

Degree of the 3-secant variety

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Abstract

In this paper, we present a formula for the degree of the 3-secant variety of a nonsingular projective variety embedded by a 5-very ample line bundle. The formula is provided in terms of Segre classes of the tangent bundle of a given variety. We use the generalized version of double point formula to reduce the calculation into the case of the 2-secant variety. Due to the singularity of the 2-secant variety, we use secant bundle as a nonsingular birational model and compute multiplications of desired algebraic cycles.

1 Introduction

Let X be a projective variety over an algebraically closed field of characteristic zero. The r -secant variety, denoted $\sigma_r(X)$, is defined as the Zariski closure of the union of r -secant hyperplanes to X in projective space \mathbb{P}^N . For example, the 3-secant variety, denoted $\sigma_3(X)$, is the Zariski closure of the union of all secant planes to X in \mathbb{P}^N . The study of secant varieties is a classic topic in algebraic geometry, with an emphasis on determining defining equations and syzygies as well as understanding the singularities of these varieties.

One traditional topic of study is the specification of the double points of linear projections of a given variety. Let $X \subset \mathbb{P}^N$ be an n -dimensional nonsingular projective variety. Let δ be the number of double point of the image of X under generic linear projection $X \rightarrow \mathbb{P}^{2n}$. In [7, Corollary 8.2.9] it is shown that δ can be represented in terms of the Segre classes of the tangent bundle T_X as:

$$2\delta = (\deg X)^2 - \sum_{k \geq 0} \binom{2n+1}{k} \deg s_k(T_X).$$

Furthermore, if the embedding of X into the projective space is 3-very ample, the double point formula provides a degree formula for the 2-secant variety $\sigma_2(X)$ for X . The double point formula is one of the corollary of [7, Theorem 8.2.8] and more generally, [7, Theorem 2.1.15]. Here we give a full statement :

Refined Bezout's theorem [7, Theorem 2.1.15] Let $V \subset \mathbb{P}^N$ be an equi-dimensional closed subscheme with $\mathcal{L} := \mathcal{O}_V(1)$. Let $\sigma_0, \dots, \sigma_d$ be global section of \mathcal{L} whose zero locus is W . Let $v^i(\underline{\sigma}, V)$ be the v -cycle in the Vogel's intersection theory and $\sigma : V \dashrightarrow \mathbb{P}^d$ be a rational map defined by global section $\sigma_0, \dots, \sigma_d$. Denote $\deg(\Gamma/\sigma(\Gamma))$ be the degree of the restriction of σ on Γ where Γ is an irreducible component of V .

$$\deg V = \sum_i \deg v^i(\underline{\sigma}, V) + \sum_{\Gamma \subset V} \deg(\Gamma/\sigma(\Gamma)) \deg \sigma(\Gamma).$$

We can represent Vogel's v -cycles in terms of the first Chern class $c_1(\mathcal{L})$ and the total Segre class of a normal cone $C_W V$ by [7, Theorem 2.4.2, Corollary 2.4.7].

In [10, Section 4] the author presents an interesting calculation of the top Segre classes of the tautological bundles associated to line bundles on Hilbert schemes of n

points on a smooth surface X . The motivation for this calculation was suggested by Donaldson with the computation of theoretical physics. The problem is as follows : let n be a positive integer and $|H|$ be a linear system that induces a map $X \rightarrow \mathbb{P}^{3n-2}$. Let $X^{[n]}$ be the Hilbert scheme of n points of X . For each zero-dimensional subscheme $\xi \in X^{[n]}$ if the map

$$H^0(\mathbb{P}^{3n-2}, \mathcal{O}_{\mathbb{P}}(1)) \rightarrow H^0(\xi, \mathcal{O}_{\xi}(1))$$

fails to be surjective, then ξ does not impose independent conditions on the linear system $|H|$. The authors computation on top Segre classes and N_n would give formulæ for the degree of all higher secant varieties of a surface under sufficient higher very ample embedding conditions.

As a generalization of above results on the degree of the 3-secant variety $\sigma_3(X)$ for arbitrary dimension of X , the main theorem is presented as follows :

Theorem 1.1 (Main theorem). *Let X be a nonsingular projective variety embedded by a 5-very ample line bundle. Let n and d be the dimension and the degree of X . Then the degree of the 3-secant variety $\sigma_3(X)$ is given by the following formula :*

$$\frac{1}{3!} \left\{ d^3 - \sum_{k=0}^n d a_{n,k} \deg s_k(T_X) + \sum_{k=0}^n \sum_{a=0}^k 2^{k-a+n+1} \binom{3n+2}{n-k} \deg(s_a(T_X) \cdot s_{k-a}(T_X)) \right\}.$$

where $a_{n,k} = \binom{2n+1}{n-k} + 2 \sum_{i=k}^n (-1)^{i-k} \binom{3n+2}{n-i} \binom{i-k+n}{n}$.

When attempting to compute the degree of the 3-secant variety for arbitrary dimension of X , it is not possible to imitate the approach in [10] by Lehn. It is because the universal family $Z_3 \subset X \times X^{[3]}$ may be singular (cf. [6]). Additionally the refined Bezout's theorem cannot be applied directly. One possible approach is to set V as the triple join $J(X, X, X)$ of X and σ as the addition map defined by $[x_0, \dots, x_N, y_0, \dots, y_N, z_0, \dots, z_N] \mapsto [x_0 + y_0 + z_0, \dots, x_N + y_N + z_N]$. The image $\sigma(V)$ is the 3-secant variety $\sigma_3(X)$. However the indeterminacy locus of σ can be non-reduced, making computation of $v^i(\underline{\sigma}, V)$ much more difficult. Therefore it is not advisable to imitate the approach in [7].

