Here r is the distance of the from the centre of the PNS, and v is the radial velocity of the wind. The total heating rate per baryon is $\dot{q}_{\rm tot} = \dot{q}_\nu + \dot{q}_w$, where the first term is due to neutrino heating and cooling (see Qian & Woosley 1996) while the second term is due to wave damping. $\dot{Y}_e$ is the rate of change in the electron fraction of the wind due to neutrino reactions (see Qian & Woosley 1996). The wave damping length and frequency are l_d and ω ($\dot{q}_w$, l_d and ω are discussed in section 3.2 below). T, c_s, ρ, and h denote the local temperature, sound speed, density, and enthalpy, respectively. W is the Lorentz factor, and $v_g = v + c_s$ represents the group velocity of the waves. G and c represent the gravitational constant and the speed of light, and $e^\Lambda = \sqrt{1 - \frac{2GM_{NS}}{rc^2}}$. Corrections from the wave stress are denoted by δf_1 and δf_2 (see section 3.1 below). Without the wave action terms, this system is the same as that of Thompson et al. (2001). We employ the equation of state of Timmes & Swesty (2000), which assumes the wind is made up of free protons, neutrons, electrons, positrons, and thermal photons. We search for solutions of these equations that pass through the critical or transonic point where f_1 and f_2 pass through zero at the the same radius.

3.1 Wave Stress

Even in the absence of damping, waves in a stellar atmosphere still exert a force on the medium through which they move. This effect is calculated using the wave action, and adds an extra stress to the momentum equation (e.g. Jacques 1977; Suzuki & Nagataki 2005). For simplicity, we derive these corrections in the non-relativistic limit.

In the absence of wave stress, the non-relativistic momentum equation for this system is

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{GM_{\rm NS}}{r^2} \tag{6}$$

Combining this with the other conservation equations yields the non-relativistic critical form equation

$$\left(v^2 - c_s^2\right)\frac{dv}{dr} = \frac{v}{r}\left(2c_s^2 - \frac{GM_{\rm NS}}{r}\right) - \frac{v}{\rho}\left[\left(\frac{\partial P}{\partial s}\right)_{\rho,Y_e}\frac{ds}{dr} + \left(\frac{\partial P}{\partial Y_e}\right)_{\rho,s}\frac{dY_e}{dr}\right] \tag{7}$$

from which we extract the non-relativistic forms of f_1 and f_2:

$$\begin{aligned} f_1 &= 1 - \left(\frac{v}{c_s}\right)^2 \\ f_2 &= \frac{GM_{\rm NS}}{c_s^2 r} - 2 \\ &\quad + \frac{r}{\rho c_s^2}\left[\left(\frac{\partial P}{\partial s}\right)_{\rho,Y_e}\frac{ds}{dr} + \left(\frac{\partial P}{\partial Y_e}\right)_{\rho,s}\frac{dY_e}{dr}\right] \end{aligned} \tag{8}$$

The non-relativistic momentum equation including corrections from wave propagation is (Jacques 1977)

$$\rho v\frac{dv}{dr} + \frac{d}{dr}\left(P + a_1\mathcal{E}\right) + \frac{\mathcal{E}}{A}\frac{dA}{dr} + \rho\frac{GM_{\rm NS}}{r^2} = 0 \tag{9}$$

where $\mathcal{E} = \frac{c_s}{v_g}\omega S$ is the energy density of the waves and $A = 4\pi r^2$. Combined with the other conservation equations, this yields a revised version of the critical form equation:

$$\begin{aligned} &\left[v^2 - c_s^2 + \frac{a_1\mathcal{E}v}{\rho v_g}\left(A_\rho\frac{c_v}{v} - 2\right)\right]\frac{dv}{dr} \\ &= \frac{v}{r}\left(2c_s^2 - \frac{GM_{\rm NS}}{r}\right) - \frac{v}{\rho}\left[\left(\frac{\partial P}{\partial s}\right)_{\rho,Y_e}\frac{ds}{dr} + \left(\frac{\partial P}{\partial Y_e}\right)_{\rho,s}\frac{dY_e}{dr}\right] \\ &\quad + \frac{a_1\mathcal{E}v}{\rho r}\left[-2A_\rho\frac{c_v}{v_g} + X_E A_s\frac{c_v}{v_g} + 2 - \frac{2}{a_1} + \frac{r}{l_d}\right] \end{aligned} \tag{10}$$

with $a_1 = \frac{1}{2}(\gamma+1)$, $c_v = c_s - v$, $X_E = \frac{r}{s}\xi_s$, $A_\rho = \left(\frac{\partial \ln c_s}{\partial \ln \rho}\right)_s$, and $A_s = \left(\frac{\partial \ln c_s}{\partial \ln s}\right)_\rho$. In the wave action terms, we have assumed a constant adiabatic index γ. Note that all terms from equation (7) are present, with an additional correction term on each side. This allows us to define corrections to the original f_1 and f_2 functions in equation (8):

$$\begin{aligned} \delta f_1 &= \frac{a_1\mathcal{E}}{\rho c_s^2}\left(2\frac{v}{v_g} - A_\rho\frac{c_v}{v_g}\right) \\ \delta f_2 &= -\frac{a_1\mathcal{E}}{\rho c_s^2}\left[\left(A_s\chi_e - 2A_\rho\right)\frac{c_v}{v_g} + 2\left(1 - \frac{1}{a_1}\right) + \frac{r}{l_d}\right] \end{aligned} \tag{11}$$

These corrections are then applied to the fully relativistic f_1 and f_2 in equation (3).

3.2 Wave Heating

Acoustic waves propagating in the wind can become non-linear and shock heat the wind. We model this shock heating via an effective damping length prescription. Wave heating will only begin when the waves steepen into shocks and begin to dissipate their energy. Mihalas & Mihalas (1984) provides an integral expression for the radial distance at which this takes place:

$$\frac{1}{4}(\gamma+1)c_s^{-1}\int_0^r u_0(r')dr' = \frac{\pi c_s}{2\omega} \tag{12}$$

where $u_0 = \sqrt{\frac{\omega S}{\rho}}$ is the amplitude of the velocity perturbation of the waves and γ is the adiabatic index of the background material. Here and elsewhere, ω represents the angular frequency (in the lab frame) of the waves. We then find the condition for shock formation to be

$$\int_0^r \sqrt{\frac{\omega S}{\rho}}dr' = \frac{2\pi c_s^2}{\omega(\gamma+1)}. \tag{13}$$

In the weak shock limit (e.g. Mihalas & Mihalas 1984), the energy density of the waves ϵ_s evolves as

$$\nabla\cdot(v_g\epsilon_s) = -\frac{m}{\pi}\omega\epsilon_s \tag{14}$$

where $m = (v/c_s)^2 - 1$ is the reduced Mach number. In a static homogeneous background, the shock can be modeled as a simple saw-tooth wave, with energy density

$$\epsilon_s = \frac{\gamma P m^2}{3(\gamma+1)^2}. \tag{15}$$

In the weak shock limit, we take $\epsilon_s = S/\omega$, which allows us to find an expression for m in terms of local quantities. Combining the wave action evolution in Eqs. (3) and (14), and assuming a constant ω, we find the dissipation length

$$l_d = \frac{\pi\gamma^2}{\gamma+1}\left(\frac{c_s^2\epsilon}{3\omega^3 S}\right)^{1/2} \tag{16}$$