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Strengthened SDP relaxation for an extended trust region subproblem with an application to optimal power flow

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Abstract

We study an extended trust region subproblem minimizing a nonconvex function over the hollow ball $r \leq \|x\| \leq R$ intersected with a full-dimensional second order cone (SOC) constraint of the form $\|x - c\| \leq b^T x - a$. In particular, we present a class of valid cuts that improve existing semidefinite programming (SDP) relaxations and are separable in polynomial time. We connect our cuts to the literature on the optimal power flow (OPF) problem by demonstrating that previously derived cuts capturing a convex hull important for OPF are actually just special cases of our cuts. In addition, we apply our methodology to derive a new class of closed-form, locally valid, SOC cuts for nonconvex quadratic programs over the mixed polyhedral-conic set $\{x \geq 0 : \|x\| \leq 1\}$. Finally, we show computationally on randomly generated instances that our cuts are effective in further closing the gap of the strongest SDP relaxations in the literature, especially in low dimensions.

Mathematics Subject Classification 90C22, 90C26

1 Introduction

The classical *trust region subproblem (TRS)* minimizes an arbitrary quadratic function over the unit Euclidean ball defined by $\|x\| \leq R$ and is solvable in polynomial-time [10]. Many authors have studied variants of TRS that incorporate additional constraints. For example, [20] also imposes the lower bound $r \leq \|x\|$. We collectively refer to variants of TRS that incorporate more general constraints as the *extended*

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TRS. In this paper, we study the following specific form of the extended TRS, which incorporates the lower bound r as well as an additional SOC (second-order cone) constraint, whose "geometry" matches the ball in the sense that its Hessian is also the identity matrix:

$$\min \quad x^T H x + 2\,g^T x \tag{1a}$$
$$\text{s.t.} \quad r \le \|x\| \le R \tag{1b}$$
$$\|x - c\| \le b^T x - a \tag{1c}$$

where $x \in \mathbb{R}^n$, $H = H^T \in \mathbb{R}^{n\times n}$, $g, c, b \in \mathbb{R}^n$, $a \in \mathbb{R}$, and $r, R \in \mathbb{R}_+$. Note that H is symmetric without loss of generality and that we have *not* scaled the problem to the unit ball (i.e., we do not assume $R = 1$) as is common in the TRS literature. The general upper bound R will be convenient for our presentation, especially in Sect. 3. The algorithm of Bienstock [3] solves (1) in polynomial time since it can be written as a nonconvex quadratic program with a fixed number of quadratic/linear constraints (in this case, four), one of which is strictly convex. However, in this paper, we are interested in developing tight convex relaxations of (1). In particular, as far as we are aware, (1) has no known tight convex relaxation.

Problem (1) includes, for example, a special case of the *two trust region subproblem*—also called the *Celis–Dennis–Tapia subproblem* [8]—in which a second ball constraint is added to TRS. In this case, $r = 0$, $b = 0$, and $a < 0$. Here, however, we are interested in the more general structure represented by (1c), which arises, for example, in the *optimal power flow problem (OPF)* as discussed in Sect. 3. More generally, the study of (1) sheds light on any nonconvex quadratically constrained quadratic program that includes a ball constraint and a second SOC constraint with identity Hessian. In Sect. 3, we will also show how this structure is relevant for the mixed polyhedral-SOC set $\{x \ge 0 : \|x\| \le R\}$. (In the concluding Sect. 6, we briefly mention an extension for handling different Hessians.)

Since (1) is a nonconvex problem, a standard approach is to approximate (1) by its so-called *Shor semidefinite programming (SDP) relaxation* [19], which is solvable in polynomial time:

$$\min \; H \bullet X + 2\,g^T x \tag{2a}$$
$$\text{s.t.} \quad r^2 \le \operatorname{tr}(X) \le R^2 \tag{2b}$$
$$\operatorname{tr}(X) - 2\,c^T x + c^T c \le bb^T \bullet X - 2\,a\,b^T x + a^2 \tag{2c}$$
$$0 \le b^T x - a \tag{2d}$$
$$Y(x, X) \succeq 0 \tag{2e}$$

where $M \bullet X := \operatorname{tr}(M^T X)$ is the trace inner product for conformal matrices and

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \tag{3}$$

is symmetric of size $(n+1) \times (n+1)$. Note that (1c) is represented as the two constraints $\|x-c\|^2 \leq (b^T x - a)^2$ and $0 \leq b^T x - a$ before lifting to (2c)–(2d). We also define

$$\mathcal{R}_{\text{shor}} := \left\{(x, X) : (x, X) \text{ satisfies } (2b)\text{–}(2e)\right\}$$

to be the feasible set of the Shor relaxation. Then (2) can be alternatively expressed as minimizing $H \bullet X + 2\, g^T x$ over $(x, X) \in \mathcal{R}_{\text{shor}}$.

Various valid inequalities can be added to (2) in order to strengthen the Shor relaxation. For example, if $v_1^T x \geq u_1$ and $v_2^T x \geq u_2$ are any two valid linear inequalities for the feasible set of (1), then the redundant quadratic constraint $(v_1^T x - u_1)(v_2^T x - u_2) \geq 0$ can be relaxed to the valid *RLT constraint* [18]:

$$v_1 v_2^T \bullet X - u_2 v_1^T x - u_1 v_2^T x + u_1 u_2 \geq 0.$$

However, since (1) does not contain explicit linear constraints, in practice one would need to separate over valid $v_1^T x \geq u_1$ and $v_2^T x \geq u_2$ to generate violated RLT constraints, but this separation is a bilinear subproblem, which does not appear to be solvable in polynomial time.

The difficulty of separating the RLT constraints when no linear constraints are explicitly given can be circumvented in the case of (1) as follows. By multiplying a valid $v_1^T x \geq u_1$ with the ball constraint $\|x\| \leq R$, we have the redundant quadratic SOC constraint $\|(v_1^T x - u_1)x\| \leq R(v_1^T x - u_1)$, which in turn yields the valid SOC constraint

$$\|X v_1 - u_1 x\| \leq R(v_1^T x - u_1) \tag{4}$$

in the lifted (x, X) space. In a similar manner, $v_1^T x \geq u_1$ can be combined with $\|x - c\| \leq b^T x - a$. These are known as *SOCRLT constraints* [5,21,22]. In fact, each SOCRLT constraint is a compact encoding of an entire collection of RLT constraints. For example, (4) captures all of the RLT constraints corresponding to $v_1^T x \geq u_1$ fixed and $v_2^T x \geq u_2$ varying over the supporting hyperplanes of $\|x\| \leq R$. Consequently, the collections of SOCRLT and RLT constraints for (1) are equivalent,[1] but in contrast to the RLT constraints, the SOCRLT constraints can be separated in polynomial-time based on the fact that TRS is polynomial-time solvable [5].

Anstreicher [1] introduced a further generalization of the SOCRLT constraints, called a *KSOC constraint*, which is based on relaxing a valid quadratic Kronecker-product matrix inequality. Specifically, the KSOC constraint is constructed from the following observations: first, defining $\text{SOC} := \{(v_0, v) : \|v\| \leq v_0\}$ to be the second-order cone, it is well-known that

$$\begin{pmatrix} v_0 \\ v \end{pmatrix} \in \text{SOC} \iff \begin{pmatrix} v_0 & v^T \\ v & v_0 I \end{pmatrix} \succeq 0;$$

[1] This differs from other papers, which often define RLT constraints only for explicitly given valid linear constraints, of which (1) has none. So, for the sake of generality, we have defined the RLT constraints allowing for *implicit* valid linear constraints.

second, it is also well-known that the Kronecker product of positive semidefinite matrices is positive semidefinite. Hence, for (1) we have the valid quadratic matrix inequality

$$\begin{pmatrix} R & x^T \\ x & R\,I \end{pmatrix} \otimes \begin{pmatrix} b^T x - a & x^T - c^T \\ x - c & (b^T x - a) I \end{pmatrix} \succeq 0.$$

After relaxing this inequality in the space (x, X), we obtain the convex KSOC constraint, which captures all SOCRLT constraints (and hence all RLT constraints) and is generally stronger [1], assuming the Shor constraints remain enforced.

Summarizing, defining $\mathcal{R}_{\text{rlt}}$ and $\mathcal{R}_{\text{socrlt}}$ to be the set of (x, X) satisfying all possible RLT and SOCRLT constraints, respectively, we have

$$\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}} \subseteq \mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{socrlt}} = \mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{rlt}}$$

where $\mathcal{R}_{\text{ksoc}}$ is the set of all (x, X) satisfying the KSOC constraint. Moreover, the first containment is proper in general. Hence, in this paper, we focus on improving the relaxation $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$. The paper [13] provides further insight into the strength of $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ relative to other techniques in the literature.

Let $\mathcal{F}$ denote the feasible set of (1), i.e., the set of all $x \in \mathbb{R}^n$ satisfying (1b)–(1c). Strengthening the SDP relaxation can alternatively be expressed as determining valid inequalities that more accurately approximate the closed convex hull

$$\mathcal{G} := \text{conv}\left\{(x, xx^T) : x \in \mathcal{F}\right\}. \tag{5}$$

Note that $\mathcal{G}$ is compact because $\mathcal{F}$ is. Moreover, because linear optimization over a compact convex set is guaranteed to attain its optimal value at an extreme point, solving (1) amounts to optimizing the linear function $H \bullet X + 2\,g^T x$ over $\mathcal{G}$. While an exact representation of $\mathcal{G}$ is unknown, there are several closely related cases in which $\mathcal{G}$ can be described exactly; see [2,7].

