b- The octahedron. It has 2^n facets, each of them is a regular simplex of volume $|F|_{n-1} = \frac{\sqrt{n}}{(n-1)!}$. Let Δ_k denote a regular k-simplex with edge length $\sqrt{2}$ (so $|\Delta_k|_k = \frac{\sqrt{k+1}}{k!}$). Then, if $n \geq 3$:

$$\frac{\text{Isop}(F)}{\text{Isop}(O_n)} = \frac{n|O_n||\partial F|}{(n-1)|\partial O_n||F|} = \frac{n}{n-1}\frac{n|\Delta_{n-2}|}{n!|F|^2}$$
$$= \frac{n}{n-1}\frac{(n-1)!^2}{n!(n-2)!}\sqrt{n-1} = \sqrt{n-1} > 1.$$

c- The cylinders. Let L be an $(n-1)$-dimensional convex body. Define $C = Conv(L, L')$ where $L' = L + te_n$ is a translate of L parallel to it. Then $|C| = t|L|$, thus :

$$\frac{\text{Isop}(L)}{\text{Isop}(C)} = \frac{n|C||\partial L|}{(n-1)|\partial C||L|} = \frac{tn}{n-1}\frac{|\partial L|}{|\partial C|} = \frac{tn}{n-1}\frac{|\partial L|}{(2|L| + t|\partial L|)}.$$

This ratio is greater than 1 for large t (as soon as $t > 2\frac{n}{n-1}(\text{Isop}(L))^{-1}$). Letting T be an appropriate affine transform, i.e. $Te_n = t_0 e_n$, and $Te_i = e_i$ $(i \leq n-1)$, where t_0 is large enough, one deduces that $b_2(C) = b_2(TC) > 1$, because $\frac{\text{Isop}(TL)}{\text{Isop}(TC)} = \frac{\text{Isop}(L)}{\text{Isop}(TC)} > 1$.

d- The half-ball. Denote $H_+ = \{x_n \geq 0\}$ a closed half-space, B_2^n the (unit) Euclidean ball, and $M = H_+ \cap B_2^n$ a half-ball. Then $|M|_n = \kappa_n/2$, $|\partial M|_{n-1} = \frac{n\kappa_n}{2} + \kappa_{n-1}$, and so $\text{Isop}(M) = 1 + \frac{2\kappa_{n-1}}{n\kappa_n}$. The unique facet of M is $F \approx B_2^{n-1}$, and so $\text{Isop}(F) = 1$. Hence $\frac{\text{Isop}(F)}{\text{Isop}(M)} < 1$.

Nonetheless, one can find an affine transform T such that $\frac{\text{Isop}(TF)}{\text{Isop}(TM)} > 1$, showing that $b_2(M) > 1$. Indeed, let $c > 1$ be large enough and let T be the unique affine transform such that $Te_n = ce_n$, and $Te_i = e_i$ for $i \leq n-1$. Then computations show that $\text{Isop}(TM) = \frac{|\partial \mathcal{E}|}{n|\mathcal{E}|} + \frac{2\kappa_{n-1}}{n\kappa_n c} < \frac{|\partial \mathcal{E}|}{n|\mathcal{E}|} + \frac{1}{c} < 1 = \text{Isop}(TF) = \text{Isop}(F)$, showing $b_2(M) = b_2(TM) < 1$. (see appendix for computational details)

In the above four examples (unit cube, octahedron O_n, cylinders, half-balls), we used Proposition 1 (or Proposition 2) to argue that $b_2(K) > 1$. For these examples, the fact that $b_2(K) > 1$ was already known : the cylinder, like the cube, is decomposable, the half-ball has (many) points of positive curvature on its boundary, and it was directly shown in [SZ15] that $b_2(O_n) = 2$ (by taking well-chosen segments). The half-ball example suggests that $\max_F \frac{\text{Isop}(F)}{\text{Isop}(K)}$ could be minimal when K is in its John's position.

As a sanity check, one may wish to compute $\max_F \frac{\text{Isop}(F)}{\text{Isop}(\Delta)}$, when Δ is an n-simplex, to check this quantity is less than 1. If $\Delta = T_n$ is a regular simplex with edge length $\sqrt{2}$, then $|T_n| = \frac{\sqrt{n+1}}{n!}$, and $|\partial T_n| = (n+1)|T_{n-1}| = (n+1)\frac{\sqrt{n}}{(n-1)!}$, so that $\text{Isop}(T_n) = \sqrt{n(n+1)}$. It results that for a regular n-simplex :

$$\max_F \frac{\text{Isop}(F)}{\text{Isop}(T_n)} = \frac{\sqrt{n(n-1)}}{\sqrt{n(n+1)}} = \sqrt{\frac{n-1}{n+1}}.$$

If $\Delta_n = Conv(0, e_1, ..., e_n)$, then $|\Delta_n| = \frac{1}{n!}$, and $|\partial \Delta_n| = \frac{n}{(n-1)!} + \frac{\sqrt{n}}{(n-1)!}$, hence $\text{Isop}(\Delta_n) = n + \sqrt{n}$. There are two kinds of facets : one is Δ_{n-1}, the other one is T_{n-1}. Their isoperimetric ratios are respectively $\text{Isop}(T_{n-1}) = \sqrt{n(n-1)}$ and $\text{Isop}(\Delta_{n-1}) = n - 1 + \sqrt{n-1} > \text{Isop}(T_{n-1})$. Therefore :

$$\max_F \frac{\text{Isop}(F)}{\text{Isop}(\Delta_n)} = \frac{\text{Isop}(\Delta_{n-1})}{\text{Isop}(\Delta_n)} = \frac{n-1+\sqrt{n-1}}{n+\sqrt{n}}.$$

Note that $\max_F \frac{\text{Isop}(F)}{\text{Isop}(\Delta_n)} > \max_F \frac{\text{Isop}(F)}{\text{Isop}(T_n)}$. One may conjecture that $\max_F \frac{\text{Isop}(F)}{\text{Isop}(T_n)} = \min_T \max_F \frac{\text{Isop}(TF)}{\text{Isop}(T\Delta_n)}$, i.e. that the (maximal) ratio is minimal when the simplex is in its John's position.

Finally, we remark that Proposition 2 yields a more concise proof of the following result, which states that having infinitely many facets, is an excluding condition.