To circumvent these issues, we propose to use the secant bundle, which was introduced by R.L.E. Schwarzenberger in [11] for the purpose of giving fiber bundle structure of secant lines. While the original construction was based on symmetric product of a given variety, rather than the Hilbert scheme of points, in this paper we adopt the convention presented in [3] and [15] where the secant bundle is a projective bundle over a Hilbert scheme of points. This approach has been used in previous works such as [3], [13], [14] and [15], where the secant bundles was used as a tool for describing the singularity and the normality of the 2-secant variety. In this paper, we utilize the birational morphism from the secant bundle to the 2-secant variety as a resolution of singularities.

In Section 2, we define the total Segre class of a cone and introduce a generalized version of double point formula. This formula allows us to compute the total Segre class $s(X, \sigma_2(X))$ in order to determine the degree of the 3-secant variety. However, the singularity of the 2-secant variety may impede this computation. To overcome this issue, we introduce the notion of higher very ampleness of a line bundle \mathcal{L} and the secant bundle in Section 3. We take the secant bundle as a nonsingular birational model for the 2-secant variety when the line bundle \mathcal{L} satisfies the higher very ampleness condition. As shown in [13], the inverse image of X under this birational morphism is isomorphic to the universal family $Z_2 \subset X \times X^{[2]}$ when \mathcal{L} is 3-very ample. This allows us to regard

Z_2 as the inverse image of X . Since the codimension of Z_2 in the secant bundle is 1, we can treat Z_2 as an effective divisor, which aids in computation. In Section 4, we derive the main theorem which provides the degree formula for the 3-secant variety by using the refined Bezout's theorem and the total Segre class computed in Section 3. Additionally, we compute the multiplicity of $\sigma_2(X)$ along X . In Section 5, we apply the main theorem to cases of curves and surfaces with explicit calculations.

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2 Generalized double point formula

The generalized version of double point formula is one of a useful application of the refined Bezout's theorem that is a fundamental theorem in intersection theory. The double point formula allows us to compute the degree of the intersection of two subvarieties of a projective space. In this paper, we will use it as a starting point to understand the complexity of the 3-secant variety in terms of the 2-secant variety.

To set the stage for the generalized version of double point formula, we first introduce some notation and conventions. Let X be an algebraic scheme over a field k and $S^\bullet := \bigoplus_{\nu} S^\nu$ be a sheaf of graded \mathcal{O}_X -algebras. We assume that the map $\mathcal{O}_X \rightarrow S^0$ is surjective, S^1 is coherent, and S^\bullet is generated by S^1 . We define the cone of S^\bullet as the relative spectrum $C := \mathbf{Spec}(S^\bullet)$ and the projection $C \rightarrow X$. The projective cone of S^\bullet is defined as the relative projective spectrum $P(C) := \mathbf{Proj}(S^\bullet)$ with the projection $P(C) \rightarrow X$. We also define $P(C \oplus 1) := \mathbf{Proj}(S^\bullet[z])$ as the projective completion of C , with projection $q : P(C \oplus 1) \rightarrow X$, and let $\mathcal{O}(1)$ be the tautological line bundle on $P(C \oplus 1)$.

Remark. Throughout this paper, we use the convention for projective bundle as in [8, Appendix B.5.5].

With these conventions in place, we can now define the Segre class of a cone.

Definition 1. (cf. [8, Chapter 4])

For a variety V , we denote the algebraic cycle corresponding to V as $[V]$. The *Segre class* of C is the class in $A_*(X)$ defined by

$$s(C) := q_*\left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [P(C \oplus 1)]\right).$$

There are two useful examples of total Segre classes of cones: when the cone is given by a vector bundle, in which case the Segre class can be interpreted as the inverse of the total Chern class, and when the cone is the normal cone of a closed embedding, in which case the Segre class is called the Segre class of the closed immersion and is denoted $s(X, Y)$.

The codimension i part of the Segre class, $s^i(C)$, of a cone C is defined in [7] as

$$s^i(C) = q_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus 1)]),$$

where $i \geq 0$. The projective completion of C is denoted as $P(C \oplus 1)$ in [8] and as $\mathbb{P}(C \oplus 1)$ in [7]. These two notations refer to the same concept. Alternatively, $s^i(C)$ can also be represented as $p_*(c_1(\mathcal{O}(1))^{i-1} \cap [\mathbb{P}C])$, where $p : \mathbb{P}C \rightarrow X$ is the projective cone of C .

The dimension i part of the Segre class $s(C)$ is denoted $s_i(C)$, and if C is equidimensional, then $s_i(C) = s^{\dim C - i}(C)$. The dimension i part of the Segre class of the normal cone of a closed immersion $X \rightarrow Y$ is denoted $s_i(C_X Y)$ or $s(X, Y)_i$.

For the better understanding of the generalized version of double point formula, we give a definition of two kind of join varieties.

Definition 2. (cf. [7, Chapter 1.3])

Let X and Y be subvarieties of \mathbb{P}^N . The embedded join of X and Y is the closure of the union of all lines connecting a point in X to a point in Y , denoted XY . The abstract or ruled join $J(X, Y)$ is the projective spectrum of the tensor product of the homogeneous coordinate rings of X and Y , denoted $J(X, Y)$.

