i- $V_n(K_1, ..., K_n) = \frac{1}{n!}\sum_{J\subset[n]}(-1)^{n-|J|}|K_J|_n$ where [2] $K_J = \sum_{i\in J} K_i$
ii- $V_n(K_1 + x_1, K_2, ..., K_n) = V_n(K_1, ..., K_n)$ (translation invariance)
iii- $V_n(K_1 + \lambda K_1', K_2, ..., K_n) = V_n(K_1, K_2, ..., K_n) + \lambda V_n(K_1', K_2, ..., K_n)$ (multilinearity)
iv- $V_n(K_{\sigma(1)}, ..., K_{\sigma(n)}) = V_n(K_1, ..., K_n)$ (symmetry in the arguments)
v- $V_n(.)$ is continuous on $(\mathcal{K}^n)^n$, with respect to Hausdorff topology.

Moreover, Minkowski proved $V_n(.)$ also enjoys the following properties :

vi- $V_n(K_1, ..., K_n) \geq 0$ (non-negativity)
vii- if $K_1 \subset K_1'$, then $V_n(K_1, K_2, ..., K_n) \leq V_n(K_1', K_2, ..., K_n)$ (monotonicity).

Hence $V_n(.)$ is a (multilinear) functional on $(\mathcal{K}^n)^n$. When the underlying dimension (i.e. the total number of arguments $V_n(.)$ takes) is clear, we may drop the subscript n, and write $V(.)$ rather than $V_n(.)$. It follows directly from the definition of mixed volumes, that $V_n(K[n]) = |K|_n$. Therefore we may slightly abuse notation, and write $V_n(K)$, or even $V(K)$, instead of $V_n(K[n])$, as this shortcut seems common in the literature. To avoid confusion, we only use this shortcut when K is indeed n-dimensional, i.e. when K is a non-degenerate convex body in $\mathbb{R}^n$.

Let $u \in \mathbb{S}^{n-1}$ be a unit vector. Denote π_u the orthogonal projection onto $u^\perp$. If $u_1, ..., u_k$ are k linearly independent unit vectors in $\mathbb{R}^n$, denote π_U the orthogonal projection onto $(u_1, ..., u_k)^\perp$. We will also need the following well-known property of mixed volumes :

viii-
$$V_n([0, u], K_2, ..., K_n) = \frac{1}{n}V_{n-1}(\pi_u K_2, ..., \pi_u K_n),$$
ix-
$$V_n([0, u_1], ..., [0, u_k], K_{k+1}, ..., K_n) = \frac{k!V_k([0, u_1], ..., [0, u_k])}{n(n-1)...(n-k+1)}V_{n-k}\left(\pi_U K_{k+1}, ..., \pi_U K_n\right).$$

(identity (ix) is deduced from (viii) by iteration).

Let $K \subset \mathbb{R}^n$ be a compact convex set. Its support function h_K, is defined on $\mathbb{R}^n$ by $h_K(x) = \max_{y\in K}\langle y, x\rangle$, where $\langle ., .\rangle$ denotes the usual scalar product on $\mathbb{R}^n$. Since $h_K(\lambda x) = \lambda h_K(x)$ for any $x \in \mathbb{R}^n$, $\lambda > 0$, we shall more often consider h_K as a function on $\mathbb{S}^{n-1}$. Note that h_K characterizes K, since $K = \bigcap_u H^-(u, h_K(u))$, where $H^-(u, b) = \{z \in \mathbb{R}^n : \langle z, u\rangle \leq b\}$.

If we fix a convex body $K \subset \mathbb{R}^n$, then there exists a unique non-negative measure S_K on $\mathbb{S}^{n-1}$, such that the following holds for any compact convex set L :

$$V_n(L, K[n-1]) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)dS_K(u). \tag{1}$$

For instance, when $K = P$ is a polytope, the following integral representation of $V_n(L, P[n-1])$ is known :

$$V_n(L, P[n-1]) = \frac{1}{n}\sum_{u\in E(P)} h_L(u)|P^u|_{n-1} \tag{2}$$

where $E(P) = \{\text{outer normal vectors of } P\}$ and $P^u = P \cap H(u, h_P(u)) = \{y \in P : \langle y, u\rangle = h_P(u)\}$ is the facet whose outer normal vector is u. This means that S_P is the discrete measure $S_P = \sum_{u\in E(P)} |P^u|_{n-1}\delta_u$, where δ_v denotes the Dirac measure at $v \in \mathbb{S}^{n-1}$.

Though the formula 1 could be taken as a definition[3] of the surface area measure S_K, one may alternatively first define S_P for polytopes, via $S_P = \sum_{u\in E(P)} |P^u|_{n-1}\delta_u$, and then define S_K for an

[2] with the convention $K_\emptyset = \emptyset$

[3] the fact that knowing $\int h_K d\mu$ for all convex bodies K, is sufficient to characterize μ, i.e. to know $\int f d\mu$ for any continuous function f on the sphere, can be easily derived for instance from Lemma 1