4 Main theorem

In order to derived the degree formula for the 3-secant variety, it is necessary to utilize each term of [7, Theorem 8.2.8]. To accomplish this, we present the following lemma.

Lemma 4.1. *Let X be a nonsingular projective variety that is embedded by a 5-very ample line bundle. Let Y be the 2-secant variety $\sigma_2(X)$. Let J be the ruled join $J(X,Y)$. Then* $\deg(J/XY)$ *is 3.*

Proof. Let w be a general point of $\sigma_3(X) \setminus \sigma_2(X)$. If there are two secant planes that contain w, 6 points of X do not satisfy the independent condition of 5-very ampleness.(cf. [5, Remark 1.7]) So, there exists a unique plane spanned by three points of X that contains w. Let x, y, and z be three distinct points of X such that their linear span contains w. Let a, b, and c be the points of intersection of $\overline{xw}$ with $\overline{yz}$, $\overline{yw}$ with $\overline{zx}$, and $\overline{zw}$ with $\overline{xy}$, respectively. Then, the three points of the ruled join $J(X,Y)$ corresponding to the ratios between $(x,w),(w,a),(y,w),(w,b)$, and $(z,w),(w,c)$ are exactly the inverse image of the rational map $J(X,Y) \dashrightarrow XY$. □

Prior to starting a proof of the main theorem, we denote the tangent sheaf of X by T_X. It follows from the definition of Segre class that $s_k(C_{\Delta(X)}(X \times X)) = s_{n-k}(T_X)$ as a Segre class of a locally free sheaf. With this notation established, we now proceed to the proof of the main theorem:

Proof of main theorem. Recall that X is a smooth projective variety of dimension n and $E \subset Z_2$ can be regarded as a projective bundle associated with the tangent bundle on X. Using equations (5), (11), and proposition 3.2, we obtain:

$$s(X, \sigma_2(X)) = q_* \left(\sum_{i \geq 0} (-1)^i (2E + \eta^*(h_1 - h_2))^i \right). \tag{14}$$

Let $P(C_{\Delta(X)}(X \times X))$ be the projective tangent cone to X and $\mathcal{O}(1)$ be the tautological line bundle. Let $g : P(C_{\Delta(X)}(X \times X)) \to X$ be the projection map. Note that $P(C_{\Delta(X)}(X \times X))$ is isomorphic to the projective bundle $\mathbb{P}(\Omega^1_X)$ where Ω^1_X is the sheaf of Kähler differentials. The total Segre classes $s(C_{\Delta(X)}(X \times X))$ and $s(T_X)$ are equal but they have different conventions for indexes: $s_k(T_X) = s_{n-k}(C_{\Delta(X)}(X \times X))$ for $0 \leq k \leq n$. As schemes, E and $P(C_{\Delta(X)}(X \times X))$ are the same and hence $E|_E = \mathcal{O}(-1)$ holds.

Remark. Recall that we use the convention for projective bundles and tautological line bundles as in [8, Appendix B.5.5].

Therefore we have

$$\eta_* E^i = \eta_*(E|_E)^{i-1} = (-1)^{i-1}\, g_*(c_1(\mathcal{O}(1))^{i-1} \cap [P(C_{\Delta(X)}(X \times X))])$$

for $i \geq 1$ and hence $\eta_* E^i = (-1)^{i-1} s_{i-n}(T_{\Delta(X))}$ for $i \geq n$. Since $\eta^*(h_1 - h_2) \cdot E$ is zero, we obtain that

$$\eta_* \sum_{l=n}^{2n} (-1)^l (2E + \eta^*(h_1 - h_2))^l = \sum_{l=n}^{2n} (-1)^l (h_1 - h_2)^l - \sum_{l=n}^{2n} 2^l s_{l-n}(T_X) \cap [X]. \tag{15}$$