For example, the ruled join of \mathbb{P}^n and \mathbb{P}^m is \mathbb{P}^{n+m+1} . The affine cone of a subvariety X is defined as the spectrum of the homogeneous coordinate ring of X . The ruled join $J(X, Y)$ consists of the closed points $[x : y]$ where x and y are points of affine cones of X and Y , respectively.

Let X and Y be two projective subvarieties of the projective space \mathbb{P}^N , and let x and y be closed points in X and Y , respectively, with coordinate representations $[x_0 : \cdots : x_N]$ and $[y_0 : \cdots : y_N]$. We define a rational map $J(X, Y) \dashrightarrow XY$ by

$$[x_0 : \cdots : x_N : y_0 : \cdots : y_N] \mapsto [x_0 - y_0 : \cdots : x_N - y_N].$$

The indeterminate locus of this map is defined by the equations $x_0 - y_0 = \cdots = x_N - y_N = 0$. We denote the degree of this rational map $J(X, Y) \dashrightarrow XY$ by $\deg(J/XY)$.

The intersection $X \cap Y$ is embedded into the product variety $X \times Y$ along its diagonal embedding. Let $C_{X \cap Y}(X \times Y)$ denote the normal cone of $X \cap Y$ to $X \times Y$. The following theorem decomposes the information of the embedded join into simpler pieces:

Theorem [7, Theorem 8.2.8] For subvarieties X, Y of \mathbb{P}^N of degree d_X , resp. d_Y and dimension n , resp. m we have

$$\deg XY \deg(J/XY) = d_X d_Y - \sum_{k \geq 0} \binom{n+m+1}{k} \deg s_k(C_{X \cap Y}(X \times Y)).$$

This formula is derived from the refined Bezout's theorem, which is based on Vogel's v and β cycle construction. For further explanations, see [7, Chapter 2] and [7, Chapter 8].

If we set $Y = \sigma_2(X)$, then the embedded join XY is the 3-secant variety $\sigma_3(X)$. Thus, in order to compute $\deg \sigma_3(X)$, we need the degree of the 2-secant variety $d_Y = \deg \sigma_2(X)$ as well as the degrees of the Segre classes $s_k(C_{\Delta(X)}(X \times \sigma_2(X)))$. Assume that $\mathcal{O}_X(1)$ is a 3-very ample line bundle. The degree of the 2-secant variety d_Y can be obtained by applying [7, Theorem 8.2.8] again:

$$2 \deg \sigma_2(X) = d_X^2 - \sum_{k \geq 0} \binom{2n+1}{k} \deg s_k(C_{\Delta(X)}(X \times X)). \quad (1)$$

If \mathcal{L} is 5-very ample, the degree of the rational map $J(X, Y) \dashrightarrow XY$ is 3. (cf. Lemma 4.1) Therefore, by the above theorem, the degree of the 3-secant variety of X can be computed as follows:

$$\frac{1}{3} \left(\frac{d_X(d_X^2 - A)}{2} - B \right) \quad (2)$$

where

$$A := \sum_{j \geq 0} \binom{2n+1}{j} \deg s_j(C_{\Delta(X)}(X \times X)) \quad (3)$$

and

$$B := \sum_{k \geq 0} \binom{3n+2}{k} \deg s_k(C_{\Delta(X)}(X \times \sigma_2(X))). \quad (4)$$

by substituting (1) into [7, Theorem 8.2.8]. The equation (3) is just a linear combination of Segre classes of tangent bundle of X . Therefore, it is sufficient to calculate the term $s(C_{\Delta(X)}(X \times \sigma_2(X)))$ in order to determine the degree of the 3-secant variety. However, the normal cone $C_{\Delta(X)}(X \times \sigma_2(X))$ has singularities along the diagonal $\Delta(X)$, which must be expressed in simpler forms for the calculation to proceed.

3 The secant bundle

In this section, we start with introducing the concepts of higher very ampleness and secant bundles to relate information of double point formula with the information of higher secant varieties. It is often necessary to consider higher very ampleness of the projective embedding in order to ensure the desirable properties of the secant bundle and the inverse image of X under the birational morphism from the secant bundle to the secant variety. (cf. [13] and [15])

Definition 3. (cf. [1] and [4])

A line bundle \mathcal{L} on a complete algebraic variety X over an algebraically closed field k is *d-very ample* if, for every zero-dimensional subscheme Z of X with length less than or equal to $d + 1$, the restriction map

$$r_Z : H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective.

Remark. Note that for two integers $d_1 \geq d_2$, if a line bundle \mathcal{L} is d_1 -very ample, \mathcal{L} is also d_2 -very ample.

A line bundle \mathcal{L} is 0-very ample if and only if it is spanned by global sections, and it is 1-very ample if and only if it is very ample. For instance, the d -uple Veronese embedding of \mathbb{P}^n is embedded by the d -very ample line bundle $\mathcal{O}(d)$, as shown in [2, Cor 2.1 and Prop 2.2].

Since information about higher secant varieties is often very complicated and difficult to work with, we define and use the secant bundle as a nonsingular birational model for the secant variety. We compute the required algebraic cycles for the degree formula of the 3-secant variety.

