Additionally, as a result of the Schwarzian inequality, we have

$$|\psi'(z)|\leq \frac{1-|\psi(z)|^2}{1-|z|^2}, \quad (z\in\mathbb{D}). \tag{2.10}$$

Moreover, from equations (2.8)-(2.10), we deduce

$$|f'(z)|\leq \left(|\psi(z)|+\frac{r(1-|\psi(z)|^2)}{(1-r^2)(\mathcal{LP}(r))}\right)|g'(z)|.$$

Substituting $|\psi(z)|=\sigma$ $(0\leq\sigma\leq 1)$, results in

$$|f'(z)|\leq \Psi(r,\sigma)|g'(z)|,$$

where

$$\Psi(r,\sigma)=\sigma+\frac{r(1-\sigma^2)}{(1-r^2)(\mathcal{LP}(r))}.$$

We need to determine $r_m\leq r^*$ so that

$$r_m=\max\{r\in[0,r^*]:\Psi(r,\sigma)\leq 1\ \forall\sigma\in[0,1]\}.$$

Equivalently if $\Phi(r,\sigma):=(1-r^2)(\mathcal{LP}(r))-r(1+\sigma)$ then we need to determine

$$r_m=\max\{r\in[0,r^*]:\Phi(r,\sigma)\geq 0\ \forall\sigma\in[0,1]\}.$$

Since $\partial\Phi/\partial\sigma=-r<0$, then $\max_{\sigma\in[0,1]}\Phi(r,\sigma)=\Phi(r,0)=:\phi_0(r)$. Further it is evident that, as $\phi_0(0)=1>0$ and $\phi_0(r^*)=-r^*<0$, then there exists $r_m\leq r^*$, a smallest positive root of the equation given in (2.7) such that $\phi_0(r)\geq 0$ for each $r\in[0,r_m]$. This completes the proof. ∎

In 2017, Peng and Zhong [22], introduced the class $\Omega\subset\mathcal{A}$, defined as

$$\Omega=\{f\in\mathcal{A}:|zf'(z)-f(z)|<1/2\}.$$

We conclude this section by determining sharp $\Omega-$radius for the class $\mathcal{F}_{\mathcal{LP}}$.

Theorem 2.21. *Let $f\in\mathcal{F}_{\mathcal{LP}}$, then $f\in\Omega$ in $|z|<r_{\mathcal{L}}\approx 0.522\ldots$ is the smallest positive root of*

$$4f_0(r)(\log((1+\sqrt{r})/(1-\sqrt{r})))^2=\pi^2$$

and

$$g_0(z)=z\left(\exp\int_0^z\frac{\mathcal{P}_0(-t)}{t}dt\right)=z+\frac{8}{\pi^2}z^2-\frac{8}{3\pi^4}(\pi^2-12)z^3+\frac{8}{135\pi^6}(1440-360\pi^2+23\pi^4)z^4-\cdots. \tag{2.11}$$

This is a sharp estimate.

Proof. Since $f\in\mathcal{F}_{\mathcal{LP}}$, then as a consequence of Remark 2.4 for $|z|=r<1$, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right|<|\mathcal{LP}(r)-1|=|\mathcal{P}_0(r)|.$$

Due to the growth theorem as mentioned in [15, Theorem 1] and Theorem 2.2, we observe that $|f(z)|\leq g_0(r)$, where $g_0(r)$ is given by (2.11). Further

$$|zf'(z)-f(z)|=|f(z)|\left|\frac{zf'(z)}{f(z)}-1\right|\leq g_0(r)|\mathcal{P}_0(r)|.$$

Thus $g_0(r)|\mathcal{P}_0(r)|\leq 1/2$ provided $|z|<r_{\mathcal{L}}\approx 0.522864$. Hence the result is established. ∎