

is in $\mathbb{Z}[x]$ and is of discriminant exactly $3^4 D$.

Consider the element $\lambda = \frac{\frac{B}{4} + \sqrt{D'}}{2} = \frac{b + \sqrt{D'}}{2} \in \mathcal{O}_{D'}$. We see that

$$\lambda \bar{\lambda} = \left(\frac{A}{4}\right)^3 = a^3$$

and

$$\lambda + \bar{\lambda} = \frac{B}{4} = b.$$

Given the polynomial $g(x) \in \mathbb{Z}(x)$ above, [9, Proposition 4.1(1)] implies that λ is a 3-virtual unit.

Denote by Λ the element of $K_{D'}^\times$:

$$\Lambda = 2^3 \lambda = \frac{2B + 8\sqrt{D'}}{2} = B + 4\sqrt{D'} \in K_{D'}^\times.$$

We recognise that Λ is the image of the point $P = (A, B)$ under the Fundamental 3-Descent Map Ψ , as this map is described for example in [7, §8.4.4] or [1]. Since by [7, Proposition 8.4.8]

$$\Psi(E_{D'}(\mathbb{Q}))/\hat{\phi}(\hat{E}_D(\mathbb{Q})) \cong \subseteq K_{D'}^\times / (K_{D'}^\times)^3,$$

then indeed, by Lemma [4.5] $\Psi(P) = \Lambda \equiv \lambda \in K_{D'}^\times / (K_{D'}^\times)^3$. Then, by [9, Proposition 4.1 (2)], $g(x)$ is irreducible over \mathbb{Q} . Finally, we see that λ is a primitive 3-virtual unit and therefore, by [9, Theorem 4.4], $g(x)$ generates a cubic field of discriminant $3^4 D$, which leads to a contradiction since we are in the escalatory case.

Case (b): Both A and B are odd. Then $g(x) \notin \mathbb{Z}[x]$ but the following polynomial $f(x)$ does have integer coefficients and it is of discriminant $8^2 3^4 D$:

$$f(x) = x^3 - 3Ax + 2B \in \mathbb{Z}[x].$$

As above, let

$$\Lambda = 2^3 \lambda = \frac{2B + 8\sqrt{D'}}{2} = B + 4\sqrt{D'} \in K_{D'}^\times / (K_{D'}^\times)^3.$$

We see that $\Lambda \bar{\Lambda} = A^3$ and $\Lambda + \bar{\Lambda} = 2B$ and therefore, by [9, Proposition 4.1, (1) and (2)], Λ is a 3-virtual unit and $f(x)$ is irreducible in $\mathbb{Q}[x]$. Furthermore, since B is odd, Λ is a primitive 3-virtual unit and again, by [9, Theorem 4.4], $f(x)$ generates a cubic field of discriminant $3^4 D$, which leads to a contradiction since we are in the escalatory case. \square

Let us remark here that if 3 divides either A or B , then $9|16D'$ which is impossible. Hence, in both Cases (a) and (b) of Proposition [4.8] above, the irreducible polynomials $g(x)$ and $f(x)$, both in $\mathbb{Z}[x]$, are in *standard* form, as this is defined in [9, Section 4.4].

5. THE CASE OF POSITIVE SQUAREFREE D AND FINAL REMARKS

A natural question to ask is what happens when we consider discriminants $D > 4$ where the same equivalence relations [1] hold. In this case, the constant term $16D' = -16 \cdot 3D$ of our elliptic curves $E_{D'}$ would be negative. By following the same steps of the proof of Proposition [3.1] and by