Lemma 2.5. *Suppose $a < 3/2$ and assume that $\zeta(\eta)$ is defined as follows:*

$$\zeta = \zeta(\eta) = \log\left(\frac{\sqrt{\eta}}{\sqrt{1-\eta}}\right) \quad \text{with } \eta = \frac{e^{-\pi\sqrt{1-2a}}}{1+e^{-\pi\sqrt{1-2a}}},$$

then $\mathcal{LP}(\mathbb{D})$ satisfies the following inclusion

$$\mathcal{D}(a, r_a) := \{\omega \in \mathbb{C} : |\omega - a| < r_a\} \subset \Omega_{\mathcal{LP}},$$

where

$$r_a = \begin{cases} \sqrt{\left(a - \dfrac{3}{2} + \dfrac{2\zeta^2}{\pi^2}\right)^2 + \dfrac{4\zeta^2}{\pi^2}}, & a \le \dfrac{1}{2} \\ \dfrac{3}{2} - a, & \dfrac{1}{2} < a < \dfrac{3}{2}, \end{cases}$$

Proof. We obtain a maximal disc centered at $(a, 0)$, where $a < 3/2$, that can be inscribed within $\Omega_{\mathcal{LP}}$. The distance from center $(a, 0)$ to the boundary $f(\partial(\mathbb{D}))$ is given by square root of

$$\mathcal{D}_a(X) := \left(a + \frac{2}{\pi^2}\left(\log\left(\frac{\sqrt{X^2}}{\sqrt{1-X^2}}\right)\right)^2 - \frac{3}{2}\right)^2 + \frac{4}{\pi^2}\left(\log\left(\frac{\sqrt{X^2}}{\sqrt{1-X^2}}\right)\right)^2,$$

where $X = \cos t$. Now the critical points of $\mathcal{D}_a(X)$ are

$$X' := \begin{cases} \pm\dfrac{e^{\frac{1}{2}\pi\sqrt{1-2a}}}{\sqrt{1+e^{\pi\sqrt{1-2a}}}}, \pm\dfrac{e^{-\frac{1}{2}\pi\sqrt{1-2a}}}{\sqrt{1+e^{-\pi\sqrt{1-2a}}}}, & \text{if } a < 1/2, \\ \pm 1/\sqrt{2}, & \text{if } 1/2 \le a < 3/2. \end{cases}$$

It can be verified that $\mathcal{D}_a''(X) > 0$ at $X = X'$, whenever $a < 3/2$. Therefore, $X = X'$ is the point of minima for $\mathcal{D}_a(X)$, which leads us to the optimal disk centered at a with radius r_a. ∎

Theorem 2.6. *Suppose $0 \le \alpha < 1$ and $-1 < B < A \le 1$, then for $f \in \mathcal{A}$, the sharp $\mathcal{F}_{\mathcal{LP}}-$radii for the classes $\mathcal{S}_p^*$, $\mathcal{S}_s^*$, Δ^*, $\mathcal{S}_\varrho^*$, $\mathcal{S}_\rho^*$, $\mathcal{S}_\wp^*$, $\mathcal{BS}^*(\alpha)$, $\mathcal{S}_{\alpha,e}^*$ and $\mathcal{S}^*(A, B)$ (see Table 1 in Appendix) are respectively given by*

(i) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_p^*) = \tanh^2(\pi/4)$.
(ii) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_s^*) = \pi/6$.
(iii) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\Delta^*) = 5/12$.
(iv) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_\varrho^*) = (\cosh^{-1}(3/2))^2$.
(v) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_\rho^*) = \sinh(1/2)$.
(vi) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_\wp^*) \approx 0.3517\ldots$.
(vii) *For $0 < \alpha < 1$, $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{BS}^*(\alpha)) = R_{\mathcal{BS}}$, where*

$$R_{\mathcal{BS}} = \begin{cases} 1/2, & \alpha = 0 \\ (\sqrt{1+\alpha} - 1)/\alpha, & 0 < \alpha < 1. \end{cases}$$

(viii) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_{\alpha,e}^*) = R_{\alpha,e}$, *where*

$$R_{\alpha,e} = \begin{cases} \log(1 - 1/2(\alpha - 1)), & 0 \le \alpha < 1 - 1/2(e-1) \\ 1, & 1 - 1/2(e-1) \le \alpha < 1. \end{cases}$$

In particular, $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_e^*) = \log(3/2)$.
(ix) $\mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}^*(A, B)) =: \tilde{R}$, *where*

$$\tilde{R} = \begin{cases} 1/(2A - 3B), & \textit{when } ((-1 < B \le (2A-1)/3) \wedge (-1 < A < 0)) \\ & \quad \vee((-1 < B < (2A-1)/3) \wedge (0 \le A \le 1)), \\ 1, & \textit{when } ((2A-1)/3 < B < A \le 1) \wedge (-1 < A < 0)) \\ & \quad \vee(((2A-1)/3 \le B < A \le 1) \wedge (0 \le A \le 1)). \end{cases}$$