With this lower bound, one can easily verify the truth of Conjecture C (and Conjecture B) for small values of ℓ. The following results are direct consequences of Theorem 1. For the reader's convenience, we give a proof.

Corollary 2.3. *For $1 \leq \ell \leq 4$, we have $N_g > 0$ for every generator g of $\mathbb{F}_q^\times$. Moreover, there exists a point $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ such that $xy \neq 0$ and hence Conjecture C holds for the case where $\ell_0 = [\mathbb{F}_p^\times : H] \leq 4$.*

Proof. As it's easy to deduce the conclusion if $\ell = 1$, we leave the verification of this case to the reader. Let's first consider the case where $\ell = 2$. Notice that in this case $p > 2$ and $\mathfrak{g}_\ell = 0$. Therefore, $\widetilde{N_g} = q+1$ by Theorem 1. It's not hard to verify that

$$N_g = \begin{cases} \widetilde{N_g} & \text{if } q \equiv 1 \pmod 4 \\ \widetilde{N_g} - 2 & \text{if } q \equiv 3 \pmod 4 \end{cases}.$$

Therefore, $N_g = q+1$ if $q \equiv 1 \pmod 4$ and $N_g = q-1$ if $q \equiv 3 \pmod 4$. In either case, we clearly have $N_g > 0$. It remains to show that there exists a point $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ such that $xy \neq 0$. Observe that there are at most four points in $\mathscr{C}_g(\mathbb{F}_q)$ with either $x = 0$ or $y = 0$. In the case where $q \equiv 1 \pmod 4$ we have $N_g = q+1 \geq 6$. It remains to look at the case where $q \equiv 3 \pmod 4$. Since $\ell = 2$ is a proper divisor of $q-1$ by assumption, we see that $q \geq 7$ and we also have $N_g = q-1 \geq 6$. Now, it's clear that there's a point $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ such that $xy \neq 0$ since $N_g > 4$ in both cases.

Next, we consider the cases where $\ell = 3$ and 4. Since $\ell > 2$, we have that $N_g = \widetilde{N_g}$ by Lemma 2.1. Suppose that $\ell = 3$. In this case $\widetilde{\mathscr{C}_g}$ is of genus one. Then the Hasse-Weil bound gives that

$$N_g \geq (q+1) - 2\sqrt{q} = (\sqrt{q}-1)^2 > 0.$$

Therefore, $N_g > 0$ for any generator g of $\mathbb{F}_q^\times$ in this case. Suppose that there exists a solution (x,y) to Equation (3) such that either $x = 0$ or $y = 0$ for $\ell = 3$. Then we get that either g or g^2 is a cube in $\mathbb{F}_q^\times$. This implies that the order of gL in the group $\mathbb{F}_q^\times/L$ divides 2 which is absurd since $|\mathbb{F}_q^\times/L| = 3$. Therefore, any $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ must satisfy $xy \neq 0$ as desired.

Assume that $\ell = 4$. A direct computation shows that for $q > 49$, we have that $N_g > 8$. Since there are at most eight solutions to Equation (3) such that either $x = 0$ or $y = 0$ for $\ell = 4$, we see that for $q > 49$ there exists $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ such that $xy \neq 0$ as asserted. It remains to check prime power numbers q satisfying $q \leq 49$ such that 4 is a proper divisor of $q-1$. Hence, we are left with eight cases where $q = 9, 13, 17, 25, 29, 37, 41, 49$ to verify. Note that if $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ with $xy \neq 0$ then $(x^{-1}, x^{-1}y) \in \mathscr{C}_{g^{-1}}(\mathbb{F}_q)$. It follows that $\mathscr{C}_g(\mathbb{F}_q)$ contains a point whose coordinates are nonzero if and only if $\mathscr{C}_{g^{-1}}(\mathbb{F}_q)$ has this property as well. Also, for any generator g' of $\mathbb{F}_q^\times$ we have $g'L \in \{gL, g^{-1}L\}$ in the case where $|\mathbb{F}_q^\times/L| = 4$. It follows that $\mathscr{C}_g(\mathbb{F}_q)$ contains a point whose coordinates are nonzero if and only if $\mathscr{C}_{g'}(\mathbb{F}_q)$ has this property as well. Hence, it suffices to show that $\mathscr{C}_g(\mathbb{F}_q)$ containing a point with nonzero coordinates for just one generator g of $\mathbb{F}_q^\times$. We