In this paper, we propose a new class of valid linear inequalities for (1) in the space (x, X), which in general strengthen $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ towards $\mathcal{G}$. Each inequality is derived from several ingredients that exploit the structure of $\mathcal{F}$: the self-duality of SOC; the RLT-type valid inequality $(R - \|x\|)(\|x\| - r) \geq 0$; and knowledge of a quadratic function $q(x)$ and a linear function $l(x)$, each of which is nonnegative over all $x \in \mathcal{F}$. We combine these ingredients to derive a valid quartic inequality, which is then relaxed to a valid quadratic inequality, which in turn yields a new valid linear inequality in (x, X).

As a small illustrative example, consider when $c = 0$ and $r = 0$, in which case $\mathcal{F}$ is defined by $\|x\| \leq R$ and $\|x\| \leq b^T x - a$. For the specific choices $q(x) = 0$ and $l(x) = 1$, our new inequality can also be derived from the following direct argument: the chain of inequalities $\|x\|^2 \leq R\|x\| \leq R(b^T x - a)$ linearizes to

$$\text{tr}(X) \leq R(b^T x - a). \tag{6}$$

The following example shows that (6) is not captured by $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$:

Example 1 Let $\mathcal{F} = \{x \in \mathbb{R}^2 : \|x\| \leq 1, \|x\| \leq 1 - x_1 - x_2\}$. Then (6) is $\operatorname{tr}(X) \leq 1 - x_1 - x_2$. Minimizing the objective $1 - x_1 - x_2 - \operatorname{tr}(X)$ over $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ yields the optimal solution

$$Y^* \approx \begin{pmatrix} 1.0000 & 0.0624 & 0.0624 \\ 0.0624 & 0.5000 & -0.3018 \\ 0.0624 & -0.3018 & 0.5000 \end{pmatrix}$$

with (approximate) optimal value -0.1248, i.e., the optimal value is negative, which demonstrates that (6) is not valid for $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$.

As far as we are aware, inequality (6) for this special case has not yet appeared in the literature. We seek in this paper, however, an even more general procedure for deriving valid inequalities using the ingredients described in the previous paragraph.

The paper is organized as follows. In Sect. 2, we present the derivation of our new valid inequalities and discuss several illustrative choices of $q(x)$ and $l(x)$. We also specialize the results to $c = 0$ and $a = 0$, a case which further enables the derivation of a similar, second type of valid linear inequality in (x, X). Then, in Sect. 3, we show that our inequalities include those introduced in [9] for the study of the OPF problem,[2] and we extend our approach to derive a new class of valid SOC constraints for $\mathcal{G}$ when $\mathcal{F}$ equals the intersection of the ball $\|x\| \leq R$ and the nonnegative orthant. Next, in Sect. 4, we prove that the separation problem for our inequalities—which can be viewed as dynamically choosing the nonnegative functions $q(x)$ and $l(x)$—is polynomial-time based on the availability of any SDP relaxation in the variables (x, X), such as the relaxations $\mathcal{R}_{\text{shor}}$ or $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$. In this sense, we are able to "bootstrap" any existing SDP relaxation for the separation subroutine to generate valid cuts. Finally, in Sect. 5, we provide computational evidence that our cuts are effective in further closing the gap between (1) and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ on randomly generated problems, especially in low dimensions. We close in Sect. 6 with a few final thoughts and directions for future research.

This paper is accompanied by the code repository https://github.com/A-Eltved/strengthened_sdr, which contains full code for the paper's examples and computational results. In addition, the first author's forthcoming Ph.D. thesis [12] will contain additional discussion and extensions.

2 New valid inequalities

In the Introduction, we discussed the valid inequality (6) for the specific case $c = 0$ and $r = 0$. Now we assume general c and r. Analogous to (6), we use $\|x\| \leq R$ and $\|x - c\| \leq b^T x - a$ along with the self-duality of SOC to obtain the following quadratic inequality:

$$\begin{pmatrix} R \\ -x \end{pmatrix}^T \begin{pmatrix} b^T x - a \\ x - c \end{pmatrix} \geq 0 \quad \Longrightarrow \quad R(b^T x - a) \geq \operatorname{tr}(X) - c^T x. \tag{7}$$

[2] Indeed, our initial motivation for this paper was the desire to understand the inequalities in [9] more fully.

Note that this inequality makes use of the equivalent constraint $\|-x\| \le R$. We seek to strengthen it further by incorporating two additional ideas.

The first idea involves exploiting the lower bound $r \le \|x\|$ and the RLT-type valid inequality $(R - \|x\|)(\|x\| - r) \ge 0$. Consider the following proposition:

Proposition 1 *Suppose $r \le \|x\| \le R$, and define $r\|x\|^{-2} := 0$ when $\|x\| = r = 0$. Then*

$$\begin{pmatrix} r + R \\ \left(1 + rR\|x\|^{-2}\right) x \end{pmatrix} \in SOC. \tag{8}$$

Proof If $r = 0$, then (8) reads $(R, x) \in \mathrm{SOC}$, which is true by assumption. So suppose $0 < r \le \|x\|$. Then we wish to prove

$$(1 + rR\|x\|^{-2})\|x\| = \|x\| + rR\|x\|^{-1} \le r + R,$$

which follows by expanding the valid expression $(R - \|x\|)(\|x\| - r) \ge 0$ and dividing by $\|x\| \ge r > 0$. □

By the proposition, analogous to (7), we have:

$$\begin{pmatrix} r + R \\ -(1 + rR\|x\|^{-2})x \end{pmatrix}^T \begin{pmatrix} b^T x - a \\ x - c \end{pmatrix} \ge 0$$
$$\iff (r + R)(b^T x - a) \ge x^T x + rR - c^T x - rR\|x\|^{-2}\, c^T x.$$

However, this inequality cannot be directly linearized in (x, X) due to the non-quadratic term $\|x\|^{-2}$. So we bound the term $r\|x\|^{-2}\, c^T x$ from above by a problem-dependent constant $[c]_{\max} \ge 0$, which satisfies $r\, c^T x \le [c]_{\max}\, x^T x$ for all $x \in \mathcal{F}$. We then have the valid linear inequality

$$(r + R)(b^T x - a) \ge \mathrm{tr}(X) + rR - c^T x - [c]_{\max} R. \tag{9}$$

Such a $[c]_{\max}$ clearly exists. For example, $[c]_{\max} = \|c\|$ works because

$$r\, c^T x \le r\|c\|\,\|x\| \le \|c\|\,\|x\|^2,$$

but naturally it is advantageous to take $[c]_{\max}$ as small as possible. One method for computing a smaller $[c]_{\max} \le \|c\|$ is binary search on $[c]_{\max}$ over the interval $[0, \|c\|]$, where at each step we check whether the optimal value of

$$\min_x \left\{ [c]_{\max}\, x^T x - r\, c^T x : \|x\| \le R,\ \|x - c\| \le b^T x - a \right\}$$

is nonnegative. The nonconvex lower bound $r \le \|x\|$ has been excluded from this subproblem to ensure convexity and polynomial-time solvability, which also ensures that the binary search is polynomial-time overall. Note that the binary search will not always return the smallest possible $[c]_{\max}$ due to the exclusion of the lower bound. Note also that, when $r = 0$ or $c = 0$, the optimal $[c]_{\max}$ equals 0.

Our second idea to improve (7) and (9) is to replace $(b^T x - a, x - c) \in \text{SOC}$ in the derivation above with another vector—but one that is still in the second-order cone. In particular, we consider the nonnegative combination

$$q_x \begin{pmatrix} R \\ x \end{pmatrix} + l_x \begin{pmatrix} b^T x - a \\ x - c \end{pmatrix} \in \text{SOC}, \tag{10}$$

where $q_x := q(x)$ is a quadratic function and $l_x := l(x)$ is a linear function, both of which are nonnegative for all $x \in \mathcal{F}$. This approach is similar to polynomial-optimization approaches such as the one pioneered in [14], which uses polynomial multipliers with limited degree to derive new, albeit redundant, constraints. Then we have the following generalization of (9):

$$\begin{pmatrix} r + R \\ -(1 + rR\|x\|^{-2})x \end{pmatrix}^T \begin{pmatrix} Rq_x + l_x(b^T x - a) \\ (q_x + l_x)x - l_x c \end{pmatrix} \geq 0$$

which rearranges and relaxes to

$$\begin{aligned}(r + R)Rq_x + (r + R)l_x(b^T x - a) &\geq (q_x + l_x)\, x^T x + rR\,(q_x + l_x) \\ &\quad - l_x c^T x - [c]_{\max}\, R\, l_x.\end{aligned}$$

Note that the right-hand side is quartic in x, and hence this inequality cannot be directly linearized in the space (x, X). Hence, we define a constant that satisfies

$$\min\{q_x + l_x : x \in \mathcal{F}\} \geq [q + l]_{\text{lowbd}} \geq 0$$

to get the valid quadratic inequality

$$\begin{aligned}(r + R)Rq_x + (r + R)l_x(b^T x - a) &\geq [q + l]_{\text{lowbd}}\, x^T x + rR\,(q_x + l_x) \\ &\quad - l_x c^T x - [c]_{\max} R\, l_x,\end{aligned} \tag{11}$$

which can be easily linearized in (x, X) as summarized in the following theorem. Note that the theorem requires only that $[q + l]_{\text{lowbd}}$ be a nonnegative lower bound on the value of $q(x) + l(x)$ over $\mathcal{F}$, but generally a larger value gives a tighter valid inequality.