In this chapter, we consider the problem of computing the degree of the 3-secant variety of a nonsingular projective variety X that is embedded by a 5-very ample line bundle \mathcal{L} . The information about higher secant varieties can be difficult to work with, so we define and use the secant bundle as a nonsingular birational model for the secant variety. We compute the required algebraic cycles for the degree formula of the 3-secant variety. We let $X^{[2]}$ be the Hilbert scheme of 2 points of X and \mathbb{P}^N be the projective space $\mathbb{P}^{H^0(X, \mathcal{L})}$. It is known that $X^{[2]}$ is smooth for any dimension of X (cf. [6]). We consider the projections $\pi_1 : X \times X^{[2]} \rightarrow X$ and $\pi_2 : X \times X^{[2]} \rightarrow X^{[2]}$, and the universal family $Z_2 \subset X \times X^{[2]}$. We let I_{Z_2} be the ideal sheaf of Z_2 on $X \times X^{[2]}$ and \mathcal{O}_{Z_2} be the structure sheaf of Z_2 .

We construct the following exact sequence of sheaves:

$$0 \rightarrow \pi_1^* \mathcal{L} \otimes I_{Z_2} \rightarrow \pi_1^* \mathcal{L} \rightarrow \pi_1^* \mathcal{L} \otimes \mathcal{O}_{Z_2} \rightarrow 0.$$

Since the restriction of π_2 to Z_2 , $\pi_2|_{Z_2} : Z_2 \rightarrow X^{[2]}$ is flat of degree 2, the sheaf $\pi_{2*}(\pi_1^* \mathcal{L} \otimes \mathcal{O}_{Z_2})$ is locally free of rank 2. Denote the tautological bundle associated with \mathcal{L} by $E_{\mathcal{L}} = \pi_{2*}(\pi_1^* \mathcal{L} \otimes \mathcal{O}_{Z_2})$. The fiber of the vector bundle $E_{\mathcal{L}}$ at a point $Z \in X^{[2]}$ is given by $H^0(X, \mathcal{L} \otimes \mathcal{O}_Z)$. Since \mathcal{L} is 1-very ample, the restriction map $H^0(X, \mathcal{L}) \rightarrow H^0(X, \mathcal{L} \otimes \mathcal{O}_Z)$ is surjective. This implies that the morphism $\pi_{2*} \pi_1^* \mathcal{L} \rightarrow E_{\mathcal{L}}$ is also surjective. As a result, we have a composition of morphisms

$$\mathbb{P}(E_{\mathcal{L}}) \rightarrow \mathbb{P}H^0(X, \mathcal{L}) \times X^{[2]} \rightarrow \mathbb{P}H^0(X, \mathcal{L}) = \mathbb{P}^N,$$

which is a closed immersion of projective varieties. In [15], it is shown that if \mathcal{L} is a 1-very ample line bundle, the image of this map is the 2-secant variety $\sigma_2(X)$. We denote this map as $r : \mathbb{P}(E_{\mathcal{L}}) \rightarrow \sigma_2(X)$. The projective bundle $\mathbb{P}(E_{\mathcal{L}})$ is known as the secant bundle of lines, and it is birational to the 2-secant variety $\sigma_2(X)$ (as shown in [3] and [15]).

It is a well-known fact that the universal family Z_2 is isomorphic to the blow-up of $X \times X$ along its diagonal $\Delta(X)$ (see [9, Remark 2.5.4]). We denote the blow-up morphism by

$$\eta : Bl_{\Delta(X)}(X \times X) \rightarrow X \times X$$

and the involution map by

$$\rho : Bl_{\Delta(X)}(X \times X) \cong Z_2 \rightarrow X^{[2]}.$$

The exceptional divisor of η on $Bl_{\Delta(X)}(X \times X)$ is denoted by E . The projections $X \times X \rightarrow X$ are denoted by pr_i . The following diagram commutes:

$$\begin{array}{ccc} Bl_{\Delta(X)}(X \times X) \cong Z_2 & \xrightarrow{\eta} & X \times X \\ \downarrow \rho & & \downarrow \\ X^{[2]} & \xrightarrow{\epsilon} & X^{(2)} \end{array}$$

where $X^{(2)}$ is the quotient $(X \times X)/S_2$ and $\epsilon : X^{[2]} \rightarrow X^{(2)}$ is the Hilbert-Chow morphism.

In [13, Lemma 1.2], it is shown that the scheme-theoretic inverse image $r^{-1}(X)$ under the map $r : \mathbb{P}(E_{\mathcal{L}}) \rightarrow \sigma_2(X)$ is isomorphic to Z_2 when \mathcal{L} is 3-very ample. Note that $X \times X^{[2]}$ is a closed subvariety of $\mathbb{P}^N \times X^{[2]}$ and hence we can regard Z_2 is a closed subvariety of the secant bundle $\mathbb{P}(E_{\mathcal{L}})$ in a natural way. By adjusting the isomorphism, we can ensure that the composition of the maps $Z_2 \rightarrow X \times X^{[2]} \rightarrow X$ corresponds to the composition of the maps $q := \text{pr}_1 \circ \eta$. From this point on, we will identify $r^{-1}(X)$ with Z_2 .

Denote by Γ_q the graph of $q : Z_2 \rightarrow X$. The product morphism $\text{id}_X \times r : X \times \mathbb{P}(E_{\mathcal{L}}) \rightarrow X \times \sigma_2(X)$ has inverse image of the diagonal $\Delta(X)$ of $X \times \sigma_2(X)$ given by the graph locus Γ_q .

