5. Final remarks

By [17, Theorem 3.17], every automorphism of the poset $(\mathcal{G}_\infty(H), \subset)$ is induced by an invertible bounded linear or conjugate-linear operator. We explain why the same statement is not proved for the restrictions of adjacency preserving transformations on connected components of $\mathcal{G}_\infty(H)$.

Let f be a bijective transformation of $\mathcal{G}_\infty(H)$ preserving the adjacency relation in both directions and let $\mathcal{C}$ be a connected component of $\mathcal{G}_\infty(H)$. Without loss of generality, we can assume that $f(\mathcal{C}) = \mathcal{C}$. Denote by $\mathcal{C}_\pm$ the set of all $X \in \mathcal{G}_\infty(H)$ such that X is a subspace of finite codimension in a certain element of $\mathcal{C}$ or X contains a certain element of $\mathcal{C}$ as a subspace of finite codimension. Then f can be uniquely extended to an automorphism of the poset $(\mathcal{C}_\pm, \subset)$ [17, Theorem 2.19]. This extension is denoted by the same symbol f.

Suppose that $f(X) = X$ for a certain $X \in \mathcal{C}$ and consider the lattice of finite-dimensional subspaces of X. The map sending every finite-dimensional subspace $M \subset X$ to $f(M^\perp \cap X)^\perp \cap X$ is a lattice automorphism. By the Fundamental Theorem of Projective Geometry, it is induced by a semilinear bijection $A : X \to X$, i.e. A is additive and there is an automorphism σ of the field of complex numbers such that $A(ax) = \sigma(a)A(x)$ for every $x \in X$ and $a \in \mathbb{C}$. It must be pointed out that A is not necessarily bounded; furthermore, the automorphism σ is not necessarily the identity or the conjugate map. Then

$$f(Y) = (A(Y^\perp \cap X))^\perp \cap X$$

for every $Y \in \mathcal{C}_\pm$ contained in X.

If A is an invertible bounded linear or conjugate-linear operator, then the restriction of f to the set of all $Y \in \mathcal{C}_\pm$ contained in X is induced by $(A^{-1})^*$ [17, Proposition 3.7].

Conversely, if this restriction is induced by a semilinear bijection $B : X \to X$, then B sends closed hyperplanes of X to closed hyperplanes which guarantees that B is a bounded linear or conjugate-linear operator (see [12] or [17, Lemma 3.12]). By [17, Proposition 3.7], A is a scalar multiple of $(B^{-1})^*$ which is impossible if A is unbounded.

So, the restriction cannot be induced by a semilinear bijection if A is unbounded. To assert that A is bounded we need to extend f on $\mathcal{G}_\infty(X)$. Only in this case, we can apply arguments used to prove [17, Theorem 3.17].

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