Theorem 1 *Let $\mathcal{F}$ be the feasible set of (1), and let $[c]_{\max} \in [0, \|c\|]$ be given such that $r\, c^T x \leq [c]_{\max} x^T x$ for all $x \in \mathcal{F}$. In addition, let $q(x) := x^T H_q x + 2\, g_q^T x + f_q$ and $l(x) := 2\, g_l^T x + f_l$ be given such that $q(x) \geq 0$ and $l(x) \geq 0$ for all $x \in \mathcal{F}$. Also, let $[q + l]_{\text{lowbd}} \geq 0$ be a valid lower bound on the sum $q(x) + l(x)$ over all $x \in \mathcal{F}$. Then the linear inequality*

$$\begin{aligned}&(r + R)R\left(H_q \bullet X + 2\, g_q^T x + f_q\right) + (r + R)\left(2\, g_l b^T \bullet X + (f_l b - 2a g_l)^T x - a f_l\right) \\ &\geq [q + l]_{\text{lowbd}}\, \text{tr}(X) + rR\left(H_q \bullet X + 2(g_q + g_l)^T x + (f_q + f_l)\right) \\ &\quad - \left(2\, g_l c^T \bullet X + f_l c^T x\right) - [c]_{\max} R(2\, g_l^T x + f_l)\end{aligned} \tag{12}$$

is valid for the convex hull $\mathcal{G}$ defined by (5).

Note that both sides of (11) contain the term $r R\, q_x$, and so the presentation of both (11) and (12) could be simplified. However, we leave these slightly unsimplified so as to facilitate our discussion in Sect. 2.2 below.

Let $\hat{r}$ be any scalar in $[0, r]$. Since $\hat{r} \le \|x\|$ is also valid for $\mathcal{F}$, we can replace r by $\hat{r}$ in (12) to obtain an alternate inequality based on $\hat{r}$. In fact, considering $\hat{r}$ to be variable in this inequality while all other quantities are fixed, we see that the inequality is linear in $\hat{r}$, which implies that all such valid inequalities over $\hat{r} \in [0, r]$ are actually dominated (implied) by the two extremes $\hat{r} = 0$ and $\hat{r} = r$. We summarize this observation in the following corollary.

Corollary 1 *Under the assumptions of Theorem 1, the infinite class of inequalities gotten by replacing r with $\hat{r} \in [0, r]$ is dominated by the two inequalities* (12) *and*

$$\begin{aligned} R^2 \left(H_q \bullet X + 2\, g_q^T x + f_q\right) + R \left(2\, g_l b^T \bullet X + (f_l b - 2a g_l)^T x - a f_l\right) \\ \ge [q + l]_{\text{lowbd}}\, \operatorname{tr}(X) - \left(2\, g_l c^T \bullet X + f_l c^T x\right) - [c]_{\max} R(2\, g_l^T x + f_l). \end{aligned} \tag{13}$$

corresponding to the extremes $\hat{r} = r$ and $\hat{r} = 0$, respectively.

2.1 Example: slab inequalities

In this subsection, we introduce a specialization of our inequalities, which we will return to in Sect. 3.2.

Suppose that we have knowledge of $s \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ such that

$$\mathcal{F} \subseteq \mathcal{S} := \{x : \lambda \le s^T x \le \mu\}, \tag{14}$$

i.e., every $x \in \mathcal{F}$ satisfies $\lambda \le s^T x \le \mu$. We call $\mathcal{S}$ a valid *slab* and, abusing notation, we refer to $\mathcal{S}$ by its tuple (λ, s, μ). For example, since $\mathcal{F}$ is bounded, for any vector s with $\|s\| = 1$, choosing $\lambda = -R$ and $\mu = R$ yields a valid slab. Given any slab (λ, s, μ), we discuss two choices of nonnegative q_x and l_x.

First, define $q_x := \mu - s^T x \ge 0$ and $l_x := s^T x - \lambda \ge 0$. Note that q_x is linear in this case, and $[q + l]_{\text{lowbd}} = q_x + l_x = \mu - \lambda$. Then (11) becomes

$$\begin{aligned} (r + R)R(\mu - s^T x) + (r + R)(s^T x - \lambda)(b^T x - a) \\ \ge (\mu - \lambda)(x^T x + r R) - (s^T x - \lambda)c^T x - [c]_{\max} R(s^T x - \lambda). \end{aligned} \tag{15}$$

Alternatively, we could also take $q_x := s^T x - \lambda$ and $l_x := \mu - s^T x$ to obtain another, similar quadratic inequality.

Second, given the slab (λ, s, μ), we may assume without loss of generality that $\lambda + \mu \ge 0$ and $\lambda^2 \le \mu^2$. To see this, we consider three cases. First, if both $\lambda, \mu \ge 0$, then the statement is clear. Second, if both $\lambda, \mu \le 0$, we can use instead the equivalent representation of $\mathcal{S}$ by $-\mu \le -s^T x \le -\lambda$. Finally, if $\lambda < 0$ and $\mu \ge 0$ with $\lambda + \mu < 0$, then we can likewise use $(-\mu, -s, -\lambda)$ instead. Now, with $\lambda + \mu \geq 0$ and $\lambda^2 \leq \mu^2$, we then define $q_x := \mu^2 - (s^T x)^2 \geq 0$ and $l_x := (\lambda + \mu)(s^T x - \lambda) \geq 0$ so that

$$\begin{aligned} q_x + l_x &= \mu^2 - (s^T x)^2 + (\lambda + \mu) s^T x - \lambda\mu - \lambda^2 \\ &= \mu^2 + (\mu - s^T x)(s^T x - \lambda) - \lambda^2 \\ &\geq \mu^2 + 0 - \lambda^2 \geq 0. \end{aligned}$$

Hence, we obtain (11) with $[q + l]_{\text{lowbd}} := \mu^2 - \lambda^2 \geq 0$.

2.2 Example: special case $c = 0$, $a = 0$, and $\lambda \geq 0$

In this subsection, we derive two cuts—see (18) below—that are closely related to the cuts just discussed in Sect. 2.1, and these will play a special role in Sect. 3.1. We assume $c = 0$ and $a = 0$, and we will use a slab (λ, s, μ) with $\lambda \geq 0$. Note that $c = 0$ implies $[c]_{\max} = 0$.

For the first cut, consider the inequality (11) with $c = 0$ and $a = 0$, which is further relaxed on the right-hand side:

$$\begin{aligned} (r + R) R q_x + (r + R) l_x b^T x &\geq [q + l]_{\text{lowbd}} x^T x + r R\,(q_x + l_x) \\ &\geq [q + l]_{\text{lowbd}} (x^T x + r R). \end{aligned} \tag{16}$$

For the second cut, we consider a pair of functions $l_x := l(x)$ and $p_x := p(x)$ that satisfy a different relationship than the previously considered l_x and q_x. Specifically, we assume linear $l_x \geq 0$ and quadratic $p_x \geq 0$, and we require $l_x - p_x \geq 0$ for all $x \in \mathcal{F}$ as well. We also define $[l - p]_{\text{lowbd}} \geq 0$ to be a lower bound on the minimum value of $l_x - p_x$ over $\mathcal{F}$. Then we have the following result.

Proposition 2 *Suppose $c = 0$, $a = 0$, and $l_x := l(x)$ and $p_x := p(x)$ are nonnegative functions on $\mathcal{F}$ such that $l_x - p_x$ is also nonnegative on $\mathcal{F}$. Then, for all $x \in \mathcal{F}$*

$$\begin{pmatrix} l_x b^T x - r p_x \\ (l_x - p_x) x \end{pmatrix} \in SOC.$$

Proof $(l_x - p_x)\|x\| = l_x \|x\| - p_x \|x\| \leq l_x b^T x - r p_x$. □

Using this proposition, the self-duality of the SOC, and Proposition 1, we have

$$\begin{pmatrix} r + R \\ -(1 + r R \|x\|^{-2}) x \end{pmatrix}^T \begin{pmatrix} l_x b^T x - r p_x \\ (l_x - p_x) x \end{pmatrix} \geq 0,$$

which rearranges and relaxes to

$$\begin{aligned} (r + R) l_x b^T x - (r + R) r p_x &\geq (l_x - p_x)\, x^T x + r R\,(l_x - p_x) \\ &\geq [l - p]_{\text{lowbd}} (x^T x + r R). \end{aligned} \tag{17}$$

Note that (17) simplifies to $R\, l_x\, b^T x \geq [l - p]_{\text{lowbd}}\, x^T x$ when $r = 0$, which is a consequence of the simpler inequality $R\, b^T x \geq x^T x$; see (6) with $a = 0$. In other words, (17) appears to be interesting only when $r > 0$.

We now consider a specific choice of q_x, l_x, and p_x for the inequalities (16) and (17) based on the slab $0 \leq \lambda \leq s^T x \leq \mu$. We choose $q_x := \mu^2 - (s^T x)^2$, $l_x := (\lambda + \mu) s^T x$, and $p_x := (s^T x)^2 - \lambda^2$ as the nonnegative functions, resulting in

$$
\begin{aligned}
q_x + l_x &= \mu^2 - (s^T x)^2 + (\lambda + \mu) s^T x \geq \mu^2 + \lambda\mu =: [q + l]_{\text{lowbd}} \\
l_x - p_x &= \lambda^2 - (s^T x)^2 + (\lambda + \mu) s^T x \geq \lambda^2 + \lambda\mu =: [l - p]_{\text{lowbd}},
\end{aligned}
$$

where the inequalities follow from the RLT inequality $(\mu - s^T x)(s^T x - \lambda) \geq 0$. Plugging these into (16)–(17), respectively, and linearizing, we obtain

$$
(r + R) R (\mu^2 - ss^T \bullet X) + (r + R)(\lambda + \mu) sb^T \bullet X \geq (\mu^2 + \lambda\mu)(\text{tr}(X) + rR) \tag{18a}
$$

$$
(r + R)(\lambda + \mu) sb^T \bullet X - (r + R) r (ss^T \bullet X - \lambda^2) \geq (\lambda^2 + \lambda\mu)(\text{tr}(X) + rR). \tag{18b}
$$

3 Applications

In this section, we explore two applications of the inequalities developed in Sect. 2. The first application shows that the valid inequalities for the optimal power flow problem (OPF) derived in [9] are in fact just special cases of our inequalities, whereas the derivation in [9] was specifically tailored to OPF. Our second application investigates the convex hull of $\mathcal{G}$, where—departing from the form of (1)—$\mathcal{F}$ equals the intersection of the ball with the nonnegative orthant, i.e., $\mathcal{F}$ possesses polyhedral aspects as well. We study this form of $\mathcal{F}$ since it is relevant for any bounded feasible set with nonnegative variables, where the bound is given by a Euclidean ball.

