COMP3910 – Combinatorial Optimisation

Coursework 1: Modelling; Simplex Method

Deadline: 1700 GMT on 14 March 2025

Award: This piece of summative coursework is worth 10% of your grade

Submission: 1) Write or type your answers inside the boxes of the current document.
2) Scan **pages 1-4** using the MyPrint portal and create a singe pdf-file. (see instructions at https://it.leeds.ac.uk/it?id=kb_article&sysparm_article=KB0012731)

Note that photos made by mobile phones or photo cameras will not be marked. You should use university printers/scanners.

3) Upload one pdf-file containing **pages 1-4** to Gradescope. (see instructions at

https://help.gradescope.com/article/ccbpppziu9-student-submit-work#submitting_a_pdf)

Use of AI: This assessment is **red** category. AI tools cannot be used.

Question 1. **[10 marks]**

A manufacturer needs to produce a health-drink satisfying the following vitamin requirements: 1 ml of the drink should contain at least 6 mg of vitamin A, 7 mg of vitamin B and 10 mg of vitamin C.

There are two concentrates, concentrate X and concentrate Y, that can be mixed with water. The contents of vitamins A, B and C in concentrate X are 15, 14, 50 mg per ml, respectively. The contents of the vitamins in concentrate Y are 15, 28, 20 mg per ml, respectively. Concentrate X costs 0.4 pence per ml, and concentrate Y costs 0.5 pence per ml. We assume that water does not cost anything. See the table below for the summary of the input data.

	Concentrate X	Concentrate Y	Minimum quantity of vitamins required
Vitamin A	15 mg/ml	15 mg/ml	6 mg/ml
Vitamin B	14 mg/ml	28 mg/ml	7 mg/ml
Vitamin C	50 mg/ml	20 mg/ml	10 mg/ml
Cost	0.4 pence/ml	0.5 pence/ml	N/A

Model the problem of finding the quantities of concentrates X and Y and the quantity of water needed to produce 1 bottle of 1000 ml of the drink, so that the cost is minimised.

The variables introduced and their meaning:

x_1 : ml of concentrate X
x_2 : ml of concentrate Y

z : cost of producing one bottle (pence)

The LP model: (*please present the model in the structured way*)

$$min\ z = 0.4x_1 + 0.5x_2$$

$$15x_1 + 15x_2 \geq 6000$$

$$14x_1 + 28x_2 \geq 7000$$

$$50x_1 + 20x_2 \geq 10000$$

$$x_1 + x_2 \leq 1000$$

$$x_1, x_2 \geq 0$$

(do not solve the formulated LP)

Provide an explanation for the constraint which ensures that "1 ml of the drink should contain at least 6 mg of vitamin A"

The constraint: $15x_1 + 15x_2 \geq 6000$ implies that at least 6000mg of the drink is present in a 1000ml bottle, which means that at least 6mg is present in 1ml of the bottle

Question 2 deals with a pair of LP problems A and B, primal and dual. Instructions are on p. **5**. **[20 marks]**

Problem A (primal):

max	$z =$	$20x_1$	$-10x_2$	....	
s. t.		$3x_1$	$-x_2$	$\leq$	8
		$4x_1$	$-3x_2$	$\leq$	4
		$x_1,$	x_2	$\geq$	0

(x1, x2)
(3,1), (5,7)
(1,0), (7,8)
(0,0)

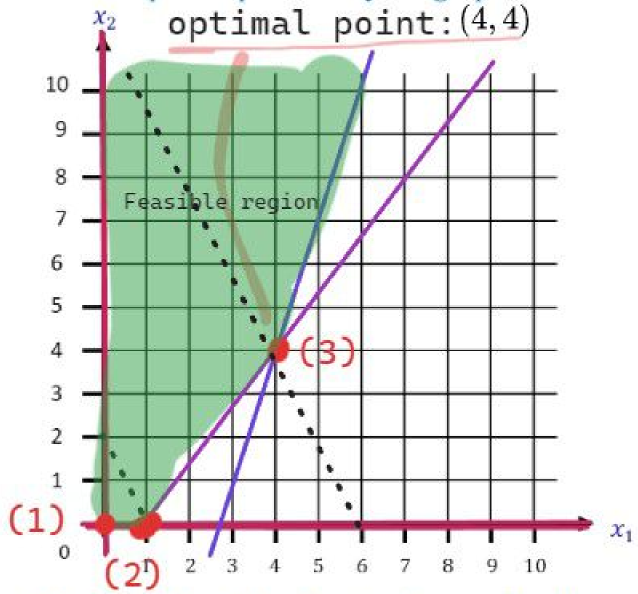
Problem B (dual):

min	$v =$	$8y_1$	$+4y_2$	.....	
s. t.		$3y_1$	$+4y_2$	$\geq$	20
		$-y_1$	$-3y_2$	$\geq$	-10
		$y_1,$	y_2	$\geq$	0

(y1, y2)
(0,5), (4,2)
(1,3), (7,1)

