

Additionally, as a result of the Schwarzian inequality, we have

$$|\psi'(z)| \leq \frac{1 - |\psi(z)|^2}{1 - |z|^2}, \quad (z \in \mathbb{D}). \quad (2.10)$$

Moreover, from equations (2.8)-(2.10), we deduce

$$|f'(z)| \leq \left( |\psi(z)| + \frac{r(1 - |\psi(z)|^2)}{(1 - r^2)(\mathcal{LP}(r))} \right) |g'(z)|.$$

Substituting  $|\psi(z)| = \sigma$  ( $0 \leq \sigma \leq 1$ ), results in

$$|f'(z)| \leq \Psi(r, \sigma) |g'(z)|,$$

where

$$\Psi(r, \sigma) = \sigma + \frac{r(1 - \sigma^2)}{(1 - r^2)(\mathcal{LP}(r))}.$$

We need to determine  $r_m \leq r^*$  so that

$$r_m = \max\{r \in [0, r^*] : \Psi(r, \sigma) \leq 1 \ \forall \sigma \in [0, 1]\}.$$

Equivalently if  $\Phi(r, \sigma) := (1 - r^2)(\mathcal{LP}(r)) - r(1 + \sigma)$  then we need to determine

$$r_m = \max\{r \in [0, r^*] : \Phi(r, \sigma) \geq 0 \ \forall \sigma \in [0, 1]\}.$$

Since  $\partial\Phi/\partial\sigma = -r < 0$ , then  $\max_{\sigma \in [0, 1]} \Phi(r, \sigma) = \Phi(r, 0) =: \phi_0(r)$ . Further it is evident that, as  $\phi_0(0) = 1 > 0$  and  $\phi_0(r^*) = -r^* < 0$ , then there exists  $r_m \leq r^*$ , a smallest positive root of the equation given in (2.7) such that  $\phi_0(r) \geq 0$  for each  $r \in [0, r_m]$ . This completes the proof. ■

In 2017, Peng and Zhong [22], introduced the class  $\Omega \subset \mathcal{A}$ , defined as

$$\Omega = \{f \in \mathcal{A} : |zf'(z) - f(z)| < 1/2\}.$$

We conclude this section by determining sharp  $\Omega$ -radius for the class  $\mathcal{F}_{\mathcal{LP}}$ .

**Theorem 2.21.** *Let  $f \in \mathcal{F}_{\mathcal{LP}}$ , then  $f \in \Omega$  in  $|z| < r_{\mathcal{L}} \approx 0.522\dots$  is the smallest positive root of*

$$4f_0(r)(\log((1 + \sqrt{r})/(1 - \sqrt{r})))^2 = \pi^2$$

and

$$g_0(z) = z \left( \exp \int_0^z \frac{\mathcal{P}_0(-t)}{t} dt \right) = z + \frac{8}{\pi^2} z^2 - \frac{8}{3\pi^4} (\pi^2 - 12) z^3 + \frac{8}{135\pi^6} (1440 - 360\pi^2 + 23\pi^4) z^4 - \dots. \quad (2.11)$$

*This is a sharp estimate.*

*Proof.* Since  $f \in \mathcal{F}_{\mathcal{LP}}$ , then as a consequence of Remark 2.4 for  $|z| = r < 1$ , we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < |\mathcal{LP}(r) - 1| = |\mathcal{P}_0(r)|.$$

Due to the growth theorem as mentioned in [15 Theorem 1] and Theorem 2.2 we observe that  $|f(z)| \leq g_0(r)$ , where  $g_0(r)$  is given by (2.11). Further

$$|zf'(z) - f(z)| = |f(z)| \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq g_0(r) |\mathcal{P}_0(r)|.$$

Thus  $g_0(r) |\mathcal{P}_0(r)| \leq 1/2$  provided  $|z| < r_{\mathcal{L}} \approx 0.522864$ . Hence the result is established. ■