Proof. For part (i), as $f \in \mathcal{S}_p^*$, then due to the geometry of the function $1+\mathcal{P}_{0,\pi}(z) = 1 + 2/\pi^2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$ it can be observed that

$$\max_{|z|=r} \operatorname{Re}(1+\mathcal{P}_{0,\pi}(z)) = 1+\mathcal{P}_{0,\pi}(r).$$

Now for $f(z)$ to lie in the class $\mathcal{F}_{\mathcal{LP}}$, we must have

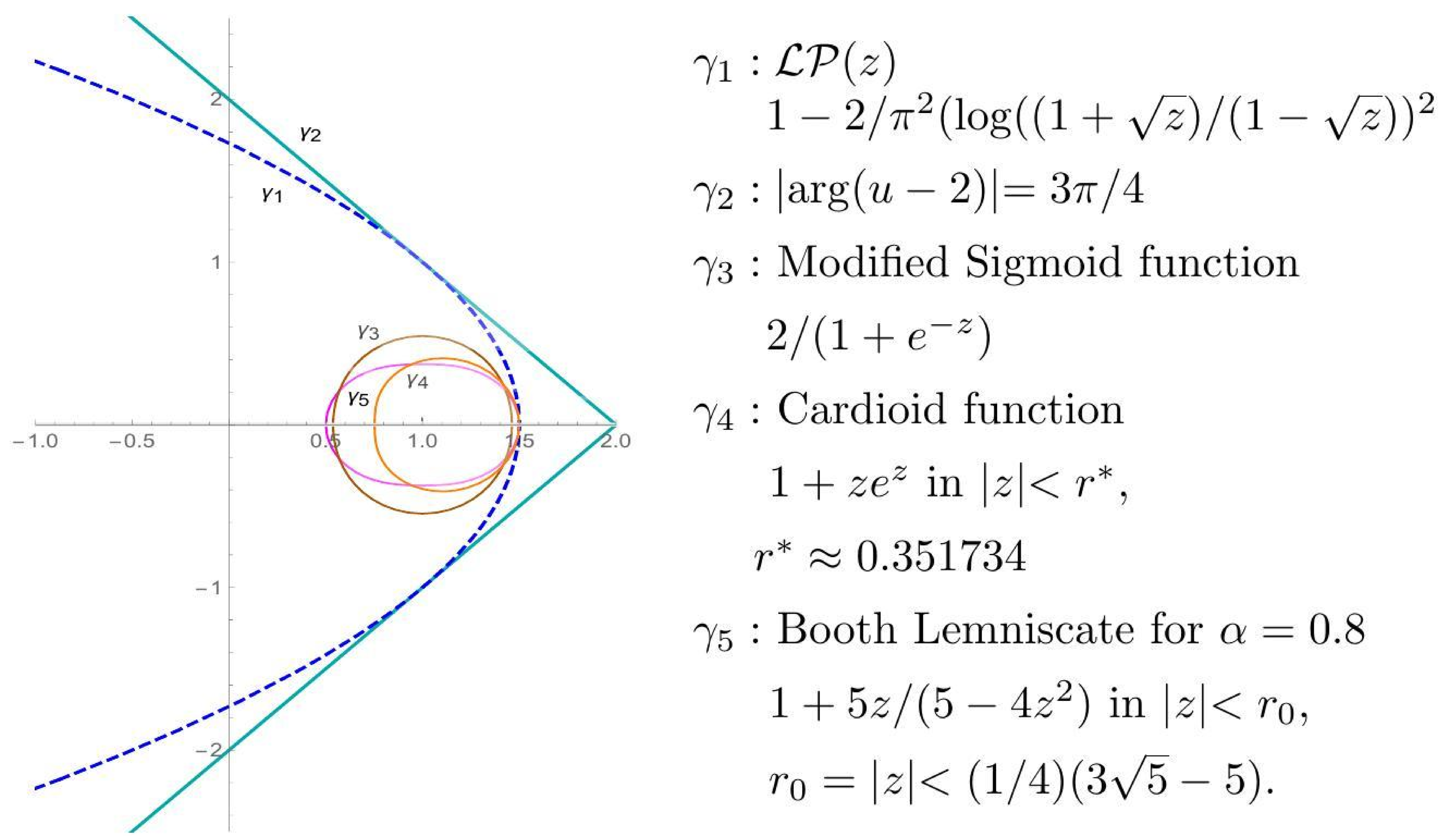

FIGURE 3. Pictorial boundary description of inclusion results pertaining to the parabolic function $\mathcal{LP}(z)$

$$1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{r}}{1-\sqrt{r}}\right)\right)^2 \leq \frac{3}{2},$$

which holds provided $r \leq \mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_p^*)$. Clearly equality in (i) is attained for the function $\tilde{f}(z)$ such that $z\tilde{f}'(z_0)/\tilde{f}(z_0) = 1+\mathcal{P}_{0,\pi}(z_0)$ at $z_0 = \mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_p^*)$. Further observe that, if $p \in \mathcal{P}[A,B] = \{p(z) : p(z) = 1+c_1z+c_2z^2\ldots \prec (1+Az)/(1+Bz), -1 \leq B < A \leq 1\}$, then for $|z|= r < 1$, it is a known fact that

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| < \frac{|A-B|r}{1-B^2r^2}. \tag{2.2}$$

Then, in view of (2.2), for part (ix) $f \in \mathcal{S}^*(A,B)$ lies in $\mathcal{F}_{\mathcal{LP}}$, if

$$\frac{(A-B)r+1-ABr^2}{1-B^2r^2} \leq \frac{3}{2}.$$

Equivalently, we can say that $(1-Br)((2A-3B)r-1) \leq 0$, provided $r \leq \mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}^*(A,B))$. Extremal function in this case is $\hat{f} \in \mathcal{A}$ satisfying $z_0\hat{f}'(z_0)/\hat{f}(z_0) = (1+Az_0)/(1+Bz_0)$, where $z_0 = \tilde{R}$. For part (iv), as $f \in \mathcal{F}_{\mathcal{LP}}$, then proceeding as before, f lies in $\mathcal{S}_\varrho^*$ if for $|z|= r$ we have $3/2 \geq \cosh\sqrt{r}$, provided $r = \mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_\varrho^*)$. Sharpness holds for the function $f_\varrho \in \mathcal{A}$ defined as $zf_\varrho'(z)/f_\varrho(z) = \cosh\sqrt{z}$. Further we know that $\max_{|z|=r} \operatorname{Re}(1+\sin z) = 1+\sin r$, then for part (ii) it is enough to find an $r < 1$ satisfying the equation $\sin r = 1/2$, thus $r = \mathcal{R}_{\mathcal{F}_{\mathcal{LP}}}(\mathcal{S}_s^*) = \pi/6$. Sharpness holds for the function $f_s(z)$ given by $zf_s'(z)/f_s(z) = 1+\sin z$. In all the subsequent parts, the proofs follow along the same lines, therefore they are omitted. ∎

Let $\mathcal{P}_\alpha$ consist of functions of the form $p(z) = 1+c_1z+c_2z^2+\ldots$, satisfying $\operatorname{Re} p(z) > \alpha$ for $0 \leq \alpha < 1$, then we say $p(z)$ is a Carathéodory function of order α. Denote $\mathcal{P}(0) =: \mathcal{P}$, commonly known as the class of Carathéodory functions. Further, assume $\mathfrak{P}_{\mathcal{LP}}$ to be the class of functions of the form $p(z) = 1+c_1z+c_2z^2+\ldots$, such that $p(z) \prec \mathcal{LP}(z)$.