AD – Blatt 2

2

1) $17 + 22 + 45 = O(1)$

$84 = O(1)$

$$\lim_{n \to \infty} \frac{1}{f(n)} = \lim_{n \to \infty} \frac{1}{17+22+45} = \frac{1}{}$$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} \geq c \cdot f(n)$$

(2)

1)

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{17+22+45}{1} = 84 < 85$$

$\Rightarrow g(n) \in \mathcal{O}(f(n))$

$\Rightarrow g(n) \in \mathcal{O}(1)$

2)

$f(n) = n^3$

$g(n) = 5n^3 + 12n^2 + 3n + 5$

$n_0 = 1$

$c = 1$

$\forall n \geq n_0 : g(n) \geq c \cdot f(n)$

I. A: $n = 1$ $\quad 5 + 12 + 3 + 5 \geq 1$

$25 \geq 1$

I. V: z.z. $\forall n \in \mathbb{N} : g(n) \geq f(n) \Rightarrow g(n+1) \geq f(n+1)$

$\cancel{5(n+1)^3 + 12(n+1)^2 + 3(n+1) + 5} \geq$

$\cancel{5n^3 + 12n^2 + 3n + 5 \geq n^2}$

$\cancel{4n^3 + 12n^2 + 3n + 5 \geq 0}$

$\cancel{\Rightarrow g(n) \in \Omega(f(n))}$

$5(n+1)^3 + 12(n+1)^2 + 3(n+1) + 5 \geq (n+1)^3$

$5(n^3 + 3n^2 + 3n + 1) + 12(n^2 + 2n + 1) + 3n + 3 + 5 \geq n^3 + 3n^2 + 3n + 1$

$5n^3 + \cancel{3n^2} + \cancel{3n} + 1 + 12n^2 + 24n + 12 + 3n + 8 \geq n^3 + \cancel{3n^2} + \cancel{3n} + \cancel{1}$

$5n^3 + 12n^2 + 3n + 5 + 24n + 14 \geq n^3$

$$g(n) + \underbrace{24n + 14}_{>0} \geq f(n)$$

gilt laut I.V.

$\square$

3) i) $g(n) = 2^{n+1}$
$f(n) = 2^n$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} =$$

$$= \lim_{n\to\infty} 2 \cdot \frac{2^n}{2^n} = 2 < 3$$

$\Rightarrow g(n) \in \mathcal{O}(f(n))$
$\Rightarrow 2^{n+1} \in \mathcal{O}(2^n)$

ii) $g(n) = 2^{2n}$
$f(n) = 2^n$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} \left(\frac{2^2}{2}\right)^n$$

$$= \lim_{n\to\infty} 2^n = \infty$$

$\Rightarrow g(n) \notin \mathcal{O}(f(n))$
$\Rightarrow 2^{2n} \notin \mathcal{O}(2^n)$

(2)

5) i) $g(n) = 2^n$

$f(n) = n!$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$\Rightarrow g(n) \in o(f(n))$

$\Rightarrow 2^n \in o(n!)$

ii)

$g(n) = n!$

$f(n) = n^n$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n!}{n^n} = 0 \Rightarrow n! \in o(n^n)$$

6) $g(n) = 6^{-5} n^{1,25}$

$f(n) = \sqrt{n}$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{\frac{1}{6^5} \cdot n^{\frac{5}{4}}}{\sqrt{n}} = \frac{1}{6^5} \lim_{n\to\infty} \left(\frac{n^{\frac{5}{2}}}{n}\right)^{\frac{1}{2}}$$

$$= \frac{1}{6^5} \cdot \lim_{n\to\infty} \left(n^{\frac{5}{2}-1}\right)^{\frac{1}{2}} = \frac{1}{6^5} \cdot \lim_{n\to\infty} n^{\frac{3}{4}} = \infty$$

$\Rightarrow g(n) \notin o(f(n))$

4) 1) z.z. $\log(n!) \in O(n \log n)$

$$\log(n!) = \sum_{i=1}^{n} \log(i) \leq \sum_{i=1}^{n} \log(n) \leq n \log n$$

$\Rightarrow \log(n!) \in O(n \cdot \log(n))$

2) z.z. $\log(n!) \in \Omega(n \log n)$

$$\log(n!) = \sum_{i=1}^{n} \log(i) \geq \sum_{i=n/2}^{n} \log(i) \geq \frac{n}{2} \cdot \log\left(\frac{n}{2}\right) = \frac{1}{2} \log(n) - \overbrace{\frac{1}{2} \log(2)}^{\text{konst.}}$$

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$\Rightarrow \log(n!) \in \Omega(n \cdot \log n) \Rightarrow \log(n!) \in \Theta(n \log n)$

③

4) Alternative aus Tut:

$$\log(n!) \hat{=} \log\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$ ✓ Stirling-Formel

$$= \dots = \underbrace{\tfrac{1}{2}\log(2\pi)}_{\in \mathcal{O}(1)} + \underbrace{\tfrac{1}{2}\log(n)}_{\in \mathcal{O}(n)} + \underbrace{n\,(\log(n) - \log(e))}_{\in \mathcal{O}(n \cdot \log n)}$$

$$\Rightarrow \log(n!) \in \mathcal{O}(n \cdot \log(n))$$

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④

3) $T(1) = 1 \quad T(2) = 2 \quad T(3) = 1$

$T(n) = 2T(n-1) + n^2$ für $n > 3$

$$T(n) = 2T(n-1) + n^2 = 2(2T(n-1-1) + (n-1)^2) + n^2$$

$$= 2(2(2T(n-1-1-1) + (n-1-1)^2) + (n-1)^2 + n^2$$

$$= \dots$$

$$= 2T(\underbrace{n-i}_{=3}) + \sum_{k=0}^{n-n} 2^k (n-k)^2$$

$$\Rightarrow i = n-3$$

Einsetzen: $T(n) = 2^{n-3}\,\underbrace{T(n(n-3))}_{=1} + \sum_{k=0}^{n-4} 2^k (n-k)^2$

