ELLIPTIC CURVES WITH POSITIVE RANK AND NO INTEGRAL POINTS

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Abstract [1]

We study the family of elliptic curves $E_{D'} : y^2 = x^3 + 16D'$, where $D' = -3D$ and $D < -4$ is a negative squarefree integer, with $3 \nmid D$, that satisfies two simple congruence conditions. By assuming finiteness of their Tate-Shafarevich group, we show that these curves must have odd rank. We then focus on the subfamily of those elliptic curves $E_{D'}$ that correspond to the fundamental discriminants D' with the property that the rank $r_3(D)$ of the 3-part of the ideal class group of $\mathbb{Q}(\sqrt{D})$ is not equal to the rank $r_3(D')$ of the 3-part of the ideal class group of $\mathbb{Q}(\sqrt{D'})$, and therefore $r_3(D) = r_3(D') + 1$ by Scholz's Reflection Theorem. By using the connection between the points of $E_{D'}(\mathbb{Q})$ and the so-called 3-virtual units of $\mathbb{Q}(\sqrt{D'})$, we show that the elliptic curves of this subfamily, even though they have non-trivial rank, they cannot have any integral points. We also discuss the case of positive squarefree integer $D > 4$ and derive a result for this case as well.

AMS Mathematics Subject Classification: Primary: 11R29, 11G05.

Keywords: Elliptic Curves, Selmer group, Rank, Ideal Class Group, Integral Points.

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1. Introduction

As it is well known, if an elliptic curve has non-trivial rank, then it has infinitely many rational points. However, as it was proved by Siegel [13], an elliptic curve can have only finitely many integral points. Siegel's theorem is not effective and given an elliptic curve, it is in general hard to determine how many integral points this curve might have, or if it has any integral points at all. Other techniques that have been developed over the years, in order to study this problem, concentrate on finding bounds for the number of these integral points. Some of these bounds depend for example on the rank of the elliptic curve, the number of distinct prime divisors of the discriminant, or the primes of bad multiplicative reduction (see for example [2], [10], [14] and references therein).

In this paper we study the family of elliptic curves of j-invariant zero $E_{D'} : y^2 = x^3 + 16D'$, where $D' = -3D$ is a positive squarefree integer that satisfies two simple congruence conditions. The basic definitions and important properties of these curves are given in Section 2. In Section 3, by assuming finiteness of their Tate-Shafarevich group, we show that these curves must have odd rank. In Section 4, we focus on the subfamily of those elliptic curves $E_{D'}$ that correspond to the fundamental discriminants D' with the property that the rank $r_3(D)$ of the 3-part of the ideal class group

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