

(ii)  $f \in \mathcal{S}_{\mathcal{RL}}^*$  in  $|z| < \tanh^2(\pi^4 \sqrt{2} \eta(1 - \sqrt{2}\eta)/2\sqrt{2}) \approx 0.283\dots$

*Remark 2.15.* From **Fig. 5** it is evident that the radii obtained in Corollary 2.14 can be further improved.

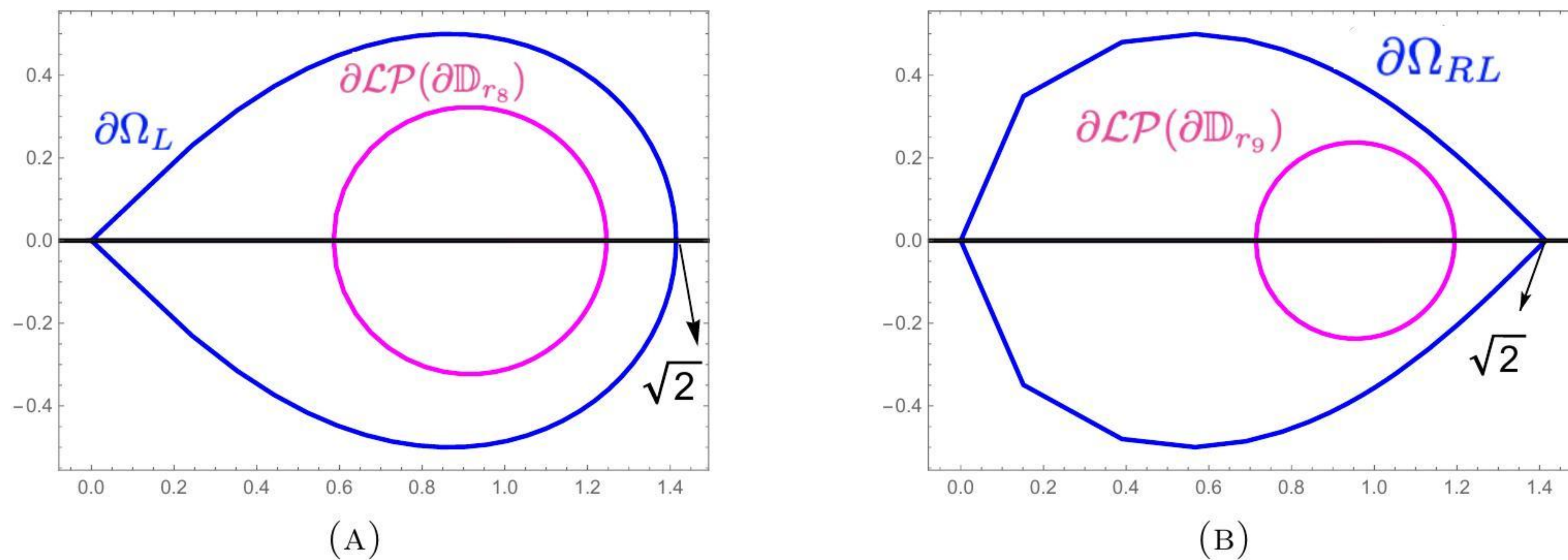


FIGURE 5. Above figures correspond to Corollary 2.14 with radii  $r_8$  and  $r_9$  given by: (a)  $r_8 \approx 0.376\dots$  (b)  $r_9 \approx 0.283\dots$

We now define the class  $\mathfrak{F}$  constructed with the help of ratios of two analytic functions  $f, g \in \mathcal{A}$ , studied by Mundalia and Kumar [21].

$$\mathfrak{F} = \left\{ f \in \mathcal{A} : \operatorname{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \ \& \ \operatorname{Re} \left( \frac{(1-z)^{1+A}g(z)}{z} \right) > 0, -1 \leq A \leq 1 \right\}.$$

In particular, for  $A = -1$  and  $A = 1$ , the class  $\mathfrak{F}$  reduces to the following classes, respectively,

$$\mathfrak{F}_1 = \left\{ f \in \mathcal{A} : \operatorname{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \ \& \ \operatorname{Re} \left( \frac{g(z)}{z} \right) > 0 \right\}$$

and

$$\mathfrak{F}_2 = \left\{ f \in \mathcal{A} : \operatorname{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \ \& \ \operatorname{Re} \left( \frac{(1-z)^2g(z)}{z} \right) > 0 \right\}.$$

For proving the next theorem, we require the following lemma of Ravichandran et al. [24].

**Lemma 2.16.** If  $p \in \mathcal{P}_n[A, B]$ , then for  $|z| = r$

$$\left| p(z) - \frac{1 - AB r^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{|A - B| r^n}{1 - B^2 r^{2n}}.$$

Particularly, if  $p \in \mathcal{P}_n(\alpha)$ , then

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1 - \alpha)r^n}{1 - r^{2n}}.$$

**Theorem 2.17.** Let  $-1 \leq A \leq 1$ , and suppose  $f \in \mathcal{F}_{\mathcal{LP}}$ , then the sharp  $\mathfrak{F}$ -radius is given by

$$\mathcal{R}_{\mathfrak{F}}(\mathcal{F}_{\mathcal{LP}}) = \frac{1}{2A + 3} \left( \sqrt{A^2 + 12A + 28} - (5 + A) \right) =: R_{\mathfrak{F}}.$$

*Proof.* Since  $f \in \mathfrak{F}$ , then by definition of the class  $\mathfrak{F}$ , we have  $f(z) = p_1(z)g(z)$  and  $g(z) = zp_2(z)(1 - z)^{-(1+A)}$ , where for each  $i = 1, 2$ ,  $p_i : \mathbb{D} \rightarrow \mathbb{C}$  are analytic functions such that  $p_i(0) = 1$  and  $\operatorname{Re} p_i(z) > 0$ . This leads to  $f(z) = zp_1(z)p_2(z)(1 - z)^{-(1+A)}$ , and as a consequence of logarithmic differentiation, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{1 + Az}{1 - z} + \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)}.$$

For each  $-1 \leq A \leq 1$ , Lemma 2.16 leads to,

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + Ar^2}{1 - r^2} \right| \leq \frac{(5 + A)r}{1 - r^2} = R. \quad (2.5)$$