arbitrary convex body K, by approximation [4] (see [Sch, Theorem 4.1.1, Theorem 4.2.1]). In this case, the integral formula 1 holds by definition for polytopes, and is deduced (in general) from continuity of mixed volumes, and of $(L \mapsto S_L)$.

Recall that if Ω is a closed subset of $\mathbb{S}^{n-1}$, and g is a continuous function on Ω, the Wulff-shape with respect to (Ω, g) is the convex body $W(\Omega, g) = \bigcap_{u\in\Omega}\{x \in \mathbb{R}^n : \langle x, u\rangle \leq g(u)\}$. Let S_K be the surface area measure of K. More specifically, if K is a convex body, Ω a closed subset of $\mathbb{S}^{n-1}$, if $supp(S_K) \subset \Omega$ and if $f : \Omega \to \mathbb{R}$ is a continuous function, then we denote $(W_t)_t = (W(\Omega, h_K + tf))_t$ the family of Wulff-shape perturbations of K associated with (Ω, f). Note that there exists $t_0 = t_0(K) < 0$ such that $V_n(W_t) > 0$ for all $t > t_0$.

When $\Omega = \mathbb{S}^{n-1}$, we denote $W(g) = W(\mathbb{S}^{n-1}, g)$ the corresponding Wulff-shapes. See for instance [SSZ2, Theorem 1.1] where Wulff-shape perturbations (with $\Omega = \mathbb{S}^{n-1}$) were used to derive a characterization of n-simplices as the only convex bodies K such that $G_K \geq 0$, where G_K is the multi-linear form on $(\mathcal{K}^n)^n$ defined by $G_K(A_1, ..., A_n) = V_n(A_1, K[n-1])V_n(K, A_2, ..., A_n) - V_n(A_1, ..., A_n)V_n(K)$.

The following theorem is known as Alexandrov's variational lemma. We refer to [Al1] for a proof, see also [Sch, Lemma 7.4.3].

Theorem 3. *Assume K is a convex body, $supp(S_K) \subset \Omega$, and $f \in \mathcal{C}(\Omega, \mathbb{R})$. For $t \in \mathbb{R}$, denote $W_t = W(\Omega, h_K + tf)$. Then $(t \mapsto V_n(W_t))$ is differentiable at 0, and*

$$\left.\frac{\mathrm{d}V_n(W_t)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{V_n(W_t) - V_n(K)}{t} = \int_{\mathbb{S}^{n-1}} f(u)dS_K(u), \tag{3}$$

Minor modifications of the proof of the above theorem, yields a similar statement in terms of (first) mixed volumes, as follows.

Lemma 1 (Alexandrov's variational lemma for mixed volume)**.** *Assume K is a convex body, $supp(S_K) \subset \Omega$, and $f \in \mathcal{C}(\Omega, \mathbb{R})$. Denote $W_t = W(\Omega, h_K + tf)$, $t \in \mathbb{R}$. Denote $V_1(t) = V_n(W_t, K[n-1])$. Then $(t \mapsto V_1(t))$ is differentiable[5] at 0, and :*

$$\left.\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{V_1(t) - V_n(K)}{t} = \frac{1}{n}\int_{\mathbb{S}^{n-1}} f(u)dS_K(u), \tag{4}$$

Fix K, Ω and f (as above), and let $t_0 = \sup\{t < 0 : |W_t|_n = 0\} < 0$. Denote $(W_t)_t$ the associated family of Wulff-shape perturbations. One can easily check that for any $u \in \mathbb{S}^{n-1}$, the map $(t \mapsto h_{W_t}(u))$ is concave on $]t_0, +\infty[$. In particular, this map is both left and right-differentiable at $t = 0$. In fact, Lemma 1 allows to draw a more precise conclusion here.

Lemma 2. *Let $(W_t)_t$ be Wulff-shape perturbations of a given convex body K, with respect to (Ω, f). Then for S_K-almost every $u \in \mathbb{S}^{n-1}$:*

$$\left.\frac{\mathrm{d}h_{W_t}(u)}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{h_{W_t}(u) - h_K(u)}{t} = f(u). \tag{5}$$

We leave a proof of this pointwise convergence lemma in appendix, see also [SSZ2, Theorem 4.1] where the statement was derived from Alexandrov's variational lemma (Theorem 3). We will need Lemma 2 below, for the proof of Proposition 1.

Finally, it will be convenient to introduce the following definition.

Definition 4. For $K \in \mathcal{K}^n$, the *Bezout constant* is given by

$$b_2(K) = \sup\frac{V_n(L_1, L_2, K[n-2])V_n(K)}{V_n(L_1, K[n-1])V_n(L_2, K[n-1])},$$

where the supremum is over pairs of convex bodies $(L_1, L_2) \in (\mathcal{K}^n)^2$.

[4] if (P_k) is a sequence of polytopes approximating K, then the sequence of measures (S_{P_k}) is tight with respect to weak topology : define S_K as the weak limit of S_{P_k}

[5] on both sides