Again, by assuming finiteness of the Tate-Shafarevich group, the non-degenerate alternating pairing on $Ш(E_{D'})[3]$ implies that this group is of even dimension as an $\mathbb{F}_3$-vector space and we obtain the congruence relation

$$r(E_{D'}) \equiv r(\mathcal{S}_3(E_{D'})) \equiv r(\mathcal{S}_\phi(E_{D'})) + r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D)) \equiv 1 \bmod 2. \tag{13}$$

The last two equivalences follow from (11) and Proposition 3.1. □

4. The Escalatory Case and No Integral Points

In the previous section we established that the elliptic curves $E_{D'}$ have odd rank. In this section, we will define a subfamily of these curves and show that this subfamily cannot have integral points.

Let us first recall a classical result of Scholz regarding the rank of the 3-part of the ideal class group of quadratic number fields. The interested reader may refer to [18, Section 10.2] for more details on Scholz's Theorem.

Theorem 4.1. ***Reflection Theorem of Scholz***
Let $d > 1$ be square-free. Let $F = \mathbb{Q}(\sqrt{d})$ and $K = \mathbb{Q}(\sqrt{-3d})$. If $3|d$ then let $K = \mathbb{Q}(\sqrt{-d/3})$. Then $r_F \leq r_K \leq r_F + 1$. □

Definition 4.2. *With notation as in Theorem 4.1, we define as* escalatory *the case where $r_K = r_F + 1$ and as* non-escalatory *the case where $r_K = r_F$.*

The terms *escalatory* and *non-escalatory* are used for example in [9, Chapter 4]. Specifically, in Section 4.10 of [9], it is shown that in the case of negative fundamental discriminant d, the escalatory case is equivalent to the non-existence of cubic fields of discriminant $3^4 d$. Translating this to our notation, we have

Remark 4.3. *If $r_3(D) = r_3(D') + 1$ then there are no cubic fields of discriminant $3^4 D$.*

We will not explain in this paper the proof of this result. We only need to mention that the proof requires the use of the so-called 3-*virtual units*, which we also use here and we define right before Proposition 4.8. These 3-virtual units live in the quadratic resolvent $K_{D'}$ and give rise to cubic extensions of K_D. The interested reader will find all the necessary theory in [9, Chapter 4] and [6, Section 5.2.2], and may find more on the relation between 3-virtual units, ideal class groups and elliptic curves in [1].

Before we go on to prove that the subfamily of elliptic curves $E_{D'}$ with $r_3(D) = r_3(D') + 1$ have no integral points, let us show the relation between the non-escalatory case $r_3(D) = r_3(D')$ and the existence of a cubic field of discriminant $3^4 D$, via elliptic curves, with the following example. Let us note that the discriminant $D = -1355$ of the example does not belong to the set of discriminants that we consider in this paper, since $-1355 \equiv 1 \bmod 3$. This is not important though since it does not affect the purpose of the example.

Example 4.4. *Consider the negative fundamental discriminant $D = -1355$. We have $r_3(D) = r_3(D') = 1$ and so we are in the non-escalatory case. To show the existence of a cubic field of discriminant equal to $3^4 D$, which is*