preserving in both directions. By Uhlhorn's theorem, g is induced by a unitary or anti-unitary operator $U_X : X \to X'$. Then

$$f(M) = g(M^{\perp} \cap X)^{\perp} \cap X' = (U_X(M^{\perp} \cap X))^{\perp} \cap X' = U_X(M)$$

for every $M \in \mathcal{G}^1(X)$. □

Lemma 5. *If $Z \in \mathcal{C}_{-1}$ is contained in $X \in \mathcal{C}_{+1}$, then*

$$f(\mathcal{S}(Z)) = \mathcal{S}(U_X(Z)).$$

Proof. We take distinct M, N belonging to $\mathcal{S}(Z) \cap \mathcal{G}^1(X)$. Then $Z = M \cap N$ and

$$f(\mathcal{S}(Z)) = \mathcal{S}(Z'),$$

where $Z' = f(M) \cap f(N)$. By Lemma 4,

$$U_X(Z) = U_X(M) \cap U_X(N) = f(M) \cap f(N) = Z'$$

which gives the claim. □

Lemma 6. *If $X, Y \in \mathcal{C}_{+1}$, then*

$$U_X(P) = U_Y(P)$$

for every 1-dimensional subspace $P \subset X \cap Y$.

Proof. Since $\mathcal{C}_{+1}$ is a connected component of $\mathcal{G}_{\infty}(H)$, there is a sequence

$$X = X_0, X_1, \ldots, X_m = Y$$

of elements of $\mathcal{C}_{+1}$, where X_{i-1}, X_i are adjacent for every $i \in \{1, \ldots, m\}$; furthermore, we can construct this sequence such that each X_i contains $X \cap Y$. For this reason, it is sufficient to consider the case when X, Y are adjacent.

If X and Y are adjacent, then $M = X \cap Y$ belongs to $\mathcal{C}$. Let P be a 1-dimensional subspace of M. Then $Z = P^{\perp} \cap M$ belongs to $\mathcal{C}_{-1}$. Since Z is contained in both X, Y, we have

$$U_X(Z) = U_Y(Z) \tag{1}$$

by Lemma 5. Then

$$U_X(P) \oplus U_X(Z) = U_X(M) = f(M) = U_Y(M) = U_Y(P) \oplus U_Y(Z)$$

and (1) implies that $U_X(P) = U_Y(P)$. □

Consider the transformation h of $\mathcal{G}_1(H)$ defined as follows: for a 1-dimensional subspace $P \subset H$ we take any $X \in \mathcal{C}_{+1}$ containing P and set $h(P) = U_X(P)$. By Lemma 6, h is well-defined. Since for any two 1-dimensional subspaces of H there is an element of $\mathcal{C}_{+1}$ containing them, h is a bijection preserving the orthogonality relation in both directions. Then h is induced by a unitary or anti-unitary operator U on H (Uhlhorn's theorem). The restriction of U to every $X \in \mathcal{C}_{+}$ is a scalar multiple of U_X. This implies that $f|_{\mathcal{C}}$ is induced by U.