Z_2 as the inverse image of X. Since the codimension of Z_2 in the secant bundle is 1, we can treat Z_2 as an effective divisor, which aids in computation. In Section 4, we derive the main theorem which provides the degree formula for the 3-secant variety by using the refined Bezout's theorem and the total Segre class computed in Section 3. Additionally, we compute the multiplicity of $\sigma_2(X)$ along X. In Section 5, we apply the main theorem to cases of curves and surfaces with explicit calculations.

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2 Generalized double point formula

The generalized version of double point formula is one of a useful application of the refined Bezout's theorem that is a fundamental theorem in intersection theory. The double point formula allows us to compute the degree of the intersection of two subvarieties of a projective space. In this paper, we will use it as a starting point to understand the complexity of the 3-secant variety in terms of the 2-secant variety.

To set the stage for the generalized version of double point formula, we first introduce some notation and conventions. Let X be an algebraic scheme over a field k and $S^{\cdot} := \bigoplus_{\nu} S^{\nu}$ be a sheaf of graded $\mathcal{O}_X$-algebras. We assume that the map $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^{\cdot}$ is generated by S^1. We define the cone of $S^{\cdot}$ as the relative spectrum $C := \mathbf{Spec}(S^{\cdot})$ and the projection $C \to X$. The projective cone of $S^{\cdot}$ is defined as the relative projective spectrum $P(C) := \mathbf{Proj}(S^{\cdot})$ with the projection $P(C) \to X$. We also define $P(C \oplus 1) := \mathbf{Proj}(S^{\cdot}[z])$ as the projective completion of C, with projection $q : P(C \oplus 1) \to X$, and let $\mathcal{O}(1)$ be the tautological line bundle on $P(C \oplus 1)$.

Remark. Throughout this paper, we use the convention for projective bundle as in [8, Appendix B.5.5].

With these conventions in place, we can now define the Segre class of a cone.

Definition 1. (cf. [8, Chapther 4])
For a variety V, we denote the algebraic cycle corresponding to V as $[V]$. The *Segre class* of C is the class in $A_*(X)$ defined by

$$s(C) := q_*(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [P(C \oplus 1)]).$$

There are two useful examples of total Segre classes of cones: when the cone is given by a vector bundle, in which case the Segre class can be interpreted as the inverse of the total Chern class, and when the cone is the normal cone of a closed embedding, in which case the Segre class is called the Segre class of the closed immersion and is denoted $s(X, Y)$.

The codimension i part of the Segre class, $s^i(C)$, of a cone C is defined in [7] as

$$s^i(C) = q_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)]),$$

where $i \geq 0$. The projective completion of C is denoted as $P(C \oplus 1)$ in [8] and as $\mathbb{P}(C \oplus 1)$ in [7]. These two notations refer to the same concept. Alternatively, $s^i(C)$ can also be represented as $p_*(c_1(\mathcal{O}(1))^{i-1} \cap [\mathbb{P}C])$, where $p : \mathbb{P}C \to X$ is the projective cone of C.