

# A Two-Variable Approach to the Two-Trust-Region Subproblem

Boshi Yang\*

Samuel Burer<sup>†</sup>

November 18, 2013

Revised: October 13, 2014

Revised: July 27, 2015

## Abstract

The trust-region subproblem minimizes a general quadratic function over an ellipsoid and can be solved in polynomial time using a semidefinite-programming (SDP) relaxation. Intersecting the feasible set with a second ellipsoid results in the two-trust-region subproblem (TTRS). Even though TTRS can also be solved in polynomial-time, existing algorithms do not use SDP. In this paper, we investigate the use of SDP for TTRS. Starting from the basic SDP relaxation of TTRS, which admits a gap, recent research has tightened the basic relaxation using valid second-order-cone inequalities. Even still, closing the gap requires more. For the special case of TTRS in dimension  $n = 2$ , we fully characterize the remaining valid inequalities, which can be viewed as strengthened versions of the second-order-cone inequalities just mentioned. We also demonstrate that these valid inequalities can be used computationally even when  $n > 2$  to solve TTRS instances that were previously unsolved using SDP-based techniques.

**Keywords:** trust-region subproblem, semidefinite programming, nonconvex quadratic programming.

**Mathematics Subject Classification:** 90C20, 90C22, 90C25, 90C26, 90C30.

---

\*Department of Mathematics, University of Iowa, Iowa City, IA, 52242-1419, USA. Email: boshi-yang@uiowa.edu.

<sup>†</sup>Department of Management Sciences, University of Iowa, Iowa City, IA, 52242-1994, USA. Email: samuel-burer@uiowa.edu.

# 1 Introduction

This paper studies the two-trust-region subproblem, called TTRS, which is the minimization of a general quadratic function over the intersection of two full-dimensional ellipsoids:

$$\begin{aligned} v^* &:= \min_{x \in \mathbb{R}^2} x^T C x + 2 c^T x \\ \text{s. t. } & (x - a_i)^T A_i (x - a_i) \leq 1 \quad i = 1, 2. \end{aligned} \tag{TTRS}$$

The data are  $n \times n$  symmetric matrices  $C, A_i$  and vectors  $c, a_i \in \mathbb{R}^n$ . Moreover, each  $A_i$  is positive definite so that each set  $E_i := \{x : (x - a_i)^T A_i (x - a_i) \leq 1\}$  is a full-dimensional ellipsoid with center  $a_i$ . Let  $F := E_1 \cap E_2$  denote the feasible region of (TTRS). If  $C$  is positive semidefinite, then TTRS is solvable in polynomial-time using second-order cone programming. So we assume that  $C$  is not positive semidefinite. TTRS was originally introduced by Celis-Dennis-Tapia [10] and hence is sometimes also called the *CDT problem*.

TTRS is a generalization of the classical *trust-region subproblem*, called TRS, that minimizes  $x^T C x + 2 c^T x$  over a single ellipsoid  $(x - a_1)^T A_1 (x - a_1) \leq 1$ , which can be assumed to be the unit ball  $\|x\| \leq 1$  without loss of generality. TRS serves as the basis of trust-region methods for nonlinear optimization [11] and, even though nonconvex, can be solved efficiently in theory and practice [13, 17, 19]. In particular, the papers [12, 23] consider the polynomial-time complexity of constructing an  $\epsilon$ -optimal solution of TRS. Moreover, the optimal value of TRS equals the optimal value of the following polynomial-time solvable semidefinite program (SDP) [19]:

$$\min \left\{ C \bullet X + 2 c^T x : A_1 \bullet X - 2 a_1^T A_1 x + a_1^T A_1 a_1 \leq 1, \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \right\}, \tag{1}$$

where the notation  $M \bullet X := \text{trace}(M^T X)$  denotes the inner product of symmetric matrices. For additional background on TRS, we refer the reader to [19, section 1.1].

In addition to TTRS, several other generalizations of TRS have been studied; see [19] for an early discussion. For example, [22] investigates the addition of a single linear inequality to TRS, which is shown to have an exact convex relaxation gotten by adding a second-order-cone constraint to (1). The paper [24] discusses an extension having two parallel, non-intersecting linear constraints, which can be solved in polynomial time by subdividing the problem into cases. Extending [24], the papers [8, 9] show that adding any number of linear constraints to TRS—as long as the constraints do not intersect in the ellipsoid—can be solved in polynomial time by adding extra linear and second-order-cone constraints to (1). Two recent papers have studied the intersection of TRS with a general polyhedron  $P$ . First,

[15] provides a sufficient condition on the data of the problem, including  $P$ , under which a simple extension of (1) is tight. Second, [6] shows that the problem with  $P$  can be solved in polynomial time as long as the number of faces of  $P$  within the ellipsoid is polynomial. The approach of [6] is combinatorial in nature by subdividing the problem into cases and, in particular, does not make use of convex relaxation.

TTRS has itself received considerable attention. Optimality conditions are studied in [18], which also discusses much of the related literature from the 1990s. A recent paper looking at global and local optimality conditions through the lens of copositivity is [7]. The papers [1, 4] study conditions when the basic SDP relaxation of TTRS, i.e., the relaxation that adds the second constraint  $A_2 \bullet X - 2a_2^T A_2 x + a_2^T A_2 a_2 \leq 1$  to (1), is tight. On the other hand, [24] develops a trajectory-following procedure that solves TTRS generally. This procedure for TTRS is not proved to be polynomial but appears to be practically quite efficient. The authors of [24] also questioned whether there might exist exact polynomial-time formulations of TTRS. The paper [8] shed some light on this question by providing valid second-order-cone constraints tightening the basic SDP relaxation of TTRS, but even these relaxations are still not tight. In Section 2, we review the known relaxations of TTRS that are based on the variables  $(x, X)$ .

Most recently, [5] has demonstrated that TTRS can be solved in polynomial-time using an algorithm of [3] to determine whether two or more quadratic forms share a common zero; see also [14]. While this establishes the polynomial complexity of TTRS, we note that the algorithm employed in [5] does not use convex relaxation (similar to [6] above). We believe it is still interesting to seek the convex relaxation that solves TTRS exactly. For example, the convex relaxation could provide insight into problems for which TTRS appears as a substructure, even when the algorithm of [5] may not be applicable. We are encouraged by the results of [5], which indicate that the goal of finding the tight convex relaxation is a reasonable one.

Hence, in this paper, our goal is to investigate which additional valid constraints in  $(x, X)$  are necessary to calculate  $v^*$  exactly. Our main theoretical contribution is a full specification of the additional constraints needed when  $n = 2$ . Section 2 and Table 1 therein provide the description of the precise inequalities required, which are strengthened (or “lifted”) versions of the inequalities previously described in [8]. Sections 3 and 4 contain the technical details and proofs. The tools that we develop help us classify the local and global behavior of all quadratics over the intersection of two ellipsoids in  $\mathbb{R}^2$ .

While our theoretical result is limited in dimension, we believe our results provide significant insight into the nature of TTRS and its solution by convex relaxation methods. Indeed, in Section 5, we show how to separate the type of valid inequalities discovered in

this paper for general  $n$ , which allows us to solve computationally instances of TTRS that were previously unsolved using SDP-based techniques.

## 1.1 Notation, terminology, and a simplifying assumption

We use standard notation for Euclidean spaces of vectors and matrices. The set  $\mathbb{R}^n$  denotes column vectors of size  $n$ , and  $\mathcal{S}^n$  denotes all symmetric  $n \times n$  matrices. The inner product of  $M, N \in \mathcal{S}^n$  is  $M \bullet N := \text{trace}(M^T N)$ . The subset  $\mathcal{S}_+^n \subseteq \mathcal{S}^n$  consists of positive semidefinite matrices, that is, symmetric matrices with all nonnegative eigenvalues. The notation  $\text{Closure}(\cdot)$  denotes the closure operation, and  $\text{Conv}(\cdot)$  denotes the convex hull. For a convex cone  $\mathcal{K}$ ,  $\text{Ext}(\mathcal{K})$  is the set of extreme rays of  $\mathcal{K}$ .

In this paper, we will deal with quadratic functions of the general form  $f(x) = x^T R x + 2r^T x + \rho$ . Without loss of generality,  $R$  is symmetric. However, we will sometimes write  $f$  using a non-symmetric representation, e.g.,  $f(x) = (\beta - \alpha^T x)(\delta - \gamma^T x) = \beta\delta - \beta\gamma^T x - \delta\alpha^T x + x^T \alpha\gamma^T x$ . This is simply for convenience and does not change the fact that the Hessian of the quadratic is assumed symmetric. Given  $f$ , we will also sometimes find it convenient to refer to the Hessian of  $f$  without specifying  $R$ . In these cases, we simply write  $\text{Hess}(f)$ .

The geometry of the feasible region  $F$  will play a significant role in our analysis, and so we define several sets describing different portions of  $F = E_1 \cap E_2$ . We assume that  $F$  is full-dimensional in  $\mathbb{R}^n$  and that  $E_1 \subsetneq E_2$  and  $E_2 \subsetneq E_1$ . In particular, we do not consider cases with  $F$  being empty, a singleton, or equal to  $E_1$  or  $E_2$ . Let  $\text{int}(E_i) := \{x \in E_i : (x - a_i)^T A_i (x - a_i) < 1\}$  be the interior of  $E_i$  and  $\text{bd}(E_i) := \{x \in E_i : (x - a_i)^T A_i (x - a_i) = 1\}$  be its boundary for  $i = 1, 2$ . We also use  $\text{int}(F)$  to denote the nonempty interior of  $F$ , and  $\text{bd}(F)$  to denote the boundary. Finally, define  $\text{vert}(F) := \text{bd}(E_1) \cap \text{bd}(E_2)$  to be the points on the boundaries of both ellipsoids; these are the *vertices*.

For the sake of simplicity, throughout this paper we assume that  $x \in \text{vert}(F)$  implies  $A_1 x - a_1$  and  $A_2 x - a_2$  are linearly independent. In other words, the constraint gradients are independent at  $x$ . Note that, when  $n = 2$  and  $F$  is full-dimensional,  $E_1 \subsetneq E_2$  and  $E_2 \subsetneq E_1$  (as assumed above), the set  $\text{vert}(F)$  is finite with cardinality 1, 2, 3, or 4. This simplifying assumption rules out the case that the cardinality of  $\text{vert}(F)$  is odd. A slightly more careful presentation is required if the assumption is not satisfied, but the major conclusions of the paper still hold.

## 2 Relaxations of TTRS

In this section, we review existing SDP relaxations in the variables  $(x, X)$  for TTRS and discuss how to tighten the relaxations using quadratic functions that are nonnegative over the feasible region  $F$ . The main point is Theorem 1, which specifies a subset of valid quadratics that close the SDP-relaxation gap completely. Technical details and the proofs are given in Sections 3–4.

### 2.1 Existing relaxations

Because (TTRS) is nonconvex, a reasonable approach is to relax it as a semidefinite program that can be solved in polynomial time [8, 21, 24]:

$$\begin{aligned} v(\text{SDP}) &:= \min C \bullet X + 2c^T x & (\text{SDP}) \\ \text{s. t. } & A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1 \quad i = 1, 2 \\ & (x, X) \in \text{PSD} \end{aligned}$$

where

$$\text{PSD} := \left\{ (x, X) : \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \right\}.$$

We call this the *basic SDP relaxation*. Note that by the Schur complement theorem,  $(x, X) \in \text{PSD}$  if and only if  $X \succeq xx^T$ . Also, when

$$\text{rank} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 1,$$

then  $X = xx^T$  and  $v(\text{SDP}) = v^*$ . However, it is well known that the optimal value of (SDP) is in general strictly less than the optimal value of (TTRS), i.e.,  $v(\text{SDP}) < v^*$ .

