Answers to 'Expectation Value and Variance

Question 1

$$X = \{0, 1, 2, 3\}$$

$$p_3 = p_0 = 0.2 \; , \; p_1 = 0.3 = p_2$$

$$E_p[X] = p_0 x_0 + p_1 x_1 + p_2 x_2 + p_3 x_3 =$$

$$0.2 \cdot 0 + 0.3 \cdot 1 + 0.3 \cdot 2 + 0.3 \cdot 3$$

$$= \underline{\underline{1.5}} = \mu$$

$$var[X] = p_1 (x_1 - \mu)^2 + p_2 (x_2 - \mu)^2 + p_3 (x_3 - \mu)^2 + p_4 (x_4 - \mu)^2$$

$$= 0.2 (0 - 1.5)^2 + 0.3 (1 - 1.5)^2 + 0.3 (2 - 1.5)^2 + 0.2 (3 - 1.5)^2 = \underline{\underline{1.05}}$$

Same question for

$q_0 = q_3 = 0.1$

$q_2 = q_1 = 0.4$

$$\mathbb{E}_q[X] = 0.1 \cdot 0 + 0.4 \cdot 1 + 0.4 \cdot 2 + 0.1 \cdot 2$$

$$= \underline{\underline{1.5}}$$

$$\mathrm{var}_q[X] = 0.1(0-1.5)^2 + 0.4(1-1.5)^2 + 0.4(2-1.5)^2 + 0.1(1-1.5)^2$$

$$= \underline{\underline{0.65}}$$

So we see that the expectation values under p and q are the same. We should expect this because the 'centre of gravity' is the same.

However the variance under q is smaller as it is less likely that events far away from the mean are produced.

Question 2

$$\text{Bern}(x|\mu) = \mu^x(1-\mu)^{1-x}$$

Find $\mathbb{E}[x]$ and $\text{var}[x]$

$x \in \{0, 1\}$

$x_0 = 0$
$x_1 = 1$

$$p_0 = \mu^0(1-\mu)^{1-0} = 1-\mu$$
$$p_1 = \mu^1(1-\mu)^{1-1} = \mu$$

$$\mathbb{E}[x] = 0 \cdot (1-\mu) + 1 \cdot \mu = \underline{\underline{\mu}}$$

$$\mathrm{var}[x] = p_0 (0-\mu)^2 + p_1 (1-\mu)^2$$

$$= (1-\mu)\mu^2 + \mu(\mu^2 - 2\mu + 1)$$

$$= \mu^2 - \mu^3 + \mu^3 - 2\mu^2 + \mu$$

$$= -\mu^2 + \mu = \underline{\underline{\mu(1-\mu)}}$$

Question 3

$$\mathrm{var}[f] = \mathbb{E}[(f(x) - \mathbb{E}[f(x)])^2]$$

Show that

$$\mathrm{var}[f] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

$$\mathbb{E}[(f(x) - \mathbb{E}[f(x)])^2] =$$

$$\sum_i p_i (f(x_i) - \mathbb{E}[f(x)])^2$$

Note that $\mathbb{E}[f(x)]$ is just a number; we write it as $f(x)$

$$\sum_i p_i \left(f(x_i) - E[f(x)] \right)^2$$

$$= \sum_i p_i \left(f(x_i) - \hat{f}(x) \right)^2$$

$$= \sum_i p_i \left[f(x_i)^2 - 2 f(x_i) \hat{f}(x) + \hat{f}(x)^2 \right]$$

$$= \sum_i p_i f(x_i)^2 - 2 \hat{f}(x) \sum p_i f(x_i) + \sum_i p_i \hat{f}(x)^2$$

$$= \sum_i p_i f(x_i) - 2 \hat{f}(x) \hat{f}(x) + \hat{f}(x)^2 \sum_i p_i$$

adds up to 1

$$= \sum_i p_i f(x_i)^2 - 2 (\hat{f}(x))^2 + \hat{f}(x)^2$$

$$= \underline{\underline{E[f^2(x)] - E[f(x)]^2}}$$

Question 4

$$\mathbb{E}[f] = \int p(x)\, f(x)\, dx$$

Calculate $\mathbb{E}[f]$

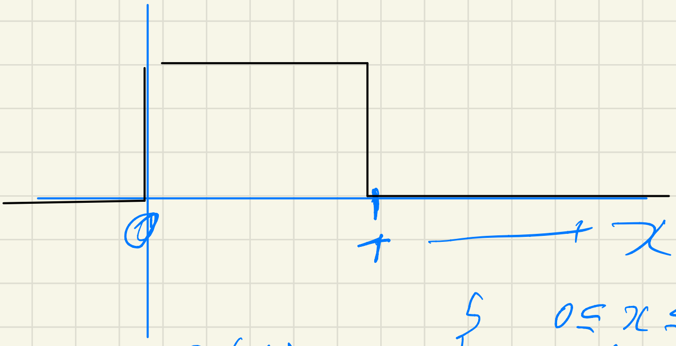
for $p(x) = U(0,1)$

(the uniform distribution between 0 and 1

$$\mathrm{var}[f] = \mathbb{E}[f(x)^2] - \mathbb{E}[f(x)]^2$$

as before

The uniform distribution looks like this:



