We construct the following exact sequence of sheaves:

$$0 \to \pi_1^*\mathcal{L} \otimes I_{Z_2} \to \pi_1^*\mathcal{L} \to \pi_1^*\mathcal{L} \otimes \mathcal{O}_{Z_2} \to 0.$$

Since the restriction of π_2 to Z_2, $\pi_2|_{Z_2} : Z_2 \to X^{[2]}$ is flat of degree 2, the sheaf $\pi_{2\,*}(\pi_1^*\mathcal{L} \otimes \mathcal{O}_{Z_2})$ is locally free of rank 2. Denote the tautological bundle associated with $\mathcal{L}$ by $E_{\mathcal{L}} = \pi_{2\,*}(\pi_1^*\mathcal{L} \otimes \mathcal{O}_{Z_2})$. The fiber of the vector bundle $E_{\mathcal{L}}$ at a point $Z \in X^{[2]}$ is given by $H^0(X, \mathcal{L} \otimes \mathcal{O}_Z)$. Since $\mathcal{L}$ is 1-very ample, the restriction map $H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L} \otimes \mathcal{O}_Z)$ is surjective. This implies that the morphism $\pi_{2*}\pi_1^*\mathcal{L} \to E_{\mathcal{L}}$ is also surjective. As a result, we have a composition of morphisms

$$\mathbb{P}(E_{\mathcal{L}}) \to \mathbb{P}H^0(X, \mathcal{L}) \times X^{[2]} \to \mathbb{P}H^0(X, \mathcal{L}) = \mathbb{P}^N,$$

which is a closed immersion of projective varieties. In [15], it is shown that if $\mathcal{L}$ is a 1-very ample line bundle, the image of this map is the 2-secant variety $\sigma_2(X)$. We denote this map as $r : \mathbb{P}(E_{\mathcal{L}}) \to \sigma_2(X)$. The projective bundle $\mathbb{P}(E_{\mathcal{L}})$ is known as the secant bundle of lines, and it is birational to the 2-secant variety $\sigma_2(X)$ (as shown in [3] and [15]).

It is a well-known fact that the universal family Z_2 is isomorphic to the blow-up of $X \times X$ along its diagonal $\Delta(X)$ (see [9, Remark 2.5.4]). We denote the blow-up morphism by

$$\eta : Bl_{\Delta(X)}(X \times X) \to X \times X$$

and the involution map by

$$\rho : Bl_{\Delta(X)}(X \times X) \cong Z_2 \to X^{[2]}.$$

The exceptional divisor of η on $Bl_{\Delta(X)}(X \times X)$ is denoted by E. The projections $X \times X \to X$ are denoted by pr_i. The following diagram commutes:

$$\begin{array}{ccc} Bl_{\Delta(X)}(X \times X) \cong Z_2 & \xrightarrow{\eta} & X \times X \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ X^{[2]} & \xrightarrow{\epsilon} & X^{(2)} \end{array}$$

where $X^{(2)}$ is the quotient $(X \times X)/S_2$ and $\epsilon : X^{[2]} \to X^{(2)}$ is the Hilbert-Chow morphism.

In [13, Lemma 1.2], it is shown that the scheme-theoretic inverse image $r^{-1}(X)$ under the map $r : \mathbb{P}(E_{\mathcal{L}}) \to \sigma_2(X)$ is isomorphic to Z_2 when $\mathcal{L}$ is 3-very ample. Note that $X \times X^{[2]}$ is a closed subvariety of $\mathbb{P}^N \times X^{[2]}$ and hence we can regard Z_2 is a closed subvariety of the secant bundle $\mathbb{P}(E_{\mathcal{L}})$ in a natural way. By adjusting the isomorphism, we can ensure that the composition of the maps $Z_2 \to X \times X^{[2]} \to X$ corresponds to the composition of the maps $q := \mathrm{pr}_1 \circ \eta$. From this point on, we will identify $r^{-1}(X)$ with Z_2.

Denote by Γ_q the graph of $q : Z_2 \to X$. The product morphism $\mathrm{id}_X \times r : X \times \mathbb{P}(E_{\mathcal{L}}) \to X \times \sigma_2(X)$ has inverse image of the diagonal $\Delta(X)$ of $X \times \sigma_2(X)$ given by the graph locus Γ_q.

According to [8, Proposition 4.2 (a)], we have that

$$r_* s(Z_2, \mathbb{P}(E_{\mathcal{L}})) = s(X, \sigma_2(X)). \tag{5}$$

and

$$(\mathrm{id}_X \times r)_* s(\Gamma_q, X \times \mathbb{P}(E_{\mathcal{L}})) = s(\Delta(X), X \times \sigma_2(X)). \tag{6}$$