The paper [8] proposes a method to strengthen (SDP). Let  $A_2^{1/2}$  be the positive definite square root of  $A_2$ , and rewrite the second ellipsoidal constraint as the second-order-cone constraint  $\|A_2^{1/2}(x - a_2)\| \leq 1$ . In addition, let  $\alpha^T x \leq \beta$  be any valid inequality that supports the first ellipsoid  $E_1$ . Then the following quadratic second-order-cone inequality is valid for  $F$ :

$$\begin{aligned} \|A_2^{1/2}(\beta x - \alpha^T x \cdot x - \beta a_2 + \alpha^T x \cdot a_2)\| &= \|A_2^{1/2}(\beta - \alpha^T x)(x - a_2)\| \\ &= (\beta - \alpha^T x) \|A_2^{1/2}(x - a_2)\| \\ &\leq \beta - \alpha^T x. \end{aligned}$$

Moreover, this inequality may be linearized via  $X$  and added to the basic SDP relaxation:

$$\|A_2^{1/2}(\beta x - X\alpha - \beta a_2 + \alpha^T x \cdot a_2)\| \leq \beta - \alpha^T x. \quad (2)$$

These inequalities are called SOCRLT constraints in [8] since their derivation is closely tied to the regular RLT (“reformulation linearization technique” [20]) constraints gotten by multiplying two valid linear constraints  $\beta - \alpha^T x \geq 0$  and  $\delta - \gamma^T x \geq 0$  and then linearizing to get  $\beta\delta - \beta \cdot \gamma^T x - \delta \cdot \alpha^T x + \alpha^T X \gamma \geq 0$ . Letting SOC denote the set of  $(x, X)$  satisfying all possible SOCRLT constraints, [8] proposes to solve

$$\begin{aligned} v(\text{SOC}) &:= \min C \bullet X + 2c^T x & (\text{SOC}) \\ \text{s. t. } & A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1 \quad i = 1, 2 \\ & (x, X) \in \text{PSD} \cap \text{SOC}. \end{aligned}$$

Although infinite in number, the SOCRLT constraints can be separated in polynomial time and hence  $v(\text{SOC})$  can be efficiently calculated. Nevertheless, [8] shows by example that  $v(\text{SDP}) < v(\text{SOC}) < v^*$  in general.

The SOCRLT constraints are defined by multiplying a supporting inequality  $\alpha^T x \leq \beta$  of  $E_1$  by the second-order-cone representation of  $E_2$ . The same relaxation value  $v(\text{SOC})$  results if the roles of  $E_1$  and  $E_2$  are switched [8]. Moreover, [8] argues that the entire collection of SOCRLT constraints is equivalent to all possible regular RLT constraints, i.e., those gotten by multiplying a supporting  $\beta - \alpha^T x \geq 0$  of  $E_1$  with a supporting  $\delta - \gamma^T x \geq 0$  of  $E_2$ . In this sense, the SOCRLT constraints are simply a different representation of the RLT constraints.

## 2.2 Exact relaxations

So the question remains: what additional valid constraints are required beyond those in (SDP) and (SOC) to close the relaxation gap?

Using the approach of [22], we claim that closing the relaxation gap is equivalent to describing the following set, which corresponds to all quadratic functions that are nonnegative over  $F$ :

$$\mathcal{K} := \{(R, r, \rho) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} : x^T R x + 2r^T x + \rho \geq 0 \ \forall x \in F\}.$$

$\mathcal{K}$  is a closed, convex cone [22]. Several classes of elements in  $\mathcal{K}$  are readily apparent. For example,  $(-A_i, A_i a_i, 1 - a_i^T A_i a_i) \in \mathcal{K}$  corresponds to the ellipsoidal constraint  $(x - a_i)^T A_i (x - a_i) \leq 1$ , and  $(\alpha\gamma^T, -\frac{1}{2}(\beta\gamma + \delta\alpha), \beta\gamma) \in \mathcal{K}$  corresponds to the RLT quadratic

$(\beta - \alpha^T x)(\delta - \gamma^T x) \geq 0$ . Consider the following optimization problem:

$$\begin{aligned} v(\mathcal{K}) &:= \min C \bullet X + 2c^T x \\ \text{s. t. } & R \bullet X + 2r^T x + \rho \geq 0 \quad \forall (R, r, \rho) \in \mathcal{K}. \end{aligned}$$

**Proposition 1.**  $v(\mathcal{K}) = v^*$ .

*Proof.* Clearly  $v(\mathcal{K}) \leq v^*$  because the optimization problem is a relaxation by dropping the constraint  $X = xx^T$ . In addition, since  $(C, c, -v^*) \in \mathcal{K}$  by definition, the constraint  $C \bullet X + 2c^T x - v^* \geq 0$  guarantees  $v(\mathcal{K}) \geq v^*$ , as desired.  $\square$

So closing the gap amounts to enforcing the constraints  $R \bullet X + 2r^T x + \rho \geq 0$  for all elements in  $\mathcal{K}$ .

Of course, the definition of  $\mathcal{K}$  is quite generic, and it would be helpful to characterize, for example, the extreme rays of  $\mathcal{K}$  or a few constraints that capture whole portions of  $\mathcal{K}$ . Indeed, the two ellipsoidal constraints  $A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1$  are fundamental, and the SOCRLT constraints capture all the RLT constraints as discussed at the end of the previous subsection. In addition, each  $(x, X)$  feasible for (SDP) satisfies  $R \bullet X + 2r^T x + \rho \geq 0$  for all  $(R, r, \rho) \in \mathcal{K}$  with  $R \succeq 0$ . This can be seen in two steps. First, each feasible  $(x, X)$  implies  $x \in F$  since

$$(x - a_i)^T A_i (x - a_i) = A_i \bullet xx^T - 2a_i^T A_i x + a_i^T A_i a_i \leq A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1$$

where the second inequality follows because  $A_i$  is positive definite and  $(x, X) \in \text{PSD}$ . Second, because  $R \succeq 0$ ,  $(x, X) \in \text{PSD}$ , and  $x \in F$ ,

$$R \bullet X + 2r^T x + \rho \geq R \bullet xx^T + 2r^T x + \rho \geq 0.$$

To fully characterize elements in  $\mathcal{K}$ , we would like to define and investigate a proper subcone  $\mathcal{G}$  of  $\mathcal{K}$  that is guaranteed to contain  $\text{Ext}(\mathcal{K})$ . In this way,  $\mathcal{G}$  gives rise to an SDP relaxation whose optimal value equals  $v^*$ :

$$\begin{aligned} v^* = v(\mathcal{G}) &:= \min C \bullet X + 2c^T x \\ \text{s. t. } & R \bullet X + 2r^T x + \rho \geq 0 \quad \forall (R, r, \rho) \in \mathcal{G}. \end{aligned}$$

The main technical approach is to show that the  $\mathcal{G}$  has a special property as described in Proposition 2 below.

We first introduce some definitions. For  $(R, r, \rho)$ , we write  $f = (R, r, \rho)$  and define the

function  $f(x) := x^T R x + 2r^T x + \rho$  acting on the vector variable  $x \in \mathbb{R}^n$ . When  $f \in \mathcal{K}$ , we say that  $f$  is *valid for  $F$* . For  $f, g \in \mathcal{K}$ , if  $f - g$  is also valid, i.e., if  $f(x) \geq g(x)$  for all  $x \in F$ , then we write  $f \succeq g$  and say that  $g$  *minorizes  $f$  over  $F$* . We note that minorization is clearly transitive, i.e.,  $f \succeq g$  and  $g \succeq h$  imply  $f \succeq h$ .

**Proposition 2.** *Let  $\mathcal{G} \subseteq \mathcal{K}$  be a cone, not necessarily convex. If, for every  $f \in \mathcal{K}$ , there exists some  $g \in \mathcal{G}$  such that  $f \succeq g$ , then  $\text{Ext}(\mathcal{K}) \subseteq \mathcal{G}$ , and so  $\text{Closure}(\text{Conv}(\mathcal{G})) = \mathcal{K}$ .*

*Proof.* We need to show every  $f \in \text{Ext}(\mathcal{K})$  is an element of  $\mathcal{G}$ . By assumption, we know  $f \succeq g$  for some  $g \in \mathcal{G}$ . If  $f \parallel g$ , i.e., there exists  $\alpha \geq 0$  such that  $f = \alpha g$ , we are done. On the other hand, when  $f \nparallel g$ , the equation  $f = (f - g) + g$  shows that  $f$  is not extreme in  $\mathcal{K}$ , a contradiction. The equation  $\text{Closure}(\text{Conv}(\mathcal{G})) = \mathcal{K}$  follows because  $\mathcal{G}$  contains all the extreme rays of  $\mathcal{K}$ , which is closed.  $\square$

From this point forward in Sections 2–4, we consider the case when  $n = 2$ . We will show computationally in Section 5 that the insights for dimension  $n = 2$  can be used more generally for  $n > 2$ .

## 2.3 Choice of valid inequalities

We state our choice of  $\mathcal{G}$  here in order to familiarize the reader with some of its features, although the insights that lead to this choice will be developed in Sections 3–4. In particular, see Theorem 1 in Section 4.

Before listing the generators of  $\mathcal{G}$ , we define several functions. First, define

$$\begin{aligned} g_{E_i}(x) &:= 1 - (x - a_i)^T A_i (x - a_i) & i = 1, 2 \\ T_{yE_i}(x) &:= 1 - (y - a_i)^T A_i (x - a_i) & \forall \text{ feasible } y \in \text{bd}(E_i), i = 1, 2. \end{aligned}$$

In words,  $g_{E_i}(x) \geq 0$  defines the ellipsoid  $E_i$ . Also,  $T_{yE_i}(x) = 0$  defines the tangent line to  $E_i$  at  $y$ , and the inequality  $T_{yE_i}(x) \geq 0$  supports  $E_i$  at  $y$ . For any two distinct points  $y, z \in \mathbb{R}^2$ , we also let  $L_{yz}$  be a linear function such that  $L_{yz}(x) = 0$  defines the unique line passing through  $y$  and  $z$ . Up to multiplication by a constant, the representation of  $L_{yz}$  is unique, and whenever we write  $L_{yz}$ , the reader may safely assume that  $y \neq z$ .

Abusing notation, we also let  $T_{yE_i}$  and  $L_{yz}$  denote the sets  $\{x : T_{yE_i}(x) = 0\}$  and  $\{x : L_{yz}(x) = 0\}$ , respectively. That is,  $T_{yE_i}$  denotes the function defining the tangent line and the tangent line itself, and similarly for  $L_{yz}$ .

We require one additional concept. Our choice of  $\mathcal{G}$  will contain valid quadratics of the form  $f + \lambda g$ , where  $g$  is itself valid and  $\lambda \in \mathbb{R}$ . Suppose that both  $f + \lambda_1 g$  and  $f + \lambda_2 g$  are



Valid Quadratic	Conditions	Nickname
$g_{E_i}$	$i \in \{1, 2\}$	ellipsoid
$f$	$\text{Hess}(f) \in \mathcal{S}_+^2$	PSD
$T_{yE_1}T_{yE_2}$	$y \in \text{vert}(F)$	vertex RLT
$T_{yE_1}T_{zE_2} + \lambda L_{yz}^2$	$y \neq z, \lambda < 0$ minimal	lifted RLT

Table 1: Our choice of valid quadratic functions generating  $\mathcal{G}$

valid such that  $\lambda_1 < \lambda_2$ . Then clearly  $f + \lambda_1 g \preceq f + \lambda_2 g$ , i.e.,  $f + \lambda_1 g$  minorizes  $f + \lambda_2 g$ , because the difference  $(\lambda_2 - \lambda_1)g$  is valid. It may happen that  $f + \lambda_1 g$  is further minorized by  $f + \lambda_0 g$  with  $\lambda_0 < \lambda_1$ . This leads to the concept of  $\lambda$  being *minimal* in  $f + \lambda g$ , which we establish in the following proposition.

