Slope inequality for an arbitrary divisor

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Abstract

Let $f : S \longrightarrow C$ be a surjective morphism with connected fibers from a smooth complex projective surface S to a smooth complex projective curve C. Let D be an arbitrary divisor on S such that $\mathrm{rk}(f_*\mathcal{O}_S(D)) > 1$. We make sense to the notion of slope inequality for D case by case. As a consequence, we prove: if $f : S \to C$ is a relatively minimal fibration with $g = g(F) \geq 2$ where F is the general fiber of f, $D = K_{S/C}$ and $N_1|_F$ (the Miyaoka divisor for the maximal destabilizing sub-vector bundle restricted to F) is nonspecial, $h^0(F, N_1|_F) > 1$, then:

$$K_{S/C}^2 \geq 4\frac{(g-1)(2g-1)}{(g-1)(2g-1)+g} \deg f_*\omega_{S/C}.$$

1 Introduction

Let $f : S \to C$ be a surjective morphism from a smooth complex projective surface S to a smooth complex projective curve C with connected fibers. We call the morphism f a fibration. Let D be a divisor on S. We consider the sheaf $\mathcal{E} = f_*\mathcal{O}_S(D)$, which is torsion free because C is a curve. Since a torsion free sheaf on curve is always locally free sheaf, $\mathcal{E}$ is locally free and its rank is $h^0(F, D|F)$ where F is a general fiber of f of genus $g(F) = g$.

The fibration f is called smooth if all its fibers are smooth, isotrivial if all its smooth fibers are isomorphic to each other and locally trivial if it is both smooth and isotrivial. Let ω_S (resp K_S) be the canonical sheaf (resp. the canonical divisor) of S, $\omega_{S/C} = \omega_S \otimes f^*\omega_C^\vee$ (resp, $K_{S/C} = K_S - f^*K_C$) the relative canonical sheaf (resp. the relative canonical divisor) where ω_C (resp. K_C) is the canonical sheaf of C (resp. the canonical divisor). In particular, if $D = K_{S/C}$, then $\mathcal{E}$ is nef vector bundle, [8], its rank is g and its degree is:

$$\deg(\mathcal{E}) := \deg(f_*\omega_{S/C}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_F).\chi(\mathcal{O}_C)$$
$$= \chi(\mathcal{O}_S) - (g-1)(b-1),$$

for $b = g(C)$. By Leray spectral sequence, we remark that:

$$h^0(C, (f_*\omega_{S/C})^\vee) = h^0(C, \mathcal{R}^1 f_*\mathcal{O}_S) = q(S) - b,$$

where $q(S) = h^1(S, \mathcal{O}_S)$ is the irregularity of the surface S.

In [7], Severi stated that if: S is a minimal smooth complex projective surface of maximal Albanese dimension, then $K_S^2 \geq 4\chi(\mathcal{O}_S)$. But the proof was not complete, the inequality was posed as a conjecture by

Reid, [22], and proved by Manetti, [16], under the assumption that the surface has ample canonical divisor. Finally, the conjecture is completely proved by Pardini, [21]. Xiao Gang, in [24], wrote a fundamental paper on fibred surfaces over curves. He discussed the geometry of the fibration where S is relatively minimal and $g(F) \geq 2$. He proved that if f is relatively minimal and not locally trivial i.e., $\deg f_*\omega_{S/C} \neq 0$ then:

$$K_{S/C}^2 \geq 4\frac{g-1}{g}\deg f_*\omega_{S/C}.$$

Recall in this setting that $K_{S/C}$ is nef divisor [20, Theorem 1.4].

Independently, Cornalba and Harris, see [6], proved the above inequality for semistable fibrations (i.e., fibrations where all the fibers are semistable curves in the sense of Deligne and Mumford). Recently Yuan Xinyi and Zhang Tong, [25], gave a new approach to prove the slope inequality by giving a sense to the relative Noether inequality and using Frobenius iteration techniques. Motivated by these, [24], [6], in addition to Fujita's fundamental papers, [8], [9], there has been interest in giving a sharp bound using the first and the second Fujita decomposition combined with a study of linear stability of the general fiber F of f.

Let us recall the first and second Fujita decomposition.

