Hello ! This is a tentative sketch of my plan for upcoming college algebra class. I like to constantly experiment and update, so some of this is a like would I did last time, while much of it is new. I tend to work my own current mathematical obsessions into what I'm teaching so that my enthusiasm for math is apparent and hopefully contagious.

As a general strategy, I plan to ask students to answer questions by giving informal constructive proofs. I will also ask them to be mathematically creative. For instance, I might ask them to create a fascinating function, or to illustrate one of the covered topics in their preferred style and medium.

I also plan to focus on the rational number system first, sticking to operations that keep us in that system, such as addition, multiplication, and inversion. Only later, after students develop confidence with this system, will I introduce irrational numbers. I imagine we will spend something like the first 3 weeks within the rational number system. I hope to discuss the rational roots theorem, which in this limited context will provide students with all possible solutions. Of course adding irrational numbers adds solutions, which typically aren't even expressible in surds, though I won't get into that with students. I will have them computationally looking for values that minimize $|f(x)|$, so that they get an informal sense of numerical methods.

I plan to prove that no rational number squared is equal to 2.[1] Of course I want to "motivate" the introduction of all the weird new concepts. Instead of just adding all the irrational numbers at once, I will likely let students play with the idea of adding only the symbol $\sqrt{2}$ and seeing what results. I remember doing this in abstract algebra, and I think most students simply learn to treat the sign $\sqrt{2}$ as a something that just squares to 2, without noticing that this is really quite a leap.

For instance, where does this new strange "irrational" number fit in with the rational numbers in terms of order ? I might sketch the basic idea of Dedekind cuts, just to hint at how rich and complex the real number system is, compared to the crystalline and simple rational number system.

I actually plan to start even before the rational number system, with positional notation. IMO, a great way to do this is to teach base 2 or binary, so that positional notation "becomes visible" again. If we include

[1] I will also linger on the strangeness of indirect proofs. In the context of the proof I will use, the existence of rational square root of 2 implies that the same positive integer is both even and odd. So the non-existence of a square root can be understood in terms of our not being able to admit one without sacrificing a prior intuitive commitment to positive integers being either even or odd. Instead of some mysterious "we can just do this and call it a proof", I want to explain proofs as convincing arguments and indirect proofs as a strange but powerful style of argument.

the radix point, which we must, then we end up discussing negative integer exponents. Especially at the art school, but not only there, I've found that students tend to be foggy on fundamental concepts. They don't really see that percentages and "decimal numbers" are also "just fractions in disguise." The irrationality of $\sqrt{2}$ is really an "arithmetical fact." So I need to teach the fundamental theorem of arithmetic in order to set up this proof.

Why bother with all of this ? In my experience, students generally have very little understanding of computer output. They don't understand that computers "lie" by telling them that $\frac{1}{3} = 0.333333333333$. And they don't understand that $\pi \neq 3.14159265359$, though that's what Google search, acting as a calculator, just told me. Of course we can't ask them to master limits, but we can give them a sense that rational numbers are the computational *basis* of a richer and stranger real number system.

Of course this has to be handled carefully. Students should not be overwhelmed, and this will *not* be a calculus class in disguise. But art students tend to like discussing elusive concepts like infinity and truth. I also tend to stress the *creativity* of the great mathematicians. "The essence of mathematics lies in its freedom."[2] In my recent class, a student was delighted with this realization. That felt good ! I think maybe that math teachers don't tend to contextualize math and say what it *is*. And this is no easy question with a final answer. My own approach lately, largely inspired by Haskell Curry, is that mathematics is the necessarily informal science of formal systems. It's "necessarily informal" in the sense that it *finally* involves "real world" beliefs about "real world" calculations in a general sense that includes *formal* proofs.

I will also spend as much time as necessary on the concept of a function. I will emphasize the differences in related approaches to this definition. For instance, in "math-platonistic" set theory, a function is often a "pre-existent" *infinite* set of ordered pairs. From a constructivist point of view, sets are more like rules for generating their elements than boxes that already contain those elements. In this context, a function is best understood as a *program.* While I lean toward constructivism myself, I discuss both approaches. I tend to include history and philosophy of math in every math class, to make mathematics more human. If students are made aware of opposed philosophical interpretations of math, they will be less likely — I hope — to mistake their confusion for a lack of ability. Indeed, confusion may even be an indicator of ability.

While it's not typical in college algebra, I like to use notation like $f : Q \to Q$ to indicate the *type* of a function that transforms a rational

[2] Cantor

number input into a rational number output. Of course this can also be understood in terms of a set of pre-existent ordered pairs. I tend to spend some time on non-numerical functions, just to emphasize the generality of the concept.

To get concrete for a moment, let me provide an example.

Let $f : Q \to Q$ be a function defined by $f(x) := x^2 - 2$. I can then ask a student to (informally) *prove* that the inequality $|f(x)| < \frac{1}{100}$ has at least both a positive and a negative solution. This is achieved, of course, by finding two of the many possible answers and checking that they satisfy the inequality. So the student is implicitly giving a constructive proof.

