where we have defined a frequency

$$\omega_k^2(\eta) = k^2 - \frac{z''}{z}. \tag{29}$$

The momentum conjugate to ξ_k is defined as $\pi_k = \xi_k'$ and we finally obtain the Hamiltonian for the scalar perturbations:

$$\mathcal{H} = \sum_k \mathcal{H}_k = \sum_k \frac{\pi_k^2}{2} + \frac{\omega_k^2(\eta)}{2}\xi_k^2. \tag{30}$$

In order to calculate the Power Spectrum, we will make the assumption that during the inflationary era the evolution is dominated by the Cosmological Constant and therefore all other components are negligible; besides, if inflation starts late enough, we will have $\rho \ll \rho_\mu$ and, as mentioned in the last section, we can neglect the correction factor in (15):

$$H^2 = \frac{\rho_\Lambda}{3} = H_s^2, \tag{31}$$

where H_s is the constant Hubble parameter of inflation. Note that, to be precise, in a pure de Sitter universe the background matter field is set to a constant value and thus, in principle, it does not make sense to speak about its perturbations. This is shown explicitly in the appearance of the slow-roll parameter ϵ, which in this limit should be vanishing; indeed in this case a Power Spectrum cannot be obtained since inflation never stops. Nonetheless, the computations can be performed by keeping the slow-roll parameter as a non-vanishing constant, and this particular case represents a very good and easy-to-compute example to derive a Power Spectrum. In this regime the conformal time acquires a precise dependence on the scale factor and the frequency ω_k greatly simplifies:

$$\eta = -\frac{1}{aH_s}, \tag{32}$$

$$\omega_k^2(\eta) = k^2 - \frac{2}{\eta^2}. \tag{33}$$

Now we can perform the quantization of the system and proceed to compute the Power Spectrum. We will first briefly present the standard Spectrum derived through the canonical quantization, and then find the modified spectrum coming from the algebra (12).

A. Standard Power Spectrum

Here we will compute the standard Power Spectrum. In the standard representation of Quantum Mechanics, the two operators corresponding to the Fourier modes will obey the standard commutation relations and will have the standard action:

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i, \tag{34}$$

$$\hat{\xi}_k\, \psi(\xi_k) = \xi_k\, \psi(\pi_k), \quad \hat{\pi}_k\, \psi(\xi_k) = i\frac{\partial}{\partial \xi_k}\psi(\xi_k). \tag{35}$$

A single Fourier mode has Hamiltonian $\mathcal{H}_k$ with a time-dependent frequency $\omega_k(\eta)$; therefore the wavefunctions $\psi(\eta, \xi_k)$ will obey a time-dependent Schrödinger equation of the form

$$i\frac{\partial}{\partial \eta}\psi(\eta, \xi_k) = \frac{1}{2}\left(-\frac{\partial^2}{\partial \xi_k^2} + \omega_k^2(\eta)\xi_k^2\right)\psi(\eta, \xi_k). \tag{36}$$

This is the Schrödinger equation of a harmonic oscillator with time-dependent frequency. The solution to such a system can be found through the method of invariants [30–32] and is a superposition of the following normalized wavefunctions:

$$\psi_n(\eta, \xi_k) = \frac{h_n(\frac{\xi_k}{f})}{\sqrt{2^n\, n!}}\, \frac{e^{-\frac{\xi_k^2}{2f^2}}}{(\pi f^2)^{\frac{1}{4}}}\, e^{i\,\frac{f'}{2f}\,\xi_k^2}\, e^{i\alpha_n}, \tag{37}$$

where $\alpha_n = \alpha_n(\eta) = -(n + \frac{1}{2})\int f^{-2} d\eta$ is a time-dependent phase, h_n are Hermite polynomials and $f = f(\eta)$ is an auxiliary function that is the solution of the following differential equation:

$$f'' + \omega_k^2 f - f^{-3} = 0. \tag{38}$$

The Spectrum for ξ_k can then be calculated by linking its perturbations to the curvature perturbations [29], yielding

$$\mathcal{P}^{\rm std}(k) = \frac{k^3}{4\pi^2}\, \frac{\langle 0|\hat{\xi}_k^2|0\rangle}{a^2\epsilon}\Bigg|_{-k\eta \ll 1} \tag{39}$$

where $\eta \to 0^-$ corresponds to $t \to +\infty$ so that $-k\eta \to 0$ is the large scale limit. The expectation value of $\hat{\xi}_k^2$ is computed on the vacuum state i.e. the ground state of the time-dependent oscillator; we therefore need to know how to express the result of $\hat{\xi}_k\psi_n$. This can be done by constructing ladder operators for the time-dependent system: they take the form

$$\hat{a}^\dagger = \frac{\frac{\hat{\xi}_k}{f} - i(f\hat{\pi}_k - f'\hat{\xi}_k)}{\sqrt{2}}, \quad \hat{a}^\dagger\psi_n = \sqrt{n+1}\, e^{i\varphi}\psi_{n+1}; \tag{40}$$

$$\hat{a} = \frac{\frac{\hat{\xi}_k}{f} + i(f\hat{\pi}_k - f'\hat{\xi}_k)}{\sqrt{2}}, \quad \hat{a}\psi_n = \sqrt{n}\, e^{-i\varphi}\psi_{n-1}; \tag{41}$$

from these we derive

$$\hat{\xi}_k\psi_n = f\left(\sqrt{\frac{n+1}{2}}\, e^{i\varphi}\psi_{n+1} + \sqrt{\frac{n}{2}}\, e^{-i\varphi}\psi_{n-1}\right), \tag{42}$$

$$\hat{\pi}_k\psi_n = i\frac{R}{f}\sqrt{\frac{n+1}{2}}\, e^{i\varphi}\psi_{n+1} - i\frac{R^*}{f}\sqrt{\frac{n}{2}}\, e^{-i\varphi}\psi_{n-1}, \tag{43}$$