We next prove that $\lim_{t\to 0^+}\frac{(h_{W_t}-h_K)(u)}{t}=f(u)$, for S_K-almost every u (the proof on the left of 0 is similar). Let $l_d(u)=\lim_{t\to 0^+}\frac{(h_{W_t}-h_K)(u)}{t}$, which is well-defined for all $u\in\mathbb{S}^{n-1}$, by concavity. Note that (by definition of W_t), $l_d(u)\le f(u)$, for all u. Let $U_\epsilon=\{u\in\mathbb{S}^{n-1}: l_d(u)<f(u)-\epsilon\}$. It is enough to show that $S_K(U_\epsilon)=0$ (for arbitrary $\epsilon>0$).

By Fatou's lemma,

$$\begin{aligned}\limsup_{t\to 0^+}\int_{U_\epsilon}\frac{(h_{W_t}-h_K)(u)}{t}dS_K(u) &\le \int_{U_\epsilon}\limsup_{t\to 0^+}\frac{(h_{W_t}-h_K)(u)}{t}dS_K(u)\\ &=\int_{U_\epsilon} l_d(u)dS_K(u)\\ &\le \int_{U_\epsilon} f(u)dS_K(u)-\epsilon S_K(U_\epsilon).\end{aligned}$$

Therefore :

$$\lim_{t\to 0^+}\frac{V_1(t)-V_0}{t}\le\frac{1}{n}\int_\Omega f(u)dS_K(u)-\frac{\epsilon}{n}S_K(U_\epsilon).$$

It follows from lemma 1 that $S_K(U_\epsilon)=0$.

Similarly, let $l_g(u)=\lim_{t\to 0^-}\frac{(h_{W_t}-h_K)(u)}{t}$, $u\in\mathbb{S}^{n-1}$, so that $l_g(u)\ge f(u)$ for all $u\in\Omega$, and let $V_\epsilon=\{u\in\mathbb{S}^{n-1}: l_g(u)>f(u)+\epsilon\}$. Fix $x_0\in int(K)$, and let $K'=K-x_0$. If $|t|$ is small enough, then a translate of $(1-C|t|)K'$ is contained in W_t, with $C=(\max_u|f|)/(\min_u h_{K'})>0$.

In other words, $\frac{(h_{W_t}-h_K)(u)}{t}\le Ch_{K'}(u)$ for all $u\in\mathbb{S}^{n-1}$, for any $t<0$ (with $|t|$ small enough). Hence, by dominated convergence,

$$\begin{aligned}\lim_{t\to 0^-}\frac{V_1-V_0}{t}&=\frac{1}{n}\int_{V_\epsilon} l_g(u)dS_K(u)+\frac{1}{n}\int_{\Omega\setminus V_\epsilon} l_g(u)dS_K(u)\\ &\ge\frac{1}{n}\int_\Omega f(u)dS_K(u)+\frac{\epsilon}{n}S_K(V_\epsilon).\end{aligned}$$

So that $S_K(V_\epsilon)=0$, by lemma 1. It follows that $l_g(u)=f(u)$ for S_K-a.e. $u\in\Omega$. $\square$

4.2. **Examples when Proposition 1 applies : computations.** We review one of the examples listed after Proposition 2.

Let M be half a Euclidean ball : $M=B_2^n\cap H_+$, with $H_+=\{x\in\mathbb{R}^n: x_n\ge 0\}$. Then

$$\mathrm{Isop}(M)=\frac{|\partial M|}{n|M|}=\frac{2|\partial M|}{n\kappa_n}=\frac{n\kappa_n}{n\kappa_n}+\frac{2\kappa_{n-1}}{n\kappa_n}=1+\frac{1}{nW_n}>1.$$

where $W_n=\int_0^{\pi/2}(\cos(\phi))^n d\phi$ (recall that $\kappa_n=2\kappa_{n-1}W_n$).

Let $T=T_a$ be the linear map such that $Te_i=e_i$ when $i\le n-1$, and $Te_n=ae_n$. Then $TB_2^n=\mathcal{E}_a$ is an ellipsoid of volume $a\kappa_n$. Moreover one may compute its surface area :

$$|\partial\mathcal{E}_a|=2(n-1)\kappa_{n-1}\int_0^{\pi/2}R_\phi^{n-1}(\cos(\phi))^{n-2}d\phi=2(n-1)\kappa_{n-1}\int_0^{\pi/2}r_\phi^{n-1}\frac{d\phi}{\cos(\phi)}.$$

The equation of the surface $\partial\mathcal{E}_a$ is $r^2+\frac{h^2}{a^2}=1$, where $h=x_n$, and $r^2=x_1^2+...+x_{n-1}^2$. The surface can be layered according to $r=R\cos(\phi)\in[0,1]$, and one may write r as $r=:\cos(\psi)$ for some $\psi\in[0,\frac{\pi}{2}]$. Then $\tan^2(\psi)=\frac{1-r^2}{r^2}=\frac{1}{a^2}\frac{h^2}{r^2}=\frac{1}{a^2}\tan^2(\phi)$, therefore (differentiating) $(1+\tan^2(\psi))d\psi=\frac{1}{a}(1+\tan^2(\phi))d\phi=\frac{1}{a}(1+a^2\tan^2(\psi))d\phi$, i.e. $d\phi=a\frac{1+\tan^2(\psi)}{1+a^2\tan^2(\psi)}d\psi=a\frac{1}{1+(a^2-1)\sin^2(\psi)}d\psi$.

Also, $\cos(\phi)=\frac{r}{R}=\frac{r}{(a^2-(a^2-1)r^2)^{1/2}}=\frac{\cos(\psi)}{(1+(a^2-1)\sin^2(\psi))^{1/2}}$.