**Proposition 3.** *Let  $f, g$  be quadratics with  $g \in \mathcal{K}$ . Suppose  $f + \bar{\lambda}g$  is valid for some  $\bar{\lambda} \in \mathbb{R}$ . Then there exists  $\lambda_{\min} \in \mathbb{R}$  such that  $f + \lambda g$  is valid if and only if  $\lambda \geq \lambda_{\min}$ .*

*Proof.* For all  $x \in F$ , define

$$\lambda_{\min}(x) := \begin{cases} -g(x)f(x)^{-1} & \text{if } f(x) \neq 0 \\ -\infty & \text{if } f(x) = 0. \end{cases}$$

That is, for each  $x$  separately,  $\lambda_{\min}(x)$  measures the smallest value of  $\lambda$  such that  $f(x) + \lambda g(x) \geq 0$ . This means in particular that  $\lambda_{\min}(x) \leq \bar{\lambda}$  since  $f + \bar{\lambda}g \in \mathcal{K}$  by assumption. Define  $\lambda_{\min} := \sup_{x \in F} \lambda_{\min}(x)$ . It is then clear that  $f + \lambda g$  is valid if and only if  $\lambda \geq \lambda_{\min}$ .  $\square$

The generators for  $\mathcal{G}$  are listed in Table 1. There are four classes of generators, each corresponding to a type of valid quadratic function. We give each class a nickname for ease of discussion. The first three classes are already known and have been incorporated into existing relaxations as discussed in Section 2.1, while the last class is new in this paper.

The first class consists of the two ellipsoid quadratics  $g_{E_i}(x) \geq 0$  for  $i \in \{1, 2\}$ , and the second class consists of all valid quadratics  $f(x) = x^T R x + 2r^T x + \rho \geq 0$  such that  $\text{Hess}(f) = 2R$  is positive semidefinite. Together, these ellipsoids and PSD quadratics give rise to the basic SDP relaxation (SDP) as discussed after Proposition 1. The third class consists of the RLT quadratics  $T_{yE_1}(x)T_{yE_2}(x) \geq 0$ , where  $y$  is a member of the vertex set  $\text{vert}(F)$ . Since the cardinality of  $\text{vert}(F)$  is at most four when  $n = 2$ , there are at most four such vertex RLT constraints.

The last class contains quadratics that are derived from valid RLT quadratics of the type  $T_{yE_1}(x)T_{zE_2}(x) \geq 0$  with  $y \neq z$ , i.e., the tangents  $T_{yE_1}$  and  $T_{zE_2}$  supporting different points

on different ellipsoids. However, here the RLT quadratic is minorized by the valid quadratic  $T_{yE_1}T_{zE_2} + \lambda L_{yz}^2$ , where  $\lambda$  is minimal and hence  $\lambda \leq 0$ . In fact we will prove later that  $\lambda < 0$ . We call these *lifted RLT quadratics* in analogy with the lifting, or strengthening, of valid inequalities in, for example, the area of linear integer programming. Figure 1 depicts a lifted RLT quadratic. In the left, 2-dimensional picture, we have graphed  $F$  and marked  $\text{vert}(F)$ . Also depicted are the tangent lines  $T_{yE_1}$  and  $T_{zE_2}$ , as well as the line  $L_{yz}$  connecting  $y$  and  $z$ . In the right, 3-dimensional picture, the value of the lifted RLT quadratic is graphed in the vertical dimension over the the boundary  $\text{bd}(F)$  of  $F$ . Note that the quadratic attains the value 0 at the points  $y$  and  $z$  as well as one of the vertices. This shows that the lifted RLT minorizes the regular RLT quadratic, which only attains 0 at  $y$  and  $z$ .

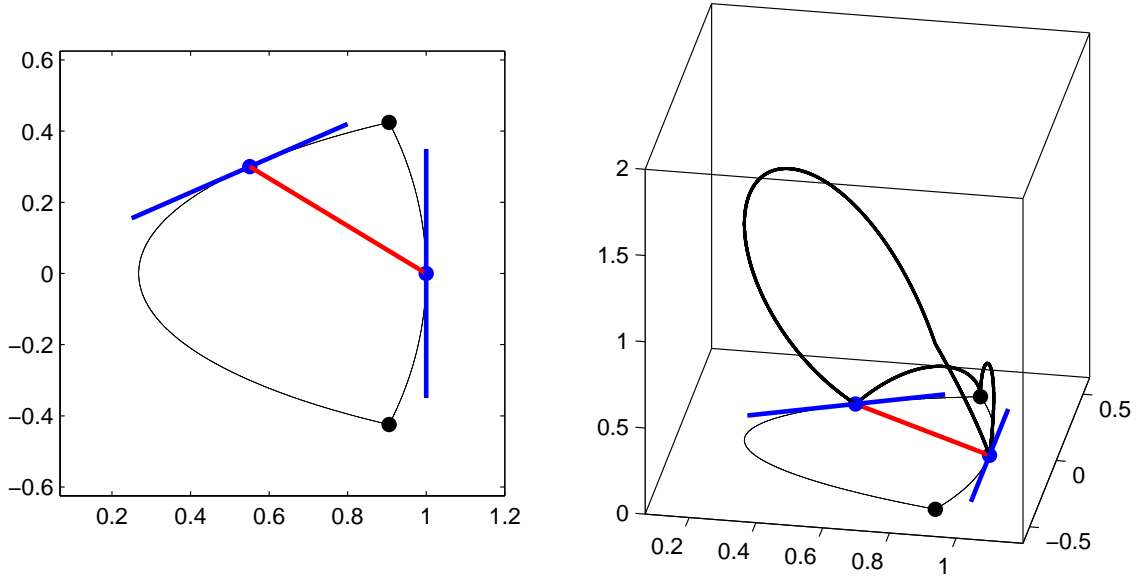


Figure 1: A lifted RLT quadratic

With the specification of  $\mathcal{G}$  given in Table 1, we introduce the following (informally specified) SDP relaxation:

$$\begin{aligned}
 v(\mathcal{G}) &:= \min C \bullet X + 2c^T x & (3) \\
 \text{s. t. } & A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1 \quad i = 1, 2 \\
 & (x, X) \in \text{PSD} \\
 & (x, X) \text{ satisfies all vertex RLT constraints} \\
 & (x, X) \text{ satisfies all lifted RLT constraints}
 \end{aligned}$$

where the various constraints are gotten by linearizing the valid quadratics. Assuming  $\mathcal{G}$

satisfies Proposition 2 (as we prove in Theorem 1 in Section 4), we have the following corollary:

**Corollary 1.** *For  $n = 2$ ,  $v(\mathcal{G}) = v^*$ .*

*Proof.* As discussed above, the first and second class of quadratics in  $\mathcal{G}$  are captured by the constraints  $A_i \bullet X - 2a_i^T A_i x + a_i^T A_i a_i \leq 1$  and  $(x, X) \in \text{PSD}$ . The third and fourth classes are incorporated directly into the SDP relaxation.  $\square$

We discuss the practical separation of the lifted RLT constraints (even when  $n > 2$ ) in Section 5.

As discussed in Section 2.1, the SOCRLT constraints can be viewed as another representation of all RLT constraints. The vertex and lifted RLT constraints in Table 1 imply all RLT constraints, and so the SOCRLT constraints are implied as well. Hence, it is unnecessary to state the SOCRLT constraints in this setting.

### 3 Local Analysis of Quadratics

We now develop some technical tools that investigate the behavior of general quadratics  $f$  on the boundary of  $F$ . In this section, we do not assume  $f$  is in  $\mathcal{K}$ , i.e., that  $f$  is valid. In Section 4, the tools and techniques of this subsection will play a vital role when we fully classify valid  $f$ .

#### 3.1 The break concept

Fix  $i \in \{1, 2\}$ , and let a feasible  $y \in \text{bd}(E_i)$  be given. Consider any locally diffeomorphic parameterization  $x(t)$  of  $\text{bd}(E_i)$  such that  $y = x(t_y)$ . That is,  $x$  locally takes open intervals in  $\mathbb{R}$  to open sub-manifolds of  $\text{bd}(E_i)$  smoothly and invertibly. Given a general quadratic  $f(x) = x^T R x + 2r^T x + \rho$  and any integer  $k \geq 0$ , we define  $(f \circ x)^{(k)}(t_y)$  to be the  $k$ -th derivative of the composition  $f \circ x$  at the point  $t_y$ . By convention, when  $k = 0$ , the derivative  $(f \circ x)^{(0)}(t_y)$  is simply the function value  $f(y)$ . We also define the *break of  $f$  at  $y$  along  $\text{bd}(E_i)$*  to be the smallest  $k$  such that  $(f \circ x)^{(k)}(t_y)$  is nonzero, i.e.,

$$\text{br}(y, \text{bd}(E_i), f) := \min\{k : (f \circ x)^{(k)}(t_y) \neq 0\}.$$

Note that the break equals  $\infty$  if all derivatives are 0, e.g., when  $f = g_{E_i}$ . In the rest of this section, we state and prove various facts about  $\text{br}(y, E_i, f)$ . To assist the reader's understanding, specific  $E_1$  and  $E_2$  are used in the facts below—instead of generic  $E_i$  and  $E_j$  with  $i, j \in \{1, 2\}$ —but this simplification does not affect the generality of our statements.

First, it is important to note that  $\text{br}(y, E_i, f)$  is independent of the locally diffeomorphic parameterization.

**Proposition 4.** *Given feasible  $y \in \text{bd}(E_1)$  and quadratic  $f$ , let  $x(t)$  and  $\hat{x}(s)$  be any two locally diffeomorphic parameterizations of  $\text{bd}(E_1)$  such that  $y = x(t_y) = \hat{x}(s_y)$ . Then*

$$\min\{k : (f \circ x)^{(k)}(t_y) \neq 0\} = \min\{k : (f \circ \hat{x})^{(k)}(s_y) \neq 0\}.$$

*Equivalently,  $\text{br}(y, E_1, f)$  is independent of the parameterization of  $\text{bd}(E_1)$ .*

*Proof.* We may write the parameterizations  $x$  and  $\hat{x}$  locally as diffeomorphisms

$$\begin{aligned} x : T &\rightarrow W, \quad t \mapsto x(t) \\ \hat{x} : S &\rightarrow W, \quad s \mapsto \hat{x}(s) \end{aligned}$$

where the open domains  $T \ni t_y$  and  $S \ni s_y$  are subsets of  $\mathbb{R}$  and the range  $W$  is a subset of  $\text{bd}(E_1)$ . Let  $\hat{x}^{-1}$  be the (local) inverse of  $\hat{x}$ . Then

$$(f \circ x)(t) = (f \circ \hat{x} \circ \hat{x}^{-1} \circ x)(t) = (f \circ \hat{x})(\hat{x}^{-1}(x(t))).$$

Letting  $g := \hat{x}^{-1} \circ x$ , we write  $(f \circ x)(t) = (f \circ \hat{x})(g(t))$ . By Faà di Bruno's formula, a generalization of chain rule, at  $t = t_y$ ,

$$(f \circ x)^{(k)}(t_y) = \sum_{i=1}^k (f \circ \hat{x})^{(i)}(s_y) \cdot B_{ki}(t_y)$$

where the functions  $B_{ki}$  are polynomials in the derivatives of  $g$ . This shows that

$$(f \circ \hat{x})^{(i)}(s_y) = 0 \quad \forall i = 1, \dots, k \quad \implies \quad (f \circ x)^{(i)}(t_y) = 0 \quad \forall i = 1, \dots, k.$$

So  $\min\{k : (f \circ x)^{(k)}(t_y) \neq 0\} \geq \min\{k : (f \circ \hat{x})^{(k)}(s_y) \neq 0\}$ . By symmetry of  $x$  and  $\hat{x}$ , the reverse inequality must also hold, which completes the proof.  $\square$

As a corollary, the break does not change under affine transformation.