3.1 Optimal power flow problem

In this subsection, we consider a result of Chen et al. [9], which provides an exact formulation for the convex hull of a nonconvex, quadratically constrained set appearing in the study of the optimal power flow (OPF) problem. In particular, the authors added two new linear inequalities to the Shor relaxation in order to capture the convex hull. Whereas these two inequalities were specifically derived for OPF, we will show that they are just special cases of (18) derived in Sect. 2.2. For additional background on convex relaxations of OPF, we refer the reader to the two-part survey [15,16]. Here, we briefly mention that the system (19) comes from considering a pair of adjacent buses with their voltage magnitude constraints ((19a)) and their voltage-angle difference constaints ((19b)); hence, L_{jj} and U_{jj} are bounds on the voltage magnitude at bus j and L_{ij} and U_{ij} are bounds on the voltage-angle difference between buses i and j.

We restate the result of Chen et al. using their notation. Let $\mathcal{J}_C \subseteq \mathbb{R}^4$ be the convex hull of the following nonconvex quadratic system:

$$L_{jj} \le W_{jj} \le U_{jj} \quad \forall\, j = 1, 2 \tag{19a}$$
$$L_{12} W_{12} \le T_{12} \le U_{12} W_{12} \tag{19b}$$
$$W_{12} \ge 0 \tag{19c}$$
$$W_{11} W_{22} = W_{12}^2 + T_{12}^2 \tag{19d}$$

where the four variables are $(W_{11}, W_{22}, W_{12}, T_{12}) \in \mathbb{R}^4$ and the data $L = (L_{11}, L_{22}, L_{12})$ and $U = (U_{11}, U_{22}, U_{12})$ satisfy $L \le U$ and $L_{jj} \ge 0$ for $j = 1, 2$. Chen et al.'s interest in this particular convex hull arose from an analysis of the OPF problem, where (19) appears as a repeated substructure. As explained in [9], $\mathcal{J}_C$ can alternatively be expressed as the following convex hull using two complex variables $z_1, z_2 \in \mathbb{C}$:

$$\mathcal{J}_C = \operatorname{conv}\left\{ \begin{pmatrix} z_1 z_1^* \\ z_2 z_2^* \\ \operatorname{Re}(z_1 z_2^*) \\ \operatorname{Im}(z_1 z_2^*) \end{pmatrix} \in \mathbb{R}^4 : \begin{array}{c} L_{jj} \le z_j z_j^* \le U_{jj} \quad \forall\, j = 1, 2 \\ L_{12}\operatorname{Re}(z_1 z_2^*) \le \operatorname{Im}(z_1 z_2^*) \le U_{12}\operatorname{Re}(z_1 z_2^*) \\ \operatorname{Re}(z_1 z_2^*) \ge 0 \end{array} \right\}. \tag{20}$$

In particular, Eq. (19d) is the usual "rank-1" condition, capturing the link between the linear variables $(W_{11}, W_{22}, W_{12}, T_{12})$ and the quadratic expressions in z_1, z_2. The authors proved that the pair of linear inequalities

$$\pi_0 + \pi_1 W_{11} + \pi_2 W_{22} + \pi_3 W_{12} + \pi_4 T_{12} \ge U_{22} W_{11} + U_{11} W_{22} - U_{11} U_{22} \tag{21a}$$
$$\pi_0 + \pi_1 W_{11} + \pi_2 W_{22} + \pi_3 W_{12} + \pi_4 T_{12} \ge L_{22} W_{11} + L_{11} W_{22} - L_{11} L_{22} \tag{21b}$$

are valid for $\mathcal{J}_C$, where

$$\begin{aligned}
\pi_0 &:= -\sqrt{L_{11} L_{22} U_{11} U_{22}} \\
\pi_1 &:= -\sqrt{L_{22} U_{22}} \\
\pi_2 &:= -\sqrt{L_{11} U_{11}} \\
\pi_3 &:= \left(\sqrt{L_{11}} + \sqrt{U_{11}}\right)\left(\sqrt{L_{22}} + \sqrt{U_{22}}\right) \frac{1 - f(L_{12}) f(U_{12})}{1 + f(L_{12}) f(U_{12})} \\
\pi_4 &:= \left(\sqrt{L_{11}} + \sqrt{U_{11}}\right)\left(\sqrt{L_{22}} + \sqrt{U_{22}}\right) \frac{f(L_{12}) + f(U_{12})}{1 + f(L_{12}) f(U_{12})}
\end{aligned}$$

and where $f(x) := (\sqrt{1 + x^2} - 1)/x$ when $x > 0$ and $f(0) := 0$. In fact, they proved that (21), when added to the Shor relaxation, is sufficient to capture $\mathcal{J}_C$:

$$\mathcal{J}_C = \left\{ (W_{11}, W_{22}, W_{12}, T_{12}) : \begin{array}{c} (19a)-(19c) \\ W_{11} W_{22} \ge W_{12}^2 + T_{12}^2 \\ (21) \end{array} \right\}.$$

Here, the convex constraint $W_{11}W_{22} \geq W_{12}^2 + T_{12}^2$ is equivalent to the regular positive-semidefinite condition.

We now relate (21) to our inequalities (18). Defining

$$\mathcal{F} := \left\{ x \in \mathbb{R}^3 : \begin{array}{c} L_{11} \leq x_1^2 + x_2^2 \leq U_{11} \\ L_{22} \leq x_3^2 \leq U_{22} \\ L_{12}x_1x_3 \leq x_2x_3 \leq U_{12}x_1x_3 \\ x_1x_3 \geq 0, \quad x_3 \geq 0 \end{array} \right\}. \tag{22}$$

and $\mathcal{G}$ by (5), the following proposition establishes an equivalence between $\mathcal{J}_C$ and $\mathcal{G}$.

Proposition 3 $\mathcal{J}_C = \{(X_{11} + X_{22}, X_{33}, X_{13}, X_{23}) : (x, X) \in \mathcal{G}\}$.

Proof Consider (20). Because the quadratic terms $z_1z_1^*$, $z_2z_2^*$, and $z_1z_2^*$ are unaffected by a rotation of $\mathbb{C}$ applied simultaneously to both z_1 and z_2, we may enforce $\mathrm{Re}(z_2) \geq 0$ and $\mathrm{Im}(z_2) = 0$ without changing the definition of $\mathcal{J}_C$. Then writing $z_1 = x_1 + ix_2$ and $z_2 = x_3$ for $x \in \mathbb{R}^3$, we thus have $\mathcal{J}_C = \operatorname{conv}\left\{(x_1^2 + x_2^2, x_3^2, x_1x_3, x_2x_3) : x \in \mathcal{F} \subseteq \mathbb{R}^3\right\}$, which proves the proposition. □

Our next proposition establishes an alternative form for $\mathcal{F}$, which matches the development in Sect. 2 except that the SOCs involve only two scalar variables, even though $\mathcal{F}$ is 3-dimensional. However, the results of Sect. 2 can easily be adapted to this case, the key point being that the Hessians of the SOCs are equal. First we need a lemma.

Lemma 1 *For $n = 2$, let $\mathcal{P} := \{x \in \mathbb{R}^2 : Ax \leq 0\}$ be a polyhedral cone with $A \in \mathbb{R}^{2\times 2}$. Then $\mathcal{P} = \{x : \|\binom{x_1}{x_2}\| \leq b^T x\}$ for some $b \in \mathbb{R}^2$.*

Proof First assume that $\mathcal{P}$ is contained in the right side of the plane, i.e., $\mathcal{P} \subseteq \{x : x_1 \geq 0\}$ and that $\mathcal{P}$ is symmetric about the x_1 axis. Then, for some $\beta \geq 0$,

$$\begin{aligned} \mathcal{P} &= \{x : x_1 \geq 0, -\beta x_1 \leq x_2 \leq \beta x_1\} \\ &= \{x : x_1 \geq 0, x_2^2 \leq \beta^2 x_1^2\} \\ &= \{x : x_1 \geq 0, x_1^2 + x_2^2 \leq (1 + \beta^2)x_1^2\} \\ &= \left\{x : \|\tbinom{x_1}{x_2}\| \leq \sqrt{1 + \beta^2}\, x_1\right\}, \end{aligned}$$

which proves the result in this case. For general $\mathcal{P}$, we may apply an orthogonal rotation to revert to the previous case, which does not affect the norm $\|\binom{x_1}{x_2}\|$ (but does change the exact form of b). □

We next state and prove the proposition. Note that the assumptions $L_{22} > 0$ and $U_{12} > L_{12}$ in the proposition are realistic for power networks: the first ensures the voltage magnitude at a bus is positive, and the second allows for a positive voltage-angle difference between the involved buses.

