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Degree of the 3-secant variety

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Abstract

In this paper, we present a formula for the degree of the 3-secant variety of a nonsingular projective variety embedded by a 5-very ample line bundle. The formula is provided in terms of Segre classes of the tangent bundle of a given variety. We use the generalized version of double point formula to reduce the calculation into the case of the 2-secant variety. Due to the singularity of the 2-secant variety, we use secant bundle as a nonsingular birational model and compute multiplications of desired algebraic cycles.

1 Introduction

Let X be a projective variety over an algebraically closed field of characteristic zero. The r-secant variety, denoted $\sigma_r(X)$, is defined as the Zariski closure of the union of r-secant hyperplanes to X in projective space $\mathbb{P}^N$. For example, the 3-secant variety, denoted $\sigma_3(X)$, is the Zariski closure of the union of all secant planes to X in $\mathbb{P}^N$. The study of secant varieties is a classic topic in algebraic geometry, with an emphasis on determining defining equations and syzygies as well as understanding the singularities of these varieties.

One traditional topic of study is the specification of the double points of linear projections of a given variety. Let $X \subset \mathbb{P}^N$ be an n-dimensional nonsingular projective variety. Let δ be the number of double point of the image of X under generic linear projection $X \to \mathbb{P}^{2n}$. In [7, Corollary 8.2.9] it is shown that δ can be represented in terms of the Segre classes of the tangent bundle T_X as:

$$2\delta = (\deg X)^2 - \sum_{k\geq 0} \binom{2n+1}{k} \deg s_k(T_X).$$

Furthermore, if the embedding of X into the projective space is 3-very ample, the double point formula provides a degree formula for the 2-secant variety $\sigma_2(X)$ for X. The double point formula is one of the corollary of [7, Theorem 8.2.8] and more generally, [7, Theorem 2.1.15]. Here we give a full statement :

Refined Bezout's theorem [7, Theorem 2.1.15] Let $V \subset \mathbb{P}^N$ be an equi-dimensional closed subscheme with $\mathcal{L} := \mathcal{O}_V(1)$. Let $\sigma_0, \cdots, \sigma_d$ be global section of $\mathcal{L}$ whose zero locus is W. Let $v^i(\underline{\sigma}, V)$ be the v-cycle in the Vogel's intersection theory and $\sigma : V \dashrightarrow \mathbb{P}^d$ be a rational map defined by global section $\sigma_0, \cdots, \sigma_d$. Denote $\deg(\Gamma/\sigma(\Gamma))$ be the degree of the restriction of σ on Γ where Γ is an irreducible component of V.

$$\deg V = \sum_i \deg v^i(\underline{\sigma}, V) + \sum_{\Gamma\subset V} \deg(\Gamma/\sigma(\Gamma)) \deg \sigma(\Gamma).$$

We can represent Vogel's v-cycles in terms of the first Chern class $c_1(\mathcal{L})$ and the total Segre class of a normal cone $C_W V$ by [7, Theorem 2.4.2, Corollary 2.4.7].

In [10, Section 4] the author presents an interesting calculation of the top Segre classes of the tautological bundles associated to line bundles on Hilbert schemes of n