CHOW'S THEOREM FOR HILBERT GRASSMANNIANS AS A WIGNER-TYPE THEOREM

MARK PANKOV, ADAM TYC

Abstract. Let H be an infinite-dimensional complex Hilbert space. Denote by $\mathcal{G}_\infty(H)$ the Grassmannian formed by closed subspaces of H whose dimension and codimension both are infinite. We say that $X, Y \in \mathcal{G}_\infty(H)$ are *ortho-adjacent* if they are compatible and $X \cap Y$ is a hyperplane in both X, Y, equivalently, the projections corresponding to X and Y commute and the rank of their difference is 2. A subset $\mathcal{C} \subset \mathcal{G}_\infty(H)$ is called a *connected component* if for any $X, Y \in \mathcal{C}$ the intersection $X \cap Y$ is of the same finite codimension in both X, Y and $\mathcal{C}$ is maximal with respect to this property. Let f be a bijective transformation of $\mathcal{G}_\infty(H)$ preserving the ortho-adjacency relation in both directions. We show that the restriction of f to every connected component of $\mathcal{G}_\infty(H)$ is induced by a unitary or anti-unitary operator or it is the composition of the orthocomplementary map and a map induced by a unitary or anti-unitary operator. Note that the restrictions of f to distinct components can be related to different operators.

1. Introduction

Two (not necessarily finite-dimensional) subspaces of a vector space are called *adjacent* if their intersection is a hyperplane in each of these subspaces. Any two adjacent subspaces are of the same dimension and codimension. Classic Chow's theorem [2] characterizes semilinear automorphisms of vector spaces over division rings as adjacency preserving transformations of Grassmannians formed by finite-dimensional subspaces. This result is a generalization of the Fundamental Theorem of Projective Geometry characterizing semilinear automorphisms as automorphisms of projective spaces. There are analogues of Chow's theorem for totally isotropic subspaces of sesquilinear forms, singular subspaces of quadratic forms and various types of matrix spaces [3, 16, 24]. An extension of Chow's theorem on Grassmannians consisting of subspaces with infinite dimension and codimension is an open problem. We refer to [17, Section 2.4] for an explanation why this problem is hard.

Chow's theorem was used to prove some Wigner-type theorems for Hilbert Grassmannians formed by finite-dimensional subspaces [4, 5, 17]. These Grassmannians are identified with the conjugacy classes of finite-rank projections. The adjacency relation can be interpreted as follows: the ranges of projections are adjacent if their difference is an operator of rank 2 (i.e. the smallest possible).

Classic Wigner's theorem [25] characterizes unitary and anti-unitary operators as quantum symmetries; see [1] for a brief history and physical background. By Gleason's theorem [9], pure states of a quantum mechanical system can be identified with rank-one projections or, equivalently, rays of a complex Hilbert space H. The

Key words and phrases. Hilbert Grassmannian, compatibility, adjacency, projection of infinite rank.