Theorem 2.7. *Let $0 \leq \alpha < 1$ and $0 \leq \gamma \leq \gamma_\alpha$, where $\gamma_\alpha = \tanh^2(\pi\sqrt{1-\alpha}/2\sqrt{2})$. If $p \in \mathfrak{P}_{\mathcal{LP}}$, then $p \in \mathcal{P}_\alpha$, i.e $p(z)$ is a Carathéodory function of order α, in the disc $|z| < \gamma_\alpha$.*

Proof. Since $p \in \mathfrak{P}_{\mathcal{LP}}$, then by definition of subordination and Schwarz Lemma, there exists an analytic function $w(z)$ with $|w(z)| \leq |z| < 1$ and $w(0) = 0$, such that $p(z) = \mathcal{LP}(w(z))$. Suppose $w(z) = Re^{i\theta}$ $(-\pi < \theta \leq \pi)$, then $|w(z)| = R \leq |z| = r < 1$. On applying Lemma 2.1 for $p \in \mathfrak{P}_{\mathcal{LP}}$, we get $\operatorname{Re} p(z) \geq \mathcal{LP}(r)$. Further $p \in \mathfrak{P}_{\mathcal{LP}}$ is Carathéodory of order α $(0 \leq \alpha < 1)$ if $\mathcal{LP}(r) \geq \alpha$, provided $r \leq \gamma_\alpha = \tanh^2(\pi\sqrt{1-\alpha}/2\sqrt{2})$. The function $f_0(z)$ given by (1.2), is the extremal. ∎

Upon replacing $p(z)$ with $zf'(z)/f(z)$ in Theorem 2.7, we deduce the next result.

Corollary 2.8. *Let $0 \leq \alpha < 1$ and $0 \leq \gamma \leq \gamma_\alpha$, where γ_α is as defined in Theorem 2.7. If $f \in \mathcal{F}_{\mathcal{LP}}$, then $f(z)$ is starlike of order α in the disc $|z| < \gamma_\alpha$. This result is sharp.*

Remark 2.9. Put $\alpha = 0$ in Theorem 2.7, we get a sharp $\mathcal{P}-$ radius for the class $\mathfrak{P}_{\mathcal{LP}}$. Infact for the class $\mathcal{F}_{\mathcal{LP}}$, Corollary 2.8 gives sharp radius of starlikeness $\gamma_0 = \tanh^2(\pi/2\sqrt{2})$. Moreover, $r = \gamma_0 < 1$ serves as the sharp radius of univalence for the class $\mathcal{F}_{\mathcal{LP}}$.

Theorem 2.10. *Assume $0 < \alpha \leq 1$, then the sharp $\mathcal{S}^*(1+\alpha z)-$ radius for the class $\mathcal{F}_{\mathcal{LP}}$ is the unique positive root $r_\alpha = \tanh^2(\pi\sqrt{\alpha}/2\sqrt{2})$ of the equation*

$$2\left(\log((1+\sqrt{r})/(1-\sqrt{r}))\right)^2 - \alpha\pi^2 = 0, \tag{2.3}$$

where α is the radius of the disc $\{\omega : |\omega - 1| < \alpha\}$.

Proof. In view of Remark 2.4, for the circle $|z| = r < 1$, we have

$$\max_{|z| \leq r < 1} |\mathcal{LP}(z)| = 1 - \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right)^2 = \mathcal{LP}(r), \tag{2.4}$$

which is a decreasing function. Infact $\mathcal{LP}(r) = 0$ if and only if $r = \tanh^2(\pi/2\sqrt{2}) \approx 0.6469\ldots$. As $f \in \mathcal{F}_{\mathcal{LP}}$, then there exists a Schwarz's function $w(z)$ with $w(0) = 0$, so that

$$\frac{zf'(z)}{f(z)} = \mathcal{LP}(w(z)).$$

Assume $w(z) = Re^{i\theta}$ where $R \leq r < 1$. Now observe that for $0 < \alpha \leq 1$, equation (2.4) yields

$$|\mathcal{LP}(z) - 1| \leq |\mathcal{LP}(R) - 1| \leq |\mathcal{LP}(r) - 1| = |\mathcal{P}_0(r)| \leq \alpha,$$

provided $r \leq \tanh^2(\pi\sqrt{\alpha}/2\sqrt{2}) = r_\alpha$. Further, at $z_0 = r_\alpha$, the function $f_0(z)$ (defined in (1.2)) such that $z{f_0}'(z)/f_0(z) = \mathcal{LP}(z)$, works as the extremal function. ∎

As a consequence of Theorem 2.10, $\mathcal{S}^*(1+\alpha z)-$ radii for some well-known Ma-Minda subclasses of starlike functions, namely, $\mathcal{S}^*_e$, $\mathcal{S}^*_s$, $\mathcal{S}^*_\varrho$, $\mathcal{S}^*_\wp$, $\mathcal{S}^*_\rho$, $\mathcal{S}^*_{SG}$ and $\mathcal{S}^*_{N_e}$ (see Table 1 in Appendix) are stated in Corollary 2.11. Moreover, sharpness of Corollary 2.11 is illustrated by **Fig.** 4.

Corollary 2.11. *Let $f \in \mathcal{A}$ belong to $\mathcal{F}_{\mathcal{LP}}$, then the following radii are sharp for the class $\mathcal{F}_{\mathcal{LP}}$, (see **Fig.** 4)*

(i) The $\mathcal{S}^*_e-$radius is $r_1 = \tanh^2(\lambda\pi)$, where $\lambda = (1/2)\sqrt{(e-1)/2e}$.
(ii) The $\mathcal{S}^*_s-$radius is $r_2 = \tanh^2(\pi/\lambda)$, where $\lambda = 2\sqrt{2\csc 1}$.
(iii) The $\mathcal{S}^*_\varrho-$radius is $r_3 = \tanh^2(\pi\lambda/2)$, where $\lambda = \sin(1/2)$.
(iv) The $\mathcal{S}^*_\wp-$radius is $r_4 = \tanh^2(\pi/2\sqrt{2e})$.
(v) The $\mathcal{S}^*_\rho-$radius is $r_5 = \tanh^2(\pi\sqrt{\lambda}/2)$, where $\lambda = (1/2)\sinh^{-1} 1$.
(vi) The $\mathcal{S}^*_{SG}-$radius is $r_6 = \tanh^2(\lambda\pi/2\sqrt{2})$, where $\lambda = \sqrt{(e-1)/(e+1)}$.
(vii) The $\mathcal{S}^*_{N_e}-$radius is $r_7 = \tanh^2(\pi/2\sqrt{3})$.