Theorem 1.1 (First Fujita decomposition for fibred surface, [8])**.** *Let $f : S \to C$ be a fibration from a smooth complex projective surface S to a smooth projective curve C. Then:*

$$f_*\omega_{S/C} = \mathcal{O}_C^{q(S)-b} \oplus \mathcal{N},$$

where $\mathcal{N}$ is a nef sub-vector bundle and $h^0(C, \mathcal{N}^\vee) = 0$.

We remark that in conclusion of Theorem 1.1, the trivial part comes from a nonzero global section of the dual of $f_*\omega_{S/C}$ i.e., from $H^0(C, \mathcal{R}^1 f_*\mathcal{O}_S)$.

Theorem 1.2 (Second Fujita decomposition for fibred surface, [9], [3], [4], [5])**.** *Let $f : S \to C$ be a fibration as above. Then:*

$$f_*\omega_{S/C} = \mathcal{A} \oplus \mathcal{U},$$

where $\mathcal{A}$ is ample sub-vector bundle and $\mathcal{U}$ is unitary flat sub-vector bundle.

In the situation of Theorem 1.2, we denote by u_f the rank of $\mathcal{U}$, and call it the *unitary rank* of the fibred surface $f : S \to C$.

A proof of the second Fujita decomposition is given by Catanese and Dettweiler, [3], [4], [5]. A more recent paper using these type of arguments is due to Riva and Stoppino [23], they proved the following inequalities:

$$K_{S/C}^2 \geq 2\frac{2g-2-m}{g-m}\deg(f_*\omega_{S/C}).$$

Here $m := \min(q_f, c_f)$, $q_f := q(s) - b$ is the *relative irregularity* of f, c_f is the *Clifford index* of f.

More precisely, we recall, [1], that the Clifford index for a curve B of genus $g(B) \geq 4$ as:

$$\mathrm{Cliff}(B) := \min\{\deg(D) - 2(\dim|D|) \mid h^0(B, D) \geq 2,\ h^1(B, D) \geq 2\}.$$

In the cases $g = 2, 3$ the Clifford index is defined to be:

- If $g=2$, $\mathrm{Cliff}(B) := 0$.
- If $g=3$, $\mathrm{Cliff}(B) := 0$ (resp. 1) if B is hyperelleptic (resp. trigonal).

Given a fibred surface $f : S \to C$, we define c_f as the Clifford index of the general fiber F. Riva and Stoppino [23] proved also this second inequality:

$$K^2_{S/C} \geq \begin{cases} 2\frac{2g-2-u_f}{g-u_f}, & \text{if } u_f \leq c_f. \\ 2\frac{(2g-2-c_f)(g-1-u_f)}{(g-u_f)(g-1-c_f)}, & \text{else.} \end{cases}$$

Konno, in [14], described directly $K^2_{S/C}$ as a sum of two parts under some strict conditions on the fibration f. More precisely, the first part is related to $\deg f_*\omega_{S/C}$ and the second one is described by the Horikawa index [12].

Now, we introduce the following notations:

- N_1 is the Miyaoka divisor [Definition 3.3] of the maximal destabilising sub-vector bundle in the Harder-Narasimhan filtration [Proposition 3.1] and:

$$\alpha = \begin{cases} \frac{g(F)}{h^0(D, D|_F)-1}, & \text{if } N_1|_F \text{ is nonspecial and } h^0(F, N_1|_F) > 1. \\ 2 & \text{else.} \end{cases}$$

- μ_f is the final slope of the filtration.
- F is the general fiber of f.
- ϵ is a suitable birational morphism.
- Z_f is the fixed part of $f_*\mathcal{O}_S(D)$.
- $N_D := \frac{2\epsilon^*D.Z_f - Z_f^2}{\deg f_*\mathcal{O}_S(D)}$, describes the negativity of D and $f_*\mathcal{O}_S(D)$ when $\deg f_*\mathcal{O}_S(D) \neq 0$.
- $N_D^{\mathcal{G}} := \frac{2\epsilon^*D.Z_f - Z_f^2}{\deg \mathcal{G}}$, for a locally free sub-sheaf $\mathcal{G} \subseteq \mathcal{E} := f_*\mathcal{O}_S(D)$ and $\deg \mathcal{G} \neq 0$.
- $d_f := N_f.F$ and $d_f^{\mathcal{G}} := N_f^{\mathcal{G}}.F$, where N_f (resp. $N_f^{\mathcal{G}}$) is the last Miyaoka divisor for $\mathcal{E}$ (resp. $\mathcal{G}$) [Definition 3.3].

