where we defined

$$R = 1 - i f f', \quad \varphi = \int \frac{d\eta}{f^2(\eta)}. \tag{44}$$

Note that these actions in terms of the ladder operators are the same if we choose the ξ_k or the π_k representation. Therefore, since $\hat{\xi}_k \psi_0 = e^{i\varphi} f \psi_1/\sqrt{2}$, we have

$$\begin{aligned} \langle 0|\hat{\xi}_k^2|0\rangle &= \int_{-\infty}^{+\infty} d\xi_k \, \psi_0^* \hat{\xi}_k^2 \psi_0 = \int_{-\infty}^{+\infty} d\xi_k \left|\hat{\xi}_k \psi_0\right|^2 = \\ &= \int_{-\infty}^{+\infty} d\xi_k \left|\frac{f\,\psi_1\, e^{i\varphi}}{\sqrt{2}}\right|^2 = \frac{f^2(\eta)}{2}. \end{aligned} \tag{45}$$

To calculate the spectrum we just need to find the expression of $f(\eta)$.

The solution to the auxiliary equation (38) can be constructed from the solutions f_1 and f_2 of the corresponding homogeneous equation:

$$f'' + \omega_k^2 f = 0, \tag{46}$$

$$f_1(\eta) = \frac{1}{\sqrt{k}} \left(\cos(k\eta) - \frac{\sin(k\eta)}{k\eta} \right), \tag{47}$$

$$f_2(\eta) = \frac{1}{\sqrt{k}} \left(\frac{\cos(k\eta)}{k\eta} + \sin(k\eta) \right). \tag{48}$$

Then the function f takes the form

$$f(\eta) = \frac{1}{\mathcal{W}} \left(A_1^2 f_1^2 + A_2^2 f_2^2 + 2 f_1 f_2 \sqrt{A_1^2 A_2^2 - \mathcal{W}^2} \right)^{\frac{1}{2}}, \tag{49}$$

where A_1, A_2 are η-independent constants and $\mathcal{W}$ is the Wronskian:

$$\mathcal{W} = f_1 f_2' - f_1' f_2 = 1. \tag{50}$$

The two constants must be set through initial conditions: we require that at the beginning of inflation, when all the modes of astrophysical interest today have a physical wavelength smaller than the Hubble radius $\frac{k}{aH} \gg 1$, the expansion of the Universe does not affect perturbations and therefore each mode behaves as a harmonic oscillator with constant frequency. Hence we impose that modes asymptotically approach Minkowskian quantum harmonic oscillators with frequency k:

$$\lim_{-k\eta \to \infty} f(\eta) = \frac{1}{\sqrt{k}}; \tag{51}$$

this is satisfied by setting $A_1^2 = A_2^2 = 1$, so that the expression for f is

$$f(\eta) = \sqrt{\frac{1 + k^2\eta^2}{k^3\eta^2}}. \tag{52}$$

Then, inserting this expression into the Spectrum, taking the large scale limit $-k\eta \ll 1$ and remembering the dependence of η on the scale factor (32), the final expression for the spectrum is

$$\begin{aligned} \mathcal{P}^{\text{std}}(k) &= \frac{k^3}{4\pi^2} \frac{f^2(\eta)}{2a^2\epsilon}\bigg|_{-k\eta\ll 1} = \\ &= \frac{H_s^2}{8\pi^2\epsilon}(1 + k^2\eta^2)\bigg|_{-k\eta\ll 1} = \frac{H_s^2}{8\pi^2\epsilon}. \end{aligned} \tag{53}$$

We have obtained the usual flat, k-independent Spectrum [2].

B. Modified Power Spectrum

Here we will derive the Power Spectrum that arises from the Fourier-transformed Mukhanov-Sasaki variable ξ_k obeying the modified algebra (12):

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i\,(1 - \mu^2 \hat{\pi}_k^2). \tag{54}$$

Due to the modified commutator depending on π_k, it will be easier to work in the momentum polarization.

By using arguments similar to those in [22, 28], if we impose that in the momentum polarization the scalar field operator acts simply differentially, we can find the action of the multiplicative momentum operator $\hat{\pi}_k \psi(\pi_k) = g(\pi_k)\psi(\pi_k)$ as

$$\frac{\mathrm{d}g}{\mathrm{d}\pi_k} = 1 - \mu^2 g^2, \quad \frac{\operatorname{arctanh}(\mu\, g)}{\mu} = \pi_k; \tag{55}$$

therefore the action of the fundamental operators is

$$\hat{\pi}_k\, \psi(\pi_k) = \frac{\tanh(\mu\,\pi_k)}{\mu}\, \psi(\pi_k), \tag{56}$$

$$\hat{\xi}_k\, \psi(\pi_k) = i\, \frac{\mathrm{d}}{\mathrm{d}\pi_k} \psi(\pi_k). \tag{57}$$

Given the action (56) for the modified operator $\hat{\pi}_k$, the Hamiltonian $\mathcal{H}_k$ for a single Fourier mode yields a time-dependent Schrödinger equation with a modified kinetic term:

$$i\, \frac{\partial}{\partial\eta} \psi(\eta, \pi_k) = \frac{1}{2} \left(\frac{\tanh^2(\mu\,\pi_k)}{\mu^2} - \omega_k^2(\eta)\, \frac{\partial^2}{\partial \pi_k^2} \right) \psi(\eta, \pi_k). \tag{58}$$

This partial differential equation (PDE) is quite difficult to solve, so we perform an expansion in powers of μ^2:

$$\frac{\tanh^2(\mu\,\pi_k)}{\mu^2} = \pi_k^2 - \mu^2\, \frac{2\pi_k^4}{3} + \mathcal{O}(\mu^4), \tag{59}$$

$$\psi(\eta, \pi_k) = \psi^0(\eta, \pi_k) + \mu^2\, \psi^1(\eta, \pi_k) + \mathcal{O}(\mu^4). \tag{60}$$