of $\mathbb{Q}(\sqrt{D})$ is not equal to the rank $r_3(D')$ of the 3-part of the ideal class group of $\mathbb{Q}(\sqrt{D'})$, and therefore $r_3(D) = r_3(D') + 1$ by Scholz's Reflection Theorem. By using the connection between the points of $E_{D'}(\mathbb{Q})$ and the so-called 3-virtual units of $\mathbb{Q}(\sqrt{D'})$, we show that the elliptic curves of this subfamily, even though they have non-trivial rank, they cannot have any integral points. In the final section we discuss the case of positive fundamental discriminant D and derive a result for this case as well.

2. Definitions and Preliminaries

The notation and definitions of this section will carry throughout the paper. For the theory and proofs of the facts that we present here, the interested reader may refer for example to [12], [15, Chapter X.4 and Appendix B], [17] and [5, Chapter IV].

For any number field $M/\mathbb{Q}$, denote by $E(M)$ the group of points of the elliptic curve E defined over M. Let $\overline{M}$ denote the algebraic closure of M. We denote by $D < -4$ every squarefree integer which satisfies the following two congruence relations

$$D \equiv 1 \bmod 4 \text{ and } D \equiv 2 \bmod 3. \tag{1}$$

We denote be D' the, also squarefree, integer $D' = -3D$ and by K_D the imaginary quadratic field $K_D = \mathbb{Q}(\sqrt{D})$ with $K_{D'} = \mathbb{Q}(\sqrt{D'})$ its so-called quadratic resolvent.

Let us note here that, given the congruence conditions in (1), the following congruences always hold:

$$D \equiv 5 \bmod 12 \text{ and } D' \equiv 9 \bmod 12.$$

We define the j-invariant zero elliptic curves

$$E_{D'} : y^2 = x^3 + 16D'. \tag{2}$$

The torsion group

$$\mathcal{T}_{D'} = \{O, (0, \pm 4\sqrt{D'})\} \subseteq E_{D'}(\overline{\mathbb{Q}})$$

is a subgroup of $E_{D'}(\overline{\mathbb{Q}})$ of order 3 and is invariant under the action of $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. The quotient of $E_{D'}$ over $\mathcal{T}_{D'}$ defines a family of isogenous elliptic curves, which are defined as

$$\hat{E}_D : Y^2 = X^3 + 16 \cdot 3^4 D. \tag{3}$$

As expected, the torsion group for these curves is

$$\hat{\mathcal{T}}_D = \{O, (0, \pm 36\sqrt{D})\} \subseteq \hat{E}_D(\overline{\mathbb{Q}}).$$

Let ϕ denote the rational 3-isogeny

$$\phi : E_{D'} \to \hat{E}_D$$

with kernel $\ker_\phi(\overline{\mathbb{Q}}) = \mathcal{T}_{D'}$. The isogeny ϕ is given by the map ([7, Proposition 8.4.3])

$$(X, Y) = \phi((x, y)) = \Big(\frac{x^3 + 4^3 3^4 D}{x^2}, \frac{y(x^3 - 2 \cdot 4^3 3^4 D)}{x^3}\Big). \tag{4}$$