In equation (10), it is deduced that

$$s(Z_2, \mathbb{P}(E_\mathcal{L})) = \sum_{i \geq 0} (-1)^i [Z_2]^{i+1}.$$

The Chow ring of the projective bundle, $A^*\mathbb{P}(E_\mathcal{L})$, is well-known, but it is simpler to compute the restriction on Z_2 rather than on the projective bundle. Therefore, the power of $[Z_2]$ is computed as follows:

$$[Z_2]^{i+1} = ([Z_2]|_{Z_2})^i$$

for $i \geq 0$. Additionally, it is straightforward to apply the push-forward r_* on the cycles on Z_2, as the image $r(Z_2)$ is X. Therefore, the equation (10) can be rewritten as:

$$s(Z_2, \mathbb{P}(E_\mathcal{L})) = \sum_{i \geq 0} (-1)^i ([Z_2]|_{Z_2})^i. \tag{11}$$

To continue, we require an expression of $[Z_2|_{Z_2}]$ as a linear combination of generators of the Chow ring A^*Z_2.

Proposition 3.2. $[Z_2]|_{Z_2} = 2E + \eta^*(h_1 - h_2)$ *in* $A^1 Z_2$ *where* $[Z_2]$ *is a cycle associated to scheme* Z_2.

Proof. We denote the canonical divisors of $X^{[2]}$ and Z_2 as $K_{X^{[2]}}$ and K_{Z_2}, respectively. The ramification divisor of ρ is E, so we have

$$\rho^* K_{X^{[2]}} = K_{Z_2} - E. \tag{12}$$

Equation (12) can be found in some literature, or it can be deduced directly from the following exact sequence of sheaves:

$$0 \to \rho^* T^*_{X^{[2]}} \to T^*_{Z_2} \to \mathcal{O}_E(-E) \to 0.$$

Let ζ be the first Chern class of the tautological line bundle of $\mathbb{P}(E_\mathcal{L})$. The normal bundle $N_{Z_2}\mathbb{P}(E_\mathcal{L})$ is represented by $\mathcal{O}_{Z_2}([Z_2])$,

$$c_1(\mathcal{O}_{Z_2}([Z_2])) = K_{Z_2} - K_{\mathbb{P}(E_\mathcal{L})}|_{Z_2}.$$

In addition, we have the following adjunction formula for the projective bundle:

$$K_{\mathbb{P}(E_\mathcal{L})} = -2\zeta + \pi^* c_1(E_\mathcal{L}) + \pi^* K_{X^{[2]}}.$$

Note that $(\pi^* K_{X^{[2]}})|_{Z_2} = \rho^* K_{X^{[2]}} = K_{Z_2} - E = K_{Z_2} - \rho^* \delta$. Then

$$c_1(\mathcal{O}_{Z_2}([Z_2])) = 2\zeta|_{Z_2} - \rho^*(H - \delta) + \rho^* \delta$$

by proposition 3.1. Recall that $c_1(\mathcal{O}_{Z_2}(Z_2) = [Z_2]_{Z_2}$ and hence we get

$$[Z_2]|_{Z_2} = 2\zeta|_{Z_2} - \rho^*(H - 2\delta). \tag{13}$$

We need information about $\zeta|_{Z_2}$. We know that Z_2 is a closed subvariety of $X \times X^{[2]} \subset \mathbb{P}H^0(X, \mathcal{L}) \times X^{[2]}$. Therefore, we can choose a suitable section of $\zeta|_{Z_2}$ whose zero locus is $(h \times X^{[2]}) \cap Z_2$.

Recall that we have an isomorphism $Z_2 \cong Bl_{\Delta(X)}(X \times X)$, which allows us to view $Z_2 \subset X \times X^{[2]} \to X$ as equivalent to $Bl_{\Delta(X)}(X \times X) \to X \times X \to X$, where the second morphism is pr_1. Thus, we obtain $\zeta|_{Z_2} = \eta^* h_1|_{Z_2}$.

Substituting this result into (13), we find that

$$[Z_2]|_{Z_2} = 2E + \eta^*(h_1 - h_2).$$

$\square$