It follows from Lemma 2 that $h_{L_t}(u_0) - h_K(u_0) > t(1-\epsilon)f(u_0) = t(1-\epsilon)$, for $0 < t < t_0(\epsilon)$. Also, $\sigma(\{u_0\}) = S(M, K[n-2], u_0) = V_{n-1}(M^{u_0}, K^{u_0}[n-2]) = V_{n-1}(B_2^{n-1}, F[n-2]) = \frac{|\partial F|_{n-2}}{n-1}$.

Therefore, when $0 < t < t_0(\epsilon)$:

$$V_n(L_t, M, K[n-2]) - V_n(M, K[n-1]) \geq (1-\epsilon)\frac{t}{n(n-1)}|\partial F|_{n-2}.$$

On the other hand, for any $t > 0$:

$$\begin{aligned} V_n(L_t, K[n-1]) - V_n(K) &= \frac{1}{n}\int (h_{L_t} - h_K)(u)dS_K(u) \\ &\leq \frac{t}{n}\int f(u)dS_K(u) < (1+\epsilon)\frac{t}{n}|F|_{n-1}. \end{aligned}$$

where the last inequality is by the choice of f. Also, by monotonicity of mixed volumes :

$$V_n(M, K[n-1]) \leq V_n(B_2^n, K[n-1]) = \frac{1}{n}|\partial K|_{n-1}.$$

We may now conclude by simple computations. For ease of notations, set $a_2 = V_n(L_t, M, K[n-2])$, $a_0 = V_n(K)$, $a_t = V_n(L_t, L[n-1])$ and $a_m = V_n(M, K[n-1])$. We shall show that $F_K(L_t, M) < 0$, which rewrites as $(a_t - a_0)a_m - (a_2 - a_m)a_0 < 0$.

Denote $b_1 = (a_2 - a_m)a_0$ and $b_2 = (a_t - a_0)a_m$, so that $F_K(L_t, M) = b_2 - b_1$. The above lower and upper bounds give us : $b_1 \geq (1-\epsilon)\frac{t}{n(n-1)}|\partial F|_{n-2}|K|_n$ and $b_2 \leq (1+\epsilon)\frac{t}{n^2}|\partial K|_{n-1}|F|_{n-1}$.

It follows that :

$$\begin{aligned} F_K(L_t, M) &\leq \epsilon\frac{t}{n}\left(\frac{|\partial F|_{n-2}|K|_n}{n-1} + \frac{|F|_{n-1}|\partial K|_{n-1}}{n}\right) - \frac{t}{n}\left(\frac{|\partial F|_{n-2}|K|_n}{n-1} - \frac{|F|_{n-1}|\partial K|_{n-1}}{n}\right) \\ &\leq \frac{t}{n}(\epsilon c - c_0) < 0. \end{aligned}$$

□

Recall that $F_K \geq 0$ is an affine-invariant property. On the other hand, the quantity

$$\sup_F \frac{\mathrm{Isop}(F)}{\mathrm{Isop}(K)} = \sup_F \frac{n|\partial F|_{n-2}|K|_n}{(n-1)|F|_{n-1}|\partial K|_{n-1}},$$

where the supremum is over the facets, is not affine-invariant. Thus, the above proposition immediately implies the following one.

Proposition 2. *Let K be a convex body such that one of its affine pairs $K' = TK$ has a facet F' satisfying :*

$$\frac{|\partial F'|_{n-2}}{(n-1)|F'|_{n-1}} > \frac{|\partial K'|_{n-1}}{n|K'|_n}.$$

Then $b_2(K) > 1$.

This yields that, if $\max_T \sup_F \frac{\mathrm{Isop}(TF)}{\mathrm{Isop}(TK)} > 1$, where the max is over $T \in O(n)$, then $b_2(K) > 1$. Let us list specific examples. Throughout, we will denote $|\cdot|$ for the Lebesgue measure (either of dimension n, $n-1$, or $n-2$).

a- The unit cube has volume 1, and so does any of its facet. Thus $|\partial C_n| = 2n$, and $\mathrm{Isop}(C_n) = \frac{|\partial C_n|}{n|C_n|} = 2$. Since each of its facets is a unit cube itself (of dimension $n-1$), they also satisfy $\mathrm{Isop}(F) = 2$. Therefore $\max_F \frac{\mathrm{Isop}(F)}{\mathrm{Isop}(C)} = 1$. But choose for T the affine transform such that $Te_i = e_i$ for all $i \neq 1$, and $Te_1 = 2e_1$. Let $C' = TC$ be the resulting box.
Then, $\frac{|\partial F|}{(n-1)|F|} = 2$ for the facet with outer normal e_1, while $\mathrm{Isop}(C') = 2 - \frac{1}{n}$. It follows that $\frac{\mathrm{Isop}(F)}{\mathrm{Isop}(C')} > 1$, and hence $b_2(C) = b_2(C') > 1$.