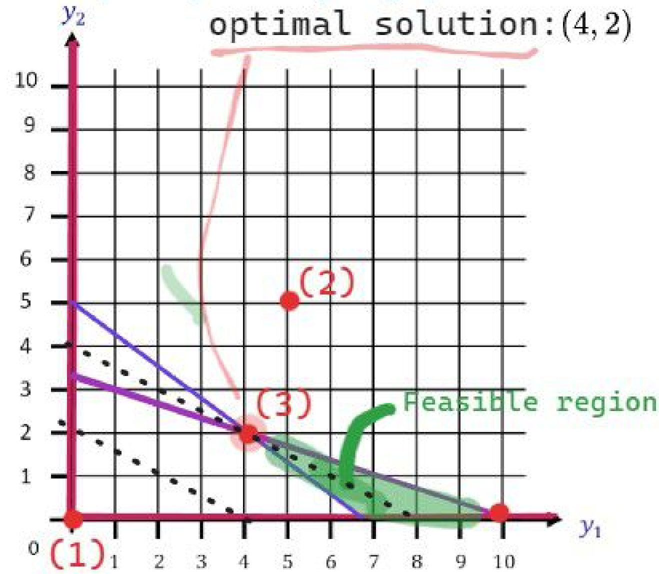
Solve the primal problem by the graphical method[a,c]

optimal point: $(4, 4)$



Solve the primal problem by the graphical method[a,c]

optimal solution: $(4, 2)$



Solve Problem A by the simplex method[b]

$$\max z = 20x_1 - 10x_2$$
$$s.t: \; 3x_1 - x_2 + s_1 = 8$$
$$4x_1 - 3x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

(1) BFS = (0,0)

1	x1	x2	
s1	3	-1	8
s2	$\boxed{4}$	-3	4 ←
z	-20 ↑	10	0

(3) BFS = (4,4)

2	s2	x2	
s1	-3/4	$\boxed{3/4}$	5 ←
x1	1/4	-3/4	1
z	5	-5 ↑	20

(2) BFS = (1,0)

3	s2	s1	
x2	-3/5	4/5	4
x1	-1/5	3/5	4
z	2	4	40

No -ve values in z-row. Solution is optimal. Terminate

The optimal solution is $x_1 = 4$, $x_2 = 4$

The optimal value of the objective function is 40

Solve problem B by the dual simplex method[b]

$(-v)\; 8y_1 + 4y_2 = 0$
$s.t: \; 3y_1 + 4y_2 - t_1 = 20$
$-y_1 - 3y_2 - t_2 = -10$
$y_1, y_2, t_1, t_2 \geq 0$

multiply by -1:
$(-v)\; 8y_1 + 4y_2 = 0$
$-3y_1 - 4y_2 + t_1 = -20$
$y_1 + 3y_2 + t_2 = 10$
$y_1, y_2, t_1, t_2 \geq 0$

(1) BFS = (0,0)

1	y1	y2	
t1	-3	$\boxed{-4}$	-20 ←
t2	1	3	10
-v	8	4 ↑	0

(2) BFS = (5,5)

2	y1	t1	
y2	3/4	-1/4	5
t2	$\boxed{-5/4}$	3/4	-5 ←
-v	5 ↑	1	-20

3	t2	t1	
y2	3/5	1/5	2
y1	-4/5	-3/5	4
-v	-4	4	-40

No -ve values in RHS col. Solution is optimal. Terminate

(3) BFS = (4,2)

The optimal solution is $y_1 = 4$, $y_2 = 2$

The optimal value of the objective function is 40

a,b,c Consider the instructions on p. 5

Question 3 **[10 marks]**

Consider again Problem B from Question 2:

Problem B

min	$v =$	$8y_1$	$+4y_2$		
s.t.		$3y_1$	$+4y_2$	$\geq$	20
		$-y_1$	$-3y_2$	$\geq$	-10
		$y_1,$	y_2	$\geq$	0

It can be solved by the two-phase simplex method.

Derive the first tableau of the 1st phase. Show all your working. *Do not solve the produced LP.*

Deriving the first tableau:

introduce surplus variables y3, y4:

$$max\,(-z) = -8y_1 - 4y_2$$

$$s.t: \quad 3y_1 + 4y_2 - y_3 = 20$$

$$-y_1 - 3y_2 - y_4 = -10$$

$$y_1, y_2, y_3, y_4 \geq 0$$

introduce artificial variables w1, w2:

$$min\,W = w_1 + w_2$$

$$s.t: \quad 3y_1 + 4y_2 - y_3 + w_1 = 20 \quad (1)$$

$$-y_1 - 3y_2 - y_4 + w_2 = -10 \quad (2)$$

$$8y_1 + 4y_2 + (-z) = 0$$

$$y_1, y_2, y_3, y_4, w_1, w_2 \geq 0$$

setup tableau with w1, w2 as basics:

$$W = w_1 + w_2 = (20 - 3y_1 - 4y_2 + y_3) + (-10 + y_1 + 3y_2 + y_4) \quad (1) + (2)$$

$$= 10 - 2y_1 - y_2 + y_3 + y_4$$

$$(-W) = -10 + 2y_1 + y_2 - y_3 - y_4$$

The tableau:

1	y1	y2	y3	y4	
w1	3	4	-1	0	20
w2	-1	-3	0	-1	-10
-z	8	4	0	0	0
-W	-2	-1	1	1	-10

Instructions for **Question 2.**

[a] For the graphical solutions for problems A and B,

- mark the solution region;
- for each line, specify two points it passes through;
- specify coordinates of the corner points of the feasible region;
- present the line for the objective function, together with the equation used to plot it, and indicate the direction in which the line should be moved;
- mark the point which defines the optimum and specify its coordinates.

[b] Use the primal simplex method for problem A in the tableau format and the dual method for problem B, also in the tableau format. Present all tableaux, indicating the choice of the pivot row, pivot column, and the pivot element. For each problem, state the final answer.

[c] After solving the problems analytically, reconsider the solutions found graphically.

- On the plot for the primal problem, mark the solutions obtained at all iterations of the primal simplex method and indicate the order in which they are obtained.
- On the plot for the dual problem, mark the solutions obtained at all iterations of the dual simplex method and indicate the order in which they are obtained.

END