By omitting pairs (j,k) of $I\times I$ satisfying $j+k=\ell$, Lemma 3.4 gives that

$$N-\varphi(\ell)(q+1)=\sum_{d\mid\ell}\sum_{\substack{1\le t\le\frac{\ell}{d},\\ (t,\frac{\ell}{d})=1}}\sum_{(j,k)\in S'(d,t)}\chi(-1)^{j+k}J(\chi^j,\chi^k)c_\ell(2j+k).$$

For $(j,k)\in S'(d,t)$, one has $(2j+k,\ell)=d$ and then $c_\ell(2j+k)=c_\ell(d)$ by Corollary 3.3. Thus,

$$N-\varphi(\ell)(q+1)=\sum_{d\mid\ell}c_\ell(d)f(d)$$

where

$$f(d)=\sum_{\substack{1\le t\le\frac{\ell}{d},\\ (t,\frac{\ell}{d})=1}}\sum_{(j,k)\in S'(d,t)}\chi(-1)^{j+k}J(\chi^j,\chi^k).$$

We need to estimate $|f(d)|$.

By definition, every pair (j,k) of $S'(d,t)$ satisfies $j+k\not\equiv 0\pmod{\ell}$ and thus $|J(\chi^j,\chi^k)|=\sqrt{q}$ by (v) of Lemma 2.2. Since $|\chi(-1)|=1$, it follows that

$$|f(d)|\le\sum_{\substack{1\le t\le\frac{\ell}{d},\\ (t,\frac{\ell}{d})=1}}|S'(d,t)|\sqrt{q}.$$

Now, we compute $|S'(d,t)|$. Observe that every pair (j,k) in $S'(d,t)$ is determined by $j\in I$ with the proviso that $j+k\neq\ell$. Thus, for $(j,k)\in I\times I$ satisfying the congruence $2j+k\equiv td\pmod{\ell}$ we have to exclude the pair (j,k) with $j\equiv td\pmod{\ell}$. Note that $td\le\ell$ while $j\le\ell-1$, this congruence can occur only when $d\lneqq\ell$ and $j=td$. Moreover, as $k\neq 0$, we also need to exclude the case where $2j\equiv td\pmod{\ell}$. This depends on the parity of ℓ. We discuss in the next paragraph to steer clear of confusing.

Suppose that ℓ is odd. Let $s\in I$ be such that $2s\equiv 1\pmod{\ell}$. Then, we need to exclude $j\in I$ such that $j\equiv std\pmod{\ell}$. If $d=\ell$, then there is no such j because $j\not\equiv 0\pmod{\ell}$. When $d\lneqq\ell$, there is exactly one $j_0\in I$ satisfying $j_0\equiv std\pmod{\ell}$. Remember that we also have to exclude the case where $j=td$. As a consequence, if ℓ is odd, then

$$|S'(d,t)|=\begin{cases}|I| & \text{if } d=\ell;\\ |I|-2 & \text{if } d\neq\ell.\end{cases}$$

Now we assume that ℓ is even. There are three cases to consider: (i) td is odd, (ii) td is even and $d\lneqq\ell$ and (iii) $t=1,d=\ell$. For case (i), since td is odd, there is no j such that $2j\equiv td\pmod{\ell}$. Only the case where $j=td$ has to be excluded. For (ii) and (iii), we have that td is even and then there is some $j_1\in I$ such that $2j_1\equiv td\pmod{\ell}$. In fact, we have $j_1\equiv\frac{td}{2}\pmod{\frac{\ell}{2}}$. If $d\lneqq\ell$, then either $j_1=\frac{td}{2}$ or $j_1=\frac{td}{2}+\frac{\ell}{2}$ and in particular, $j_1\neq td$ in this case. If $d=\ell$, then $t=1$ and $j_1=\frac{\ell}{2}$. We conclude that

$$|S'(d,t)|=\begin{cases}|I|-1 & \text{if } d=\ell;\\ |I|-1 & \text{if } d\neq\ell \text{ and } td \text{ is odd;}\\ |I|-3 & \text{otherwise.}\end{cases}$$