We next prove that $\lim_{t\to 0^+} \frac{(h_{W_t}-h_K)(u)}{t} = f(u)$, for S_K-almost every u (the proof on the left of 0 is similar). Let $l_d(u) = \lim_{t\to 0^+} \frac{(h_{W_t}-h_K)(u)}{t}$, which is well-defined for all $u \in \mathbb{S}^{n-1}$, by concavity. Note that (by definition of W_t), $l_d(u) \le f(u)$, for all u. Let $U_\epsilon = \{u \in \mathbb{S}^{n-1} : l_d(u) < f(u) - \epsilon\}$. It is enough to show that $S_K(U_\epsilon) = 0$ (for arbitrary $\epsilon > 0$).

By Fatou's lemma,

$$\begin{aligned}\limsup_{t\to 0^+} \int_{U_\epsilon} \frac{(h_{W_t}-h_K)(u)}{t} dS_K(u) &\le \int_{U_\epsilon} \limsup_{t\to 0^+} \frac{(h_{W_t}-h_K)(u)}{t} dS_K(u)\\ &= \int_{U_\epsilon} l_d(u) dS_K(u)\\ &\le \int_{U_\epsilon} f(u) dS_K(u) - \epsilon S_K(U_\epsilon).\end{aligned}$$

Therefore :

$$\lim_{t\to 0^+} \frac{V_1(t) - V_0}{t} \le \frac{1}{n}\int_\Omega f(u) dS_K(u) - \frac{\epsilon}{n} S_K(U_\epsilon).$$

It follows from lemma 1 that $S_K(U_\epsilon) = 0$.

Similarly, let $l_g(u) = \lim_{t\to 0^-} \frac{(h_{W_t}-h_K)(u)}{t}$, $u \in \mathbb{S}^{n-1}$, so that $l_g(u) \ge f(u)$ for all $u \in \Omega$, and let $V_\epsilon = \{u \in \mathbb{S}^{n-1} : l_g(u) > f(u) + \epsilon\}$. Fix $x_0 \in int(K)$, and let $K' = K - x_0$. If $|t|$ is small enough, then a translate of $(1 - C|t|)K'$ is contained in W_t, with $C = (\max_u |f|)/(\min_u h_{K'}) > 0$.

In other words, $\frac{(h_{W_t}-h_K)(u)}{t} \le C h_{K'}(u)$ for all $u \in \mathbb{S}^{n-1}$, for any $t < 0$ (with $|t|$ small enough). Hence, by dominated convergence,

$$\begin{aligned}\lim_{t\to 0^-} \frac{V_1 - V_0}{t} &= \frac{1}{n}\int_{V_\epsilon} l_g(u) dS_K(u) + \frac{1}{n}\int_{\Omega\setminus V_\epsilon} l_g(u) dS_K(u)\\ &\ge \frac{1}{n}\int_\Omega f(u) dS_K(u) + \frac{\epsilon}{n} S_K(V_\epsilon).\end{aligned}$$

So that $S_K(V_\epsilon) = 0$, by lemma 1. It follows that $l_g(u) = f(u)$ for S_K-a.e. $u \in \Omega$. □

4.2. **Examples when Proposition 1 applies : computations.** We review one of the examples listed after Proposition 2.

Let M be half a Euclidean ball : $M = B_2^n \cap H_+$, with $H_+ = \{x \in \mathbb{R}^n : x_n \ge 0\}$. Then

$$\mathrm{Isop}(M) = \frac{|\partial M|}{n|M|} = \frac{2|\partial M|}{n\kappa_n} = \frac{n\kappa_n}{n\kappa_n} + \frac{2\kappa_{n-1}}{n\kappa_n} = 1 + \frac{1}{nW_n} > 1.$$

where $W_n = \int_0^{\pi/2} (\cos(\phi))^n d\phi$ (recall that $\kappa_n = 2\kappa_{n-1}W_n$).

Let $T = T_a$ be the linear map such that $Te_i = e_i$ when $i \le n-1$, and $Te_n = ae_n$. Then $TB_2^n = \mathcal{E}_a$ is an ellipsoid of volume $a\kappa_n$. Moreover one may compute its surface area :

$$|\partial\mathcal{E}_a| = 2(n-1)\kappa_{n-1}\int_0^{\pi/2} R_\phi^{n-1}(\cos(\phi))^{n-2} d\phi = 2(n-1)\kappa_{n-1}\int_0^{\pi/2} r_\phi^{n-1}\frac{d\phi}{\cos(\phi)}.$$

The equation of the surface $\partial\mathcal{E}_a$ is $r^2 + \frac{h^2}{a^2} = 1$, where $h = x_n$, and $r^2 = x_1^2 + ... + x_{n-1}^2$. The surface can be layered according to $r = R\cos(\phi) \in [0,1]$, and one may write r as $r =: \cos(\psi)$ for some $\psi \in [0, \frac{\pi}{2}]$. Then $\tan^2(\psi) = \frac{1-r^2}{r^2} = \frac{1}{a^2}\frac{h^2}{r^2} = \frac{1}{a^2}\tan^2(\phi)$, therefore (differentiating) $(1 + \tan^2(\psi))d\psi = \frac{1}{a}(1 + \tan^2(\phi))d\phi = \frac{1}{a}(1 + a^2\tan^2(\psi))d\phi$, i.e. $d\phi = a\frac{1+\tan^2(\psi)}{1+a^2\tan^2(\psi)} d\psi = a\frac{1}{1+(a^2-1)\sin^2(\psi)} d\psi$.

Also, $\cos(\phi) = \frac{r}{R} = \frac{r}{(a^2-(a^2-1)r^2)^{1/2}} = \frac{\cos(\psi)}{(1+(a^2-1)\sin^2(\psi))^{1/2}}$.