announce that Conjecture B has been verified to hold for prime $p \leq 2^{30}$. Applying Corollary B, we are able to extend the range of prime numbers p such that Conjecture B holds.

For $\ell \in \mathbb{N}$, let

$$b(\ell) = (2^{\omega(\ell)}(\ell - 3 - \delta) + 2)^2 - 2$$

be the lower bound appearing in Theorem A. Suppose that p is a prime number such that the index ℓ_0 of H in $\mathbb{F}_p^\times$ satisfying $b(\ell_0) \leq 2^{30}$, then either $p \leq 2^{30}$ or $p > 2^{30} \geq b(\ell_0)$. It follows that Conjecture C holds for this prime number p. This leads to the question about the integer ℓ such that $b(\ell) \leq 2^{30}$. The answer depends on the number of distinct prime divisors of ℓ. For $\omega(\ell) < 4$, one can check that $b(\ell) \leq 2^{30}$ if one of the following conditions holds:

$$\begin{cases} \ell < 16411 \text{ with } \omega(\ell) = 1, \\ \ell < 8197 \text{ with } \omega(\ell) = 2, \\ \ell < 4100 \text{ with } \omega(\ell) = 3. \end{cases}$$

Moreover, one also has $b(\ell) \leq 2^{30}$ whenever $\ell < 2070$. As a consequence, we have the following result.

Theorem 5.1. Conjecture C *holds for primes p with ℓ_0 satisfying one of the following conditions*

$$\begin{cases} \ell_0 < 16411 \textit{ with } \omega(\ell_0) = 1, \\ \ell_0 < 8197 \textit{ with } \omega(\ell_0) = 2, \\ \ell_0 < 4100 \textit{ with } \omega(\ell_0) = 3, \\ \ell_0 < 2070. \end{cases}$$

For an integer ℓ such that $b(\ell) > 2^{30}$, we consider the set of prime numbers between 2^{30} and $b(\ell)$. Let

$$P(\ell) = \left\{ \text{primes } p > 2^{30} \mid [\mathbb{F}_p^\times : \langle -1, 2 \rangle] = \ell \text{ and } p < b(\ell) \right\}.$$

For prime numbers in $P(\ell)$, we verify Conjecture C by the aid of computer for the computations. For instance, we obtain that $P(2070) = \varnothing$ and hence Conjecture C holds for prime numbers p with $\ell_0 = 2070$. By computer search, there are 423 primes in the union of $P(\ell)$ for $2070 \leq \ell \leq 3000$. The largest prime number in the union is 7324065841 with $\ell = 2730$. We have checked that Conjecture C holds for these prime numbers.

Theorem 5.2. Conjecture C *holds for primes p such that* $[\mathbb{F}_p^\times : \langle -1, 2 \rangle] \leq 3000$.

For the solvability of Equation (3), we have the following simple observation.

Proposition 5.3. *Let ℓ be a proper divisor of $q-1$ and $\ell' \mid \ell$. Suppose that Equation* (3) *is solvable over $\mathbb{F}_q$ for exponent ℓ then it is also solvable for exponent ℓ'.*

Proposition 5.3 leads to the following consideration for $q = p$, a prime number, and $\ell = \frac{p-1}{2}$ in Equation (3). In this case, $L = \{\pm 1\}$ and it suffices to consider the four possibilities $\pm g^2 \pm g + 1 = 0$ where $g \in \mathbb{N}$ is a primitive root modulo p (i.e. g is a generator of $\mathbb{F}_p^\times$). Clearly, $g^2 \pm g + 1 = 0$ if and only if $g^3 = \pm 1$, and then $p - 1 = 3, 6$.