Proposition 4 *Suppose $L_{22} > 0$ and $U_{12} > L_{12}$. Then the feasible set, $\mathcal{F}$, defined by* (22) *satisfies*

$$\mathcal{F} = \left\{ x \in \mathbb{R}^3 : \begin{array}{c} \sqrt{L_{11}} \le \left\| \binom{x_1}{x_2} \right\| \le \sqrt{U_{11}} \\ \left\| \binom{x_1}{x_2} \right\| \le b_1 x_1 + b_2 x_2 \\ \sqrt{L_{22}} \le x_3 \le \sqrt{U_{22}} \end{array} \right\}$$

where b_1 and b_2 uniquely solve the system

$$\begin{pmatrix} 1 & L_{12} \\ 1 & U_{12} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1 + L_{12}^2} \\ \sqrt{1 + U_{12}^2} \end{pmatrix}.$$

Proof The assumption $L_{22} > 0$ implies $x_3 > 0$, which in turn implies

$$\mathcal{F} = \left\{ x \in \mathbb{R}^3 : \begin{array}{c} L_{11} \le x_1^2 + x_2^2 \le U_{11} \\ \sqrt{L_{22}} \le x_3 \le \sqrt{U_{22}} \\ L_{12} x_1 \le x_2 \le U_{12} x_1 \\ x_1 \ge 0 \end{array} \right\}.$$

Next, the assumption $U_{12} > L_{12}$ makes $x_1 \ge 0$ redundant, and clearly the first constraint in $\mathcal{F}$ is equivalent to $\sqrt{L_{11}} \le \|\binom{x_1}{x_2}\| \le \sqrt{U_{11}}$.

To complete the proof, we claim that $L_{12} x_1 \le x_2 \le U_{12} x_1$ is equivalent to the SOC constraint $\|\binom{x_1}{x_2}\| \le b_1 x_1 + b_2 x_2$. Indeed, it is clear that the set defined by these two linear inequalities is a polyhedral cone with the two extreme rays $r^1 = \binom{1}{L_{12}}$ and $r^2 = \binom{1}{U_{12}}$. So, by Lemma 1, the set is SOC-representable in the form $\|\binom{x_1}{x_2}\| \le b_1 x_1 + b_2 x_2$ for some $b \in \mathbb{R}^2$. In particular, the extreme rays r^j must satisfy $\|r^j\| = b^T r^j$. By plugging in the values of r^1 and r^2, we get the 2×2 linear system defining b, as desired. Note that the 2×2 matrix is invertible because its determinant $U_{12} - L_{12}$ is positive. $\square$

Based on Propositions 3 and 4, we now prove that (21) is simply (18) tailored to the OPF case.

Theorem 2 *Inequalities* (21) *are the inequalities* (18) *tailored to system* (19).

Proof By Proposition 3, we can translate (21a) to the variables (x, X). After collecting terms, (21a) becomes

$$(\pi_0 + U_{11}U_{22}) + (\pi_1 - U_{22})(X_{11} + X_{22}) + (\pi_2 - U_{11})X_{33} + \pi_3 X_{13} + \pi_4 X_{23} \ge 0. \quad (23)$$

Using Proposition 4, consider (18a) with the following replacements:

$$x \leftarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \;\; r \leftarrow \sqrt{L_{11}}, \;\; R \leftarrow \sqrt{U_{11}}, \;\; \lambda \leftarrow \sqrt{L_{22}}, \;\; s^T x \leftarrow x_3, \;\; \mu \leftarrow \sqrt{U_{22}}.$$

This results in the following valid inequality:

$$\left(\sqrt{L_{22}U_{22}} + U_{22}\right) \frac{X_{11} + X_{22} + \sqrt{L_{11}U_{11}}}{\sqrt{L_{11}} + \sqrt{U_{11}}}$$
$$\leq \left(\sqrt{L_{22}} + \sqrt{U_{22}}\right) (b_1 X_{13} + b_2 X_{23}) + (U_{22} - X_{33})\sqrt{U_{11}}.$$

Simple, although tedious, algebraic manipulations establish that this inequality is precisely (23). A similar argument establishes that (21b) corresponds to (18b).[3] □

We also verified numerically that (21) is not captured by $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ in this case.

3.2 Intersection of the ball and nonnegative orthant

As stated in the Introduction, the critical feature of $\mathcal{F}$ studied in this paper is its intersection of the ball with a second SOC-representable set, which shares the Hessian identity matrix. However, there are of course many other forms of $\mathcal{F}$ that can be of interest in practice. For example, when $\mathcal{F}$ is the nonnegative orthant, then $\mathcal{G}$ is the completely positive cone, which can be used to model many NP-hard problems as linear conic programs [4]. Another common case is when $\mathcal{F}$ is a box, e.g., the set $[0, 1]^n$ [6].

Let us examine the case in which $\mathcal{F}$ is the intersection of the nonnegative orthant and the unit ball. For general n, define $\mathcal{F} := \{x \geq 0 : \|x\| \leq 1\} \subseteq \mathbb{R}^n$. Since

$$x \in \mathcal{F} \quad \Rightarrow \quad \|x\| \leq \|x\|_1 = e^T x,$$

we have

$$\mathcal{F} \subseteq \{x : \|x\| \leq 1, \|x\| \leq e^T x\}, \tag{24}$$

and for $n = 2$, one can actually show that (24) is an equation. Since $\mathcal{F}$ is a subset of the nonnegative orthant, any inequality, which is valid for the completely positive cone, is also valid for $\mathcal{F}$, but here we focus on the implied structure in (24). Section 2 applies with $r = 0$, $R = 1$, $c = 0$, $b = e$, and $a = 0$. In particular, the constraints $\text{tr}(X) \leq 1$ and $\text{tr}(X) \leq e^T x$ are valid for $\mathcal{G}$; see the Introduction and inequality (6).

We can strengthen $\text{tr}(X) \leq 1$ and $\text{tr}(X) \leq e^T x$ using the slab inequalities of Sect. 2.1. Geometrically, given any $s \in \mathbb{R}^n$ with $s \geq 0$ and $\|s\| = 1$, we have the slab $\lambda := 0 \leq s^T x \leq 1 =: \mu$, which is valid for $\mathcal{F}$:

$$0 \leq s^T x \leq \|s\| \|x\| = \|x\| \leq 1.$$

After linearization, inequality (15) in this case reads $1 - s^T x + s^T X e \geq \text{tr}(X)$. Moreover, if we switch the role of q_x and l_x in (15)—recall that q_x is linear for slabs—then we have $s^T x + e^T x - s^T X e \geq \text{tr}(X)$. Rearranging, we write these two inequalities as

[3] We provide Matlab code for these manipulations in the file `chenetal/verify_chenetal.m` at the website https://github.com/A-Eltved/strengthened_sdr.

$$\mathrm{tr}(X) \le 1 + s^T(Xe - x) \tag{25a}$$
$$\mathrm{tr}(X) \le e^T x - s^T(Xe - x). \tag{25b}$$

Letting s vary over its constraints $\|s\| = 1$ and $s \ge 0$, we derive a compact SOC-representation of this class of inequalities over various domains of $\mathcal{G}$.

Theorem 3 *Let* (I, J) *be a partition of the index set* $\{1, \dots, n\}$*, and define the domain*

$$\mathcal{D}_{IJ} := \left\{ (x, X) : \begin{array}{l} [Xe - x]_I \ge 0 \\ [Xe - x]_J \le 0 \end{array} \right\}.$$

Then the following SOC constraints are locally valid for $\mathcal{G}$ *on* $\mathcal{D}_{IJ}$*:*

$$\mathrm{tr}(X) \le 1 - \|[Xe - x]_J\| \tag{26a}$$
$$\mathrm{tr}(X) \le e^T x - \|[Xe - x]_I\|. \tag{26b}$$

Moreover, (26) *imply all valid inequalities* (15) *derived from slabs of the form* $0 \le s^T x \le 1$*, where* s *is any vector satisfying* $\|s\| = 1$ *and* $s \ge 0$*.*

Proof Consider the constraints (25), and for notational convenience, define $y := Xe - x$. Because $s \ge 0$, the quantity $s^T y$ on the right-hand side of (25a) breaks into $s_I^T y_I \ge 0$ and $s_J^T y_J \le 0$ on $\mathcal{D}_{IJ}$. By minimizing the right-hand side of (25a) with respect to s, we achieve the tightest cut corresponding to $s = (s_I, s_J) = (0, -y_J/\|y_J\|)$, which yields $\mathrm{tr}(X) \le 1 - \|y_J\|$, as desired. A similar argument for (25b) yields $\mathrm{tr}(X) \le e^T x - \|y_I\|$. □

We remark that, when I is empty, inequality (26b) reduces to the inequality $\mathrm{tr}(X) \le e^T x$ over D_{IJ}. Similarly, when J is empty, (26a) is $\mathrm{tr}(X) \le 1$.

In practice, one idea for using Theorem 3 is as follows. For a given relaxation in (x, X), solve the relaxation to obtain an optimal solution $(\bar{x}, \bar{X})$. Then define the partition (I, J) and corresponding domain D_{IJ} according to $\bar{X}e - \bar{x}$. Then, if either of the inequalities in (26) is violated, we can derive a violated supporting hyperplane of the SOC constraint. After adding the violated linear inequality to the current relaxation, which is globally valid because it is linear, we can resolve and repeat the process.

We close this section with an example showing that the cuts derived above are not implied by $\mathcal{R}_{\mathrm{shor}} \cap \mathcal{R}_{\mathrm{ksoc}}$.

Example 2 Let $n = 2$, and consider $I = \{1, 2\}$ and $J = \emptyset$. Then $\mathrm{tr}(X) \le e^T x - \|Xe - x\|$ is valid on the domain $\mathcal{D}_{IJ} = \{(x, X) : Xe - x \ge 0\}$. In particular, $\mathrm{tr}(X) \le e^T x - u^T(Xe - x)$ for all vectors u satisfying $\|u\| = 1$, and taking $u = e_1$, we have $\mathrm{tr}(X) \le e^T x - [Xe - x]_1$, which is globally valid since it is linear. Minimizing $e^T x - [Xe - x]_1 - \mathrm{tr}(X)$ over $\mathcal{R}_{\mathrm{shor}} \cap \mathcal{R}_{\mathrm{ksoc}}$ yields the optimal value -0.088562, indicating that $\mathcal{R}_{\mathrm{shor}} \cap \mathcal{R}_{\mathrm{ksoc}}$ does not capture this valid constraint.