According to [8, Proposition 4.2 (a)], we have that

$$r_* s(Z_2, \mathbb{P}(E_{\mathcal{L}})) = s(X, \sigma_2(X)). \quad (5)$$

and

$$(\text{id}_X \times r)_* s(\Gamma_q, X \times \mathbb{P}(E_{\mathcal{L}})) = s(\Delta(X), X \times \sigma_2(X)). \quad (6)$$

Proposition [8, Proposition 4.2 (a)] Let $f : Y' \rightarrow Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \rightarrow X$ the induced morphism

(a) If f is proper, Y irreducible, and f maps each irreducible component of Y' onto Y , then

$$g_*(s(X', Y')) = \deg(Y'/Y)s(X, Y).$$

Equation (6) can be used to calculate (4). Consider the following closed immersions:

$$\Gamma_q \subset X \times Z_2 \subset X \times \mathbb{P}(E_{\mathcal{L}})$$

Since each term is nonsingular, each of these closed immersions is a regular immersion. Therefore, we have the following exact sequence of normal bundles:

$$0 \rightarrow N_{\Gamma_q}(X \times Z_2) \rightarrow N_{\Gamma_q}(X \times \mathbb{P}(E_{\mathcal{L}})) \rightarrow N_{X \times Z_2}(X \times \mathbb{P}(E_{\mathcal{L}}))|_{\Gamma_q} \rightarrow 0 \quad (7)$$

After simplification, we obtain the following:

$$\begin{aligned} s(N_{\Gamma_q}(X \times Z_2)) &= (\Gamma_q \rightarrow X)^* s(T_X) = (\text{id} \times r)^* s(T_{\Delta(X)}) \\ s(N_{X \times Z_2}(X \times \mathbb{P}(E_{\mathcal{L}}))|_{\Gamma_q}) &= (\Gamma_q \rightarrow Z_2)^* s(N_{Z_2} \mathbb{P}(E_{\mathcal{L}})) \end{aligned}$$

Using (7), we get:

$$s(N_{\Gamma_q}(X \times \mathbb{P}(E_{\mathcal{L}}))) = (\text{id} \times r)^* s(T_{\Delta(X)}) \cdot (\Gamma_q \rightarrow Z_2)^* s(N_{Z_2} \mathbb{P}(E_{\mathcal{L}})). \quad (8)$$

Note that $(\text{id} \times r|_{\Gamma_q})_* \circ (\Gamma_q \rightarrow Z_2)^*$ is $(\Delta(X) \rightarrow X)^* \circ r_*$ on Chow rings. From (6) and (8), we get the following :

$$s(\Delta(X), X \times \sigma_2(X)) = (\Delta(X) \rightarrow X)^* (s(T_X) \cdot r_* s(N_{Z_2} \mathbb{P}(E_{\mathcal{L}})) \cap [X]) \quad (9)$$

So, it remains to compute the total Segre class $s(\mathcal{N}_{Z_2} \mathbb{P}(E_{\mathcal{L}}))$. Since $r^{-1}(X)$ can be regarded as an effective divisor of $\mathbb{P}(E_{\mathcal{L}})$, we can compute $s(Z_2, \mathbb{P}(E_{\mathcal{L}}))$ by [8, Cor 4.2.2] as follow:

$$s(Z_2, \mathbb{P}(E_{\mathcal{L}})) = \frac{[Z_2]}{1 + [Z_2]}. \quad (10)$$

In order to proceed, it is necessary to express the term $[Z_2]$ in terms of the tautological line bundle ζ of $\mathbb{P}(E_{\mathcal{L}})$ and $\pi^* \beta$, where β is a divisor on $X^{[2]}$ (cf. Fulton 2013, Chapter 3.3). This is achieved by calculating the first Chern class of $E_{\mathcal{L}}$ in proposition 3.1.

Let h be the divisor corresponding to a line bundle \mathcal{L} on X . We denote the pullback of h under the i -th projection as h_i . The morphism ρ is an involution map, so $\rho_* \eta^* h_1 = \rho_* \eta^* h_2$. We define $H = \rho_* \eta^* h_1 = \rho_* \eta^* h_2$ and $\delta = \frac{1}{2} \rho_* E$. The following proposition may be known to experts but I could not find appropriate references, so I will provide a proof.

Proposition 3.1. $c_1(E_{\mathcal{L}}) = H - \delta$

Proof. Let $\pi : \mathbb{P}(E_{\mathcal{L}}) \rightarrow X^{[2]}$ be the projection map of a projective bundle. Consider the normal sheaf $\mathcal{N} := \mathcal{N}_{Z_2/X \times X^{[2]}}$ and the closed immersion $j : Z_2 \rightarrow X \times X^{[2]}$ with the composition $q : Z_2 \rightarrow X$ of η and the first projection. The morphism $\pi_1|_{Z_2} : Z_2 \rightarrow X^{[2]}$ is a finite flat morphism, and $\pi_{2*}(\pi_1^* \mathcal{L} \otimes \mathcal{O}_{Z_2})$ is a locally free sheaf on $X^{[2]}$, so by Grauert's theorem, all higher direct images $R^i \pi_{2*}(\pi_1^* \mathcal{L} \otimes \mathcal{O}_{Z_2})$ vanish for $i \geq 1$. Let T_X be the tangent sheaf of X .

Since j is an affine morphism, there are no higher direct images and we obtain followings :

$$\mathrm{ch}E_{\mathcal{L}} = \pi_{2*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2}) \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X)$$

and

$$\mathrm{ch}j_*\mathcal{O}_{Z_2} = j_*((\mathrm{td}\mathcal{N})^{-1})$$

by applying the Grothendieck-Riemann-Roch theorem. Note that $\pi_2 \circ j = \rho$.

