is in $\mathbb{Z}[x]$ and is of discriminant exactly 3^4D.

Consider the element $\lambda = \frac{\frac{B}{4}+\sqrt{D'}}{2} = \frac{b+\sqrt{D'}}{2} \in \mathcal{O}_{D'}$. We see that

$$\lambda\bar{\lambda} = (\frac{A}{4})^3 = a^3$$

and

$$\lambda + \bar{\lambda} = \frac{B}{4} = b.$$

Given the polynomial $g(x) \in \mathbb{Z}(x)$ above, [9, Proposition 4.1(1)] implies that λ is a 3-virtual unit.

Denote by Λ the element of $K_{D'}^{\times}$:

$$\Lambda = 2^3\lambda = \frac{2B + 8\sqrt{D'}}{2} = B + 4\sqrt{D'} \in K_{D'}^{\times}.$$

We recognise that Λ is the image of the point $P = (A, B)$ under the Fundamental 3-Descent Map Ψ, as this map is described for example in [7, §8.4.4] or [1]. Since by [7, Proposition 8.4.8]

$$\Psi(E_{D'}(\mathbb{Q}))/\hat{\phi}(\hat{E}_D(\mathbb{Q})) \cong\subseteq K_{D'}^{\times}/(K_{D'}^{\times})^3,$$

then indeed, by Lemma 4.5, $\Psi(P) = \Lambda \equiv \lambda \in K_{D'}^{\times}/(K_{D'}^{\times})^3$. Then, by [9, Proposition 4.1 (2)], $g(x)$ is irreducible over $\mathbb{Q}$. Finally, we see that λ is a primitive 3-virtual unit and therefore, by [9, Theorem 4.4], $g(x)$ generates a cubic field of discriminant 3^4D, which leads to a contradiction since we are in the escalatory case.

Case (b): Both A and B are odd. Then $g(x) \notin \mathbb{Z}[x]$ but the following polynomial $f(x)$ does have integer coefficients and it is of discriminant 8^23^4D:

$$f(x) = x^3 - 3Ax + 2B \in \mathbb{Z}[x].$$

As above, let

$$\Lambda = 2^3\lambda = \frac{2B + 8\sqrt{D'}}{2} = B + 4\sqrt{D'} \in K_{D'}^{\times}/(K_{D'}^{\times})^3.$$

We see that $\Lambda\bar{\Lambda} = A^3$ and $\Lambda + \bar{\Lambda} = 2B$ and therefore, by [9, Proposition 4.1, (1) and (2)], Λ is a 3-virtual unit and $f(x)$ is irreducible in $\mathbb{Q}[x]$. Furthermore, since B is odd, Λ is a primitive 3-virtual unit and again, by [9, Theorem 4.4], $f(x)$ generates a cubic field of discriminant 3^4D, which leads to a contradiction since we are in the escalatory case. □

Let us remark here that if 3 divides either A or B, then $9|16D'$ which is impossible. Hence, in both Cases (a) and (b) of Propositon 4.8 above, the irreducible polynomials $g(x)$ and $f(x)$, both in $\mathbb{Z}[x]$, are in *standard* form, as this is defined in [9, Section 4.4].

5. The case of positive squarefree D and Final Remarks

A natural question to ask is what happens when we consider discriminants $D > 4$ where the same equivalence relations (1) hold. In this case, the constant term $16D' = -16 \cdot 3D$ of our elliptic curves $E_{D'}$ would be negative. By following the same steps of the proof of Proposition 3.1 and by