the individual entries of an $m \times m$ symmetric matrix, are constructed via RLT by forming the nonnegative outer product $(b-Ax)(b-Ax)^T \geq 0$ and then linearizing xx^T to X.

Non-Intersecting Constraints

Let $\mathcal{G} = \mathcal{F} \cap \mathcal{H}$, where $\mathcal{F} \subseteq \mathbb{R}^n$ is nonempty and closed (but not necessarily quadratically defined) and

$$\mathcal{H} = \{ x \in \mathbb{R}^n \mid x^T Ax + 2a^T x + \alpha \leq 0 \} \neq \emptyset$$

is defined by one quadratic constraint. It is clear that $\overline{C}(\mathcal{G}) \subseteq \overline{C}(\mathcal{F}) \cap \overline{C}(\mathcal{H}) = \overline{C}(\mathcal{F}) \cap S(\mathcal{H})$. In [7], Joyce and Yang show moreover that equality holds, i.e., $\overline{C}(\mathcal{G}) = \overline{C}(\mathcal{F}) \cap S(\mathcal{H})$, when the boundary of $\mathcal{H}$ is completely contained in $\mathcal{F}$, i.e.,

$$\overline{C}(\mathcal{G}) = \overline{C}(\mathcal{F}) \cap \{ (x, X) \mid \ A \bullet X + 2a^T x + \alpha \leq 0 \} .$$

This result is a generalization of [15], which proves the result when $\mathcal{F}$ is quadratically defined and bounded.

Conclusions

The (closed) quadratic hull plays an essential role in the convexification of nonconvex QCQP. Despite its importance, there are only a few cases for which the quadratic hull is known explicitly, in particular, as semidefinite representable problems. Additional novel techniques are needed to generate strong valid inequalities for quadratic hulls.

See also

- ▶ Copositive Optimization
- ▶ Duality Gaps in Nonconvex Optimization
- ▶ Duality for Semidefinite Programming
- ▶ Quadratic Integer Programming: Complexity and Equivalent Forms
- ▶ Quadratic Programming over an Ellipsoid
- ▶ Reformulation-Linearization Technique for Global Optimization

References

1. Anstreicher KM, Burer S (2010) Computable representations for convex hulls of low-dimensional quadratic forms. Math Program 124(1):33–43
2. Burer S (2015) A gentle, geometric introduction to copositive optimization. Math Program 151(1):89–116
3. Burer S, Anstreicher KM (2013) Second-order-cone constraints for extended trust-region subproblems. SIAM J Optim 23(1):432–451
4. Burer S, Yang B (2015) The trust region subproblem with non-intersecting linear constraints. Math Program 149(1–2):253–264
5. Eichfelder G, Povh J (2013) On the set-semidefinite representation of nonconvex quadratic programs over arbitrary feasible sets. Optim Lett 7(6):1373–1386
6. Jin Q, Tian Y, Deng Z, Fang S-C, Xing W (2013) Exact computable representation of some second-order cone constrained quadratic programming problems. J Oper Res Soc China 1(1):107–134
7. Joyce A, Yang B (2023) Convex hull results on quadratic programs with non-intersecting constraints. Math Program 1–20. https://doi.org/10.1007/s10107-023-01985-x
8. Lu C, Fang S-C, Jin Q, Wang Z, Xing W (2011) KKT solution and conic relaxation for solving quadratically constrained quadratic programming problems. SIAM J Optim 21(4):1475–1490
9. Pardalos PM, Vavasis SA (1991) Quadratic programming with one negative eigenvalue is NP-hard. J Glob Optim 1(1):15–22
10. Pong TK, Wolkowicz H (2014) The generalized trust region subproblem. Comput Optim Appl 58(2):273–322
11. Sherali HD, Adams WP (2013) A reformulation-linearization technique for solving discrete and continuous nonconvex problems. Nonconvex Optimization and Its Applications. Springer Science & Business Media, Springer, US, 31
12. Sturm JF, Zhang S (2003) On cones of nonnegative quadratic functions. Math Oper Res 28(2):246–267
13. Wang S, Xia Y (2015) Strong duality for generalized trust region subproblems: S-lemma with interval bounds. Optim Lett 9(6):1063–1073
14. Xia Y, Wang S, Sheu R-L (2016) S-lemma with equality and its applications. Math Program 156(1):513–547
15. Yang B, Anstreicher K, Burer S (2018) Quadratic programs with hollows. Math Program 170(2):541–553
16. Ye Y, Zhang S (2003) New results on quadratic minimization. SIAM J Optim 14(1):245–267