By the projection formula, we have:

$$\begin{aligned} \mathrm{ch}E_{\mathcal{L}} &= \pi_{2*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2}) \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X) \\ &= \pi_{2*}(j_* \mathrm{ch}(\mathrm{td}\mathcal{N})^{-1} \pi_1^* \mathrm{ch}\mathcal{L} \pi_1^* \mathrm{td}T_X) \\ &= (\pi_{2*} \circ j_*)(\mathrm{ch}(\mathrm{td}\mathcal{N})^{-1} j^* \pi_1^* \mathrm{ch}\mathcal{L} j^* \pi_1^* \mathrm{td}T_X) \\ &= \rho_*((\mathrm{td}\mathcal{N})^{-1} q^* \mathrm{ch}\mathcal{L} q^* \mathrm{td}T_X). \end{aligned}$$

Consider the sheaf sequence

$$0 \rightarrow \mathcal{N}^* \rightarrow q^*\Omega_X \rightarrow \mathcal{O}_E(-E) \rightarrow 0$$

as given in [6, Lemma 2.1]. (Note that the morphism q in proposition 3.1 and in [6] are different morphisms.) Taking dual of this sequence, we obtain :

$$0 \rightarrow q^*T_X \rightarrow \mathcal{N} \rightarrow \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \rightarrow 0.$$

Therefore we have

$$(\mathrm{td}\mathcal{N})^{-1} = (q^* \mathrm{td} T_X)^{-1} \cdot (\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}))^{-1}.$$

To evaluate $\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2})$, we consider the exact sequence :

$$0 \rightarrow \mathcal{O}_{Z_2}(-2E) \rightarrow \mathcal{O}_{Z_2}(-E) \rightarrow \mathcal{O}_E(-E) \rightarrow 0.$$

Taking the dual of this sequence, we have :

$$0 \rightarrow \mathcal{O}_{Z_2}(E) \rightarrow \mathcal{O}_{Z_2}(2E) \rightarrow \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \rightarrow 0.$$

Therefore, we obtain

$$\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) = \frac{2E}{1 - e^{-2E}} / \frac{E}{1 - e^{-E}} = \frac{2}{1 + e^{-E}}$$

and

$$\begin{aligned} \mathrm{ch}E_{\mathcal{L}} &= \rho_* \left(q^* \mathrm{ch}\mathcal{L} \cdot \frac{1 + e^{-E}}{2} \right) \\ &= \rho_* \left(\left(1 + \eta^* h_1 + \frac{1}{2!} \eta^* h_1^2 + \frac{1}{3!} \eta^* h_1^3 + \dots \right) \cdot \left(1 - \frac{1}{2}E + \frac{1}{4}E^2 - \frac{1}{12}E^3 + \dots \right) \right) \\ &= \rho_* \left(1 + \left(\eta^* h_1 - \frac{1}{2}E \right) + \left(\frac{1}{2} \eta^* h_1^2 - \frac{1}{2} \eta^* h_1 \cdot E + \frac{1}{4}E^2 \right) + \dots \right) \\ &= 2 + (H - \delta) + \left(\frac{1}{2} \rho_* \eta^* h_1^2 - \frac{1}{2} \rho_* (\eta^* h_1 \cdot E) + \frac{1}{4} \rho_* E^2 \right) + \dots \end{aligned}$$

So we obtain $c_1(E_{\mathcal{L}}) = H - \delta$. □

In equation (10), it is deduced that

$$s(Z_2, \mathbb{P}(E_{\mathcal{L}})) = \sum_{i \geq 0} (-1)^i [Z_2]^{i+1}.$$

The Chow ring of the projective bundle, $A^*\mathbb{P}(E_{\mathcal{L}})$, is well-known, but it is simpler to compute the restriction on Z_2 rather than on the projective bundle. Therefore, the power of $[Z_2]$ is computed as follows:

$$[Z_2]^{i+1} = ([Z_2]|_{Z_2})^i$$

for $i \geq 0$. Additionally, it is straightforward to apply the push-forward r_* on the cycles on Z_2 , as the image $r(Z_2)$ is X . Therefore, the equation (10) can be rewritten as:

$$s(Z_2, \mathbb{P}(E_{\mathcal{L}})) = \sum_{i \geq 0} (-1)^i ([Z_2]|_{Z_2})^i. \quad (11)$$

To continue, we require an expression of $[Z_2]|_{Z_2}$ as a linear combination of generators of the Chow ring A^*Z_2 .

Proposition 3.2. $[Z_2]|_{Z_2} = 2E + \eta^*(h_1 - h_2)$ in A^1Z_2 where $[Z_2]$ is a cycle associated to scheme Z_2 .

Proof. We denote the canonical divisors of $X^{[2]}$ and Z_2 as $K_{X^{[2]}}$ and K_{Z_2} , respectively. The ramification divisor of ρ is E , so we have

$$\rho^*K_{X^{[2]}} = K_{Z_2} - E. \quad (12)$$

Equation (12) can be found in some literature, or it can be deduced directly from the following exact sequence of sheaves:

$$0 \rightarrow \rho^*T_{X^{[2]}}^* \rightarrow T_{Z_2}^* \rightarrow \mathcal{O}_E(-E) \rightarrow 0.$$

Let ζ be the first Chern class of the tautological line bundle of $\mathbb{P}(E_{\mathcal{L}})$. The normal bundle $N_{Z_2}\mathbb{P}(E_{\mathcal{L}})$ is represented by $\mathcal{O}_{Z_2}([Z_2])$,

$$c_1(\mathcal{O}_{Z_2}([Z_2])) = K_{Z_2} - K_{\mathbb{P}(E_{\mathcal{L}})|_{Z_2}}.$$