4 Separation

In this section, we argue that the inequalities (12)–(13) given by Theorem 1 and Corollary 1 are separable in polynomial time. To state this result precisely, we assume that $[c]_{\max}$ has already been pre-computed and that a fixed convex relaxation of the convex hull $\mathcal{G}$ defined by (5) is available. For convenience, we write this fixed convex relaxation

$$\mathcal{R} := \left\{(x, X) : Y(x, X) \in \widehat{\mathcal{R}}\right\} \supseteq \mathcal{G},$$

where $Y(x, X)$ is given by (3) and $\widehat{\mathcal{R}}$ is a closed, convex cone in the space of $(n + 1) \times (n + 1)$ symmetric matrices. In particular, $\mathcal{R}$ is just the slice of $\widehat{\mathcal{R}}$ with the top-left corner of Y set to 1. Then the relaxation of (1) over $\mathcal{R}$ can be stated as $\min\{H \bullet X + 2\, g^T x : (x, X) \in \mathcal{R}\}$ with dual

$$\max\left\{y : \begin{pmatrix} -y & g^T \\ g & H \end{pmatrix} \in \widehat{\mathcal{R}}^*\right\}$$

where $\widehat{\mathcal{R}}^*$ is the dual cone of $\widehat{\mathcal{R}}$. We state this general form for ease of notation and to make evident that one can choose different $\mathcal{R}$ in computation. For example, one could take $\mathcal{R} = \mathcal{R}_{\text{shor}}$ at one extreme or $\mathcal{R} = \mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ at the other.

In fact, to separate (12)–(13) we will use the following observation concerning $\mathcal{R}$, $\widehat{\mathcal{R}}$, and $\widehat{\mathcal{R}}^*$:

Observation *Given a quadratic function $q(x) := x^T H_q x + 2\, g_q^T x + f_q$, if there exists $y \in \mathbb{R}$ such that*

$$\begin{pmatrix} -y + f_q & g_q^T \\ g_q & H_q \end{pmatrix} \in \widehat{\mathcal{R}}^*,$$

then $q(x) \geq y$ for all $x \in \mathcal{F}$.

This observation follows by weak duality because y is a lower bound on the optimal relaxation value of $H_q \bullet X + 2\, g_q^T x + f_q$ over $(x, X) \in \mathcal{R}$, which is itself a lower bound on the minimum value of $q(x)$ over $x \in \mathcal{F}$. As a result, the following system guarantees that the conditions of Theorem 1 on $q(x)$ and $l(x)$ hold, where (H_q, g_q, f_q), (g_l, f_l), and $[q + l]_{\text{lowbd}}$ are the variables:

$$\begin{pmatrix} f_q & g_q^T \\ g_q & H_q \end{pmatrix} \in \widehat{\mathcal{R}}^*, \quad \begin{pmatrix} f_l & g_l^T \\ g_l & 0 \end{pmatrix} \in \widehat{\mathcal{R}}^*, \tag{27a}$$

$$[q + l]_{\text{lowbd}} \geq 0, \quad \begin{pmatrix} -[q + l]_{\text{lowbd}} + f_q + f_l & (g_q + g_l)^T \\ g_q + g_l & H_q \end{pmatrix} \in \widehat{\mathcal{R}}^*. \tag{27b}$$

Then, separation amounts to optimizing the linear function in (12)—or (13) as the case may be—over (27) for fixed values of (x, X). However, before we state the exact separation problem for (12), we require one additional assumption, namely that $\mathcal{F}$ is full-dimensional, i.e., there exists $\hat{x} \in \mathcal{F}$ such that $\|\hat{x}\| < R$ and $\|\hat{x} - c\| < b^T\hat{x} - a$. In this case, it is well known that $\mathcal{G}$ and hence $\mathcal{R}$ are also full-dimensional in (x, X)-space. In particular, $(\hat{x}, \hat{x}\hat{x}^T) \in \text{int}(\mathcal{G}) \subseteq \text{int}(\mathcal{R})$, and hence

$$\hat{Y} := \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix}\begin{pmatrix} 1 \\ \hat{x} \end{pmatrix}^T \in \text{int}(\widehat{\mathcal{R}}).$$

It thus follows that $\widehat{\mathcal{R}}^* \cap \{J : \hat{Y} \bullet J \leq 1\}$ is a bounded truncation of $\widehat{\mathcal{R}}^*$.[4] This truncation is important so that the separation problem presented below has a bounded feasible set and thus has a well-defined optimal value.

We are now ready to state the separation subproblem for (12) given fixed values $(\bar{x}, \bar{X})$ of the variables (x, X):

$$\min\ (r+R)R\left(H_q \bullet \bar{X} + 2\,g_q^T\bar{x} + f_q\right) + (r+R)\left(2\,g_l b^T \bullet \bar{X} + (f_l b - 2a g_l)^T\bar{x} - a f_l\right) \tag{28a}$$

$$- [q+l]_{\text{lowbd}}\,\text{tr}(\bar{X}) - rR\left(H_q \bullet \bar{X} + 2(g_q + g_l)^T\bar{x} + (f_q + f_l)\right)$$

$$+ \left(2\,g_l c^T \bullet \bar{X} + f_l c^T\bar{x}\right) + [c]_{\max} R(2\,g_l^T\bar{x} + f_l) \tag{28b}$$

$$\text{s.t.}\quad (27) \tag{28c}$$

$$\hat{Y} \bullet \begin{pmatrix} f_q & g_q^T \\ g_q & H_q \end{pmatrix} \leq 1, \quad \hat{Y} \bullet \begin{pmatrix} f_l & g_l^T \\ g_l & 0 \end{pmatrix} \leq 1. \tag{28d}$$

The subproblem for (13) is similar—just replace r with 0.

We remark that system (27) could be simplified in certain cases. For example, if $r = 0$ and hence $\mathcal{F}$ is convex, then it is not difficult to see that the second condition of (27a), which ensures that $l(x)$ is nonnegative over $\mathcal{F}$, could be replaced by a dual system based on $\mathcal{F}$ alone, not on $\mathcal{R}$. One could also simplify by forcing additional structure on $q(x)$ and $l(x)$. For example, one could separate against the slabs $\lambda \leq s^T x \leq \mu$ introduced in Sect. 2.1 by forcing $(H_q, g_q, f_q) = (0, -\frac{1}{2}s, \mu)$, $(g_l, f_l) = (\frac{1}{2}s, -\lambda)$, and $[q+l]_{\text{lowbd}} = \mu - \lambda$, in which case (27b) is automatically satisfied.

The following example demonstrates the separation procedure, whose implementation will be discussed in the next section:

Example 3 Consider the 2-dimensional problem

$$\begin{aligned} \min\ & -x_1^2 - x_2^2 - 1.1x_1 - x_2 \\ \text{s.t. } & \|x\| \leq 1 \\ & \|x\| \leq 1 - x_1 - x_2 \end{aligned}$$

with $H = -I$, $g = (-0.55, -0.5)$, $r = 0$, $R = 1$, $a = -1$, $b = (-1, -1)$, and $c = (0, 0)$ in (1). All values reported here are truncated from the computations and therefore

[4] Indeed, for any closed, convex cone $\mathcal{K}$ and dual cone $\mathcal{K}^*$, given $\hat{x} \in \text{int}(\mathcal{K})$, we claim the truncation $\mathcal{K}^* \cap \{s : \hat{x}^T s \leq 1\}$ is bounded. Specifically, its recession cone $\mathcal{K}^* \cap \{s : \hat{x}^T s \leq 0\} = \{0\}$. If not, then some nonzero $\tilde{s} \in \mathcal{K}^*$ satisfies $\hat{x}^T\tilde{s} \leq 0$. Because $\hat{x}$ is interior, for sufficiently small $\epsilon > 0$, the point $\tilde{x} := \hat{x} - \epsilon\tilde{s}$ satisfies $\tilde{x} \in \mathcal{K}$ and $\tilde{x}^T\tilde{s} < 0$. However, this contradicts the fact that $\tilde{s} \in \mathcal{K}^*$.

approximate. The optimal value of $\min\{H \bullet X + 2g^T x : (x, X) \in \mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}\}$ is -1.1431 with optimal solution

$$\bar{x} = \begin{pmatrix} 0.2922 \\ -0.1783 \end{pmatrix}, \ \bar{X} = \begin{pmatrix} 0.4963 & -0.3210 \\ -0.3210 & 0.5037 \end{pmatrix}.$$

Solving the separation subproblem at $(\bar{x}, \bar{X})$, we obtain the cut corresponding to

$$\begin{aligned}
q_1(x) &= x^T \begin{pmatrix} -0.3812 & 0 \\ 0 & -0.3812 \end{pmatrix} x + 2 \begin{pmatrix} -0.5578 \\ -0.5531 \end{pmatrix}^T x + 0.8563, \\
l_1(x) &= 2 \begin{pmatrix} 0.3462 \\ 0.3608 \end{pmatrix}^T x + 1, \\
[q_1 + l_1]_{\text{lowbd}} &= 1.42.
\end{aligned}$$

We add the corresponding cut, resolve to obtain a new $(\bar{x}, \bar{X})$, and repeat this loop two more times, resulting in the cuts

$$\begin{aligned}
q_2(x) &= x^T \begin{pmatrix} -0.7065 & 0.1719 \\ 0.1719 & -0.4368 \end{pmatrix} x + 2 \begin{pmatrix} -0.7808 \\ -0.7278 \end{pmatrix}^T x + 1, \\
l_2(x) &= 2 \begin{pmatrix} 0.3442 \\ 0.3626 \end{pmatrix}^T x + 1, \\
[q_2 + l_2]_{\text{lowbd}} &= 1.155, \\
q_3(x) &= x^T \begin{pmatrix} -0.6296 & 0.2398 \\ 0.2398 & -0.4512 \end{pmatrix} x + 2 \begin{pmatrix} -0.7868 \\ -0.7580 \end{pmatrix}^T x + 1, \\
l_3(x) &= 2 \begin{pmatrix} 0.3479 \\ 0.3591 \end{pmatrix}^T x + 1, \\
[q_3 + l_3]_{\min} &= 1.149.
\end{aligned}$$

We finally obtain the rank-1, and hence optimal, solution

$$Y(x^\star, X^\star) = \begin{pmatrix} 1 & 0.7071 & -0.7071 \\ 0.7071 & 0.5 & -0.5 \\ -0.7071 & -0.5 & 0.5 \end{pmatrix}$$

with objective value -1.0707. We note that, even though the procedure generates three cuts, the last cut is actually enough to recover the rank-1 solution. Moreover, running this procedure starting from $\mathcal{R}_{\text{shor}}$ instead of $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$, we also get the same optimal $(x^\star, X^\star)$ after adding 16 cuts.