[Beobachtung: $T(n) \in \Omega(2^n)$, $T(n) \in \mathcal{O}(2^n \cdot n^?)$?]

$$\sum_{k=0}^{n-4} 2^k (n-k)^2 = 2^0 \cdot n^2 + 2^1 \cdot (n-1)^2 + \dots + 2^{n-4} \cdot 4^2$$

$$= \sum_{k=4}^{n} 2^{n-k} k^2 = \sum_{k=4}^{n} \frac{2^n}{2^k} k^2 - 2^n \sum_{k=4}^{n} \frac{k^2}{2^k}$$

$\rightarrow$ Quotientenkriterium: $q = \lim \frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{(n+1)}} \cdot \frac{2^n}{n^2} = \lim \frac{n^2+2n+1}{2n^2} = \frac{1}{2} \in \mathcal{O}(1)$

$\Rightarrow T(n) \in \Omega(2^n)$

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4

1) $T(1) = 1$

$\forall n > 1: \quad T(n) = 4T(\frac{n}{2}) + n$

$$T(n) = 4T(\tfrac{n}{2}) + n = 4(4T(\tfrac{\frac{n}{2}}{2}) + \tfrac{n}{2}) + n$$
$$= 4(4(4T(\tfrac{n}{8}) + \tfrac{n}{4}) + \tfrac{n}{2}) + n$$
$$= 4T(\tfrac{n}{2^i}) + \sum_{k=0}^{n-1} 4^k (\tfrac{n}{2^k})$$

$$\frac{n}{2^i} = 1 \iff n = 2^i \iff i = \log n$$

Einsetzen: $T(n) = 4T(\frac{n}{2^{\log n}}) + \sum_{k=0}^{n-1} 2^{2k} \frac{n}{2^k}$
$$= 4 \cdot 1 + \frac{1}{4}\sum_{k=0}^{n-1} n = 4 \cdot 1 + \frac{(n-1)(n-2)}{8}$$
$$\Rightarrow T(n) \in \Theta(n^2)$$

2) $T(1) = 1$

$\forall n > 1: \quad T(n) = 2T(n/4) + \sqrt{n}$

$$T(n) = 2T(\tfrac{n}{4}) + \sqrt{n} = 2(2T(\tfrac{n}{16}) + \sqrt{\tfrac{n}{4}}) + \sqrt{n}$$
$$= 2(2(2T(\tfrac{n}{4^3}) + \sqrt{\tfrac{n}{4^2}}) + \sqrt{\tfrac{n}{4}}) + \sqrt{n}$$
$$= 2T(\tfrac{n}{4^i}) + \sum_{k=0}^{n-1} 2^k \sqrt{\tfrac{n}{4^{k-1}}}$$

$$\frac{n}{4^i} = 1 \iff n = 4^i \iff i = \log_4 n$$

Einsetzen: $T(n) = 2T(\frac{n}{4^{\log_4 n}}) + \sum_{k=0}^{n-1} 2^k \cdot \frac{\sqrt{n}}{2^{k-1}}$
$$= 2 \cdot 1 + 2\sum_{k=0}^{n-1} \sqrt{n} = 2 + (n-1)\sqrt{n}$$
$$\Rightarrow T(n) \in \Theta(n\sqrt{n})$$

AD - Blatt 2 13.10.19

$1+\frac{1}{\varphi} = 1 + \frac{2}{1+\sqrt{5}}$

(3) z.z. $\forall n \in \mathbb{N}$: $f(n) = \frac{\varphi^n - \hat{\varphi}^n}{\sqrt{5}} \Rightarrow f(n+1) = \frac{\varphi^{n+1} - \hat{\varphi}^{n+1}}{\sqrt{5}}$

I.A.: $n=1$

$f(1) = 1$

$f(1) = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} = 1$ ✓

I.V. $f(n) = \frac{\varphi^{n} - \hat{\varphi}^{n}}{\sqrt{5}} \Rightarrow f(n+1) = \frac{\varphi^{n+1} - \hat{\varphi}^{n+1}}{\sqrt{5}}$

I.S.

$f(n+1) = f(n) + f(n-1) = \frac{\varphi^n - \hat{\varphi}^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \hat{\varphi}^{n-1}}{\sqrt{5}}$

$= \frac{1}{\sqrt{5}}(\varphi^n - \hat{\varphi}^n + \varphi^{n-1} - \hat{\varphi}^{n-1})$

$= \frac{\varphi^n}{\sqrt{5}}\left(1 + \frac{1}{\varphi}\right) - \frac{\hat{\varphi}^n}{\sqrt{5}}\left(1 + \frac{1}{\hat{\varphi}}\right)$

Der goldene Schnitt:

$\frac{a}{b} = \frac{a+b}{a} \Leftrightarrow \frac{a}{b} - 1 = \frac{b}{a}$ $\quad \frac{a}{b} = \varphi$

$\Leftrightarrow \varphi - 1 = \frac{1}{\varphi} \Leftrightarrow \frac{1}{\varphi} + 1 = \varphi$

$\frac{\varphi^n}{\sqrt{5}} \cdot (\varphi) - \frac{\hat{\varphi}^n}{\sqrt{5}}(\hat{\varphi}) = \frac{\varphi^{n+1} - \hat{\varphi}^{n+1}}{\sqrt{5}}$