In addition, we have the following adjunction formula for the projective bundle:

$$K_{\mathbb{P}(E_{\mathcal{L}})} = -2\zeta + \pi^*c_1(E_{\mathcal{L}}) + \pi^*K_{X^{[2]}}.$$

Note that $(\pi^*K_{X^{[2]}})|_{Z_2} = \rho^*K_{X^{[2]}} = K_{Z_2} - E = K_{Z_2} - \rho^*\delta$. Then

$$c_1(\mathcal{O}_{Z_2}([Z_2])) = 2\zeta|_{Z_2} - \rho^*(H - \delta) + \rho^*\delta$$

by proposition 3.1. Recall that $c_1(\mathcal{O}_{Z_2}(Z_2)) = [Z_2]|_{Z_2}$ and hence we get

$$[Z_2]|_{Z_2} = 2\zeta|_{Z_2} - \rho^*(H - 2\delta). \quad (13)$$

We need information about $\zeta|_{Z_2}$. We know that Z_2 is a closed subvariety of $X \times X^{[2]} \subset \mathbb{P}H^0(X, \mathcal{L}) \times X^{[2]}$. Therefore, we can choose a suitable section of $\zeta|_{Z_2}$ whose zero locus is $(h \times X^{[2]}) \cap Z_2$.

Recall that we have an isomorphism $Z_2 \cong Bl_{\Delta(X)}(X \times X)$, which allows us to view $Z_2 \subset X \times X^{[2]} \rightarrow X$ as equivalent to $Bl_{\Delta(X)}(X \times X) \rightarrow X \times X \rightarrow X$, where the second morphism is pr_1 . Thus, we obtain $\zeta|_{Z_2} = \eta^*h_1|_{Z_2}$.

Substituting this result into (13), we find that

$$[Z_2]|_{Z_2} = 2E + \eta^*(h_1 - h_2).$$

□

4 Main theorem

In order to derived the degree formula for the 3-secant variety, it is necessary to utilize each term of [7, Theorem 8.2.8]. To accomplish this, we present the following lemma.

Lemma 4.1. *Let X be a nonsingular projective variety that is embedded by a 5-very ample line bundle. Let Y be the 2-secant variety $\sigma_2(X)$. Let J be the ruled join $J(X, Y)$. Then $\deg(J/XY)$ is 3.*

Proof. Let w be a general point of $\sigma_3(X) \setminus \sigma_2(X)$. If there are two secant planes that contain w , 6 points of X do not satisfy the independent condition of 5-very ampleness. (cf. [5, Remark 1.7]) So, there exists a unique plane spanned by three points of X that contains w . Let x, y , and z be three distinct points of X such that their linear span contains w . Let a, b , and c be the points of intersection of \overline{xw} with \overline{yz} , \overline{yw} with \overline{zx} , and \overline{zw} with \overline{xy} , respectively. Then, the three points of the ruled join $J(X, Y)$ corresponding to the ratios between $(x, w), (w, a), (y, w), (w, b)$, and $(z, w), (w, c)$ are exactly the inverse image of the rational map $J(X, Y) \dashrightarrow XY$. \square

Prior to starting a proof of the main theorem, we denote the tangent sheaf of X by T_X . It follows from the definition of Segre class that $s_k(C_{\Delta(X)}(X \times X)) = s_{n-k}(T_X)$ as a Segre class of a locally free sheaf. With this notation established, we now proceed to the proof of the main theorem:

Proof of main theorem. Recall that X is a smooth projective variety of dimension n and $E \subset Z_2$ can be regarded as a projective bundle associated with the tangent bundle on X . Using equations (5), (11), and proposition 3.2, we obtain:

$$s(X, \sigma_2(X)) = q_* \left(\sum_{i \geq 0} (-1)^i (2E + \eta^*(h_1 - h_2))^i \right). \quad (14)$$

Let $P(C_{\Delta(X)}(X \times X))$ be the projective tangent cone to X and $\mathcal{O}(1)$ be the tautological line bundle. Let $g : P(C_{\Delta(X)}(X \times X)) \rightarrow X$ be the projection map. Note that $P(C_{\Delta(X)}(X \times X))$ is isomorphic to the projective bundle $\mathbb{P}(\Omega_X^1)$ where Ω_X^1 is the sheaf of Kähler differentials. The total Segre classes $s(C_{\Delta(X)}(X \times X))$ and $s(T_X)$ are equal but they have different conventions for indexes: $s_k(T_X) = s_{n-k}(C_{\Delta(X)}(X \times X))$ for $0 \leq k \leq n$. As schemes, E and $P(C_{\Delta(X)}(X \times X))$ are the same and hence $E|_E = \mathcal{O}(-1)$ holds.

Remark. Recall that we use the convention for projective bundles and tautological line bundles as in [8, Appendix B.5.5].

