danger of confusion. We fix a character χ of order ℓ. Then we have

$$N_g = q + \sum_{1\le j,k\le \ell-1} \chi^j(-g^{-2})\chi^k(-g^{-1})J(\chi^j,\chi^k) \tag{5}$$

where

$$J(\chi^j,\chi^k) = \sum_{a\in\mathbb{F}_q} \chi^j(a)\chi^k(1-a)$$

is a *Jacobi sum* with respect to χ^j and χ^k. The following properties of Jacobi sums are useful.

Lemma 2.2 ([LN97, Theorem 5.19, 5.21, 5.22])**.** *Let λ,ψ be two extended characters of $\mathbb{F}_q$.*

(i) $J(\lambda,\psi) = J(\psi,\lambda)$;
(ii) $J(\varepsilon,\varepsilon) = q$;
(iii) $J(\lambda,\varepsilon) = 0$ *if* $\lambda \neq \varepsilon$;
(iv) $J(\lambda,\lambda^{-1}) = -\lambda(-1)$ *if* $\lambda \neq \varepsilon$;
(v) $|J(\lambda,\psi)| = \sqrt{q}$ *if λ,ψ and $\lambda\psi$ are all nontrivial.*

Note that $|\chi^i(a)| = 1$ for all $a\in\mathbb{F}_q^\times$. By (iv) and (v) of Lemma 2.2, one has the following estimate of N_g from (5)

$$|N_g - q| \le M_0 + M_1\sqrt{q}$$

where M_0 (resp. M_1) is the number of pairs (j,k) with $\chi^j\chi^k = \varepsilon$ (resp. $\chi^j\chi^k \neq \varepsilon$). Observe that $M_0 = \ell-1$ and $M_1 = (\ell-1)(\ell-2)$. Thus, if

$$q > (\ell-1) + (\ell-1)(\ell-2)\sqrt{q}, \tag{6}$$

then $N_g > 0$. Consequently, for q large enough (for example $q > (\ell-1)^4$), one has $N_g > 0$ for any $g\in\mathbb{F}_q^\times$.

For the numbers of rational solutions to equations over finite fields, the Hasse-Weil bound [Wei48] provides more precise information than the crude estimate given above.

Theorem 1 (Hasse-Weil bound)**.** *Let $\mathscr{C}$ be a non-singular, absolutely irreducible projective curve over $\mathbb{F}_q$ and let $N_{\mathscr{C}} = |\mathscr{C}(\mathbb{F}_q)|$ be the number of $\mathbb{F}_q$-rational points of $\mathscr{C}$. Then,*

$$|N_{\mathscr{C}} - (q+1)| \le 2\mathfrak{g}\sqrt{q}$$

where $\mathfrak{g}$ is the genus of $\mathscr{C}$.

Applying the Hasse-Weil bound to $\widetilde{\mathscr{C}_g}$, we see that

$$|\widetilde{N_g} - (q+1)| \le (\ell-1)(\ell-2)\sqrt{q}$$

since the genus of $\widetilde{\mathscr{C}_g}$ is $\mathfrak{g}_\ell = (\ell-1)(\ell-2)/2$ by the degree-genus formula [Har77]. Consequently, $\widetilde{N_g} > 0$ for any generator g of $\mathbb{F}_q^\times$ provided that $q+1 > (\ell-1)(\ell-2)\sqrt{q}$ and therefore $\widetilde{\mathscr{C}_g}(\mathbb{F}_q)$ is non-empty if $q \ge (\ell-1)^2(\ell-2)^2$.