The remaining calculation simply involves taking the first projections, which yields:

$$\begin{aligned}(\mathrm{pr}_1)_*(h_1-h_2)^l &= \sum_{a=0}^{l}(-1)^{l-a}\binom{l}{a}h^a\cdot(\mathrm{pr}_1)_*\mathrm{pr}_2^*\,h^{l-a}\\ &= (-1)^n\binom{l}{l-n}h^{l-n}\cdot d[X]. \qquad (l\geq n)\end{aligned}$$

By substituting (15) into (14) we have :

$$\begin{aligned}s(X,\sigma_2(X)) &= \sum_{l=n}^{2n}(-1)^{n+l}\binom{l}{n}\cdot d\,h^{l-n}-\sum_{l=n}^{2n}2^l s_{l-n}(T_X)\cap[X]\\ &= \sum_{i=0}^{n}(-1)^i\binom{i+n}{n}\cdot d\,h^i-\sum_{i=0}^{n}2^{i+n}s_i(T_X)\cap[X].\end{aligned}$$

From equation (9), we obtain:

$$\begin{aligned}s(\Delta(X),X\times\sigma_2(X)) &= s(T_X)\cap s(X,\sigma_2(X))\\ &= \sum_{i=0}^{n}\sum_{a+b=i}\{(-1)^b\binom{b+n}{n}\,d\,h^b\cdot s_a(T_X)\cap[X]-2^{b+n}s_a(T_X)s_b(T_X)\cap[X]\}.\end{aligned}$$

by omitting the pull back $(\Delta(X)\to X)^*$. Therefore, we obtain the main formula from equation (2), which completes the proof. □

Corollary. *The multiplicity of $\sigma_2(X)$ along X is $d-2^n$.*

Proof. The multiplicity of $\sigma_2(X)$ along X is the coefficient of $[X]$ in the class $s(X,\sigma_2(X))$. (cf. [8, Chapter 4.3]) □

5 Examples

5.1 In the case of curves

Consider the case where C is a smooth projective curve. Let g be the genus of a curve C. The degree of each term of Segre class T_C is given by $\deg s_0(T_C)=d$ and $\deg s_1(T_C)=2g-2$. By substituting those terms in the main formula we get :

$$\deg\sigma_3(C)=\frac{1}{3!}(d^3-9d^2+26d+24-6dg-24g)$$

which can be found in [12, Proposition 1].

5.2 In the case of surfaces

Consider the case where S is a smooth projective surface. Let K be the canonical divisor of S, let $d=h^2$, let $\pi=h\cdot K$, let $\kappa=K^2$, and let $e=c_2$ be the topological Euler characteristic. The total Segre class of the tangent sheaf T_S of S has degree 0, 1, and 2 terms, which are given by:

$$s_0(T_S)=[S],\ s_1(T_S)=K,\ s_2(T_S)=\kappa-e.$$

By substituting those terms in the main formula we get :

$$\deg\sigma_3(S)=\frac{1}{3!}(d^3-30d^2+224d-3d(5\pi+\kappa-e)+192\pi+56\kappa-40e)$$