Memorandum: April 14, 2021

To: Dr. Mark Colton
From: Spencer Mitton
Subject: Study of Apollo Spacecraft Attitude Dynamics

This memo entails the results of studying the dynamics of the Apollo space craft. Two different situations specifically were studied, one in which the dynamics are controlled by thrusters and a second in which the dynamics are actuated by control moment gyros.

For the Command Service Module (CSM) of the Apollo spacecraft, the inertia tensor was calculated from the data given in figure 3 of the handout for the center of mass for the CSM. A simplified inertia tensor was calculated in which the center of mass for the command module, service module, and propellants were assumed to all be on the same x-axis. Both inertia tensors are shown below.

$$I_{CSM} = \begin{bmatrix} 40820.0 & 1538.0 & -3179.0 \\ 1538.0 & 90590.0 & 128.6 \\ -3179.0 & 128.6 & 98740.0 \end{bmatrix} kg* m^2 \qquad I_{CSM,simplified} = \begin{bmatrix} 40482 & 0 & 0 \\ 0 & 90358 & 0 \\ 0 & 0 & 98637 \end{bmatrix} kg* m^2$$

To actually model the attitude of the CSM, I found the equation of motions (EOMs) by rearranging the equations for Euler Angles (equations 1-3). Rearranging got me the matrix relation shown in equation 4 in which the angular velocity of the spacecraft is related to the velocity and torques about each axis of the spacecraft. The last three EOMs for the spacecraft are given in equation 5.

Equations 1-3:

$$I_{xx}\dot{\omega}_x - I_{xy}(\dot{\omega}_y - \omega_x\omega_z) - I_{xz}(\dot{\omega}_z + \omega_x\omega_y) - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}({\omega_y}^2 - {\omega_z}^2) = M_x$$
$$I_{yy}\dot{\omega}_y - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) - I_{xy}(\dot{\omega}_x + \omega_z\omega_y) - (I_{zz} - I_{xx})\omega_x\omega_z - I_{xz}({\omega_z}^2 - {\omega_x}^2) = M_y$$
$$I_{zz}\dot{\omega}_z - I_{xz}(\dot{\omega}_x - \omega_y\omega_z) - I_{yz}(\dot{\omega}_y + \omega_x\omega_z) - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}({\omega_x}^2 - {\omega_y}^2) = M_z$$

Equation 4:

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = {I_{CSM}}^{-1}\left(\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} - \begin{bmatrix} I_{xy}\omega_x\omega_z - I_{xz}\omega_x\omega_y - (I_{yy} - I_{zz})\omega_y\omega_z - I_{yz}({\omega_y}^2 - {\omega_z}^2) \\ I_{yz}\omega_x\omega_y - I_{xy}\omega_z\omega_y - (I_{zz} - I_{xx})\omega_x\omega_z - I_{yxz}({\omega_z}^2 - {\omega_x}^2) \\ I_{xz}\omega_y\omega_z - I_{yz}\omega_x\omega_z - (I_{xx} - I_{yy})\omega_x\omega_y - I_{xy}({\omega_x}^2 - {\omega_y}^2) \end{bmatrix}\right)$$

Equation 5:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dfrac{\omega_y \sin\phi + \omega_z \cos(\phi)}{\cos(\theta)} \\ \omega_y \cos(\phi) + \omega_z \sin(\phi) \\ \dfrac{\omega_y \sin(\theta)\sin(\phi) + \omega_z \sin(\theta)\cos(\phi) + \omega_x}{\cos(\theta)} \end{bmatrix}$$

With the EOMs and a torque input, the attitude of the spacecraft could be modeled over time. For model situation 1 for task C.b (see Table 1), and over a time span of 30 seconds, the spacecraft attitude is shown in Figure 1. In this model, the moments for the spacecraft are given. To find the attitude over time, the EOMs that were found (see above), were numerically integrated. The maximum and minimum values for the velocities and angles of the spacecraft for this simulation are shown in Table 1.