so $p(x) = \begin{cases} 0 \le x \le 1 : p(x) = 1 \\ \text{elsewhere } p(x) = 0 \end{cases}$

First note that the area under the curve is one.
so $p(x)$ is a proper probability density

Now

$$E[x] = \int_{-\infty}^{\infty} x\,p(x)\,dx$$

$$= \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big|_0^1 = \tfrac{1}{2}$$

as you'd expect

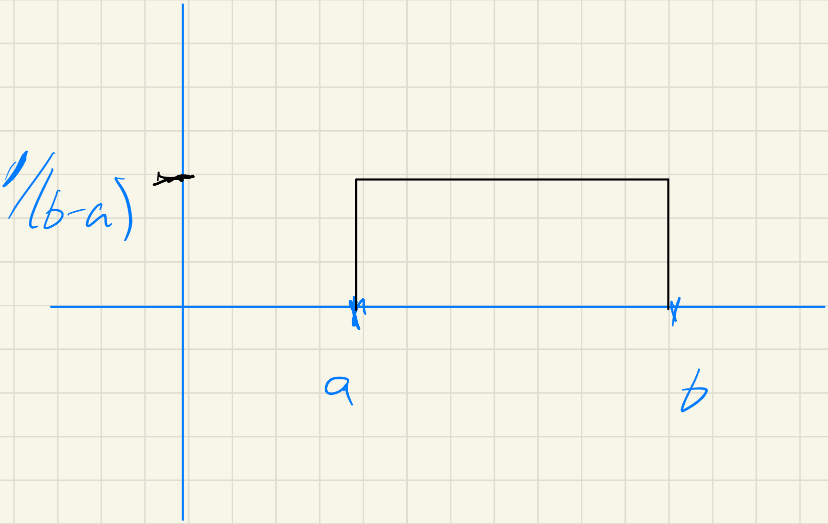
$$\mathrm{var}[x] = \int_{-\infty}^{\infty} p(x)\,(x-\tfrac{1}{2})^2\,dx$$

$$= \int_0^1 (x-\tfrac{1}{2})^2\,dx \qquad u = x - \tfrac{1}{2}$$

$$du = dx$$

$$= \int_{-1/2}^{1/2} u^2\,du = \tfrac{1}{3}u^3\Big|_{-1/2}^{1/2} = \tfrac{1}{3}\left(\tfrac{2}{8}\right) = \underline{\underline{\tfrac{1}{12}}}$$

The general case $U(a,b)$ goes similar but you have to be careful about $p(x)$



In this case $p(x) = \frac{1}{b-a}$

Since the area under the curve must be 1.

So $\mathbb{E}[x] =$

$$\int_{-\infty}^{\infty} P(x)\,dx =$$

$$\frac{1}{(b-a)} \int_a^b x\,dx =$$

$$\frac{1}{(b-a)} \; \tfrac{1}{2} x^2 \Big|_a^b =$$

$$\tfrac{1}{2} \frac{(b^2 - a^2)}{(b-a)} = \tfrac{1}{2} \frac{(b+a)(b-a)}{(b-a)} =$$

$\tfrac{1}{2}(b+a)$ which again is

as you'd expect.

$\mathrm{var}[x] =$

$$\int_{-\infty}^{\infty} p(x)\left(x - \tfrac{1}{2}(a+b)\right)^2 dx$$

$$= \frac{1}{(b-a)} \int_a^b \left(x - \tfrac{1}{2}(a+b)\right)^2 dx$$

write $\mu = \tfrac{1}{2}(a+b)$

$$= \frac{1}{b-a} \int_{a-\mu}^{b-\mu} u^2 \, du$$

$$= \frac{1}{(b-a)} \cdot \frac{1}{3} u^3 \Big|_{a-\mu}^{b-\mu}$$

$$= \frac{1}{(b-a)} \cdot \tfrac{1}{3}\left\{(b-\mu)^3 - (a-\mu)^3\right\}$$

longer calculation

$$= \underline{\underline{\frac{1}{12}(b-a)^2}}$$