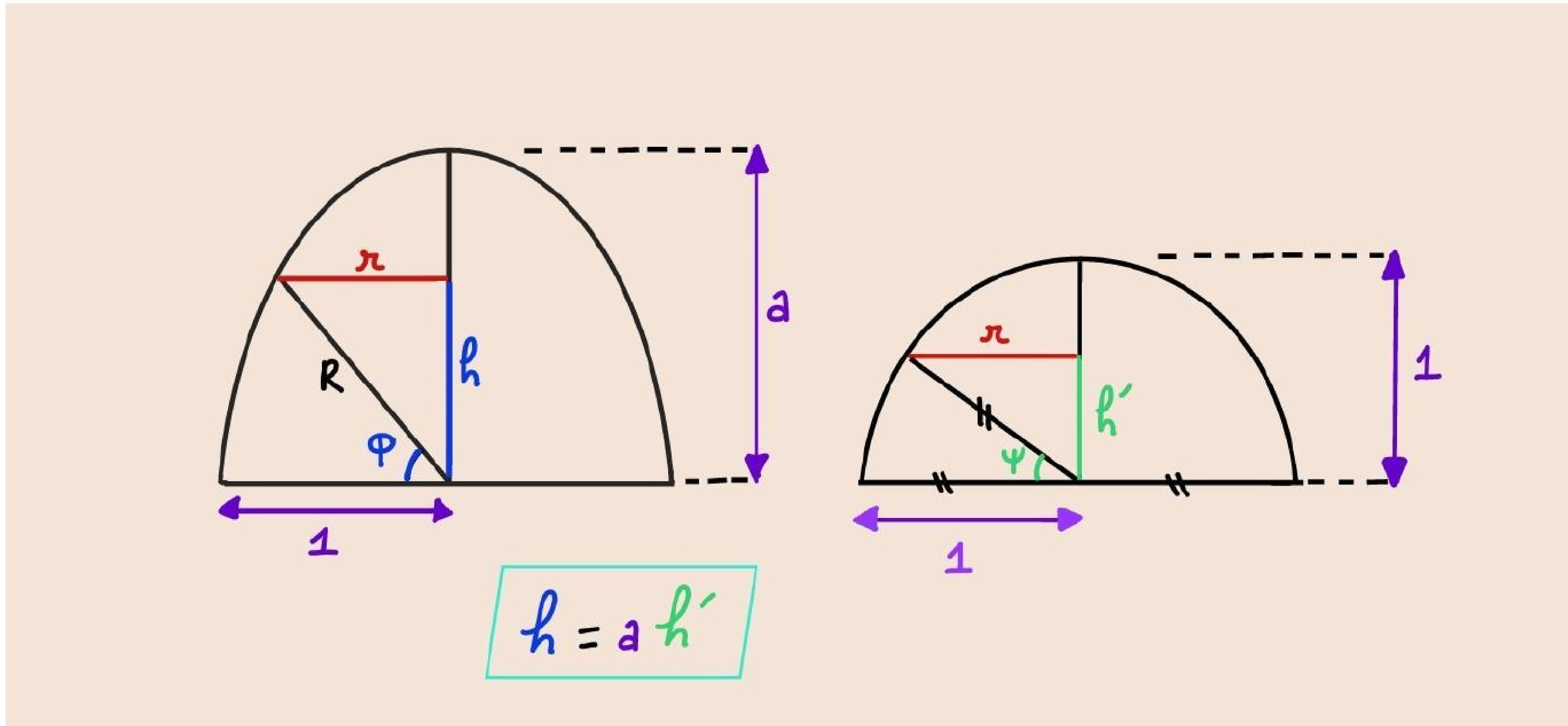

Changing the parametrization in ϕ, into a parametrization in ψ, yields for the surface area :

$$|\partial\mathcal{E}_a| = 2(n-1)\kappa_{n-1}\int_0^{\pi/2}(\cos(\psi))^{n-2}\frac{a}{(1+(a^2-1)\sin^2(\psi))^{1/2}}d\psi.$$

When $a=1$, one recovers $|\partial\mathcal{E}_1| = 2(n-1)\kappa_{n-1}W_{n-2} = 2\kappa_{n-1}nW_n = n\kappa_n$. When $a>1$, one gets $|\partial\mathcal{E}_a| = \lambda(an\kappa_n)$, where $\lambda\in(a^{-1},1)$ is defined via :

$$\lambda := \int_0^{\pi/2}(\cos(\psi))^{n-2}\frac{1}{(1+(a^2-1)\sin^2(\psi))^{1/2}}d\psi\left(\int_0^{\pi/2}(\cos(\psi))^{n-2}d\psi\right)^{-1}.$$

Hence TM, the image of the half-ball M under T, is a half-ellipsoid whose "isoperimetric ratio" is given by :

$$\mathrm{Isop}(TM) = \frac{|\partial\mathcal{E}_a|}{n|\mathcal{E}_a|} + \frac{2\kappa_{n-1}}{n|\mathcal{E}_a|} = \lambda + \frac{1}{anW_n}.$$

The unique face F of TM satisfies $\mathrm{Isop}(F)=1$: hence TM satisfies the condition of Prop 2, as soon as a is such that : $\lambda < 1-\frac{1}{anW_n}$.

If $n=2$ and $a\geq 3$, then $1-\frac{1}{anW_n} = 1-\frac{2}{a\pi} > \frac{3}{4}$, and one can easily check $\lambda<\frac{3}{4}$, so that $\mathrm{Isop}(TM)<1$ holds. Since $(\psi\mapsto(1+(a^2-1)\sin^2(\psi)))$ is increasing in $\psi\in[0,\pi/2]$, one may upper bound $\lambda=\lambda_a$:

$$\lambda = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{(1+(a^2-1)\sin^2(\psi))^{1/2}}d\psi < \frac{2}{\pi}\left(\frac{\pi}{4}+\frac{\pi}{4}\left(\frac{2}{a^2+1}\right)^{1/2}\right) < \frac{3}{4} \quad (\text{if } a\geq 3).$$

If $n\geq 3$, then we use the estimate $\frac{\pi}{2(n+1)} < W_n^2 < \frac{\pi}{2n}$, yielding

$$nW_n > \sqrt{\frac{\pi n}{2}}\left(1+\frac{1}{n}\right)^{-1/2} > \sqrt{\frac{\pi n}{2}}\left(1-\frac{1}{2n}\right) \geq \frac{5}{6}\sqrt{\frac{\pi n}{2}} > \sqrt{n},$$

and so $1-\frac{1}{anW_n} > 1-\frac{1}{a\sqrt{n}}$.

One may check that if $a\geq 4n\sqrt{n}$, then $\lambda=\lambda_a<1-\frac{1}{a\sqrt{n}}$. Indeed, letting $w_\psi := \cos(\psi)^{n-2}$, similarly as in the 2-dimensional case, one may upper bound λ by splitting the integral :

$$\begin{aligned}\lambda W_{n-2} &= \int_0^{\pi/2n}(1+(a^2-1)\sin^2(\psi))^{-1/2}w_\psi d\psi + \int_{\pi/2n}^{\pi/2}(1+(a^2-1)\sin^2(\psi))^{-1/2}w_\psi d\psi \\ &< \frac{\pi}{2n} + \frac{\pi}{2}\frac{n}{(n^2+a^2-1)^{1/2}}\end{aligned}$$