The point of framing things in terms of informal proofs is to emphasize the logical basis of mathematics. This logic is not usually made explicit until students take higher-level math classes, which most of them never take. This is unfortunate, because the study of math is a great opportunity to practice careful and definite reasoning.

Note also that the provided domain of the function clarifies the *kind* of number allowed to count as a solution. I hope students will enjoy the puzzle-like nature of equations and inequalities. To talk in terms of proof will hopefully remind students to check that their "solutions" are indeed solutions.

If the goal is understood to be a proof, this calculation is crucial — it *is* the proof, basically. Techniques for *finding* solutions are one thing, and verification is another. A person can use a brute force search in some cases and be sure that they have indeed found a solution, even if they lack a general technique.

I like calculators with "function tables" that allow students to do this, but I have had problems in the past getting students to actually obtain the specified calculator. If our class is small, the admin may just buy the calculators for the class, which I'll hold on to for the *next* class. I already have 4 or 5 calculators (the TI-36X Pro) that would be great. It's very powerful for its cost of $\approx$ 20 dollars. Another option, which I'm still considering, is to use an online Python REPL. The great thing about Python, as you may know, is that it handles a rational number type of arbitrary size. So it can handle and multiply numbers like $\frac{123098345093434}{3803409343}$ easily. Discussing floating point numbers can be avoided, if this path is taken. But, in general, I do tend to talk about the "facts" of computation. I think is justified because students have easy access to Wolfram Alpha, etc. The problem is not a lack of calculators but a lack of understanding of calculators. So I try to give students a fair amount of big picture stuff on the intersection of computer science and mathematics, largely in passing.

Of course we will cover a fair amount traditional material, such as polynomials (especially linear and quadratic functions) and exponential functions. I may focus more on 2^x than e^x, since the exponential function doesn't make much sense without the calculus that demonstrates its importance. Of course I want to discuss logs as the inverses of exponential functions. I plan on teaching 2^x in a way that includes recursion. If $f(0) = 1$ and $f(x+1) = 2f(x)$, then $f(1) = 2f(0) = 2$ and $f(2) = 2f(1) = 4$, and so on. So x is the number of doublings of 1.

Because it's fascinating to me and usually to students, I also want to teach Pascal's triangle. This is useful for expanding $(x + y)^n$, and the "n choose k" interpretation sets up a detour into the "stars and bars" method of combinatorics.

A less typical inclusion is a brief discussion of cardinality and enumeration, which students usually never see unless they are math majors. It gives them a taste of some wild math and also helps them understand bijections as a very special kind of function. Art students tend to enjoy this. I will definitely include a way to enumerate the rational numbers. There's a very nice relationship between the Calkin-Wilf tree and hyperbinary numbers that we'll probably explore. I covered this material recently at a nursing school, and the students liked it. I've actually included at least a little infinite set theory in all of my college algebra classes at KyCAD, and it's went well each time.

A less mathematical and more social issue is grading. I'm pretty lenient. I really just want students to show up and try, which includes spending some time on homework. I've never had to fail a student, though I have occasionally supported withdrawal from the class by students who can't find the time or focus to complete enough work to pass.

My goal, which I have been lucky enough to achieve in most cases, is to encourage the development of a community feeling in the classroom, so that students (believe it or not) *like* being there. So I embrace a controlled amount of joking around and socialization, as this helps make the total experience something that students value. I tend to think of mathematics as a language, and I think the way to learn a language is to use it. So getting students working together and tutoring one another is ideal. In my last class, I had one outgoing student "take my place" at the whiteboard and explain concepts to other students. This was a joy to see, because the student was also learning the joy of teaching/communicating math, which is such an important part of math. I half-joked with her that she'd end up a math teacher.

Some students at KyCAD have problems with attendance. I try to minimize this tendency by working attendance as a serious factor of the final grade. I also try to keep students off screens. This is a challenge ! Because I don't want to stop a good mathematical conversation with those paying attention to police the screen use of those not paying attention. I may insist this time that students have only pen and paper and calculator on the table/desk, which will maybe (?) be more effective than a vague no-screens rule.

I'm still picking out a book. It'll be some open/free book that students can use as a supplement. Frankly we probably won't rely on it much. Lately I've been using Python to generate homework and solutions. I might use HW problems from the text, if that turns out to be convenient. But I usually write up lecture notes in LaTeX. Homework (especially if generated with Python) is sometimes provided as text files. I tend to insist that students turn in handwritten solutions. Quizzes will include written-answer questions that are graded leniently but help me understand where students are conceptually.

I'd be glad to hear any ideas you may have for the class. I'll give them serious consideration. I'm also glad to expand on anything discussed above and link to references for any of the proposed non-traditional topics. I'm looking forward to teaching this class, and I expect that you will also enjoy the KyCAD environment and spirit. It's my favorite place to teach by far.