**Corollary 2.** *Let  $\mathcal{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an invertible, affine transformation. Given feasible  $y \in \text{bd}(E_1)$  and quadratic  $f$ , define  $\hat{y} := \mathcal{A}(y)$ ,  $\hat{E}_1 := \mathcal{A}(E_1)$ , and  $\hat{f} := f \circ \mathcal{A}^{-1}$ . Then  $\text{br}(y, E_1, f) = \text{br}(\hat{y}, \hat{E}_1, \hat{f})$ .*

*Proof.* Suppose  $x$  is a locally diffeomorphic parametrization of  $\text{bd}(E_1)$  and  $y = x(t_y)$ . Then  $\hat{x} := \mathcal{A} \circ x$  is a locally diffeomorphic parametrization of  $\text{bd}(\hat{E}_1)$  and  $\hat{y} = \hat{x}(t_y)$ . By the preced-

ing proposition,  $\text{br}(y, E_1, f)$  and  $\text{br}(\hat{y}, \hat{E}_1, \hat{f})$  may be calculated by examining the derivatives of  $(f \circ x)(t)$  and  $(\hat{f} \circ \hat{x})(t)$  at  $t = t_y$ , respectively. In addition, for all  $t$  in the domain of  $x$ , it holds that

$$(\hat{f} \circ \hat{x})(t) = (f \circ \mathcal{A}^{-1} \circ \mathcal{A} \circ x)(t) = (f \circ x)(t),$$

i.e.,  $\hat{f} \circ \hat{x}$  and  $f \circ x$  are the same function. It follows that  $\text{br}(\hat{y}, \hat{E}_1, \hat{f}) = \text{br}(y, E_1, f)$ .  $\square$

The following proposition describes how breaks behave under sums and products of quadratics.

**Proposition 5.** *Given feasible  $y \in \text{bd}(E_1)$  and quadratics  $f$  and  $g$ , define  $a := \text{br}(y, E_1, f)$  and  $b := \text{br}(y, E_1, g)$ . It holds that:*

(i) *for any  $\lambda, \mu \in \mathbb{R}$ ,  $\text{br}(y, E_1, \lambda f + \mu g) \geq \min\{a, b\}$  with equality if  $a \neq b$  and  $\lambda\mu \neq 0$ ;*

(ii) *if  $f$  and  $g$  are linear, then  $\text{br}(y, E_1, fg) = a + b$ .*

*Proof.* To prove (i), we define  $h := \lambda f + \mu g$  and apply the definition of  $\text{br}(y, E_1, h)$ . Without loss of generality, assume  $\min\{a, b\} = a$ . Note that, for all  $k \geq 0$ ,  $(h \circ x)^{(k)}(t_y) = \lambda(f \circ x)^{(k)}(t_y) + \mu(g \circ x)^{(k)}(t_y)$ . Hence,

$$k < a \implies (h \circ x)^{(k)}(t_y) = \lambda \cdot 0 + \mu \cdot 0 = 0.$$

Hence,  $\text{br}(y, E_1, h) \geq a$ , as claimed. If, in addition,  $a < b$  and  $\mu \neq 0$ , then

$$k = a \implies (h \circ x)^{(k)}(t_y) = \lambda \cdot 0 + \mu \cdot (f \circ x)^{(k)}(t_y) \neq 0,$$

which proves  $\text{br}(y, E_1, h) = a$ .

To prove (ii), we apply the product rule to derive that, for all  $k \geq 0$ ,

$$((fg) \circ x)^{(k)}(t_y) = \sum_{j=0}^k \binom{k}{j} \cdot (f \circ x)^{(k-j)}(t_y) \cdot (g \circ x)^{(j)}(t_y).$$

When  $k < a + b$ , for all  $j \leq k$ , it must hold that  $k - j < a$  or  $j < b$ . Hence, every term in the above summand is zero by assumption. When  $k = a + b$ , the only nonzero summand corresponds to  $j = b$  and equals  $\binom{k}{b} \cdot (f \circ x)^{(a)}(t_y) \cdot (g \circ x)^{(b)}(t_y) \neq 0$ . We have thus shown that  $\text{br}(y, E_1, fg) = a + b$ .  $\square$

The break concept will be significant because, as we will see in the sequel, feasible zeros  $y \in \text{bd}(E_i)$  of  $f$ , which are also local minimizers of  $f$  over  $F$ , often have relatively high breaks, e.g.,  $\text{br}(y, E_i, f)$  may equal 2, 3, 4, or even  $\infty$ . When this occurs, we can use the zero

$f$	$\text{br}(y, E_1, f)$	$\text{br}(y, E_2, f)$ when $y \in \text{vert}(F)$	$\text{br}(z, E_2, f)$
$g_{E_1}$	$\infty$	$\geq 1$	(not needed)
$T_{yE_1}^2$	4	2	(not needed)
$T_{yE_1}T_{yE_2}$	3	3	(not needed)
$T_{yE_1}L_{yz}$	3	(not needed)	1
$T_{yE_1}T_{zE_2}$	2	(not needed)	2
$L_{yz}^2$	2	2	2

Table 2: Breaks for feasible  $y \in \text{bd}(E_1)$ ,  $z \in \text{bd}(E_2)$  such that  $y \neq z$  for various quadratics  $f$ .

derivatives indicated by the break as a tool for classifying all quadratic functions possessing that particular break at  $y$ . For example, if  $y \in \text{bd}(E_1)$  with  $\text{br}(y, E_1, f) = 3$  and  $y = x(t_y)$ , then the three equations  $f(y) = (f \circ x)^{(1)}(t_y) = (f \circ x)^{(2)}(t_y) = 0$  give three linear equations that  $f = (R, r, \rho)$  must satisfy.

### 3.2 Breaks for specific quadratics

In Section 4, we will require a simple and effective way to check, given two valid quadratics  $f$  and  $g$ , whether some positive multiple of  $g$  minorizes  $f$ . As it will turn out, the technique will depend heavily on the zeros and breaks of  $f$  and  $g$ . Hence, in this subsection we pre-calculate the breaks for some specific quadratics as reference for later use; see Table 2.

We first look at the breaks of some linear functions, which are the building blocks of most of the quadratics in Table 2.

**Lemma 1.** *Let feasible  $y \in \text{bd}(E_1)$  and  $z, w \in \mathbb{R}^2$  with  $T_{yE_1}(z) \neq 0$ ,  $L_{zw}(y) \neq 0$  be given. Then  $\text{br}(y, E_1, T_{yE_1}) = 2$ ,  $\text{br}(y, E_1, L_{yz}) = 1$  and  $\text{br}(y, E_1, L_{zw}) = 0$ .*

*Proof.* By Proposition 4 and Corollary 2, we may assume without loss of generality that  $E_1$  is the unit ball  $\{x : \|x\| \leq 1\}$ . Let  $x(t) := (\cos t, \sin t)^T$  be the standard parameterization of  $\text{bd}(E_1)$ , and suppose  $t_y$  satisfies  $y = x(t_y)$ . Also let  $T_{yE_1}(x)$  and  $L_{yz}(x)$  be represented as  $1 - y^T x$  and  $l_0 - l^T x$ .

We use the specific form of  $T_{yE_1}$  to calculate the derivatives  $(T_{yE_1} \circ x)^{(k)}(t)$  explicitly:

$$\begin{aligned}(T_{yE_1} \circ x)^{(0)}(t) &= 1 - \cos t_y \cos t - \sin t_y \sin t \\(T_{yE_1} \circ x)^{(1)}(t) &= \cos t_y \sin t - \sin t_y \cos t \\(T_{yE_1} \circ x)^{(2)}(t) &= \cos t_y \cos t + \sin t_y \sin t.\end{aligned}$$

Evaluated at  $t_y$ , we have  $(T_{yE_1} \circ x)^{(0)}(t_y) = (T_{yE_1} \circ x)^{(1)}(t_y) = 0$  and  $(T_{yE_1} \circ x)^{(2)}(t_y) = 1 \neq 0$ . So  $\text{br}(y, E_1, T_{yE_1}) = 2$ . For  $L_{yz}$ ,

$$\begin{aligned}(L_{yz} \circ x)^{(0)}(t) &= l_0 - l_1 \cos t - l_2 \sin t \\(L_{yz} \circ x)^{(1)}(t) &= l_1 \sin t - l_2 \cos t.\end{aligned}$$

Evaluated at  $t_y$ , we have  $(L_{yz} \circ x)^{(0)}(t) = l_0 - l_1 y_1 - l_2 y_2 = L_{yz}(y) = 0$ . Since  $L_{yz} \nparallel T_{yE_1}$ ,  $(L_{yz} \circ x)^{(1)}(t) = l_1 y_2 - l_2 y_1 \neq 0$ . So  $\text{br}(y, E_1, L_{yz}) = 1$ . Finally, as  $(L_{zw} \circ x)^{(0)}(t_y) = L_{zw}(y) \neq 0$ ,  $\text{br}(y, E_1, L_{zw}) = 0$ .  $\square$

Intuitively, Lemma 1 tells us that a tangent line has break 2 at its support point, while a secant line has break 1. A line that does not pass through  $y$  at all has break 0.

Since all but one of the quadratics in Table 2 are products of linear functions, Lemma 1 and Proposition 5(ii) provide an easy formula to calculate the breaks for those quadratics. For example, suppose  $y \in \text{vert}(F)$ ,  $z \in \text{bd}(E_2) \cap F$ , and  $f = T_{yE_1} L_{yz}$  are given. Then

$$\begin{aligned}\text{br}(y, E_1, f) &= \text{br}(y, E_1, T_{yE_1}) + \text{br}(y, E_1, L_{yz}) = 2 + 1 = 3 \\ \text{br}(z, E_2, f) &= \text{br}(z, E_2, T_{yE_1}) + \text{br}(z, E_2, L_{yz}) = 0 + 1 = 1.\end{aligned}$$

Lastly, since ellipsoidal constraints are not products of linear functions, we handle their breaks separately in the following lemma.

**Lemma 2.** *Let feasible  $y \in \text{bd}(E_1)$  be given. Then  $\text{br}(y, E_1, g_{E_1}) = \infty$ . If in addition,  $y \in \text{vert}(F)$ , then  $\text{br}(y, E_2, g_{E_1}) \geq 1$ .*

*Proof.*  $\text{br}(y, E_1, g_{E_1})$  is clearly  $\infty$  as  $g_{E_1}$  is zero and constant along  $E_1^1$ . If  $y \in \text{vert}(F)$ , then  $g_{E_1}(y) = 0$ , so  $\text{br}(y, E_2, g_{E_1}) \geq 1$ .  $\square$

### 3.3 Quadratics for specific breaks

While the previous subsection provides the breaks of some specific quadratics, in this subsection we look for quadratics that satisfy specific breaks. As we mentioned in Section 3.1, the

higher the breaks a quadratic  $f = (R, r, \rho)$  has, the more constrained the entries of  $(R, r, \rho)$  become. Based on specified breaks, we can often classify the form of  $f$  with the help of Table 2. When we ultimately characterize all valid  $f \in \mathcal{K}$  in Section 4, we will divide the proof into different cases with respect to different breaks. The results contained in this subsection help us deduce the forms of the quadratics in each case.

We assume throughout this subsection that  $f$  is defined by  $f(x) = x^T R x + 2 r^T x + \rho$  for  $(R, r, \rho) \in \mathcal{S}^2 \times \mathbb{R}^2 \times \mathbb{R}$ . We also define the zeros of  $f$  in  $F$  (or “null” points) by

$$N := N(f) := \{x \in F : f(x) = 0\}.$$

We do not assume that  $f$  is valid.

Lemma 3 and Propositions 6 and 7 below consider the case when  $f$  has a zero at a vertex with relatively high breaks.

