The remaining calculation simply involves taking the first projections, which yields:

$$\begin{aligned}(\mathrm{pr}_1)_*(h_1 - h_2)^l &= \sum_{a=0}^{l}(-1)^{l-a}\binom{l}{a}h^a \cdot (\mathrm{pr}_1)_*\mathrm{pr}_2^*\, h^{l-a} \\ &= (-1)^n\binom{l}{l-n}h^{l-n}\cdot d[X]. \qquad (l \geq n)\end{aligned}$$

By substituting (15) into (14) we have :

$$\begin{aligned}s(X,\sigma_2(X)) &= \sum_{l=n}^{2n}(-1)^{n+l}\binom{l}{n}\cdot d\, h^{l-n} - \sum_{l=n}^{2n} 2^l s_{l-n}(T_X)\cap[X] \\ &= \sum_{i=0}^{n}(-1)^i\binom{i+n}{n}\cdot d\, h^i - \sum_{i=0}^{n}2^{i+n}s_i(T_X)\cap[X].\end{aligned}$$

From equation (9), we obtain:

$$\begin{aligned}s(\Delta(X), X\times\sigma_2(X)) &= s(T_X)\cap s(X,\sigma_2(X)) \\ &= \sum_{i=0}^{n}\sum_{a+b=i}\{(-1)^b\binom{b+n}{n}\, d\, h^b\cdot s_a(T_X)\cap[X] - 2^{b+n}s_a(T_X)s_b(T_X)\cap[X]\}.\end{aligned}$$

by omitting the pull back $(\Delta(X)\to X)^*$. Therefore, we obtain the main formula from equation (2), which completes the proof. □

Corollary. *The multiplicity of $\sigma_2(X)$ along X is $d - 2^n$.*

Proof. The multiplicity of $\sigma_2(X)$ along X is the coefficient of $[X]$ in the class $s(X,\sigma_2(X))$. (cf. [8, Chapter 4.3]) □

5 Examples

5.1 In the case of curves

Consider the case where C is a smooth projective curve. Let g be the genus of a curve C. The degree of each term of Segre class T_C is given by $\deg s_0(T_C) = d$ and $\deg s_1(T_C) = 2g-2$. By substituting those terms in the main formula we get :

$$\deg\sigma_3(C) = \frac{1}{3!}(d^3 - 9d^2 + 26d + 24 - 6dg - 24g)$$

which can be found in [12, Proposition 1].

5.2 In the case of surfaces

Consider the case where S is a smooth projective surface. Let K be the canonical divisor of S, let $d = h^2$, let $\pi = h\cdot K$, let $\kappa = K^2$, and let $e = c_2$ be the topological Euler characteristic. The total Segre class of the tangent sheaf T_S of S has degree 0, 1, and 2 terms, which are given by:

$$s_0(T_S) = [S],\ s_1(T_S) = K,\ s_2(T_S) = \kappa - e.$$

By substituting those terms in the main formula we get :

$$\deg\sigma_3(S) = \frac{1}{3!}(d^3 - 30d^2 + 224d - 3d(5\pi + \kappa - e) + 192\pi + 56\kappa - 40e)$$