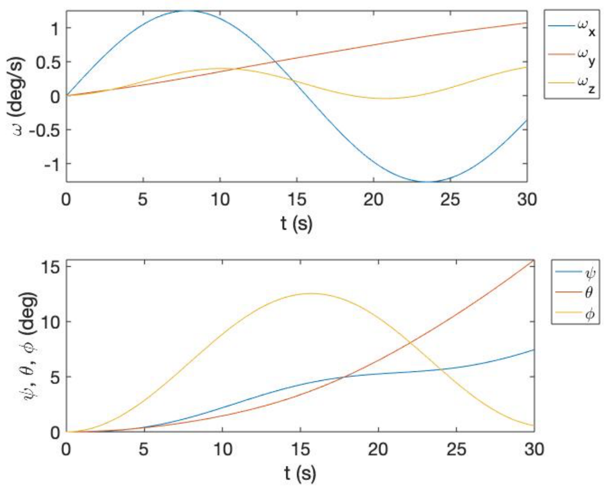


Figure 1: Attitude of spacecraft for model situation 1 when dynamics are controlled by thrusters. Torque equations are found in Table 1. The simulation shown in this graph used the general moment of inertia. The simplified moment of inertia produced nearly identical results.

The other simulation asked about for the CSM under thruster control was BBQ mode. The graphs for the CSM in BBQ mode are shown in Figure 2. The torques required for BBQ mode are $M_x = 0$, $M_y = -.9685$ N-m, and $M_z = -.4684$ N-m. These constant torques are necessary to keep the CSM in constant rotation about only the x-axis. If 0 N-m is applied about each axis, the CSM does not rotate constantly (see Figure 3).

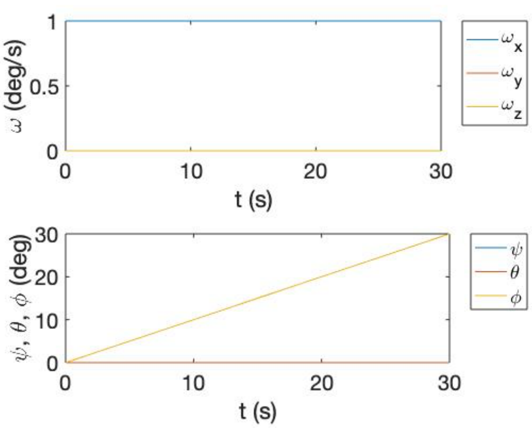


Figure 2: Graph of the CSM in BBQ mode (when it rotates at a constant velocity about its x-axis). In this instance of BBQ mode, the CSM rotates at 1 deg/s. The moments required to produce these dynamics are $M_x = 0$, $M_y = -.9685$ N-m, & $M_z = -.4684$ N-m.

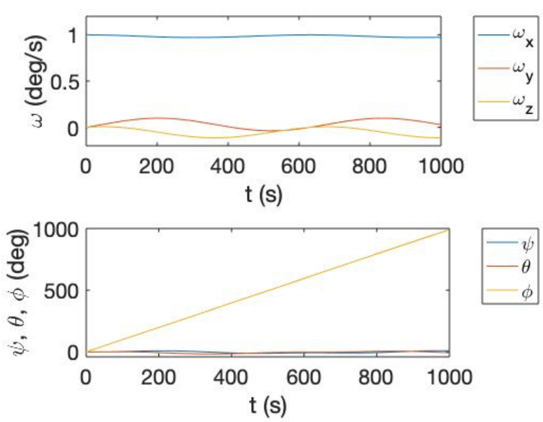


Figure 3: Graph of the CSM with 0 N-m of torque being applied on each axis, graphed for 1000s.

After the dynamics of the CSM were modeled with specific torque inputs, another situation was studied in which control moment gyros (CMGs) were used as torque actuators. The torques created by the CMGs are shown below.