**Lemma 3.** *Let  $f$  and  $y \in \text{vert}(F)$  be given. Let  $z \in \mathbb{R}^2$  be arbitrary such that  $T_{yE_1}(z) \neq 0$  and  $T_{yE_2}(z) \neq 0$ . If  $\text{br}(y, E_1, f) \geq 2$  and  $\text{br}(y, E_2, f) \geq 2$ , then there exists  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$  such that*

$$f = \alpha_1 T_{yE_1}^2 + \alpha_2 T_{yE_1} T_{yE_2} + \alpha_3 L_{yz}^2. \quad (4)$$

*Proof.* The zero equation  $f(y) = 0$  implies

$$y_1^2 R_{11} + 2 y_1 y_2 R_{21} + y_2^2 R_{22} + 2 y_1 r_1 + 2 y_2 r_2 + \rho = 0. \quad (5)$$

Using the definition of breaks, the inequalities  $\text{br}(y, E_1, f) \geq 2$  and  $\text{br}(y, E_2, f) \geq 2$  imply

$$\begin{aligned} 0 &= (f \circ x)'(t_y) = \nabla f(y)^T x'(t_y), \\ 0 &= (f \circ \bar{x})'(\bar{t}_y) = \nabla f(y)^T \bar{x}'(\bar{t}_y), \end{aligned}$$

where  $x'(t_y)$  and  $\bar{x}'(\bar{t}_y)$  are tangent vectors at  $y$  along  $\text{bd}(E_1)$  and  $\text{bd}(E_2)$ . Since  $y \in \text{vert}(F)$ , the simplifying assumption of Section 1.1 implies that  $x'(t_y)$  and  $\bar{x}'(\bar{t}_y)$  are linearly independent. So  $\nabla f(y) = 0$ , i.e.,

$$y_1 R_{11} + y_2 R_{21} + r_1 = 0, \quad (6)$$

$$y_1 R_{21} + y_2 R_{22} + r_2 = 0. \quad (7)$$

Considering  $(R, r, \rho)$  to be unknown, equations (5–7) thus provide three homogeneous equations in  $(R, r, \rho)$ . Moreover, these three equations can easily be seen to be linearly independent. Since  $R$  is symmetric, there are a total of six unknowns. So the space of solutions in  $(R, r, \rho)$  satisfying (5–7) has dimension three.



Table 2 provides three solutions  $(R, r, \rho)$  as suggested in the decomposition (4) of  $f$ . Specifically, each of the three component functions  $T_{yE_1}^2$ ,  $T_{yE_1}T_{yE_2}$  and  $L_{yz}^2$  has break at least 2 at  $y$  with respect to both  $\text{bd}(E_1)$  and  $\text{bd}(E_2)$ . It remains to show that the three solutions are independent.

Suppose (4) satisfies  $f = 0$ , and define the affine function  $M := \alpha_1 T_{yE_1} + \alpha_2 T_{yE_2}$ . Then  $f = T_{yE_1}M + \alpha_3 L_{yz}^2 = 0$ . Since  $T_{yE_1} \neq 0$ , it is clear that  $\alpha_3 = 0$  and then  $M = 0$ . Since  $T_{yE_1}$  and  $T_{yE_2}$  are clearly independent,  $\alpha_1 = \alpha_2 = 0$  as well.  $\square$

**Proposition 6.** *Let  $f \neq 0$  and distinct  $y, z, w \in N \cap \text{bd}(F)$  be given. Suppose  $y \in \text{vert}(F)$ . If  $\text{br}(y, E_1, f) \geq 2$  and  $\text{br}(y, E_2, f) \geq 2$ , then  $f = \bar{\alpha} L_{yz} L_{zw}$  for some  $\bar{\alpha} \in \mathbb{R}$ . As a consequence,  $f$  is not valid.*

*Proof.* Apply Lemma 3 to write  $f$  as (4). As  $T_{yE_1}(z) \neq 0$  and

$$0 = 0 - 0 = f(z) - \alpha_3 L_{yz}^2(z) = T_{yE_1}(z) (\alpha_1 T_{yE_1}(z) + \alpha_2 T_{yE_2}(z)),$$

we have  $(\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2})(z) = 0$ . Note that  $\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2}$  corresponds to a line passing through  $y$  and  $z$ . So there exists  $\tilde{\alpha} \in \mathbb{R}$  such that  $\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2} = \tilde{\alpha} L_{yz}$ . Now

$$f = \tilde{\alpha} T_{yE_1} L_{yz} + \alpha_3 L_{yz}^2 = L_{yz} (\tilde{\alpha} T_{yE_1} + \alpha_3 L_{yz}).$$

Using  $L_{yz}(w) \neq 0$  and a similar argument as just applied, there exists  $\bar{\alpha} \in \mathbb{R}$  such that  $\tilde{\alpha} T_{yE_1} + \alpha_3 L_{yz} = \bar{\alpha} L_{yw}$ . Then  $f = \bar{\alpha} L_{yz} L_{yw}$ , and  $f \neq 0$  implies that  $\bar{\alpha} \neq 0$ . Since the lines  $L_{yz}$  and  $L_{yw}$  geometrically divide  $F$  into three parts and  $f$  cannot have the same sign on all three parts,  $f$  is not valid.  $\square$

**Proposition 7.** *Let  $f \neq 0$  and  $y \in \text{vert}(F)$  be given. If  $\text{br}(y, E_1, f) \geq 3$  and  $\text{br}(y, E_2, f) \geq 2$ , then there exist  $\alpha_1, \alpha_2 \in \mathbb{R}$  such that  $f = \alpha_1 T_{yE_1}^2 + \alpha_2 T_{yE_1} T_{yE_2}$ . In addition, if there exists  $z \in N \cap \text{bd}(F)$  with  $z \neq y$ , then  $f = \hat{\alpha} T_{yE_1} L_{yz}$  for some  $\hat{\alpha} \in \mathbb{R}$ , and as a consequence,  $f$  is not valid.*

*Proof.* Apply Lemma 3 to write  $f$  as (4). Note that  $\text{br}(y, E_1, T_{yE_1}^2) = 4$ ,  $\text{br}(y, E_1, T_{yE_1} T_{yE_2}) = 3$ , and  $\text{br}(y, E_1, L_{yz}^2) = 2$  by Table 2. Proposition 5(i) thus implies  $\alpha_3 = 0$ . Rewriting  $f$  as  $T_{yE_1}(\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2})$ , we can use the same technique as in the proof of Proposition 6 to prove the second statement of the proposition.  $\square$

Next, Lemma 4 and Proposition 8 allow us to characterize quadratics with specific breaks at zeros on both ellipsoids.

**Lemma 4.** *Let  $f$  and feasible  $y \in \text{bd}(E_1) \cap N$  and  $z \in \text{bd}(E_2) \cap N$  with  $y \neq z$  be given. Then there exist  $\beta_1, \beta_2, \beta_3, \beta_4$  such that*

$$f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L_{yz}^2 + \beta_3 T_{zE_2} L_{yz} + \beta_4 T_{yE_1} L_{yz}. \quad (8)$$

*Proof.* We take the same approach as the proof of Lemma 3, but in this case, the equations  $f(y) = f(z) = 0$  are just two independent equations that limit the dimension of solutions  $f$  to 4. The four functions  $T_{yE_1} T_{zE_2}$ ,  $L_{yz}^2$ ,  $T_{zE_2} L_{yz}$ , and  $T_{yE_1} L_{yz}$  are clearly solutions, so (8) holds as long as the four are independent.

So suppose that  $f$  as presented in (8) satisfies  $f = 0$ , and define the affine function  $M := \beta_2 L_{yz} + \beta_3 T_{zE_2} + \beta_4 T_{yE_1}$ . Then  $f = L_{yz} M + \beta_1 T_{yE_1} T_{zE_2}$ . It is clear that  $L_{yz} M$  and  $\beta_1 T_{yE_1} T_{zE_2}$  are linearly dependent if and only if  $M = 0$  and  $\beta_1 = 0$ . Thus,  $0 = M(y) = \beta_3 T_{zE_2}(y)$ , which implies  $\beta_3 = 0$  since  $T_{zE_2}(y) \neq 0$ . Similarly  $\beta_4 = 0$ , and so finally  $\beta_2 = 0$  as well.  $\square$

**Proposition 8.** *Let  $f$  and feasible  $y \in \text{bd}(E_1) \cap N$  and feasible  $z \in \text{bd}(E_2) \cap N$  with  $y \neq z$  be given. If  $\text{br}(y, E_1, f) \geq 2$  and  $\text{br}(z, E_2, f) \geq 2$ , then there exist  $\beta_1, \beta_2$  such that  $f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L_{yz}^2$ .*

*Proof.* Apply Lemma 4 to write  $f$  as (8). According to Table 2,  $\text{br}(z, E_2, T_{yE_1} L_{yz}) = 1$ ,  $\text{br}(z, E_2, T_{yE_1} T_{zE_2}) = \text{br}(z, E_2, L_{yz}^2) = 2$  and  $\text{br}(z, E_2, T_{zE_2} L_{yz}) = 3$ . As  $\text{br}(z, E_2, f) \geq 2$ , it holds that  $\beta_4 = 0$  by Lemma 5(i). By symmetry,  $\beta_3 = 0$  as desired.  $\square$

## 4 Global Analysis of Valid Quadratics

As discussed in Section 2, Proposition 2 is the key result required for our choice of  $\mathcal{G}$ . In this section, we argue in Theorem 1 that Proposition 2 does indeed hold, i.e., we show that every quadratic function  $f \in \mathcal{K}$  can be minorized by some  $g \in \mathcal{G}$ .

The following lemma gives conditions under which a valid  $f \in \mathcal{K}$  can be perturbed to a valid  $f + \lambda g$  for some  $\lambda < 0$ , where  $g$  is also valid. In other words,  $f$  can be minorized by  $-\lambda g$ . The key insight is to compare the zeros and breaks of  $f$  and  $g$ .

**Lemma 5.** *Let  $f, g \in \mathcal{K}$ , and suppose  $\text{Hess}(f) \notin \mathcal{S}_+^2$  and  $N(f) \subseteq N(g)$  with  $|N(f)|$  finite. In particular,  $N(f) \subseteq \text{bd}(F)$ . Suppose also that  $\text{br}(y, E_i, f) \leq \text{br}(y, E_i, g)$  for all  $y \in \text{bd}(E_i) \cap F \cap N(f)$ ,  $i = 1, 2$ . Then  $f + \lambda g$  is valid for some  $\lambda < 0$ .*

*Proof.* Since the Hessian of  $f$  is not positive semidefinite, there exists a small  $\lambda_1 < 0$  such that  $\text{Hess}(f + \lambda_1 g) \notin \mathcal{S}_+^2$ . We will require  $\lambda_1 \leq \lambda < 0$ , in which case  $f + \lambda g$  will attain its global minimum over  $F$  in the boundary  $\text{bd}(F)$ .

Next let  $y \in \text{bd}(E_i) \cap F \cap N(f)$ . Suppose  $r = \text{br}(y, E_i, f) \leq \text{br}(y, E_i, g)$ . In the intersection of  $F$ ,  $\text{bd}(E_i)$  and a sufficiently small open neighborhood  $O(y) \subseteq \mathbb{R}^2$  of  $y$ , we have the Taylor approximations

$$\begin{aligned}(f \circ x)(t) &= \frac{1}{r!} \cdot (f \circ x)^{(r)}(t_y) \cdot (t - t_y)^r + \mathcal{O}((t - t_y)^{r+1}), \\ (g \circ x)(t) &= \frac{1}{r!} \cdot (g \circ x)^{(r)}(t_y) \cdot (t - t_y)^r + \mathcal{O}((t - t_y)^{r+1}),\end{aligned}$$

where  $x(t)$  is any parametrization of  $\text{bd}(E_i)$ ,  $y = x(t_y)$  and  $\mathcal{O}((t - t_y)^{r+1})$  expresses terms of  $t - t_y$  with degree at least  $r + 1$ . Since  $(f \circ x)^{(r)}(t_y) \neq 0$ , there exists a small  $\lambda_y < 0$  such that  $((f + \lambda_y g) \circ x)^{(r)}(t_y)$  is nonzero with the same sign as  $(f \circ x)^{(r)}(t_y)$ . Therefore,  $f(z) + \lambda_y g(z) \geq 0$  for all  $z \in F \cap \text{bd}(E_i) \cap O(y)$ , because  $(f + \lambda_y g) \circ x$  and  $f \circ x$  have the same local behavior around  $t_y$ . In words,  $f + \lambda_y g$  is locally valid around  $y$ . We will also require  $\lambda_y \leq \lambda < 0$ .

