Since j is an affine morphism, there are no higher direct images and we obtain followings :

$$\mathrm{ch}E_{\mathcal{L}} = \pi_{2\,*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2})\ \pi_1^*\mathrm{ch}\mathcal{L}\ \pi_1^*\mathrm{td}T_X)$$

and

$$\mathrm{ch}j_*\mathcal{O}_{Z_2} = j_*((\mathrm{td}\mathcal{N})^{-1})$$

by applying the Grothendieck-Riemann-Roch theorem. Note that $\pi_2 \circ j = \rho$.

By the projection formula, we have:

$$\begin{aligned}
\mathrm{ch}E_{\mathcal{L}} &= \pi_{2\,*}(\mathrm{ch}(j_*\mathcal{O}_{Z_2})\ \pi_1^*\mathrm{ch}\mathcal{L}\ \pi_1^*\mathrm{td}T_X) \\
&= \pi_{2\,*}(j_*\mathrm{ch}(\mathrm{td}\mathcal{N})^{-1}\ \pi_1^*\mathrm{ch}\mathcal{L}\ \pi_1^*\mathrm{td}T_X) \\
&= (\pi_{2\,*} \circ j_*)(\mathrm{ch}(\mathrm{td}\mathcal{N})^{-1}\ j^*\pi_1^*\mathrm{ch}\mathcal{L}\ j^*\pi_1^*\mathrm{td}T_X) \\
&= \rho_*((\mathrm{td}\mathcal{N})^{-1}\ q^*\mathrm{ch}\mathcal{L}\ q^*\mathrm{td}T_X).
\end{aligned}$$

Consider the sheaf sequence

$$0 \to \mathcal{N}^* \to q^*\Omega_X \to \mathcal{O}_E(-E) \to 0$$

as given in [6, Lemma 2.1]. (Note that the morphism q in proposition 3.1 and in [6] are different morphisms.) Taking dual of this sequence, we obtain :

$$0 \to q^*T_X \to \mathcal{N} \to \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \to 0.$$

Therefore we have

$$(\mathrm{td}\mathcal{N})^{-1} = (q^*\mathrm{td}\ T_X)^{-1} \cdot (\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}))^{-1}.$$

To evaluate $\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2})$, we consider the exact sequence :

$$0 \to \mathcal{O}_{Z_2}(-2E) \to \mathcal{O}_{Z_2}(-E) \to \mathcal{O}_E(-E) \to 0.$$

Taking the dual of this sequence, we have :

$$0 \to \mathcal{O}_{Z_2}(E) \to \mathcal{O}_{Z_2}(2E) \to \mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) \to 0.$$

Therefore, we obtain

$$\mathrm{td}\mathcal{E}xt^1(\mathcal{O}_E(-E), \mathcal{O}_{Z_2}) = \frac{2E}{1-e^{-2E}} \Big/ \frac{E}{1-e^{-E}} = \frac{2}{1+e^{-E}}$$

and

$$\begin{aligned}
\mathrm{ch}E_{\mathcal{L}} &= \rho_*\left(q^*\mathrm{ch}\mathcal{L} \cdot \frac{1+e^{-E}}{2}\right) \\
&= \rho_*\left(\left(1 + \eta^*h_1 + \frac{1}{2!}\eta^*h_1^2 + \frac{1}{3!}\eta^*h_1^3 + \cdots\right) \cdot \left(1 - \frac{1}{2}E + \frac{1}{4}E^2 - \frac{1}{12}E^3 + \cdots\right)\right) \\
&= \rho_*\left(1 + \left(\eta^*h_1 - \frac{1}{2}E\right) + \left(\frac{1}{2}\eta^*h_1^2 - \frac{1}{2}\eta^*h_1 \cdot E + \frac{1}{4}E^2\right) + \cdots\right) \\
&= 2 + (H-\delta) + \left(\frac{1}{2}\rho_*\eta^*h_1^2 - \frac{1}{2}\rho_*(\eta^*h_1 \cdot E) + \frac{1}{4}\rho_*E^2\right) + \cdots .
\end{aligned}$$

So we obtain $c_1(E_{\mathcal{L}}) = H - \delta$. □