employing the same result of Satgé for this case of negative constant term, we obtain

$$r(\mathcal{S}_{\phi}(E_{D'})) = r_3(D') = r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D)).$$

Hence, from the same discussion that follows Proposition 3.1, including the proof of Corollary 3.2, we obtain the following parity relation between the rank of our curves with positive squarefree D and their Selmer group

$$r(E_{D'}) \equiv 2r_3(D') \equiv 0 \bmod 2.$$

So the rank of our elliptic curves must be even in this case and might even be zero. The equivalent of Remark 4.3 for $D > 4$ is that there are no cubic fields of discriminant $3^4 D$ when we are in the non-escalator case, i.e. when $r_3(D) = r_3(D')$. Since Lemma 4.5 and Proposition 4.8 hold for this case as well, we see that the subfamily of curves $E_{D'}$ with $D > 4$ and $r_3(D) = r_3(D')$ has no integral points.

A final question, which constitutes the topic of an upcoming paper of ours, concerns the study of the the group $\text{Ш}(E_{D'})[\phi]$, for both $D < -4$ and $D > 4$. By studying the exact sequence in (9) and by following a constructive approach via binary quadratic forms, similar to that in [1], we show how to construct curves that violate the local-global principle in the case that $\text{Ш}(E_{D'})[\phi]$ is not empty.

References

[1] E. Agathocleous: On the Selmer group and rank of a family of elliptic curves and curves of genus one violating the Hasse principle, (Under Review) (2022).

[2] M. Bhargava, A. Shankar, T. Taniguchi, F. Thorne, J. Tsimerman, Y. Zhao: (2017) Bounds on 2-torsion in class groups of number fields and integral points on elliptic curves, https://doi.org/10.48550/arXiv.1701.02458.

[3] W. Bosma, J. Cannon and C. Playoust: The Magma algebra system. I. The user language, *J. Symbolic Comput.*, **24** (1997), 235–265.

[4] J. W. S. Cassels, Arithmetic on curves of genus 1, IV, Proof of the Hauptvermutung, *J. Reine Angew. Math.* **211**, (1962), 95-112.

[5] J.W.S. Cassels and A. Fröhlich: Algebraic Number Theory Proceedings of an Instructional Conference Organized by the London Mathematical Society (a NATO Advanced Study Institute) with the Support of the International Mathematical Union, London Mathematical Society, 2010.

[6] H. Cohen: *Advanced Topics in Computational Number Theory*, Springer-Verlag New York, 2000.

[7] H. Cohen: *Number Theory Volume I: Tools and Diophantine Equations*, Springer-Verlag, 2007.

[8] D.A. Cox, *Primes of the form $x^2 + ny^2$: Fermat, Class Field Theory and Complex Multiplication*, Wiley, 1989.

[9] S. Hambleton and H. Williams: *Cubic Fields with Geometry*, Springer Nature Switzerland Ag, 2018.