Therefore we have

$$\eta_* E^i = \eta_*(E|_E)^{i-1} = (-1)^{i-1} g_*(c_1(\mathcal{O}(1))^{i-1} \cap [P(C_{\Delta(X)}(X \times X))])$$

for $i \geq 1$ and hence $\eta_* E^i = (-1)^{i-1} s_{i-n}(T_{\Delta(X)})$ for $i \geq n$. Since $\eta^*(h_1 - h_2) \cdot E$ is zero, we obtain that

$$\eta_* \sum_{l=n}^{2n} (-1)^l (2E + \eta^*(h_1 - h_2))^l = \sum_{l=n}^{2n} (-1)^l (h_1 - h_2)^l - \sum_{l=n}^{2n} 2^l s_{l-n}(T_X) \cap [X]. \quad (15)$$

The remaining calculation simply involves taking the first projections, which yields:

$$\begin{aligned} (\text{pr}_1)_*(h_1 - h_2)^l &= \sum_{a=0}^l (-1)^{l-a} \binom{l}{a} h^a \cdot (\text{pr}_1)_* \text{pr}_2^* h^{l-a} \\ &= (-1)^n \binom{l}{l-n} h^{l-n} \cdot d[X]. \quad (l \geq n) \end{aligned}$$

By substituting (15) into (14) we have :

$$\begin{aligned} s(X, \sigma_2(X)) &= \sum_{l=n}^{2n} (-1)^{n+l} \binom{l}{n} \cdot d h^{l-n} - \sum_{l=n}^{2n} 2^l s_{l-n}(T_X) \cap [X] \\ &= \sum_{i=0}^n (-1)^i \binom{i+n}{n} \cdot d h^i - \sum_{i=0}^n 2^{i+n} s_i(T_X) \cap [X]. \end{aligned}$$

From equation (9), we obtain:

$$\begin{aligned} s(\Delta(X), X \times \sigma_2(X)) &= s(T_X) \cap s(X, \sigma_2(X)) \\ &= \sum_{i=0}^n \sum_{a+b=i} \{(-1)^b \binom{b+n}{n} d h^b \cdot s_a(T_X) \cap [X] - 2^{b+n} s_a(T_X) s_b(T_X) \cap [X]\}. \end{aligned}$$

by omitting the pull back $(\Delta(X) \rightarrow X)^*$. Therefore, we obtain the main formula from equation (2), which completes the proof. \square

Corollary. *The multiplicity of $\sigma_2(X)$ along X is $d - 2^n$.*

Proof. The multiplicity of $\sigma_2(X)$ along X is the coefficient of $[X]$ in the class $s(X, \sigma_2(X))$. (cf. [8, Chapter 4.3]) \square

5 Examples

5.1 In the case of curves

Consider the case where C is a smooth projective curve. Let g be the genus of a curve C . The degree of each term of Segre class T_C is given by $\deg s_0(T_C) = d$ and $\deg s_1(T_C) = 2g - 2$. By substituting those terms in the main formula we get :

$$\deg \sigma_3(C) = \frac{1}{3!} (d^3 - 9d^2 + 26d + 24 - 6dg - 24g)$$

which can be found in [12, Proposition 1].

5.2 In the case of surfaces

Consider the case where S is a smooth projective surface. Let K be the canonical divisor of S , let $d = h^2$, let $\pi = h \cdot K$, let $\kappa = K^2$, and let $e = c_2$ be the topological Euler characteristic. The total Segre class of the tangent sheaf T_S of S has degree 0, 1, and 2 terms, which are given by:

$$s_0(T_S) = [S], \quad s_1(T_S) = K, \quad s_2(T_S) = \kappa - e.$$

By substituting those terms in the main formula we get :

$$\deg \sigma_3(S) = \frac{1}{3!} (d^3 - 30d^2 + 224d - 3d(5\pi + \kappa - e) + 192\pi + 56\kappa - 40e)$$

which can be found in [10, Section 4].

We can directly calculate the total Segre classes $s(S, \sigma_2(S))$. Note that $[Z_2]|_{Z_2}$ is represented by $2E + \eta^*(h_1 - h_2)$. This yields the followings :

$$\begin{aligned} ([Z_2]|_{Z_2})^2 &= 4E^2 + 4E \eta^*(h_1 - h_2) + \eta^*(h_1 - h_2)^2 \\ ([Z_2]|_{Z_2})^3 &= 8E^3 + 12E^2 \eta^*(h_1 - h_2) + 6E \eta^*(h_1 - h_2)^2 + \eta^*(h_1 - h_2)^3 \\ ([Z_2]|_{Z_2})^4 &= 16E^4 + 32E^3 \eta^*(h_1 - h_2) + 24E^2 \eta^*(h_1 - h_2)^2 + 8E \eta^*(h_1 - h_2)^3 + \eta^*(h_1 - h_2)^4 \end{aligned}$$

Since $\eta_* E^l = (-1)^{l-1} s_{l-2}(T_{\Delta(S)})$ and $\Delta(S) \cdot (h_1 - h_2) = 0$, we obtain :

$$\begin{aligned} \eta_*([Z_2]|_{Z_2})^2 &= -4[\Delta(S)] + (h_1 - h_2)^2 & q_*([Z_2]|_{Z_2})^2 &= (d-4)[S] \\ \eta_*([Z_2]|_{Z_2})^3 &= 8s_1(T_{\Delta(S)}) + (h_1 - h_2)^3 & q_*([Z_2]|_{Z_2})^3 &= 8s_1(T_S) + 3dh \\ \eta_*([Z_2]|_{Z_2})^4 &= -16s_2(T_{\Delta(S)}) + 6h_1^2 h_2^2 & q_*([Z_2]|_{Z_2})^4 &= -16s_2(T_S) + 6d^2 \end{aligned}$$

The total Segre class $s(S, \sigma_2(S))$ is the summation of the three terms on the right-hand side of the equation. Therefore, the total Segre class $s(\Delta(S), S \times \sigma_2(S))$ is given by:

$$(d-4)[S] + ((d-12)s_1(T_S) - 3dh) + (-8s_1(T_S)^2 + (d-20)s_2(T_S) - 3dhs_1(T_S) + 6d^2).$$

Using [7, Theorem 8.2.8], we obtain the correct formula for $\deg \sigma_3(S)$ again.

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