Now consider  $f + \lambda g$  over the complement  $Q := \text{bd}(F) \setminus \bigcup_{y \in N(f)} O(y)$ . Because  $\{O(y)\}$  is a finite collection of open sets containing the zeros of  $f$ ,  $Q$  is compact and  $\min_{x \in Q} f(x)$  is positive. Hence, there exists  $\lambda_Q < 0$  such that  $f + \lambda_Q g$  is valid over  $Q$ . We will also require  $\lambda_Q \leq \lambda < 0$ .

Based on the previous three paragraphs, we take  $\lambda$  to be the maximum of  $\lambda_1$ ,  $\lambda_Q$ , and  $\lambda_y$  for all  $y \in N(f)$ . This proves the existence of  $\lambda < 0$  such that  $f + \lambda g$  is valid.  $\square$

For a valid  $f \in \mathcal{K}$  with  $\text{Hess}(f) \notin \mathcal{S}_+^2$ , non-vertex zeros, i.e., zeros in  $\text{bd}(F) \setminus \text{vert}(F)$ , have different break properties compared to vertex zeros. In particular, the following result shows that non-vertex zeros have even breaks.

**Lemma 6.** *Let  $f \in \mathcal{K}$  with  $\text{Hess}(f) \notin \mathcal{S}_+^2$ , and suppose  $y \in N(f)$  and  $y \in \text{bd}(E_1) \setminus \text{vert}(F)$ . Then  $\text{br}(y, E_1, f)$  is even, and in particular,  $\text{br}(y, E_1, f) \geq 2$ .*

*Proof.* Let  $x(t)$  be the parameterization of  $\text{bd}(E_1)$  such that  $y = x(t_y)$ . Since  $f$  is valid,  $f(y) = 0$ , and  $y \notin \text{vert}(F)$ , the one-dimensional function  $(f \circ x)(t)$  has a local minimum at  $t_y$  in an open neighborhood containing  $t_y$ . Using standard calculus, this implies  $\text{br}(y, E_1, f)$  is even, and thus, no less than 2.  $\square$

We are finally ready to state our main theorem that Proposition 2 holds for our choice of  $\mathcal{G}$ .

**Theorem 1.** *Every  $f \in \mathcal{K}$  satisfies  $f \succeq g$  for some  $g \in \mathcal{G}$ , where  $\mathcal{G}$  is given by Table 1.*

*Proof.* If  $\text{Hess}(f) \in \mathcal{S}_+^2$ , then the theorem holds true as  $f \succeq f$  and  $f \in \mathcal{G}$ . So assume  $\text{Hess}(f) \notin \mathcal{S}_+^2$ , in which case  $N := N(f) \subseteq \text{bd}(F)$ . We define

$$\max \text{br}(f) := \max\{\text{br}(\hat{y}, E_i, f) : \hat{y} \in \text{bd}(E_i) \cap F, i = 1, 2\},$$

possibly  $-\infty$  (if  $|N| = 0$ ) or  $\infty$ . That is,  $\max \text{br}(f)$  is the maximum break of  $f$  at its zeros measured along the corresponding ellipsoid boundaries. Choose any  $y \in E_i$  such that  $\max \text{br}(f) = \text{br}(y, E_i, f)$ . Without loss of generality, we assume  $i = 1$ . Then define

$$\max 2 \text{br}(f) := \max\{\text{br}(\hat{z}, E_2, f) : \hat{z} \in \text{bd}(E_2) \cap F\},$$

possibly  $-\infty$  (if  $E_2$  has no zeros) or  $\infty$ . That is,  $\max 2 \text{br}(f)$  is the maximum break of  $f$  measured with respect to the ellipsoid at which  $\max \text{br}(f)$  is not obtained. Also let  $z \in E_2$  satisfy  $\max 2 \text{br}(f) = \text{br}(z, E_2, f)$ . Note that  $\max 2 \text{br}(f) \leq \max \text{br}(f)$ . The proof of the theorem considers cases based on  $|N|$ ,  $\max \text{br}(f)$ , and  $\max 2 \text{br}(f)$ .

First suppose  $\max 2 \text{br}(f) \leq 1$ . The contrapositive of Lemma 6 implies that  $z \in \text{vert}(F)$ , which in turn implies  $N \subseteq \text{bd}(E_1)$ . If  $|N| = \infty$ , then  $f \in \mathcal{G}$  as a nonnegative multiple of  $g_{E_1} \in \mathcal{G}$ . If  $|N| < \infty$  (including  $N = \emptyset$ ), applying Lemma 5 with the breaks of  $g_{E_1}$  in Table 2, we see that  $f$  is minorized by  $\lambda g_{E_1} \in \mathcal{G}$  for some  $\lambda > 0$ , as desired.

So we may assume  $\max \text{br}(f) \geq \max 2 \text{br}(f) \geq 2$ . We consider two cases: (i)  $z \neq y$ ; (ii)  $z = y$ . For (i), Proposition 8 implies  $f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L_{yz}^2$  for some  $\beta_1, \beta_2 \in \mathbb{R}$ . If  $\beta_1 = 0$ , then  $\beta_2 \geq 0$  because  $f \in \mathcal{K}$ , but this contradicts our assumption that  $\text{Hess}(f) \notin \mathcal{S}_+^2$ . So  $\beta_1 \neq 0$ , and in fact,  $\beta_1 > 0$ ; otherwise,  $f$  would be invalid along the line  $L_{yz}$ . Hence, after scaling to  $\beta_1 = 1$ ,  $f$  is minorized by a lifted-RLT member of  $\mathcal{G}$ .

For case (ii),  $y = z \in \text{vert}(F)$ , and we consider two subcases: (a)  $\max \text{br}(f) \geq 3$ , and (b)  $\max \text{br}(f) = 2$ . For subcase (a), Proposition 7 implies that  $f = \alpha_1 T_{yE_1}^2 + \alpha_2 T_{yE_1} T_{yE_2}$  for some  $\alpha_1, \alpha_2 \in \mathbb{R}$ . If either  $\alpha_1$  or  $\alpha_2$  is zero, then  $f \in \mathcal{K}$  implies that the other parameter is nonnegative. Since we assume  $\text{Hess}(f) \notin \mathcal{S}_+^2$ , the only possible case is that  $\alpha_1 = 0 < \alpha_2$ , so  $f$  is generated by a vertex RLT quadratic in  $\mathcal{G}$ . If both  $\alpha_1$  and  $\alpha_2$  are nonzero, then by the first part of Proposition 5 and the break information of  $T_{yE_1}^2$  and  $T_{yE_1} T_{yE_2}$  shown in Table 2,  $\text{br}(y, E_1, f) = 3$  and  $\text{br}(y, E_2, f) = 2$ . Since the second part of Proposition 7 implies that  $f$  is not valid unless  $|N| = 1$ ,  $f$  then can be minorized by a positive multiple of  $T_{yE_1}^2$  by appealing to Lemma 5.

Finally, for subcase (b), the contrapositive of Proposition 6 implies that  $|N| \leq 2$ . Then by Lemma 5,  $f$  can be minorized by a positive multiple of  $L_{yz}^2$ , which completes the proof.  $\square$

## 5 The Separation Problem and Computational Results

The goal of this section is to demonstrate how to use the lifted-RLT constraints of Table 1 in practice. Since there are an infinite number of such constraints, the separation problem is key. We first discuss separation of lifted-RLT constraints when  $n = 2$  and then extend the

technique to  $n > 2$ . We end the section with some computational tests.

Given a point  $(\bar{x}, \bar{X})$ , which satisfies some lifted-RLT constraints, we envision separation—finding a lifted-RLT constraint violating  $(\bar{x}, \bar{X})$ —as consisting of two steps. First choose  $y \in \text{bd}(E_1)$  and  $z \in \text{bd}(E_2)$ . Then calculate the minimal  $\lambda_{\min} < 0$  such that  $T_{yE_1}T_{zE_2} + \lambda_{\min}L_{yz}^2$  is valid. We begin by discussing the calculation of  $\lambda_{\min}$ .

### 5.1 For $n = 2$ , calculating $\lambda_{\min}$ given $(y, z)$

For  $n = 2$  and given  $y \in \text{bd}(E_1)$  and  $z \in \text{bd}(E_2)$ , we argue that  $\lambda_{\min}$  can be calculated with high precision.

Recall the definitions of  $T_{yE_1}$ ,  $T_{zE_2}$ , and  $L_{yz}$  given at the beginning of Section 2.3, and for any  $\lambda < 0$ , define

$$q_\lambda := T_{yE_1}T_{zE_2} + \lambda L_{yz}^2.$$

Note that  $\text{Hess}(q_\lambda) \notin \mathcal{S}_+^2$ , so that the global minimizers of  $q_\lambda$  over  $F$  are contained in  $\text{bd}(F)$ . Since  $\text{bd}(F) \subseteq \text{bd}(E_1) \cup \text{bd}(E_2)$ , it follows that  $q_\lambda$  is valid over  $F$  if and only if it is simultaneously valid over both  $\text{bd}(E_1) \cap F$  and  $\text{bd}(E_2) \cap F$ . We next will argue that the validity of  $\text{bd}(E_i) \cap F$  for each  $i = 1, 2$  can be determined easily, so that checking validity of  $q_\lambda$  over  $F$  is easy. As a consequence, a simple bisection procedure over  $\lambda$  can be used to calculate  $\lambda_{\min}$ .

We discuss only validity over  $\text{bd}(E_1) \cap F$  since the second case is similar. Also, for simplification but without loss of generality, we assume  $E_1$  equals the unit ball  $\mathcal{B} := \{x \in \mathbb{R}^2 : x^T x \leq 1\}$ . The following lemma provides the key insight for our approach. Note that, when applying the lemma below,  $w$  will play a role different than  $z$ , although  $w$  could be equal to  $z$ .

