where ζ_n is a primitive *n*-th root of 1 in \mathbb{C} . Studying on cyclotomic polynomials, O. Hölder [Höl36] showed that the sum $c_n(m)$ has a nice closed form in terms of the Euler and Möbius functions. Denote φ the Euler's totient function and μ the Möbius function. We present it in the following lemma where the right-hand side is also called *von Sterneck function* [Ste03]. A proof is given below to the readers for convenience. For different proofs, one refers to [AA53], [Mol52] and [HW08] Theorem 272].

Lemma 3.2. *Let* $n \in \mathbb{N}$ *and* $m \in \mathbb{Z}$. *Then*

$$c_n(m) = \mu\left(\frac{n}{(n,m)}\right) \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,m)}\right)}.$$

Proof. First of all, suppose that m = 1. Recall an elementary formula that

$$\sum_{k|r} \mu(k) = \begin{cases} 1 & \text{if } r = 1; \\ 0 & \text{if } r > 0, \end{cases}$$

for $r \in \mathbb{N}$ ([Bur11], Theorem 6.6]). Then

$$c_n(1) = \sum_{t=1}^n \zeta_n^t \sum_{k|(t,n)} \mu(k) = \sum_{k|n} \mu(k) \sum_{\substack{1 \le t \le n, \\ k|(t,n)}} \zeta_n^t = \sum_{k|n} \mu(k) \sum_{\substack{1 \le t \le \frac{n}{k}}} (\zeta_n^k)^{t'}$$

where $t' = \frac{t}{k}$. Since ζ_n^k is a primitive $\frac{n}{k}$ -th root of 1, the last sum gives 1 if k = n and 0 if k < n. Hence, $c_n(1) = \mu(n)$.

For general $m \in \mathbb{Z}$, we rewrite $c_n(m)$ as

$$c_n(m) = \sum_{t \in (\mathbb{Z}/n\mathbb{Z})^{\times}} z^t$$

where $z = \zeta_n^m$. Let $d = \frac{n}{(n,m)}$. Recall that $(\mathbb{Z}/n\mathbb{Z})^{\times} = M \cdot N$ where M and N are given in Lemma 3.1 Note that $z = \zeta_n^m$ is a primitive d-th root of 1 since $\left(d, \frac{m}{(n,m)}\right) = 1$. Hence,

$$c_n(m) = \sum_{t_1 \in M} \sum_{t_2 \in N} z^{t_1 t_2} = \sum_{t_1 \in M} |N| z^{t_1} = |N| c_d(1).$$

Now, the result follows from $|N| = \frac{\varphi(n)}{\varphi(d)}$ and $c_d(1) = \mu(d)$ by the first paragraph.

Note that (n,m) = (n,(n,m)). Thus, we have the following immediate consequence.

Corollary 3.3. Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Then $c_n(m) = c_n((n, m))$.

As a consequence, we note that if m' is an integer such that $m' \equiv m \pmod{n}$, then we conclude from Corollary 3.3 that $c_n(m') = c_n(m)$.

The following decomposition of a Cartesian product is useful for counting pairs of integers. Roughly speaking, the product below is partitioned by parallel lines on $\mathbb{Z} \times \mathbb{Z}$.

Lemma 3.4. *Let* $n \in \mathbb{N}$ *and let* $I = \{1, 2, ..., n-1\}$. *For every* $a, b \in \mathbb{Z}$, *we have*

$$I \times I = \biguplus \ \biguplus \{(x,y) \in I \times I \mid ax + by \equiv td \pmod{n}\}.$$

$$d \mid n \text{ } 1 \leq t \leq \frac{n}{d},$$

$$(t, \frac{n}{d}) = 1$$