5 Computational results

To quantify the practical effect of the cuts proposed in Theorem 1 and Corollary 1, we embed the separation subproblem described in Sect. 4 in a straightforward implementation to solve random instances of the form (1). We consider two relaxations to "bootstrap" the separation procedure: $\mathcal{R}_{\text{shor}}$ and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$. We will denote by $\mathcal{R}_{\text{cuts}}$ the points (x, X) satisfying the added cuts, so that our improved relaxations will be expressed as $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{cuts}}$ and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}} \cap \mathcal{R}_{\text{cuts}}$.

We implement our experiments in Matlab 9.6 (R2019a) using CVX [11] to model the relaxations and MOSEK 9.1 [17] to solve them. We run the problem instances on a single core of an Intel Xeon E5-2650v4 processor using a maximum of 2 GB memory. We do not report complete run times because we are most interested in the strength of the added cuts, but we do report the number of cuts added to measure the overall effort. Recall that calculating a single cut requires solving the separation problem (28) described in Sect. 4, which in essence involves three copies of the current bootstrap relaxation—$\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{cuts}}$ or $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}} \cap \mathcal{R}_{\text{cuts}}$—since the separation problem includes three sets of variables constrained to be in the dual cone of the bootstrap relaxation as described by (27). However, to give the reader a sense of the run times, consider the following: for an instance of our largest dimension, $n = 10$, solving $\mathcal{R}_{\text{shor}}$ took approximately 0.6 s, solving $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ required about 50 s, and solving a single separation problem for $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ took approximately 64 s. We note that our implementation is rudimentary and makes no effort to take advantage of, for example, any particular problem structure or sparsity, so these times can probably be improved significantly.

We generate a single random instance by fixing the dimension n and generating random data a, b, c, r, R, H, g in such a way that (1) is feasible with a known interior point $\hat{x}$, which is also randomly generated. In short, we first set $R = 1$ without loss of generality, generate r uniformly in $[0, R]$, generate $\hat{x}$ uniformly in $\{x : r \leq \hat{x} \leq R\}$, generate b, c, H, g with entries i.i.d. standard normal, and finally set $a := b^T \hat{x} - \|\hat{x} - c\| - \theta$, where θ is uniform in $[0, 1]$ so that $\mathcal{F}$ has a nonempty interior.[5] Recall that $\hat{x}$ is required for the separation procedure as discussed in Sect. 4. Before running the separation procedure for an instance, we compute $[c]_{\max}$ by a binary search on $[c]_{\max}$ over the interval $[0, \|c\|]$ as discussed in Sect. 2. Then, when running the overall algorithm, we consider the current relaxation's optimal solution $(\bar{x}, \bar{X})$ to be *separated* if: the objective value of the separation subproblem (28) is less than $\tau_{\text{sep}} = -10^{-5}$; or the optimal value of the separation subproblem for the inequalities (13) in Corollary 1, i.e., (28) with $r = 0$, is less than τ_{sep}. If $(\bar{x}, \bar{X})$ is indeed separated, we add the resulting cut represented by the data $(H_q, g_q, f_q, g_l, f_l, [q + l]_{\text{lowbd}})$ to the current bootstrap relaxation, optimize for a new point to be separated, and repeat. The overall loop stops when the current $(\bar{x}, \bar{X})$ is not separated with tolerance τ_{sep}.

[5] We refer the reader to our GitHub site (https://github.com/A-Eltved/strengthened_sdr) for the full random-generation procedure.

Regarding a given relaxation and its optimal solution $(\bar{x}, \bar{X})$, we say the relaxation is *exact* if $Y(\bar{x}, \bar{X})$ satisfies

$$\frac{\lambda_1(Y(\bar{x}, \bar{X}))}{\lambda_2(Y(\bar{x}, \bar{X}))} > \tau_{\text{rank}}, \tag{29}$$

where $\lambda_1(M)$ denotes the largest eigenvalue of M, $\lambda_2(M)$ denotes the second largest eigenvalue of M, and $\tau_{\text{rank}} > 0$ is a tolerance, which we choose to be 10^4 in our implementation, ensuring that $Y(\bar{x}, \bar{X})$ is numerically rank-1. We define the *gap* as the difference between the optimal value of (1) and the relaxation optimal value. Note that an exact relaxation implies a gap of 0.

After running the algorithm on a particular instance, we classify the instance into one of two categories: *exact initial* or *inexact initial*, when the initial bootstrap relaxation is exact or inexact, respectively. Furthermore, we break all inexact-initial instances into one of three subcategories: *improved*, when the initial relaxation gap is improved but not completely closed to 0; *closed*, when the relaxation becomes exact after adding one or more cuts (i.e, the resulting $(\bar{x}, \bar{X})$ satisfies (29)); and *no improvement*, when no cuts are successfully added to improve the gap, i.e., the separation routine does not help. (Actually, in the tables below, we will not directly report information about the exact-initial and no-improvement instances, as these details will be implicitly available from the other categories.)

We conduct these experiments for several values of n and many randomly generated instances. In addition, we also consider special cases where some of the data a, b, c, r, R is fixed to zero in order to assess whether the cuts are more effective in these special cases. In particular, we consider the following three cases: the general case, where no data is fixed *a priori* to zero; the special case with $r = a = 0$ and $c = 0$; and the case of the TTRS (two trust region subproblem) with $r = 0$ and $b = 0$. For each of these cases, we generate 15,000 instances for each dimension $2 \le n \le 10$, and we solve each instance twice, once bootstrapping from $\mathcal{R}_{\text{shor}}$ and once from $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$.

For the improved and closed instances, we report the average number of cuts added. Also for the improved instances, we report the average gap closure in percentage terms, i.e., we report the average relative gap closure. Since we do not actually know the optimal value of (1) for the improved instances, to approximate the relative gap closure from above, we calculate a local minimum value, v_{local}, by taking the lowest value of the quadratic objective function gotten by running Matlab's `fmincon` with 100 random initial points. The relative gap for the instance is then calculated as

$$\text{relative gap closure} = \frac{v_{\text{relax final}} - v_{\text{relax initial}}}{v_{\text{local}} - v_{\text{relax initial}}} \times 100\%,$$

where $v_{\text{relax initial}}$ is the optimal value of the initial relaxation and $v_{\text{relax final}}$ is the optimal value of the final relaxation. Note that a larger gap closure corresponds to a stronger relaxation, i.e., a larger gap closure is better.

Table 1 Results for the $\mathcal{R}_{\text{shor}}$ bootstrap relaxation on 15,000 random general instances for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure (%)	Closed	Avg cuts
2	2923	1188	15	51	264	4
3	2582	761	17	46	175	7
4	2161	422	10	40	53	7
5	1801	416	10	36	46	9
6	1583	265	12	36	29	8
7	1360	186	11	36	10	11
8	1091	140	14	39	15	7
9	1029	107	12	34	4	15
10	896	86	13	30	4	11

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

5.1 The general case

We consider 15,000 random instances for each dimension $2 \le n \le 10$ and report the results separately for the $\mathcal{R}_{\text{shor}}$ and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ bootstrap relaxations in Tables 1 and 2, respectively.

In Table 1, we see that our cuts improve the $\mathcal{R}_{\text{shor}}$ relaxation in many instances. For $n = 2$, it improves more than a third of the inexact instances, and it closes the gap for about 9%. As the dimension goes up, these proportions go down, suggesting that our cuts are more effective in lower dimensions.

Table 2 shows that $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ is generally quite strong for instances of the form (1). Especially for larger n, the number of inexact instances is small, and the ability of our cuts to improve or close the gaps is limited. In particular, for $n \ge 4$ our cuts do not improve any of the inexact instances, which again suggests that the cuts are most helpful in lower dimensions.

5.2 Special case: $r = a = 0$ and $c = 0$

We next consider the special case when $\mathcal{F}$ equals $\{x \in \mathbb{R}^n : \|x\| \le 1, \|x\| \le b^T x\}$ with $b \in \mathbb{R}^n$. Note that, by rotating the feasible space, we may assume without loss of generality that b lies in the direction of e, the all ones vector. In particular, we generate instances with $b = \beta e$, where $\beta \in [1/\sqrt{n}, 1/\sqrt{n} + 2n]$. The choice of this interval for β is based on the following observation: for $\beta < 1/\sqrt{n}$ the feasible space $\mathcal{F}$ is empty; for $\beta = 1/\sqrt{n}$ the feasible space $\mathcal{F}$ has no interior; for $\beta \to \infty$, the constraint $\|x\| \le b^T x$ resembles the half space $0 \le e^T x$.

