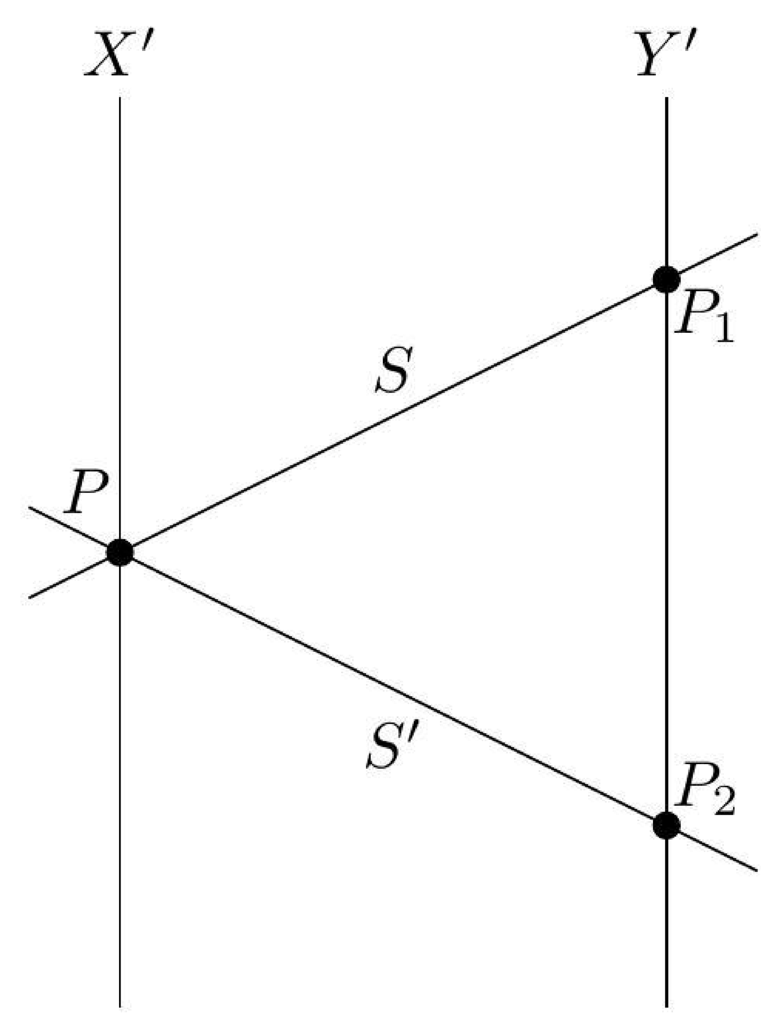

FIGURE 1. The intersections of X', Y' with S and S'

Show that S and S' are compatible to both X', Y'. The following statement is well-known [17, Lemma 1.14]: if a closed subspace $A \subset H$ is compatible to closed subspaces $B, C \subset H$, then A is compatible to $B \cap C$, $\overline{B+C}$ and $B^{\perp}$. This implies that Z is compatible to $X \cap Y$ and $X+Y$ (since Z is compatible to X, Y). Then Z is compatible to

$$M = (X+Y) \cap (X \cap Y)^{\perp}$$

and, consequently, to $X' = X \cap M$. So, X' is compatible to Z and M (X' is contained in M) which means that X' is compatible to $S = Z \cap M$. For the same reason, X' is compatible to S' and Y' is compatible to both S, S'.

To complete our proof we need to show that X' and Y' are orthogonal.

Recall that $P = S \cap S'$ is a 1-dimensional subspace of X'. Let Q be the unique 1-dimensional subspace of X' orthogonal to P. Since X' is compatible to S and S', we obtain that Q is orthogonal to S and S'. Therefore, Q is orthogonal to $S+S'$ and, consequently, to $Y' = P_1 + P_2 \subset S + S'$.

Show that P is orthogonal to Y'. Let Q_i, $i = 1, 2$ be the 1-dimensional subspace of Y' orthogonal to P_i. Then $Q_1 \neq Q_2$, since $P_1 \neq P_2$. Furthermore, Q_1 is orthogonal to S (since S and Y' are compatible) and, similarly, Q_2 is orthogonal to S'. This means that Q_1, Q_2 both are orthogonal to $P = S \cap S'$ and $Y' = Q_1 + Q_2$ is orthogonal to P.

So, $X' = P + Q$ is orthogonal to Y'. This implies that X and Y are compatible. □

Lemma 3. *For distinct $X, Y \in \mathcal{G}_{\infty}(H)$ the following conditions are equivalent:*

1. *X, Y are adjacent (not necessarily ortho-adjacent);*
2. *there are infinitely many $Z \in \mathcal{G}_{\infty}(H)$ ortho-adjacent to both X, Y such that there are infinitely many $Z' \in \mathcal{G}_{\infty}(H)$ ortho-adjacent to X, Y, Z.*

Proof. (1) $\Rightarrow$ (2). Since $(X+Y)^{\perp}$ is infinite-dimensional, for every 1-dimensional $P \subset (X+Y)^{\perp}$ there are infinitely many 1-dimensional subspaces $Q \subset (X+Y)^{\perp}$ orthogonal to P. For any such P and Q the ortho-adjacent subspaces

$$P + (X \cap Y), Q + (X \cap Y)$$

are ortho-adjacent to each of X, Y.