expected according to [9, Section 4.10], we can look for an irreducible polynomial of discriminant 3^4D *or equivalently for rational solutions* u, v *such that*

$$\mathrm{disc}(x^3 - ux + v) = 4u^3 - 27v^2 = 3^4D \iff (4v)^2 = (\frac{4}{3}u)^3 - 3 \cdot 16D.$$

Therefore, we looked for rational points on the elliptic curve

$$E_{D'} : y^2 = x^3 + 16D'.$$

The integral point $P = (64, 572)$ *gives* $u = 48$ *and* $v = 143$ *and indeed the cubic polynomial* $g(x) = x^3 - 48x + 143$ *is irreducible in* $\mathbb{Q}[x]$ *and has discriminant equal to* 3^4D.

Lemma 4.5. *Let* $D < -4$ *be any squarefree integer which satisfies the congruence relations in (1). Let* $E_{D'}$ *be the elliptic curves that we have defined above. If there is an* integral *point* $P \in E_{D'}(\mathbb{Z})$, *then this point cannot be the image of any point* $Q = (\hat{x}, \hat{y}) \in \hat{E}_D(\mathbb{Q})$.

Proof. We can write Q as

$$Q = (\hat{x}, \hat{y}) = (\frac{X}{Z^2}, \frac{Y}{Z^3}) = (\frac{x}{3^{(2c-a)}z^2}, \frac{y}{3^{(3c-b)}z^3}),$$

where $a, b, c \geq 0$, $3^a||X$, $3^b||Y$, $3^c||Z$, and the symbol '$||$' means *divides exactly*. Since $Q \in \hat{E}_D(\mathbb{Q})$ we must have

$$3^{(2b-3a)}y^2 = x^3 + 3^{(6c-3a+4)}16Dz^6.$$

We immediately see that we cannot have $6c - 3a + 4 = 0$ because, if either a or b is not zero then $3|4$ which is absurd, and if $a = c = 0$ then $4 = 0$, also absurd.

Case (a): Let us call Case (a) the case where $a = b = c = 0$.

If at least one of the a, b or c is not zero, we need to examine the following cases:

Case (b): If $6c - 3a + 4 = 2b - 3a$ (which implies that $2b - 3a \neq 0$), then $b = 3c + 2$ and therefore $\hat{y} = \frac{3^2 y}{z^3}$. Plugging it into the equation of $\hat{E}_D$ this implies that $\hat{x} = \frac{3^2 x}{z^2}$. In this case we must also have that $a = 2c + 2$.

Case (c): If $6c - 3a + 4 \neq 2b - 3a$ and $2b - 3a \neq 0$, then we arrive at a contradiction since we always have that 3 must divide one of the terms y, x or $16 \cdot Dz^6$, hence this case cannot happen.

Case (d): If $2b - 3a = 0$ (and therefore $6c - 3a + 4 \neq 2b - 3a$) then $b = 3a/2$ and a must be even.

Assume now that $P = \hat{\phi}(Q)$, where $\hat{\phi}$ is defined as ([7, Proposition 8.4.3]):

$$(A, B) = \hat{\phi}(\hat{x}, \hat{y}) = \Big(\frac{\hat{x}^3 + 4^3 3^4 D}{9\hat{x}^2}, \frac{\hat{y}(\hat{x}^3 - 2 \cdot 4^3 3^4 D)}{27\hat{x}^3}\Big). \tag{14}$$

Substituting Q in (14), the y-coordinate in particular gives the relation

$$3^{(3+3c-b)}Bx^3z^3 + 3^{(4+6c-3a)}2^7yz^6D = yx^3. \tag{15}$$

In Case (a), $a = b = c = 0$ and equation (15) implies that $3|yx^3$, impossible. Hence, if Case (a) holds, then P cannot be the image of any point $Q \in \hat{E}_D(\mathbb{Q})$.