**Lemma 7.** *Suppose  $y, w \in \text{bd}(\mathcal{B})$  with  $y \neq w$ . Then  $T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x) = L_{yw}^2(x)$  for all  $x \in \text{bd}(\mathcal{B})$  when  $n = 2$ .*

*Proof.* In this case,  $T_{y\mathcal{B}}(x) = 1 - y^T x$  and  $T_{w\mathcal{B}}(x) = 1 - w^T x$ . Also  $L_{yw}^2(x) = (u^T(x - y))^2$ , where  $u$  is a unit vector that is perpendicular to  $w - y$ . We take  $u = (y + w)/\|y + w\|$ , and by an orthogonal rotation, we assume without loss of generality that  $y = (1, 0)^T$ . Assuming

$x^T x = 1$  and using  $y^T y = w^T w = 1$ , we have

$$\begin{aligned}
L_{yw}^2(x) &= (u^T(x - y))^2 = \frac{((y + w)^T(x - y))^2}{\|y + w\|^2} = \frac{((1 + w_1)(x_1 - 1) + w_2 x_2)^2}{2(1 + w_1)} \\
&= \frac{1}{2}(1 + w_1)(x_1 - 1)^2 + w_2(x_1 - 1)x_2 + \frac{(1 - w_1^2)(1 - x_1^2)}{2(1 + w_1)} \\
&= (1 - x_1) \left( \frac{1}{2}(1 + w_1)(1 - x_1) - w_2 x_2 + \frac{1}{2}(1 - w_1)(1 + x_1) \right) \\
&= (1 - x_1)(1 - w_1 x_1 - w_2 x_2) = (1 - y^T x)(1 - w^T x) \\
&= T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x).
\end{aligned}$$

□

By construction, the line  $L_{yz}$  passes through  $y \in \text{bd}(\mathcal{B})$  and  $z \in \text{bd}(E_2)$ . Geometrically,  $L_{yz}$  must also intersect  $\text{bd}(\mathcal{B})$  in a second point, say  $w \in \text{bd}(\mathcal{B})$ . ( $w$  may equal  $z$  when  $z \in \text{vert}(F)$ .) In addition,  $L_{yz} = L_{yw}$ . Then Lemma 7 shows that

$$\begin{aligned}
x \in \text{bd}(\mathcal{B}) \quad \implies \quad q_\lambda(x) &= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda L_{yz}^2(x) \\
&= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda L_{yw}^2(x) \\
&= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x) \\
&= T_{y\mathcal{B}}(x) \cdot (T_{zE_2}(x) + \lambda T_{w\mathcal{B}}(x)).
\end{aligned}$$

In words,  $q_\lambda$  restricted to  $\text{bd}(\mathcal{B})$  can be expressed as the product of two linear functions,  $T_{y\mathcal{B}}$  and  $l_\lambda := T_{zE_2} + \lambda T_{w\mathcal{B}}$ . Since  $T_{y\mathcal{B}}$  is valid over  $\text{bd}(\mathcal{B}) \cap F$  and zero only at a single point,  $q_\lambda$  is valid over  $\text{bd}(\mathcal{B}) \cap F$  if and only if  $l_\lambda$  is valid over  $\text{bd}(\mathcal{B}) \cap F$ , that is, if and only if

$$\begin{aligned}
v(S_\lambda^1) &:= \min_{x \in E_2} l_\lambda(x) \\
\text{s. t. } &x^T x = 1, x \in E_2
\end{aligned} \tag{S_\lambda^1}$$

is nonnegative. So we have reduced the validity of  $q_\lambda$  to the validity of  $l_\lambda$  over  $\text{bd}(\mathcal{B}) \cap F$ .

We claim that, in turn, the validity of  $l_\lambda$  holds if and only if the optimal value of

$$\begin{aligned}
v(S_\lambda^2) &:= \min_{x \in E_2} 1 - x^T x \\
\text{s. t. } &l_\lambda(x) \leq 0, x \in E_2
\end{aligned} \tag{S_\lambda^2}$$

is nonnegative.

**Proposition 9.** *For all  $\lambda \leq 0$ ,  $v(S_\lambda^1) \geq 0$  if and only if  $v(S_\lambda^2) \geq 0$ .*

*Proof.* ( $\Leftarrow$  contrapositive) If  $v(S_\lambda^1) < 0$ , then there exists  $x \in \text{bd}(\mathcal{B}) \cap E_2$  such that  $l_\lambda(x) < 0$ . We consider two cases:  $x \in \text{int}(E_2)$  and  $x \in \text{bd}(E_2)$ . In the first case, we can perturb  $x$  to  $\hat{x}$  such that  $\hat{x}^T \hat{x} > 1$ ,  $\hat{x} \in \text{int}(E_2)$  and  $l_\lambda(\hat{x}) < 0$ . This implies  $v(S_2) < 0$ . In the second case,  $x \in \text{vert}(F)$ . We can then perturb  $x$  to  $\hat{x}$  such that  $\hat{x} \in \text{bd}(\mathcal{B})$ ,  $\hat{x} \in \text{int}(E_2)$ , and  $l_\lambda(\hat{x}) < 0$ . Then the first case applies to  $\hat{x}$ .

( $\Rightarrow$ ) Define the convex feasible set of  $(S_\lambda^2)$  to be  $R_\lambda^2 := \{x : l_\lambda(x) \leq 0\} \cap E_2$ . If  $R_\lambda^2 \subseteq \mathcal{B}$ , then  $v(S_\lambda^2) \geq 0$ . So suppose  $R_\lambda^2 \not\subseteq \mathcal{B}$  and consider two subcases: (i)  $R_\lambda^2$  crosses  $\text{bd}(\mathcal{B})$ ; (ii)  $R_\lambda^2$  is completely outside of  $\text{int}(\mathcal{B})$ . For subcase (i),  $R_\lambda^2$  must be full-dimensional. So we clearly have points satisfying  $x \in \text{bd}(\mathcal{B})$ ,  $l_\lambda(x) < 0$ , and  $x \in E_2$ . However, this is inconsistent with the assumption  $v(S_\lambda^1) \geq 0$ .

For subcase (ii), we consider three mutually exclusive and collectively exhaustive alternatives: (a)  $\lambda < 0$  and  $T_{w\mathcal{B}}(z) = 0$ ; (b)  $\lambda < 0$  and  $T_{w\mathcal{B}}(z) > 0$ ; and (c)  $\lambda = 0$ . If (a), then  $z \in \text{vert}(F)$  and  $w = z$ . Note that  $\{x : l_\lambda(x) = 0\} \subseteq \{x : T_{zE_2}(x)T_{w\mathcal{B}}(x) \geq 0\} = TC_z(F) \cup -TC_z(F)$ , where  $TC_z(F)$  is the tangent cone of  $F$  at  $z$ . Then  $\lambda < 0$  implies that  $l_\lambda$  intersects  $\text{int}(F)$ , a contradiction. If (b), then  $l_\lambda$  evaluated at  $z$  equals  $T_{zE_2}(z) + \lambda T_{w\mathcal{B}}(z) = \lambda T_{w\mathcal{B}}(z) < 0$ . Then we can perturb  $z$  to  $\hat{z}$  such that  $\hat{z} \in \text{int}(F)$  and  $l_\lambda(\hat{z}) < 0$ , again a contradiction. So in fact (c) is the only true alternative, in which case  $l_\lambda = T_{zE_2}$ ,  $R_\lambda^2 = \{z\}$ , and  $v(S_\lambda^2) \geq 0$ .  $\square$

Note that calculating  $v(S_\lambda^2)$  is a trust region subproblem with one linear constraint, which has been proved tractable in [22].

To illustrate Proposition 9 and the calculation of  $\lambda_{\min}$ , we consider a geometric example in Figure 2. Let  $R_\lambda^1$  and  $R_\lambda^2$  be the feasible regions of  $(S_\lambda^1)$  and  $(S_\lambda^2)$ , which are bold and shaded, respectively. In the left-most picture,  $\lambda = -0.3$ , and  $v(S_\lambda^1) > 0$  because  $R_\lambda^1$  lies entirely on the nonnegative side of  $l_\lambda$ , while  $v(S_\lambda^2) > 0$  because  $R_\lambda^2 \subseteq \text{int}(\mathcal{B})$ . As  $\lambda$  decreases,  $l_\lambda$  rotates clockwise around point  $p$ , and first intersects  $R_\lambda^1$  at point  $q$ , as shown in the middle picture (in which  $\lambda = -0.37$ ). For this  $\lambda$ ,  $(S_\lambda^1)$  and  $(S_\lambda^2)$  have the same minimizer  $q$  and minimum value 0, and moreover  $\lambda = \lambda_{\min}$ . If  $\lambda$  continues to decrease past  $\lambda_{\min}$ , then  $q$  lies on the negative side of  $l_\lambda$ , and thus  $v(S_\lambda^1) < 0$  as shown in the right-most picture with  $\lambda = -0.46$ . Also, since  $R_\lambda^2 \setminus \mathcal{B} \neq \emptyset$ , it holds that  $v(S_\lambda^2) < 0$ .

## 5.2 For any $n$ , choosing $(y, z)$

Now we propose a heuristic way to choose  $y \in \text{bd}(E_1)$  and  $z \in \text{bd}(E_2)$  upon which to base the lifted-RLT constraint of the previous subsection. Our idea is based on looking for a violated SOCRLT constraint as described in [8]. Note that, by the discussion at the end of Section 2.1, if there exists a violated SOCRLT, then there exists a violated RLT constraint,

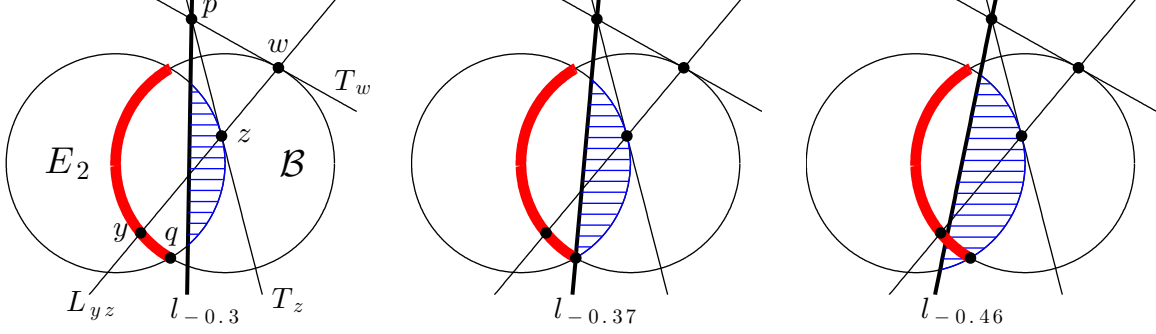


Figure 2: An example to illustrate Proposition 9

in which case there also exists a violated lifted-RLT constraint. On the other hand, the converse does not hold, and accordingly we only propose this procedure when a violated SOCRLT is found.

Suppose that  $(\bar{x}, \bar{X})$  is our current solution. Based on  $(\bar{x}, \bar{X})$  we solve the SOCRLT-separation problem as discussed in section 5 of [8]. If a violated SOCRLT is found, we use the solution of the separation problem to choose  $y$ . In particular, the separation problem always yields a distinguished  $y \in \text{bd}(E_1)$ , which is the “support point” of the SOCRLT constraint. Moreover, adding the violated SOCRLT to the current relaxation and resolving guarantees that the new SOCRLT subsequently becomes active, which in turn yields a  $z \in \text{bd}(E_2)$ . In terms of the discussion in Section 2 and the SOCRLT constraint (2), the formula for  $z$  is

$$z := \frac{\beta \hat{x} - \hat{X} \alpha}{\beta - \alpha^T \hat{x}},$$

where  $(\hat{x}, \hat{X})$  is optimal after the new SOCRLT constraint has been added. Please note that the SOCRLT constraints are used only for generating  $(y, z)$  and are never added to the current relaxation.

### 5.3 When $n > 2$

We next discuss a generalization of the lifted-RLT constraints for general  $n$ . Let  $y \in \text{bd}(E_1) \cap F$  and  $z \in \text{bd}(E_2) \cap F$  with  $y \neq z$  be given. To generalize the function  $L_{yz}^2(x)$  in dimension 2, our idea is to consider functions of the type  $M_{yz}(x) := (x - z)^T H(x - z)$  where  $H$  satisfies  $H \succeq 0$  and  $(y - z)^T H(y - z) = 0$ . This ensures that  $M_{yz}(x) \geq 0$  and  $M_{yz}(y) = M_{yz}(z) = 0$  in analogy with  $L_{yz}^2(x)$ . Then with  $y, z$ , and  $M$  chosen, we search for the most negative  $\lambda_{\min} < 0$  such that

$$T_{yE_1} T_{zE_2} + \lambda_{\min} M_{yz} \geq 0 \tag{9}$$



is valid for  $F$ . We are unsure whether this class of lifted-RLT constraints closes the relaxation gap for general  $n$ , but we propose the following heuristic to generate such cuts in practice.

