With D satisfying the conditions in (1), let $r(\mathcal{S}_\phi(E_{D'}))$ and $r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D))$ denote the rank, as $\mathbb{F}_3$-vector spaces, of the Selmer groups $\mathcal{S}_\phi(E_{D'})$ and $\mathcal{S}_{\hat{\phi}}(\hat{E}_D)$, relative to the isogenies ϕ and $\hat{\phi}$, of the curves $E_{D'}$ and $\hat{E}_D$ respectively, over $\mathbb{Q}$. Denote by $r_3(D)$ and $r_3(D')$ the rank of the 3-part of the ideal class group $\mathcal{CL}(K_D)$ and $\mathcal{CL}(K_{D'})$ of K_D and $K_{D'}$ respectively. In the next section we will compute the precise rank for the Selmer groups $\mathcal{S}_\phi(E_{D'})$ and $\mathcal{S}_{\hat{\phi}}(\hat{E}_D)$ and obtain a parity result regarding the rank of the curves $E_{D'}$.

3. On the 3-Selmer group and rank of the elliptic curves $E_{D'}$

By employing the results of Satgé [12, Section 3], we compute below the precise rank for the Selmer groups $\mathcal{S}_\phi$ and $\mathcal{S}_{\hat{\phi}}$:

Proposition 3.1. *With D satisfying the congruence conditions in (1), the rank of the Selmer groups $\mathcal{S}_\phi(E_{D'})$ and $\mathcal{S}_{\hat{\phi}}(\hat{E}_D)$ of the curves $E_{D'}$ and $\hat{E}_D$ are as follows:*

$$r(\mathcal{S}_\phi(E_{D'})) = r_3(D')$$

$$r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D)) = r_3(D') + 1.$$

Proof. Our elliptic curves $E_{D'}$ have a constant term equal to $16D' > 0$. With D' squarefree and with $2^4||16D'$, Lemma 3.1 in [12] is vacuously true. Now $3||16D'$ and, given the congruence condition $D \equiv 2 \bmod 3$, we have that $-16D \equiv 1 \bmod 3$. Therefore, from Proposition 3.2(1) of [12] we have that $r(\mathcal{S}_\phi(E_{D'})) = r_3(D')$. Finally, since $16D' > 0$, Proposition 3.3.(1) of [12] gives $r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D)) = r_3(D') + 1$. □

As in Remark 2.3, we denote by $\mathcal{S}_3(\hat{E}_D)$ and $\mathcal{S}_3(E_{D'})$ the 3-Selmer group of the corresponding elliptic curves. Its rank will be denoted by $r(\mathcal{S}_3(E_{D'}))$ and similarly for $\hat{E}_D$. We now consider the exact sequence ([11, Corollary 1])
(10)

$$0 \to \frac{\hat{E}_D(\mathbb{Q})[\hat{\phi}]}{\phi(E_{D'}(\mathbb{Q})[3])} \to \mathcal{S}_\phi(E_{D'}) \to \mathcal{S}_3(E_{D'}) \to \mathcal{S}_{\hat{\phi}}(\hat{E}_D) \to \frac{\text{Ш}(\hat{E}_D)[\hat{\phi}]}{\phi(\text{Ш}(E_{D'})[3])} \to 0.$$

Since our curves have no rational 3-torsion points, the first term of (10) is trivial. As it is known, because of the non-degenerate alternating pairing on $\frac{\text{Ш}(\hat{E}_D)[\hat{\phi}]}{\phi(\text{Ш}(E_{D'})[3])}$ (defined by Cassels in [4]), this last term is an even-dimensional $\mathbb{F}_3$-vector space (assuming finiteness of the Tate-Shafarevich group). Therefore, we obtain the following result regarding the parity of the rank of the 3-Selmer group and the ranks of the two Selmer groups $\mathcal{S}_\phi(E_{D'})$ and $\mathcal{S}_{\hat{\phi}}(\hat{E}_D)$:

$$r(\mathcal{S}_3(E_{D'})) \equiv r(\mathcal{S}_\phi(E_{D'})) + r(\mathcal{S}_{\hat{\phi}}(\hat{E}_D)) \bmod 2. \tag{11}$$

Corollary 3.2. *The elliptic curves $E_{D'}$ have odd rank.*

Proof. Given Remark 2.3, the exact sequence analogous to (9) is

$$0 \to \hat{E}_D(\mathbb{Q})/3(E_{D'}(\mathbb{Q})) \to \mathcal{S}_3(E_{D'}(\mathbb{Q})) \to \text{Ш}(E_{D'}(\mathbb{Q}))[3] \to 0. \tag{12}$$