subspaces P, P', Q, Q' contains a vector from the basis B, the subspaces

$$Z_1 = P' \oplus (X \cap Y) \oplus Q \text{ and } Z_2 = P \oplus (X \cap Y) \oplus Q'$$

are spanned by subsets of B and ortho-adjacent to each of X, Y, Z.

Consider any $Z' \in \mathcal{G}_\infty(H)$ ortho-adjacent to each of X, Y, Z. The subspaces X, Y, Z, Z' are mutually compatible and there is an orthonormal basis B' such that all these subspaces are spanned by subsets of B'. Then $X \cap Y, X', Y'$ are also spanned by subsets of B' and each of the 1-dimensional subspaces P, P', Q, Q' contains a vector from B'. The subspace Z' contains $X \cap Y$, the subspaces $X' \cap Z'$, $Y' \cap Z'$ are 1-dimensional and

$$Z' = (X' \cap Z') \oplus (X \cap Y) \oplus (Y' \cap Z')$$

(Z' is ortho-adjacent to X and Y). Recall that

$$Z = P \oplus (X \cap Y) \oplus Q.$$

Since $X' \cap Z'$ is a 1-dimensional subspace of X' containing a vector from B', it coincides with P or P'. Similarly, $Y' \cap Z'$ coincides with Q or Q'. Then Z and Z' are ortho-adjacent only when Z' is Z_1 or Z_2. □

Lemma 2. *Let X, Y be elements of $\mathcal{G}_\infty(H)$ whose intersection is of codimension 2 in both X, Y. If there are ortho-adjacent $Z, Z' \in \mathcal{G}_\infty(H)$ such that each of them is ortho-adjacent to both X, Y, then X, Y are compatible.*

Proof. We assert that $Z \cap Z'$ is contained in X or Y. If $Z \cap Z'$ is not contained in X, then $Z \cap X$ and $Z' \cap X$ are distinct hyperplanes of X and their sum is X. Similarly, if $Z \cap Z' \not\subset Y$, then $Z \cap Y$ and $Z' \cap Y$ are distinct hyperplanes of Y whose sum is Y. Then $Z + Z'$ contains both X, Y. Since $Z \cap X$ and $Z' \cap X$ are subspaces of codimension 2 in $Z + Z'$, the subspace X is a hyperplane of $Z + Z'$. For the same reason, Y is a hyperplane of $Z + Z'$. Then X, Y are adjacent which is impossible.

Without loss of generality, we can assume that $Z \cap Z'$ is contained in X. Then $Z \cap Z'$ is a hyperplane of X. The orthogonal complement of $X \cap Y$ in $X + Y$ is 4-dimensional and we denote this subspace by M. Then

$$X' = X \cap M \text{ and } Y' = Y \cap M$$

are 2-dimensional subspaces whose intersection is 0. Observe that each of Z, Z' contains $X \cap Y$ and is contained in $X + Y$ which implies that

$$S = Z \cap M \text{ and } S' = Z' \cap M$$

are distinct 2-dimensional subspaces. Since $Z \cap Z'$ is a hyperplane of X and $X \cap Y$ is contained in $Z \cap Z'$,

$$Z \cap Z' \cap M = S \cap S'$$

is a 1-dimensional subspace of X' which will be denoted by P. The subspaces

$$Z = (X \cap Y) \oplus S, \quad Z' = (X \cap Y) \oplus S'$$

are ortho-adjacent to $Y = (X \cap Y) \oplus Y'$ and, consequently, S and S' intersect Y' in certain 1-dimensional subspaces P_1 and P_2, respectively. The subspaces P_1, P_2 are distinct (since $S = P + P_1$ and $S' = P + P_2$ are distinct).