Let $\lambda_t = V_1(t)/V_0$, and $\mu_t = V_{n-1}(t)/V_n(t)$ (which are both well-defined and positive, as long as $t > t_0$). Then, using Brunn-Minkowski's inequality :

$$(V_1 - V_0)\sum_{r=0}^{n-1} \lambda_t^r = \frac{V_1^n - V_0^n}{V_0^{n-1}} \geq \frac{V_n V_0^{n-1} - V_0^n}{V_0^{n-1}} = V_n - V_0. \tag{BM1}$$

And symmetrically :

$$(V_{n-1} - V_n)\sum_{r=0}^{n-1} \mu_t^r = \frac{V_{n-1}^n - V_n^n}{V_n^{n-1}} \geq \frac{V_0 V_n^{n-1} - V_n^n}{V_n^{n-1}} = V_0 - V_n. \tag{BM2}$$

Therefore (combining the above two)

$$V_1 - V_0 \geq \alpha_t(V_n - V_{n-1}) \quad \text{where } \alpha_t = \frac{\sum_{r=0}^{n-1} \mu_t^r}{\sum_{r=0}^{n-1} \lambda_t^r}.$$

Since $W_t \to K$ as $t \to 0$ (for the Hausdorff distance), continuity of $V_n(.)$ implies that $V_k(t) \to V_0$ (for all $k \leq n$), while (weak) continuity of $(L \mapsto S_L)$ implies that $S_{W_t} \to S_K$ (for the weak topology on Radon measures on the sphere). Therefore $\alpha_t \to 1$, while

$$\begin{aligned}\frac{V_n - V_{n-1}}{t} &= \frac{1}{n}\int_\Omega \frac{(h_{W_t} - h_K)(u)}{t} dS_{W_t}(u) \\ &= \frac{1}{n}\int_\Omega f(u) dS_{W_t}(u) \to \frac{1}{n}\int_\Omega f(u) dS_K(u)\end{aligned}$$

where we used that $h_{W_t}(u) = h_K(u) + tf(u)$ for S_{W_t}-a.e. $u \in \Omega$. Hence,

$$\liminf_{t\to 0^+} \frac{V_1 - V_0}{t} \geq \left(\lim_{t\to 0^+} \alpha_t\right)\left(\lim_{t\to 0^+} \frac{V_n - V_{n-1}}{t}\right) = \frac{1}{n}\int_\Omega f(u) dS_K(u).$$

Similarly, for all $t_0 < t < 0$: $\frac{V_1 - V_0}{t} \geq \frac{1}{n}\int_\Omega f(u)dS_K(u)$, while combining **BM1** and **BM2** yields

$$\limsup_{t\to 0^+} \frac{V_1 - V_0}{t} \leq \left(\lim_{t\to 0^-} \alpha_t\right)\left(\lim_{t\to 0^-} \frac{V_n - V_{n-1}}{t}\right) = \frac{1}{n}\int_\Omega f(u) dS_K(u).$$

□

We recall the statement of the (almost-sure) pointwise convergence Lemma 2 (which we used in the proof of Proposition 1), and provide a simple proof here.

Lemma 4. *Let $(W_t)_t$ be Wulff-shape perturbations of a given convex body K, with respect to (Ω, f). Then for S_K-almost every $u \in \mathbb{S}^{n-1}$:*

$$\left.\frac{\mathrm{d}h_{W_t}(u)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0} \frac{h_{W_t}(u) - h_K(u)}{t} = f(u). \tag{6}$$

Proof. First, recall that, for all $u \in \mathbb{S}^{n-1}$, the map $(t \mapsto h_{W_t}(u))$ is concave on $]t_0, +\infty[$, where $t_0 = t_0(K, \Omega, f) = \inf\{t \in \mathbb{R} : |W_t|_n > 0\}$.

Indeed, Let $t_0 < t_1 < t_2$, and let $\lambda \in (0, 1)$. Let $x \in (1-\lambda)W_{t_1} + \lambda W_{t_2}$, and let $u \in \Omega$. Let $x_i \in W_{t_i}$, $i = 1, 2$, such that $x = (1-\lambda)x_1 + \lambda x_2$. Denote $t_\lambda := (1-\lambda)t_1 + \lambda t_2$. Then

$$\langle x, u\rangle = (1-\lambda)\langle x_1, u\rangle + \lambda\langle x_2, u\rangle \leq (1-\lambda)h_{W_{t_1}}(u) + \lambda h_{W_{t_2}}(u) \leq h_K(u) + t_\lambda f(u).$$

As this holds for any $u \in \Omega$, one concludes that $(1-\lambda)W_{t_1} + \lambda W_{t_2} \subset W_{t_\lambda}$, from which concavity (on the domain $]t_0, +\infty[$) of $(t \mapsto h_{W_t}(u))$,for any fixed $u \in \mathbb{S}^{n-1}$, readily follows.

In particular, for any $u \in \mathbb{S}^{n-1}$, the map $(t \mapsto h_{W_t}(u))$ is concave on a neighborhood of $t = 0$, implying that $\lim_{t\to 0^+} \frac{(h_{W_t} - h_K)(u)}{t}$ and $\lim_{t\to 0^-} \frac{(h_{W_t} - h_K)(u)}{t}$, exist for any $u \in \mathbb{S}^{n-1}$. We now explain how to deduce that right and left derivative (at $t = 0$) coincide, for S_K-a.e. $u \in \mathbb{S}^{n-1}$, from lemma 1.