Similar to Tables 1 and 2 of the previous subsection, Tables 3 and 4 contain the results of our separation algorithm on 15,000 randomly generated instances for each dimension, where Table 3 corresponds to $\mathcal{R}_{\text{shor}}$ and Table 4 to $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$. Contrary to what we saw in the general case in Tables 1 and 2, there does *not* seem to be a drop

Table 2 Results for the $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ bootstrap relaxation on the same 15,000 random general instances as depicted in Table 1 for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure	Closed	Avg cuts
2	251	40	13	45%	3	3
3	84	5	36	48%	0	–
4	44	0	–	–	0	–
5	16	0	–	–	0	–
6	6	0	–	–	0	–
7	7	0	–	–	0	–
8	2	0	–	–	0	–
9	3	0	–	–	0	–
10	3	0	–	–	0	–

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

Table 3 Results for the $\mathcal{R}_{\text{shor}}$ bootstrap relaxation on 15,000 random instances with $r = a = 0$ and $c = 0$ for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure (%)	Closed	Avg cuts
2	7744	2755	22	82	4988	2
3	7635	914	23	86	6495	3
4	7736	395	13	83	6966	3
5	7709	401	4	81	6596	3
6	7584	402	5	67	7182	3
7	7648	185	5	87	7463	3
8	7614	131	8	89	7483	3
9	7566	77	7	93	7489	2
10	7552	44	7	89	7508	2

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

in the proportion of instances where the cuts help as n increases. Overall, our cuts seem to be quite effective in this special case.

Specifically for $n = 2$, the results in Table 4 suggest that $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}} \cap \mathcal{R}_{\text{cuts}}$ is tight, i.e., it captures the convex hull $\mathcal{G}$. To test this further, we generated an additional 110,000 instances with $n = 2$. The $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ relaxation was exact for 109,938 of these, and our cuts closed the gap for the remaining 62 instances with an average of 3 cuts added. Our computational experience thus motivates a conjecture:

Conjecture 1 *For the 2-dimensional feasible space $\mathcal{F} := \{x \in \mathbb{R}^2 : \|x\| \le 1, \|x\| \le b^T x\}$ with arbitrary $b \in \mathbb{R}^2$, $\mathcal{R}_{shor} \cap \mathcal{R}_{ksoc} \cap \mathcal{R}_{cuts}$ equals the convex hull $\mathcal{G}$ defined in* (5).

In addition, in Sect. 3.2, for $n = 2$ and $b = e$, we proposed the locally valid cuts (26), which were derived from slabs of a particular form. (Note that these cuts would

Table 4 Results for the $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ bootstrap relaxation on the same 15,000 random instances as depicted in Table 3 with $r = a = 0$ and $c = 0$ for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure	Closed	Avg cuts
2	15	0	–	–	15	2
3	50	7	43	37%	30	2
4	36	4	78	75%	28	2
5	29	0	–	–	27	3
6	15	3	8	88%	12	3
7	13	2	4	57%	11	2
8	12	0	–	–	12	2
9	6	0	–	–	5	1
10	6	0	–	–	5	3

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

not necessarily be valid for a different scaling $b = \beta e$.) By generating many random objectives, we were able to find 100 additional instances, which were *not* solved exactly by $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$, and then separated just these locally valid cuts—instead of the more general cuts represented by $\mathcal{R}_{\text{cuts}}$. All 100 instances were solved exactly, i.e., achieved the tolerance τ_{rank}. We believe this is strong evidence to support the following conjecture as well:

Conjecture 2 *For the 2-dimensional feasible space $\mathcal{F} := \{x \in \mathbb{R}^2 : \|x\| \le 1, \|x\| \le e^T x\} = \{x \ge 0 : \|x\| \le 1\}$, the constraints defined by $\mathcal{R}_{shor} \cap \mathcal{R}_{ksoc}$ intersected with the locally valid cuts* (26) *capture the convex hull $\mathcal{G}$ defined in* (5).

5.3 Special case: TTRS ($b = 0$ and $r = 0$)

Setting $b = 0$ and $r = 0$ in (1) with $a < 0$ to ensure feasibility, we explore the two-trust-region subproblem (TTRS). We generate 15,000 random instances of this type for each dimension $2 \le n \le 10$ and bootstrap from the $\mathcal{R}_{\text{shor}}$ and $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ relaxations. The results are shown in Tables 5 and 6. The trends in these tables are similar to what we saw in the general case in Sect. 5.1. In particular, our cuts are less effective in higher dimensions.

We catalog the following example showing an explicit case for $n = 2$ in which our cuts close the gap for TTRS compared to just applying $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$.

Example 4 Consider the instance with $n = 2$, $r = 0$, $R = 1$, $a = -0.77$, and

$$b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} -0.38 \\ 0.18 \end{pmatrix}, \quad H = \begin{pmatrix} -1.32 & 0.21 \\ 0.21 & -0.81 \end{pmatrix}, \quad g = \begin{pmatrix} -0.25 \\ 0.05 \end{pmatrix}.$$

The (approximate) optimal value of $\min\{H \bullet X + 2\, g^T x : (x, X) \in \mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}\}$ is -0.9087 and the solution is not rank-1. Solving the separation problem starting from

Table 5 Results for the $\mathcal{R}_{\text{shor}}$ bootstrap relaxation on 15,000 random TTRS instances for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure	Closed	Avg cuts
2	1404	364	16	33%	86	4
3	1287	172	15	27%	34	4
4	985	79	12	27%	20	5
5	745	34	9	22%	7	3
6	508	14	7	22%	3	2
7	454	4	5	25%	2	3
8	347	5	8	58%	0	–
9	293	0	–	–	1	2
10	251	1	4	2%	0	–

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

Table 6 Results for the $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ bootstrap relaxation on the same 15,000 random TTRS instances as depicted in Table 5 for each dimension n

n	Inexact initial	Improved	Avg cuts	Avg gap closure	Closed	Avg cuts
2	31	4	20	24%	0	–
3	78	7	43	29%	1	7
4	63	3	55	19%	0	–
5	34	1	59	6%	0	–
6	22	0	–	–	0	–
7	16	0	–	–	0	–
8	14	0	–	–	0	–
9	6	0	–	–	0	–
10	4	0	–	–	0	–

The columns *Inexact initial*, *Improved*, and *Closed* report the number of instances out of 15,000 in each category

this relaxation, we obtain the (approximate) cut corresponding to

$$g_l = \begin{pmatrix} 1.8633 \\ -0.8826 \end{pmatrix}, \quad f_l = 4.1236, \quad [q + l]_{\text{lowbd}} = 1.2604,$$
$$H_q = \begin{pmatrix} -4.9035 & 0.0000 \\ 0.0000 & -4.9035 \end{pmatrix}, \quad g_q = \begin{pmatrix} -1.8633 \\ 0.8826 \end{pmatrix}, \quad f_q = 2.0403.$$

Solving the relaxation with this cut, results in the (numerically) rank-1 solution

$$Y(x^\star, X^\star) = \begin{pmatrix} 1.0000 & -0.9065 & 0.4223 \\ -0.9065 & 0.8217 & -0.3828 \\ 0.4223 & -0.3828 & 0.1783 \end{pmatrix}$$

with (approximate) optimal value -0.8943.

6 Conclusions

In this paper, we have derived a new class of valid linear inequalities for SDP relaxations of problem (1). These cuts are separable in polynomial time, which, by the equivalence of separation and optimization (up to an $\epsilon > 0$ optimality tolerance), ensures that the SDP relaxation enforcing all of these inequalities is polynomial-time solvable. We have also shown that a special case of our cuts has been applied by Chen et al. [9] to obtain the convex hull of an important substructure arising in the OPF problem. In addition, we have extended our methodology to derive new, locally valid, second-order-cone cuts for nonconvex quadratic programs over the mixed polyhedral-conic set $\{x \geq 0 : \|x\| \leq 1\}$. Using specific examples as well as computational experiments, we have demonstrated that the new class of valid inequalities strengthens the strongest known SDP relaxation, $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$, especially in low dimensions.

For the specific 2-dimensional feasible set $\mathcal{F} = \{x \in \mathbb{R}^2 : \|x\| \leq 1, x \leq b^T x\}$, our computational experiments indicate that our cuts intersected with $\mathcal{R}_{\text{shor}} \cap \mathcal{R}_{\text{ksoc}}$ capture the relevant convex hull $\mathcal{G}$. We leave this as a conjecture requiring further research. Furthermore, when $b = e$, we also conjecture that the locally valid cuts (26), which are derived from slabs, are by themself enough to capture $\mathcal{G}$. For general $\mathcal{F}$, however, our cuts do not close the gap fully, and so there remains room for improvement.

One limitation of our approach is the assumption that the SOC constraint (1c) shares the identity Hessian with the hollow ball (1b). If instead we are presented with a general SOC constraint $\|Jx - c\| \leq b^T x - a$, where $J \in \mathbb{R}^{n \times n}$ is arbitrary, one idea would be to bound

$$
\begin{aligned}
b^T x - a &\geq \|Jx - c\| \\
&\geq \|x - c\| - \|x - Jx\| \\
&= \|x - c\| - \|(I - J)x\| \\
&\geq \|x - c\| - \sqrt{\lambda_{\max}[(I - J)^T (I - J)]}\, R,
\end{aligned}
$$

which yields the valid constraint $\|x - c\| \leq b^T x - \left(a - \sqrt{\lambda_{\max}[(I - J)^T (I - J)]} R\right)$, to which our methodology can be applied. Additional options for handling arbitrary Hessians can be considered by refining the deriviations of Sect. 2.

Further opportunities for future research include streamlining the separation subroutine, investigating the effectiveness of our cuts in higher dimensions, and examining other applications where the structure of (1) appears. Also, the idea of using the self-duality of a cone to derive valid linear cuts could be applied to other self-dual cones or possibly even non-self-dual cones.

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