Here I explore something like intuitionist real numbers. I want to discuss two types.

First we have total recursive functions $f : N \to \{0, 1\}$.

For instance, we might have $x = 0101010101...$ or $x = \overline{01}$

Note that the pattern is understood to repeat deterministically. If we let $N = \{0, 1, 2, ...\}$, then we can say that $f(n) = 0$ when n is even and $f(n) = 1$ when n is odd. So f is the characteristic function of the odd numbers.

We can even identity this "computable real number" *with* this computable subset of N. We can actually think in terms of computable properties *instead* of "sets." IMO, this makes more sense as sets become infinite.

We want to do some calculus with these weird real numbers, so we need order and distance. We basically use alphabetic order, where 0 comes before 1, so $\overline{001} < \overline{100}$.

We've looked at very simple "periodic" real numbers that are easy to order, but there's no limit on the complexity of these numbers, so we might have to wait a long time to make a decision. Indeed, there is no limit on how long we might have to wait. Computation requires time, material, and energy. Where **exactly** is π ? We have millions or billions of digits, but its location remains endless if shrinkingly indeterminate. Even though we have finite programs for it.

Of course $x = y$ if $\forall n \; x(n) = y(n)$. Actually we might want to use $x \equiv y$ instead of $x = y$. Why ? Because x and y are usually two different programs. So we are only determining that they "do the same thing" in some sense. We might reserve $x = y$ for the case when we can scan the programs and see that they really are identical, even at the level of source code.

Note that we, in general, have to work hard to decide if $x = y$. This requires a proof, based on the programs. If we only have access to output, we can never rule out some n_0 such that $x(n_0) \neq y(n_0)$.

Our situation is eased somewhat by a definition of distance. Let n be the first input, counting from 0, at which x and y disagree. Then $|x - y| = 2^{-n}$.

For instance $|\overline{10} - \overline{11}| = 2^{-1} = \frac{1}{2}$. Also $|\overline{1} - \overline{01}| = 2^0 = 1$.

Note that 1 is the maximum distance between our computable real numbers. And we don't care what happens after their first divergence in output.

It turns out that all functions $f : R \to R$ are continuous !

But we have to define functions $f : R \to R$ to make sense for these weird real numbers.

Note that we only ever have access to "prefixes" of real numbers. So we define infinite sequences, but we only ever "touch" finite sequences of "output so far."

A function $f : R \to R$ is therefore — intuitively — a computable function from prefixes to prefixes, from finite sequences to finite sequences. Of course technically it is one more real number, one more computable sequence.

I will use $[x]_4 = 10101$ to indicated that x starts with the finite sequence 10101. The 4 indicates that we are looking at the first five values, starting from 0.

Let $x = \overline{1010111}$ and $y = \overline{10101}$.

Then $[x]_5 = [y]_5 = 10101$, but $x \neq y$.

Let's say that *we* know this but that someone else hasn't seen the programs for x and y. All they know is that $[x]_5 = [y]_5 = 10101$ and that these prefixes are generated by unknown valid "programs."

Let's define $f : R \to R$ as a bit-inverter. For instance $f([01]_2) = [10]_2$.

So $f([x]_5) = f([y]_5) = [01010]_5$. Just like us, this function doesn't in general know what lies ahead. All it can do is react deterministically on prefixes.

Note that $|x - y| = 2^{-5} = \frac{1}{32}$. Also $|f(x) - f(y)| < 2^{-4} = \frac{1}{16}$. In fact, we know enough to calculate $|f(x) - f(y)|$ exactly, but even without knowing the programs for x and y, we could determine this from the prefixes above.

Functions can't "react" to differences they haven't seen yet. Note that this means that we can't create functions like $f(x) = \overline{0}$ if $x > \overline{0}$ and $f(x) = \overline{1}$ if $x = \overline{0}$. Technically, we *could* define functions like this and *try* to prove $x >= 0$ from the source code, *if* we have it. But in this case the function is useless until we get a proof.

But here we prefer functions that can act immediately on prefixes. This makes more sense as we look at the "freer" version of these numbers.

I haven't proved but only explained why functions $f : R \to R$ are continuous. Let's say that $[x]_{1000} = [y]_{1000}$, so that $x(n) = y(n)$ for $0 < n < 999$.

Then, for any function $f : R \to R$, we have $|f(x) - f(y)| <= 2^{-1000}$.

This is because f has to treat equal prefixes equally. For this approach, "time is real."

OK, so we know that the computable subsets of N are countable. So there are countably many computable real numbers of this type. There are also countably many functions on real numbers. A tacit assumption is that we are working with what we believe are total functions, even if we have no total function to check this for us.

What if we loosen this constraint ?

Let's imagine a signal from outer space as a sequence of bits. At any point our satellites have gathered only a finite number of bits. Realistically, the transmission could just stop, we want to model that transmission as unbounded.

Another example: let's just flip a coin over and over again. We record the results in the lengthening list.

Here we don't have a program for the real number. We just decide to organize bits in a sequence that we allow to get longer and longer.

Do we manage to get an uncountable infinity of real numbers finally ? With positive measure ? Some might say so. To me we always in fact have a finite amount of information, so I might — as others have — speak instead of a continuum that is becoming rather than become. So we don't have an eternal static set of real numbers, which is already uncountable. We are idealizing the outputs of processes, allowing them to go on **indefinitely.**

Now we can *really* see why $f(x) = \overline{0}$ if $x > \overline{0}$ and $f(x) = \overline{1}$ if $x = \overline{0}$ is an invalid function. You might say it only *becomes* sensible restriction with this freer and stranger conception of real numbers. At some point it *may* become clear that $x > \overline{0}$, but there is no limit on how long we might have to wait. We also have no program to examine. We really do just have to wait. The values of x don't even come at some definite rate, like 1 bit per second or even 1 bit per year.

This is an "anti-Platonism" real number system. The continuum is "liquid" and indeterminate.

All functions $f : R \to R$ that handle these liquid numbers at all "have" to be continuous. They operate on prefixes and they can't see around the corner. The future doesn't exist yet. Let f be our bit inverter above. Let x be our message from outerspace. Then $f(x)$ is itself a real number in the process of being created. We might have some other function g and consider $g(f(x))$. Then $g(f(x))$ is still being created. So is $f(x) + g(x)$ if we define addition as the logical **or** of prefixes.

OK, but what if we want something more recognizable ? Why not "becoming" sequences of rational numbers ? This can also be done. We just need pick something like fundamental or Cauchy sequences of rational numbers. This can be done in many ways.

We might require that $f(n+2) < \frac{1}{2}|f(n+1) - f(n)|$. Of course we don't have the program available. So, if we get a $f(n+2)$ that violates this rule, we can just repeat $f(n+1)$ forever. This allows us to handle any becoming list of rational numbers. We "force" convergence on the unruly sequences, so that they even become rational real numbers.

What I don't like about this is that we have free numbers that have boringly settled down. So we could skip values. We just throw away the badly behaved $f(n+2)$ until something acceptable comes along. If we take this path, then it's not clear that there *are* rational real numbers.

Probably most would accept *computable* fundamental sequences of rational numbers as a "subset" or "subtype" of these freer numbers.

How useful are these things ? I don't know. I like them as a somewhat plausible approach toward a genuine (intuitively satisfying) continuum. They are "realistic" about our human situation, you might say. The future isn't here yet. Reality is indeterminate.