Further, for each $|z|= r \leq R_{\mathfrak{F}}$, one can observe that

$$\frac{1}{2} \leq 1 \leq a = \frac{1+Ar^2}{1-r^2} \leq \frac{1+AR_{\mathfrak{F}}^2}{1-R_{\mathfrak{F}}^2} < \frac{3}{2}. \tag{2.6}$$

Infact inequalities (2.5) and (2.6) yields the inequality,

$$\frac{(5+A)r}{1-r^2} \leq \frac{3}{2} - \frac{1+Ar^2}{1-r^2},$$

provided $r \leq R_{\mathfrak{F}}$. Due to Lemma 2.5, it is clear that the disc $|u-a|< R$ lies in $\Omega_{\mathcal{LP}}$. Further, at $z_0 = R_{\mathfrak{F}}$ the function $f_{\mathfrak{F}}(z)$ defined as $f_{\mathfrak{F}}(z) = z(1+z)^2/(1-z)^{3+A}$ acts as the extremal function. ∎

Corollary 2.18. *Let $f \in \mathcal{F}_{\mathcal{LP}}$, then sharp $\mathfrak{F}_1-$ radius and $\mathfrak{F}_2-$ radius for the class $\mathcal{F}_{\mathcal{LP}}$ are respectively given as*

(i) $\mathcal{R}_{\mathfrak{F}_1}(\mathcal{F}_{\mathcal{LP}}) = \sqrt{17} - 4 \approx 0.123...$
(ii) $\mathcal{R}_{\mathfrak{F}_2}(\mathcal{F}_{\mathcal{LP}}) = (\sqrt{41} - 6)/5 \approx 0.080...$

Theorem 2.19. *Let $\delta = (\pi\sqrt{\beta-1}/\sqrt{2})$, where $1 < \beta < 3/2$, and suppose $f \in \mathcal{F}_{\mathcal{LP}}$, then $\mathcal{M}(\beta)-$ radius is $r_\beta = 1 + 2\,(\cot\delta)^2 - 2|\sec\delta/(\tan^2\delta)|$.*

Proof. From Lemma 2.1, it can be viewed that

$$\operatorname{Re}\mathcal{LP}(z) \leq \mathcal{LP}(-r) = 1 - \frac{2}{\pi^2}\left(\log\left(\frac{1+i\sqrt{r}}{1-i\sqrt{r}}\right)\right)^2 = 1 + \frac{2}{\pi^2}\left(\tan^{-1}\left(\frac{2\sqrt{r}}{1-r}\right)\right)^2.$$

As $f \in \mathcal{F}_{\mathcal{LP}}$, then assume that $zf'(z)/f(z) = p(z)$. Due to the above inequality $\operatorname{Re} p(z) \leq \mathcal{LP}(-r)$. Moreover, $\mathcal{LP}(-r) \leq \beta$ provided $r \leq r_\beta$, where r_β is the root of the equation $(1-\beta)\pi^2 + 2(\tan^{-1}(2\sqrt{r}/(1-r)))^2 = 0$ for $1 < \beta < 3/2$. Equality here occurs for the function $f_0 \in \mathcal{A}$, given by (1.2). ∎

If $f(z)$ and $g(z)$ be analytic functions in $|z|< r$, then $f(z)$ is said to be majorized by $g(z)$, denoted as $f(z) \ll g(z)$, in $|z|< r$, if $|f(z)|\leq |g(z)|$ in $|z|< r$. Equivalently, a function $f(z)$ is said to be be majorized by $g(z)$, if there exists an analytic $\Psi(z)$ with $|\Psi(z)|\leq 1$ in $\mathbb{D}$ and $f(z) = \Psi(z)f(z)$ for all $z \in \mathbb{D}$. For recent update on majorization for starlike and convex function, see [4, 5]. In the next theorem, we determine sharp majorization radius for the class $\mathcal{F}_{\mathcal{LP}}$.

Theorem 2.20. *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{F}_{\mathcal{LP}}$. Further assume that $f(z)$ is majorized by $g(z)$ in $\mathbb{D}$, i.e $f(z) \ll g(z)$, then for $|z|\leq r_m \approx 0.4220\ldots$,*

$$|f'(z)|\leq |g'(z)|,$$

where r_m is the unique positive root of the following equation

$$2\pi^2 r - (1-r^2)(\pi^2 - 2(\log((1+\sqrt{r})/(1-\sqrt{r})))^2) = 0. \tag{2.7}$$

Proof. Suppose $0 \leq r < r^* = \tanh^2(\pi/2\sqrt{2}) \approx 0.646\ldots$, then due to Remark 2.9 we conclude that, $g \in \mathcal{F}_{\mathcal{LP}}$ qualifies to be a Ma-Minda type function in $|z|< r^*$. Further let $w(z)$ be a Schwarz function in $\mathbb{D}$ with $w(0) = 0$, then by definition of subordination,

$$\frac{zg'(z)}{g(z)} = \mathcal{LP}(w(z)).$$

Note that for each $|z|= r < 1$, the inequality $|\mathcal{LP}(w(z))|\leq |\mathcal{LP}(r)|$ holds. Now for $|z|= r < r^*$, we obtain

$$\left|\frac{g(z)}{g'(z)}\right| = \frac{|z|}{|\mathcal{LP}(z)|} \leq \frac{r}{1-|\mathcal{P}_0(r)|} = \frac{r}{\mathcal{LP}(r)}. \tag{2.8}$$

As $f(z)$ is majorized by $g(z)$ in $\mathbb{D}$, we find from the definition of majorization,

$$f(z) = \psi(z)g(z).$$

Upon differentiating the above equality and suitable rearrangement of terms, we obtain

$$f'(z) = g'(z)\left(\psi'(z)\frac{g(z)}{g'(z)} + \psi(z)\right). \tag{2.9}$$