Proof. Suppose $z = re^{i\alpha}$, where $-\pi < \alpha \leq \pi$, then for $|z|= r < 1$,

$$\begin{aligned}
\operatorname{Re}(\mathcal{P}_0(z)) &= -\frac{2}{\pi^2}\left\{\operatorname{Re}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right\}\\
&= -\frac{2}{\pi^2}\left(\log\left(\sqrt{\frac{\mu_1(r,c)}{\mu_2(r,c)}}\right)\right)^2 + \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{1-c^2}\sqrt{r}}{1-r}\right)\right)^2\\
&=: \mathcal{G}(r,c).
\end{aligned}$$

where $c := \cos(\alpha/2)$ and

$$\mu_i(r,c) := \begin{cases} 1+r+2c\sqrt{r}, & i=1,\\ 1+r-2c\sqrt{r}, & i=2.\end{cases}$$

Observe that $c \in [-1,1]$, infact it is easy to check that $\partial\mathcal{G}(r,c)/\partial c = 0$ if and only if $c = 0$, also $\partial^2\mathcal{G}(r,0)/\partial c^2 < 0$, which leads to

$$\max_{c\in[-1,1]} \mathcal{G}(r,c) = \mathcal{G}(r,0) = \mathcal{P}_0(-r) = 1 + \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{r}}{1-r}\right)\right)^2. \tag{2.1}$$

Moreover, for each $R \leq r < 1$, equation (2.1) leads to $\mathcal{G}(r,0) \geq \mathcal{P}_0(r) = \mathcal{G}(r,1)$. Since $\mathcal{G}(r,0)$ is an increasing function, whereas $\mathcal{G}(r,1)$ is a decreasing function of r, this leads to the inequality $\mathcal{G}(r,1) < \mathcal{G}(r,0)$, for each $r < 1$. Hence the required bound is achieved. ∎

As a consequence of Lemma 2.1 and [15, Theorem 2.1 & Corollary 2.2], we obtain the Growth and Covering Theorems for the class $\mathcal{F}_{\mathcal{LP}}$.

Theorem 2.2. *Let $f \in \mathcal{F}_{\mathcal{LP}}$, then the following holds*

I. (Growth Theorem) For $|z|= r < 1$, let

$$\max_{|z|=r} \operatorname{Re}\mathcal{P}_0(z) = \mathcal{P}_0(-r) \text{ and } \min_{|z|=r} \operatorname{Re}\mathcal{P}_0(z) = \mathcal{P}_0(r),$$

then for $|z|= r < 1$ the following sharp inequality holds

$$r\exp\left(\int_0^r \frac{\mathcal{P}_0(t)}{t}dt\right) \leq |f(z)| \leq r\exp\left(\int_0^r \frac{\mathcal{P}_0(-t)}{t}dt\right).$$

II. (Covering Theorem) Suppose $\min_{|z|=r} \operatorname{Re}\mathcal{P}_0(z) = \mathcal{P}_0(r)$ and $f \in \mathcal{F}_{\mathcal{LP}}$. Let f_0 be given by (1.2), then $f(z)$ is a rotation of f_0 or $\{w \in \mathbb{D} : |w| \leq -f_0(-1)\} \subset f(\mathbb{D})$, where $-f_0(-1) = \lim_{r\to 1} -f_0(-1)$.

Remark 2.3. (See **Fig.** 3) If $f(z)$ is of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, belongs to $\mathcal{F}_{\mathcal{LP}}$, then

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - 2\right)\right| > \frac{3\pi}{4}.$$

It can be verified that the equation of tangent corresponding to $\Gamma : y^2 = 1+2(1-x)$ is given by $y = \pm(x-2)$. Infact these tangents intersect the parabola Γ at the points $(1, \pm 1)$. Therefore it can be observed that the convex region $\Omega_{\mathcal{LP}}$ lies in the sector $|\arg(\omega - 2)| > 3\pi/4$. Hence this gives a sharp argument estimate for functions lying in the class $\mathcal{F}_{\mathcal{LP}}$.

Remark 2.4. Due to Lemma 2.1, for $|z|= r < 1$, we have $\mathcal{LP}(r) \leq \operatorname{Re}\mathcal{LP}(z) \leq \mathcal{LP}(-r)$ and infact $\max_{|z|=r}|\mathcal{LP}(z)| = |\mathcal{LP}(r)| = |\mathcal{P}_0(r)|$.

2.2. **Radius Problems for the class $\mathcal{F}_{\mathcal{LP}}$.** Based on the definition of the class $\mathcal{F}_{\mathcal{LP}}$ and pictorial representation of $\mathcal{LP}(\partial\mathbb{D})$ (see **Fig.** 2), we have $\max_{|z|\leq 1} \operatorname{Re}(\mathcal{LP}(z)) = \mathcal{LP}(-1) = 3/2$. This means $\operatorname{Re} zf'(z)/f(z) < 3/2$, thus $f \in \mathcal{F}_{\mathcal{LP}}$ may or may not be a univalent function. Therefore it is an interesting problem to establish the largest radius $r_0 < 1$ such that each $f \in \mathcal{F}_{\mathcal{LP}}$ is starlike in $|z| \leq r_0$. In this section, we study some radius results for the class $\mathcal{F}_{\mathcal{LP}}$ along with the classes $\mathcal{S}^*(\phi)$ and $\mathcal{F}(\psi)$ for some special choices of $\phi(z)$ and $\psi(z)$, as mentioned in Table 1 (see Appendix) and equation (1.1), respectively. Here below we provide a lemma that yields a maximal disc that can be subscribed within the parabolic region $\Omega_{\mathcal{LP}}$.