(2) $\Rightarrow$ (1). Suppose that X, Y are not adjacent. Then $X \cap Y$ is of codimension 2 in both X, Y. Since there is a pair of ortho-adjacent elements of $\mathcal{G}_\infty(H)$ which are ortho-adjacent to both X, Y, Lemma 2 implies that X and Y are compatible. Then, by Lemma 1, for every $Z \in \mathcal{G}_\infty(H)$ ortho-adjacent to both X, Y there are precisely two elements of $\mathcal{G}_\infty(H)$ ortho-adjacent to X, Y, Z which contradicts our assumption. Therefore, X, Y are adjacent. □

4. Proof of Theorem 1

Let f be a bijective transformation of $\mathcal{G}_\infty(H)$ preserving the ortho-adjacency in both directions. Lemma 3 shows that f also is adjacency preserving in both directions. Then f preservers the family of subsets maximal with respect to the property that any two distinct elements are adjacent. By [17, Proposition 2.14], every such subset is of one of the following types:

- the star $\mathcal{S}(X)$, $X \in \mathcal{G}_\infty(H)$ which consists of all closed subspaces of H containing X as a hyperplane;
- the top $\mathcal{G}^1(X)$, $X \in \mathcal{G}_\infty(H)$.

Let $\mathcal{C}$ be a connected component of $\mathcal{G}_\infty(H)$. Denote by $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ the sets of all $X \in \mathcal{G}_\infty(H)$ such that X contains a certain element of $\mathcal{C}$ as a hyperplane or X is a hyperplane in a certain element of $\mathcal{C}$, respectively. Then $X \in \mathcal{G}_\infty(H)$ belongs to $\mathcal{C}_{+1}$ if and only if the top $\mathcal{G}^1(X)$ is contained in $\mathcal{C}$; similarly, X belongs to $\mathcal{C}_{-1}$ if and only if the star $\mathcal{S}(X)$ is contained in $\mathcal{C}$. Note that $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ are connected components of $\mathcal{G}_\infty(H)$. By [17, Theorem 2.19], one of the following possibilities is realized:

(1) f sends every top $\mathcal{G}^1(X)$, $X \in \mathcal{C}_{+1}$ to a top and every star $\mathcal{S}(Y)$, $Y \in \mathcal{C}_{-1}$ to a star;
(2) f transfers all tops $\mathcal{G}^1(X)$, $X \in \mathcal{C}_{+1}$ to stars and all stars $\mathcal{S}(Y)$, $Y \in \mathcal{C}_{-1}$ to tops.

In the case (2), the composition of the orthocomplementary map and $f|_\mathcal{C}$ sends tops to tops and stars to stars. Therefore, it is sufficient to show that $f|_\mathcal{C}$ is induced by a unitary or anti-unitary operator in the case (1).

Consider the case (1).

Lemma 4. *Let $X \in \mathcal{C}_{+1}$. If $f(\mathcal{G}^1(X)) = \mathcal{G}^1(X')$, then there is a unitary or anti-unitary operator $U_X : X \to X'$ such that*

$$f(M) = U_X(M)$$

for every $M \in \mathcal{G}^1(X)$.

Proof. Consider the bijective map $g : \mathcal{G}_1(X) \to \mathcal{G}_1(X')$ defined as follows:

$$g(P) = f(P^\perp \cap X)^\perp \cap X'$$

for every 1-dimensional subspace $P \subset X$. Observe that 1-dimensional subspaces $P, Q \subset X$ are orthogonal if and only if

$$P^\perp \cap X, Q^\perp \cap X \in \mathcal{G}^1(X)$$

are ortho-adjacent; the same holds for 1-dimensional subspaces of X'. This implies that g is orthogonality preserving in both directions, since f is ortho-adjacency