For a given current solution  $(\bar{x}, \bar{X})$ , we choose  $(y, z)$  exactly as described in Section 5.2, which works for general  $n$ . Then we search for a matrix  $H$  such that  $2H$  will serve as the Hessian of  $M_{yz}$ . To guide our choice of  $H$ , we first examine the linearized form of (9):

$$[\beta\delta - \beta\gamma^T x - \delta\alpha^T x + \alpha^T X\gamma] + \lambda_{\min} [H \bullet X - 2y^T Hx + y^T Hy] \geq 0,$$

where  $T_{yE_1}(x) := \beta - \alpha^T x$  and  $T_{zE_2}(x) := \delta - \gamma^T x$ . Given this form and keeping in mind that  $\lambda_{\min} < 0$  is yet to be determined—it will depend on  $H$ —a reasonable choice for  $H$  is one that maximizes  $H \bullet \bar{X} - 2y^T H\bar{x} + y^T Hy$ . This will increase the chance that the linearized form is ultimately violated when plugging in  $(\bar{x}, \bar{X})$  and  $\lambda_{\min}$ , i.e., that we will be able to find a good cut. So we solve

$$\begin{aligned} \max_H \quad & H \bullet \bar{X} - 2y^T H\bar{x} + y^T Hy \\ \text{s. t.} \quad & H(y - z) = 0 \\ & \text{trace}(H) = 1 \\ & H \succeq 0. \end{aligned}$$

Because  $H \succeq 0$ , the constraint  $H(y - z) = 0$  is equivalent to  $(y - z)^T H(y - z) = 0$ . Also, the normalization constraint  $\text{trace}(H) = 1$  simply bounds the feasible region.

Given  $y, z$ , and  $H$ , it remains to calculate  $\lambda_{\min}$ . Unfortunately, for general  $n$ , we do not know how to calculate  $\lambda_{\min}$  exactly, but we can calculate an upper bound  $\lambda_{\min} \leq \lambda_{\text{upper}} < 0$  as follows. Without loss of generality, we assume that  $E_1$  equals the unit ball  $\mathcal{B}$ , and recall that when  $n = 2$ , Lemma 7 allows us to rewrite  $L_{yz}^2 = T_{y\mathcal{B}}T_{w\mathcal{B}}$  in the restricted domain  $\text{bd}(\mathcal{B})$ . When  $n > 2$ , simple examples show that the analog of Lemma 7 does not hold. However, we try a similar idea by looking for  $\alpha \geq 1$  such that  $M_{yz}(x) \leq \alpha T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x)$  for all  $x \in \text{bd}(\mathcal{B})$ , where  $w \in \text{bd}(\mathcal{B})$  lies on the line connecting  $y$  and  $z$  (just as in Section 5.1). Note that  $M_{yz}(w) = 0$ . In fact, the smallest such  $\alpha_{\min}$  can be calculated by bisection on  $\alpha$  using the solution of the following (equality constrained) TRS problem:

$$\begin{aligned} \min \quad & \alpha T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x) - M_{yz}(x) \\ \text{s. t.} \quad & x \in \text{bd}(\mathcal{B}). \end{aligned}$$

The basic decision in the bisection routine is as follows: if the optimal value is negative, then we decrease  $\alpha$ ; otherwise, we increase  $\alpha$ . After  $\alpha_{\min}$  is determined, we then follow the ideas

of Section 5.1 to calculate a minimum  $\lambda < 0$  guaranteeing that  $T_{y\mathcal{B}}(T_{zE_2} + \lambda T_{w\mathcal{B}}) \geq 0$  is valid on  $F$ . Finally, we define  $\lambda_{\text{upper}} := \lambda/\alpha_{\min}$  so that

$$\begin{aligned} 0 &\leq T_{y\mathcal{B}}(T_{zE_2} + \lambda T_{w\mathcal{B}}) \\ &= T_{y\mathcal{B}}T_{zE_2} + \lambda T_{y\mathcal{B}}T_{w\mathcal{B}} \\ &= T_{y\mathcal{B}}T_{zE_2} + \lambda_{\text{upper}}\alpha_{\min}T_{y\mathcal{B}}T_{w\mathcal{B}} \\ &\leq T_{y\mathcal{B}}T_{zE_2} + \lambda_{\text{upper}}M_{yz}, \end{aligned}$$

showing that  $T_{y\mathcal{B}}T_{zE_2} + \lambda_{\text{upper}}M_{yz}$  is valid on  $F$ .

## 5.4 Computational Tests

For  $n = 2$ , consider the following instance of TTRS with concentric ellipsoids centered at 0:

$$\begin{aligned} v^* &:= \min_{x \in \mathbb{R}^n} x^T C x + 2 c^T x \\ \text{s. t. } &x^T x \leq 1 \\ &x^T A_2 x \leq 1 \end{aligned}$$

where

$$C = \begin{pmatrix} -3/5 & \sqrt{6}/4 \\ \sqrt{6}/4 & -2/5 \end{pmatrix}, \quad c = -\frac{1}{2} \begin{pmatrix} \sqrt{6}/2 \\ 1 \end{pmatrix}, \quad A_2 = \frac{1}{2} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since  $C$  is not positive semidefinite, we know that an optimal solution must occur on the boundary. By carefully tracing the boundary of the feasible region, we can verify that  $x^* = (1, 1)^T/\sqrt{2}$  is a global optimal solution with optimal value  $v^* \approx -1.4608$ . Numerical results show that the optimal value of the relaxation with only SOCRLT constraints is  $-1.5$ , a gap of 2.74%. However, if we construct a lifted-RLT constraint based on  $y = (0, 1)^T$  and  $z = (\sqrt{6}/2, 0)^T$ , we can calculate  $\lambda_{\min} \approx 0.2986$ . In fact, with some care, one can see that  $\lambda_{\min}$  is determined by  $x^*$  being the third zero of the lifted-RLT quadratic along with  $y$  and  $z$ . In this case, we have the following precise formula for  $\lambda_{\min}$ :

$$\lambda_{\min} = \frac{20 + 5(\sqrt{6} - 2\sqrt{3} - 2\sqrt{2})}{18 + 4(\sqrt{6} - 2\sqrt{3} - 2\sqrt{2})} \approx 0.2986.$$

After adding the lifted-RLT constraint, we get the exact optimal value  $v^*$  with  $X^* \approx x^*(x^*)^T$ .

For  $n = 2$ , we also tested our approach on the example in Section 5.2 of [8]. The relaxation gap is closed after six lifted-RLT cuts in the sense that the optimal solution  $(1, x^T; x, X)$  has numerical rank 1.

$n$	% solved by basic SDP	% solved by adding SOC-RLT cuts to basic SDP <sup>1</sup>	additional % solved by adding heuristic lifted-RLT cuts to basic SDP	% still unsolved
5	92.2	4.0	2.0	1.8
10	24.6	68.3	2.2	4.9
20	4.1	85.3	4.0	6.6

Table 3: Numerical Results on TTRS instances from [8]

For  $n > 2$ , we also solve the instances generated in [8], where 1,000 instances of (TTRS) were generated for each of  $n = 5, 10, 20$ . All tests were performed on a Macintosh OS X desktop with 3.2 GHz Quad-core Intel Core i5 processor and 8GB 1600MHz DDR3 SDRAM using YALMIP [16], Matlab, and Mosek [2]. For the three different values of  $n$ , [8] found that 41, 70 and 104 instances could not be solved by adding SOCRLT constraints to the basic SDP relaxation. (In [8], an instance was regarded as solved if the relative gap between the relaxation value and the feasible value gotten by extracting  $x$  is less than  $10^{-4}$ .) Applying our heuristic lifted-RLT constraints to the 215 previously unsolved instances, we can solve 82—about 38%—of them; see Table 3. However, the CPU time increases dramatically when separating the lifted-RLT cuts. For example, on a particular  $n = 20$  instance, it takes 4.2 seconds to solve with only the SOCRLT constraints, while the same instance with lifted-RLT cuts requires 91.9 seconds. In general, our lifted-RLT process takes more than ten times longer on average than solving with only SOCRLT constraints.

## Acknowledgments

The authors are in debt to two anonymous referees and the associate editor for extremely helpful suggestions and insights.

## References

- [1] W. Ai and S. Zhang. Strong duality for the CDT subproblem: a necessary and sufficient condition. *SIAM J. Optim.*, 19(4):1735–1756, 2008.
- [2] M. ApS. *TheMOSEKoptimization toolbox for MATLAB manual. Version 7.1 (Revision 28)*., 2015.

---

<sup>1</sup>This column is slightly different from [8] because we use a different SDP solver in the experiment.

- [3] A. I. Barvinok. Feasibility testing for systems of real quadratic equations. *Discrete Comput. Geom.*, 10(1):1–13, 1993.
- [4] A. Beck and Y. C. Eldar. Strong duality in nonconvex quadratic optimization with two quadratic constraints. *SIAM J. Optim.*, 17(3):844–860 (electronic), 2006.
- [5] D. Bienstock. A note on polynomial solvability of the CDT problem. Preprint, Department of Industrial Engineering and Operations Research, Columbia University, New York, NY, 2013. Revised February 2015.
- [6] D. Bienstock and A. Michalka. Polynomial solvability of variants of the trust-region subproblem. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, chapter 28, pages 380–390.
- [7] I. M. Bomze and M. L. Overton. Narrowing the difficulty gap for the CelisDennisTapia problem. *Mathematical Programming*, 151(2):459–476, 2015.
- [8] S. Burer and K. M. Anstreicher. Second-order-cone constraints for extended trust-region subproblems. *SIAM Journal on Optimization*, 23(1):432–451, 2013.
- [9] S. Burer and B. Yang. The trust region subproblem with non-intersecting linear constraints. *Mathematical Programming*, 149(1-2):253–264, 2015.
- [10] M. R. Celis, J. E. Dennis, and R. A. Tapia. A trust region strategy for nonlinear equality constrained optimization. In *Numerical Optimization, 1984 (Boulder, Colo., 1984)*, pages 71–82. SIAM, Philadelphia, PA, 1985.
- [11] A. R. Conn, N. I. M. Gould, and P. L. Toint. *Trust-Region Methods*. MPS/SIAM Series on Optimization. SIAM, Philadelphia, PA, 2000.
- [12] M. Fu, Z.-Q. Luo, and Y. Ye. Approximation algorithms for quadratic programming. *J. Combinatorial Optimization*, 2:29–50, 1998.
- [13] N. I. M. Gould, S. Lucidi, M. Roma, and P. L. Toint. Solving the trust-region subproblem using the Lanczos method. *SIAM J. Optim.*, 9(2):504–525 (electronic), 1999.
- [14] D. Grigoriev and D. V. Pasechnik. Polynomial-time computing over quadratic maps. I. Sampling in real algebraic sets. *Comput. Complexity*, 14(1):20–52, 2005.
- [15] V. Jeyakumar and G. Li. Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. *Mathematical Programming*, 2013. DOI 10.1007/s10107-013-0716-2.
- [16] J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.
- [17] J. J. Moré and D. C. Sorensen. Computing a trust region step. *SIAM J. Sci. Statist. Comput.*, 4(3):553–572, 1983.

- [18] J.-M. Peng and Y.-X. Yuan. Optimality conditions for the minimization of a quadratic with two quadratic constraints. *SIAM J. Optim.*, 7(3):579–594, 1997.
- [19] F. Rendl and H. Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Math. Programming*, 77(2, Ser. B):273–299, 1997.
- [20] H. D. Sherali and W. P. Adams. *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*. Kluwer, 1997.
- [21] N. Shor. Quadratic optimization problems. *Soviet Journal of Computer and Systems Science*, 25:1–11, 1987. Originally published in *Tekhnicheskaya Kibernetika*, 1:128–139, 1987.
- [22] J. F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Math. Oper. Res.*, 28(2):246–267, 2003.
- [23] Y. Ye. A new complexity result on minimization of a quadratic function with a sphere constraint. In C. Floudas and P. Pardalos, editors, *Recent Advances in Global Optimization*. Princeton University Press, Princeton, NJ, 1992.
- [24] Y. Ye and S. Zhang. New results on quadratic minimization. *SIAM J. Optim.*, 14(1):245–267 (electronic), 2003.