Proof. If $j+k=\ell$, then $J(\chi^j,\chi^k)=-\chi(-1)^j$ by (iv) of Lemma 2.2. It follows that

$$\sum_{\substack{1\le j,k\le \ell-1,\\ j+k=\ell}} \chi(-1)^{j+k}\chi(g^{-1})^{(2j+k)t}J(\chi^j,\chi^k) = -\sum_{j=1}^{\ell-1}\chi(-g^{-t})^j.$$

Note that the kernel of χ is L. If $-g^{-t}\in L$, then $g^{-t}L=-L$. This gives $\ell=|\mathbb{F}_q^\times/L|\le 2$ since $g^{-t}L$ also generates $\mathbb{F}_q^\times/L$ as $1\le t\le \ell$ and $(t,\ell)=1$. This is not our case and thus $\chi(-g^{-t})\ne 1$. Hence, $\sum_{j=1}^{\ell-1}\chi(-g^{-t})^j=-1$ and the result follows. □

Now, we are ready to prove our main theorem.

Theorem 4.3. *Let q be a power of a prime and let ℓ be a proper divisor of $q-1$. If*

$$q\ge (2^{\omega(\ell)}(\ell-3-\delta)+2)^2-2$$

where $\omega(\ell)$ is the number of distinct prime divisors of ℓ and

$$\delta=\begin{cases}1 & \text{if } 4\mid \ell,\\ 0 & \text{otherwise,}\end{cases}$$

then there is a generator g of $\mathbb{F}_q^\times$ such that $N_g>0$.

Proof. By Proposition 4.1, it is enough to consider the subsum

$$N=\sum_{\substack{1\le t\le \ell,\\ (t,\ell)=1}} N_{g^t}$$

where g is a fixed generator of $\mathbb{F}_q^\times$. Lemma 4.2 gives that

$$N=\varphi(\ell)(q+1)+\sum_{\substack{1\le j,k\le \ell-1,\\ j+k\ne\ell}}\chi(-1)^{j+k}J(\chi^j,\chi^k)z(j,k)\quad\text{where}$$

$$z(j,k)=\sum_{\substack{1\le t\le \ell,\\ (t,\ell)=1}}\chi(g^{-1})^{(2j+k)t}.$$

Note that $\chi(g^{-1})=\zeta_\ell$ is a primitive ℓ-th root of 1. Therefore,

$$z(j,k)=\sum_{\substack{1\le t\le \ell,\\ (t,\ell)=1}}\zeta_\ell^{(2j+k)t}=c_\ell(2j+k),$$

is a Ramanujan's sum and

$$N=\varphi(\ell)(q+1)+\sum_{\substack{1\le j,k\le \ell-1,\\ j+k\ne\ell}}\chi(-1)^{j+k}J(\chi^j,\chi^k)c_\ell(2j+k).$$

Let $I=\{1,2,\dots,\ell-1\}$ and for positive integer $d\mid\ell$ and integer t with $1\le t\le \ell/d$ such that $(t,d/\ell)=1$, we set

$$S'(d,t)=\{(j,k)\in I\times I\mid 2j+k\equiv td\pmod{\ell}\text{ and } j+k\ne\ell\}.$$