Theorem 5 (Theorem 4.2 in [SSZ2]). *Let K be a convex body with infinitely many facets. Then $b_2(K) > 1$.*

Proof. If K has infinitely many facets, then infinitely many of them will satisfy $\mathrm{Isop}(F) > \mathrm{Isop}(K)$, so that Theorem 5 follows from Proposition 1.

Let C be a d-dimensional convex body. Then

$$\mathrm{Isop}(C) = \frac{1}{d}\frac{|\partial C|_{d-1}}{|C|_d} = \frac{1}{d}\frac{|\partial C|_{d-1}}{|C|_d^{d-1/d}}|C|_d^{-1/d} \geq \frac{1}{d}\frac{|\partial B_2^d|_{d-1}}{|B_2^d|_d^{d-1/d}}|C|_d^{-1/d} = \frac{\kappa_d^{1/d}}{|C|_d^{1/d}},$$

where we used the isoperimetric inequality, and where κ_d denotes the volume of the d-dimensional euclidean ball.

It follows that $\mathrm{Isop}(F) \to +\infty$ when $|F|_{n-1} \to 0$. Since a convex body only has finite surface area measure, it follows that if K has infinitely many facets, then all but finitely many of them will satisfy $\mathrm{Isop}(F) > \mathrm{Isop}(K)$. □

We conclude this section with the following question.

Question 1: Let P be a polytope, other than a simplex. Denote $\mathcal{F}_{n-1}(P)$ the set of its facets. Do we necessarily have $\max_{T\in O(n)} \max_{F\in\mathcal{F}_{n-1}(P)} \frac{\mathrm{Isop}(F)}{\mathrm{Isop}(P)} > 1$?

In words : (if P is not a simplex) does there always exist an affine transform T such that $P' = TP$ has at least one facet F' satisfying $\mathrm{Isop}(F') > \mathrm{Isop}(P')$?

If the answer is positive, then it would yield another proof of Theorem 1 (by applying Proposition 1).

4. Appendix

4.1. **Wulff Shape Lemma.** We recall the statement of Alexandrov's variational lemma for mixed volumes, Lemma 1, and then provide a proof.

Lemma 3 (Alexandrov's variational lemma, mixed volume version). *Assume K is a convex body, $supp(S_K) \subset \Omega$, and $f \in \mathcal{C}(\Omega, \mathbb{R})$. Denote $W_t = W(\Omega, h_K + tf)$, $t \in \mathbb{R}$. Denote $V_1(t) = V_n(W_t, K[n-1])$. Then $(t \mapsto V_1(t))$ is differentiable[7] at 0, and :*

$$\left.\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{V_1(t) - V_n(K)}{t} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} f(u)dS_K(u),$$

Let K be a convex body, Ω a closed subset of $\mathbb{S}^{n-1}$ *determining*[8] K, meaning $K = \bigcap_{u\in\Omega} H^-(u, h_K(u))$, and let f be a continuous [9] function defined on Ω. Recall we denote $W_t = W(\Omega, h_K + tf)$ the associated Wulff-shape perturbations : W_t is a convex body if $t > t_0$.

Following Alexandrov's notations (see [Al1]), we denote $V_k(t) = V_n(W_t[k], K[n-k])$ (in particular $V_0(t) = V_n(K)$ does not depend on t). We prove $\frac{V_1(t)-V_0(t)}{t} \to \frac{1}{n}\int_\Omega f(u)dS_K(u)$ separately for $t \to 0^+$ then for $t \to 0^-$. For sake of clarity, we may omit some dependence from the notations : V_1 means $V_1(t)$, and similarly V_k stands for $V_k(t)$.

Proof. For all t, note that (by definition of W_t) $V_1 - V_0 = \frac{1}{n}\int_\Omega (h_{W_t} - h_K)(u)dS_K(u) \leq \frac{t}{n}\int_\Omega f(u)dS_K(u)$. In particular, $\limsup_{t\to 0^+} \frac{V_1(t)-V_0}{t} \leq \frac{1}{n}\int_\Omega f(u)dS_K(u)$.

[7] on both sides

[8] we are following Schneider's terminology [Sch]

[9] (f measurable and bounded on Ω would do as well)