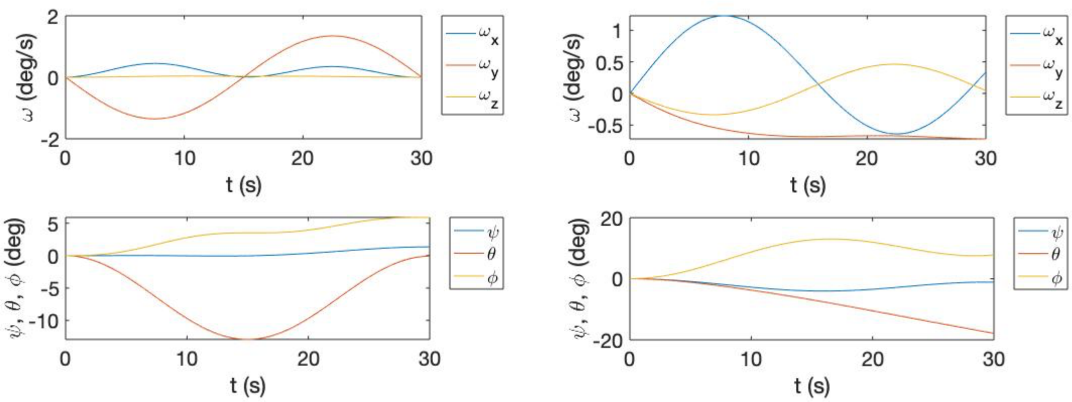
Equations 6-8:

$$M_x = -cos(\theta_3)\,\dot{\theta}_3 I_R \Omega + sin\,(\theta_1)\dot{\theta}_1 I_R \Omega$$
$$M_y = -cos(\theta_1)\,\dot{\theta}_1 I_R \Omega + sin\,(\theta_2)\dot{\theta}_2 I_R \Omega$$
$$M_z = -cos(\theta_2)\,\dot{\theta}_2 I_R \Omega + sin\,(\theta_3)\dot{\theta}_3 I_R \Omega$$

These equations were found by finding the angular momentum of each gyro and then taking the time derivative to find the moment it caused on the spacecraft as its angular momentum vector rotated. The derivative of the angular momentum for each gyro was found by the transport theorem. Since the angular velocity of the gyros was assumed to be constant, the transport theorem only produced a cross product of the angular momentum vector and frame angular velocity.

After the reaction moments for each gyro were found and added together to produce one reaction moment vector, I used the rest of my code from modeling the CSM under thruster control. To model the attitude of the CSM, only an input theta is needed for each gyro. Two situations for tasks E.b and E.c. were explored in which the dynamics of the CSM were modeled given an input theta for each CMG. The graphs of each situation are given in Figures 3,4. The equations for theta as well as the maximum angles and angular velocities for each situation are found in Table 1, situations 2 and 3.

The graph for situation 2, task E.b, shows that there is no moment being applied about the z axis, which matches what should happen given the equations for the CMGs (equations 6-8). Examining these equations shows that if θ_1 and θ_2 are held constant, the most torque is applied about the y-axis, and then a smaller torque is applied about the x-axis.



Figures 4,4 : Attitude of CSM for model situation 2 (left) and situation 3 (right). Task E.b is shown on the left. Task E.c is shown on the right.

Table 1: Maximum Euler angles and angular speeds for the CSM under three different simulations. Simulation one (row 1) is for given moments, assuming thrusters for control of the space craft. Simulations 2 and 3 (rows 2,3), are from simulating the spacecraft being actuated by control moment gyros. Moments are in N-m, angles are in degrees, and angular-velocities are in degrees/s.

Inputs to Model	$\omega_{x,max}$	$\omega_{x,min}$	$\omega_{y,max}$	$\omega_{y,min}$	$\omega_{z,max}$	$\omega_{z,min}$	ψ_{max}	ψ_{min}	θ_{max}	θ_{min}	ϕ_{max}	ϕ_{min}
$M_x = 176\cos(.2t)$ $M_y = 54$ $M_z = 98\sin(.3t)$	1.2499	-1.2704	1.0722	0	.4202	-.0425	7.4577	0	15.6084	0	12.5465	0
$\theta_1 = 15\sin\left(\frac{\pi t}{15}\right)$ $\theta_2 = 0, \theta_3 = 0$	.4493	-.0012	1.3434	-1.3510	.0391	0	1.3546	-.0617	0	-12.9314	5.8940	0
$\theta_1 = \frac{15}{1+e^{-.3t}}$ $\theta_2 = 15\sin\left(\frac{\pi t}{15}\right)$ $\theta_3 = 15\sin\left(\frac{\pi t}{15}\right)$	1.2288	-.6416	0	-.7206	.4618	-.3370	0	-4.0171	0	-17.8655	12.9980	0