

Since j is an affine morphism, there are no higher direct images and we obtain followings :

$$\mathrm{ch}E_{\mathcal{L}} = \pi_{2*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2}) \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X)$$

and

$$\mathrm{ch}j_*\mathcal{O}_{Z_2} = j_*((\mathrm{td}\mathcal{N})^{-1})$$

by applying the Grothendieck-Riemann-Roch theorem. Note that $\pi_2 \circ j = \rho$.

By the projection formula, we have:

$$\begin{aligned} \mathrm{ch}E_{\mathcal{L}} &= \pi_{2*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2}) \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X) \\ &= \pi_{2*}(j_*\mathrm{ch}(\mathrm{td}\mathcal{N})^{-1} \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X) \\ &= (\pi_{2*} \circ j_*)(\mathrm{ch}(\mathrm{td}\mathcal{N})^{-1} j^* \pi_1^* \mathrm{ch}\mathcal{L} j^* \pi_1^* \mathrm{td}T_X) \\ &= \rho_*((\mathrm{td}\mathcal{N})^{-1} q^* \mathrm{ch}\mathcal{L} q^* \mathrm{td}T_X). \end{aligned}$$

Consider the sheaf sequence

$$0 \rightarrow \mathcal{N}^* \rightarrow q^*\Omega_X \rightarrow \mathcal{O}_E(-E) \rightarrow 0$$

as given in [6 Lemma 2.1]. (Note that the morphism q in proposition 3.1 and in [6] are different morphisms.) Taking dual of this sequence, we obtain :

$$0 \rightarrow q^*T_X \rightarrow \mathcal{N} \rightarrow \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \rightarrow 0.$$

Therefore we have

$$(\mathrm{td}\mathcal{N})^{-1} = (q^* \mathrm{td} T_X)^{-1} \cdot (\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}))^{-1}.$$

To evaluate $\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2})$, we consider the exact sequence :

$$0 \rightarrow \mathcal{O}_{Z_2}(-2E) \rightarrow \mathcal{O}_{Z_2}(-E) \rightarrow \mathcal{O}_E(-E) \rightarrow 0.$$

Taking the dual of this sequence, we have :

$$0 \rightarrow \mathcal{O}_{Z_2}(E) \rightarrow \mathcal{O}_{Z_2}(2E) \rightarrow \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \rightarrow 0.$$

Therefore, we obtain

$$\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) = \frac{2E}{1 - e^{-2E}} / \frac{E}{1 - e^{-E}} = \frac{2}{1 + e^{-E}}$$

and

$$\begin{aligned} \mathrm{ch}E_{\mathcal{L}} &= \rho_* \left(q^* \mathrm{ch}\mathcal{L} \cdot \frac{1 + e^{-E}}{2} \right) \\ &= \rho_* \left(\left(1 + \eta^* h_1 + \frac{1}{2!} \eta^* h_1^2 + \frac{1}{3!} \eta^* h_1^3 + \cdots \right) \cdot \left(1 - \frac{1}{2}E + \frac{1}{4}E^2 - \frac{1}{12}E^3 + \cdots \right) \right) \\ &= \rho_* \left(1 + \left(\eta^* h_1 - \frac{1}{2}E \right) + \left(\frac{1}{2} \eta^* h_1^2 - \frac{1}{2} \eta^* h_1 \cdot E + \frac{1}{4}E^2 \right) + \cdots \right) \\ &= 2 + (H - \delta) + \left(\frac{1}{2} \rho_* \eta^* h_1^2 - \frac{1}{2} \rho_* (\eta^* h_1 \cdot E) + \frac{1}{4} \rho_* E^2 \right) + \cdots. \end{aligned}$$

So we obtain $c_1(E_{\mathcal{L}}) = H - \delta$. □