where ζ_n is a primitive n-th root of 1 in $\mathbb{C}$. Studying on cyclotomic polynomials, O. Hölder [Höl36] showed that the sum $c_n(m)$ has a nice closed form in terms of the Euler and Möbius functions. Denote φ the Euler's totient function and μ the Möbius function. We present it in the following lemma where the right-hand side is also called *von Sterneck function* [Ste03]. A proof is given below to the readers for convenience. For different proofs, one refers to [AA53], [Mol52] and [HW08, Theorem 272].

Lemma 3.2. *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then*

$$c_n(m) = \mu\left(\frac{n}{(n,m)}\right)\frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)}.$$

Proof. First of all, suppose that $m = 1$. Recall an elementary formula that

$$\sum_{k|r}\mu(k) = \begin{cases} 1 & \text{if } r = 1; \\ 0 & \text{if } r > 0, \end{cases}$$

for $r \in \mathbb{N}$ ([Bur11, Theorem 6.6]). Then

$$c_n(1) = \sum_{t=1}^{n}\zeta_n^t \sum_{k|(t,n)}\mu(k) = \sum_{k|n}\mu(k)\sum_{\substack{1\le t\le n,\\ k|(t,n)}}\zeta_n^t = \sum_{k|n}\mu(k)\sum_{1\le t'\le \frac{n}{k}}(\zeta_n^k)^{t'}$$

where $t' = \frac{t}{k}$. Since ζ_n^k is a primitive $\frac{n}{k}$-th root of 1, the last sum gives 1 if $k = n$ and 0 if $k < n$. Hence, $c_n(1) = \mu(n)$.

For general $m \in \mathbb{Z}$, we rewrite $c_n(m)$ as

$$c_n(m) = \sum_{t\in(\mathbb{Z}/n\mathbb{Z})^\times} z^t$$

where $z = \zeta_n^m$. Let $d = \frac{n}{(n,m)}$. Recall that $(\mathbb{Z}/n\mathbb{Z})^\times = M \cdot N$ where M and N are given in Lemma 3.1. Note that $z = \zeta_n^m$ is a primitive d-th root of 1 since $\left(d, \frac{m}{(n,m)}\right) = 1$. Hence,

$$c_n(m) = \sum_{t_1\in M}\sum_{t_2\in N} z^{t_1t_2} = \sum_{t_1\in M}|N|z^{t_1} = |N|c_d(1).$$

Now, the result follows from $|N| = \frac{\varphi(n)}{\varphi(d)}$ and $c_d(1) = \mu(d)$ by the first paragraph. □

Note that $(n,m) = (n,(n,m))$. Thus, we have the following immediate consequence.

Corollary 3.3. *Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then $c_n(m) = c_n((n,m))$.*

As a consequence, we note that if m' is an integer such that $m' \equiv m \pmod{n}$, then we conclude from Corollary 3.3 that $c_n(m') = c_n(m)$.

The following decomposition of a Cartesian product is useful for counting pairs of integers. Roughly speaking, the product below is partitioned by parallel lines on $\mathbb{Z}\times\mathbb{Z}$.

Lemma 3.4. *Let $n \in \mathbb{N}$ and let $I = \{1,2,\dots,n-1\}$. For every $a,b \in \mathbb{Z}$, we have*

$$I\times I = \biguplus_{d|n}\ \biguplus_{\substack{1\le t\le \frac{n}{d},\\ (t,\frac{n}{d})=1}} \{(x,y)\in I\times I \mid ax+by \equiv td \pmod{n}\}.$$