We state our main result:

Theorem 1.3. *Let $f : S \to C$ be a fibration from a smooth complex projective surface to a smooth complex projective curve, let D be an arbitrary divisor on S such that $\mathrm{rk}\,\mathcal{E} = \mathrm{rk}(f_*\mathcal{O}_S(D)) > 1$ Then:*

- *If D is nef and $\mu_f \geq 0$ we have:*

$$D^2 \geq \frac{2\alpha d_f}{d_f + \alpha} \deg f_*\mathcal{O}_S(D).$$

- *If D is nef, $\mu_f < 0$ and $\exists\, \mathcal{G} \subsetneq \mathcal{E}$ nef locally free sub-sheaf, we have:*

$$D^2 \geq \frac{2\alpha d_f^{\mathcal{G}}}{d_f^{\mathcal{G}} + \alpha} \deg \mathcal{G}.$$

- *If ($\mu_f < 0$ and there is no nef locally free sub-sheaf of $\mathcal{E}$) or (D is not nef and ($\mu_f > 0$ or ($\mu_f = 0$ and $\mathcal{E}$ is not semi-stable))), we have:*

$$D^2 \geq \left(\frac{2\alpha d_f}{d_f + \alpha} + N_D\right) \deg f_* \mathcal{O}_S(D).$$

- *If D is not nef, $\mu_f < 0$ and $\exists\, \mathcal{G} \subsetneq \mathcal{E}$ nef locally free sheaf such that $\mu_f^{\mathcal{G}} > 0$ or ($\mu_f^{\mathcal{G}} = 0$ and $\mathcal{G}$ is not sem-stable) then:*

$$D^2 \geq \left(\frac{2\alpha d_f^{\mathcal{G}}}{d_f^{\mathcal{G}} + \alpha} + N_D^{\mathcal{G}}\right) \deg \mathcal{G}.$$

- *Otherwise, if $D^2 < 0$ and $\forall\, \mathcal{G} \subseteq \mathcal{E}$ nef $\Longrightarrow \deg \mathcal{G} = 0$. There is no sense to the slope inequality.*

For the particular case that f is relatively minimal and $D = K_{S/C}$, $N_1|_F$ is nonspecial and $h^0(F, N_1|_F) > 1$. The Theorem 1.3 takes the following form:

Corollary 1.4. *Let $f : S \to C$ be a relatively minimal fibration with $g(F) \geq 2$ and $D = K_{S/C}$, if $N_1|_F$ is nonspecial and $h^0(F, N_1|_F) > 1$. Then:*

$$K_{S/C}^2 \geq 4\frac{(g-1)(2g-1)}{(g-1)(2g-1)+g} \deg f_* \omega_{S/C}.$$

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In the next sections, we let $f : S \to C$ be a fibration, from a smooth projective surface S to a smooth projective curve C, and we let D be a divisor on S.

2 Rational map to a projective bundle

Let $\mathcal{F} \subseteq f_* \mathcal{O}_S(D)$ be a locally free sub-sheaf of rank $r_{\mathcal{F}}$. There exist always the following commutative diagrams:

$$\begin{tikzcd} S \arrow[r, dashed, "\psi"] \arrow[dr, "f"'] & \mathbb{P}_C(f_* \mathcal{O}_S(D)) \arrow[d, "\pi"] \\ & C \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r, dashed, "\psi_{\mathcal{F}}"] \arrow[dr, "f"'] & \mathbb{P}_C(\mathcal{F}) \arrow[d, "\pi_{\mathcal{F}}"] \\ & C \end{tikzcd}$$

In the above, $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ (resp $\mathbb{P}_C(\mathcal{F})$) is the projective bundle of one dimensional quotients (Grothendieck's notations) of $f_*\mathcal{O}_S(D)$ (resp of $\mathcal{F}$), the morphism π (resp $\pi_{\mathcal{F}}$) is the projective morphism from $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ to C (resp from $\mathbb{P}_C(\mathcal{F})$ to C). The maps ψ and $\psi_{\mathcal{F}}$ are rational and defined by the following evaluation maps:

$$f^*f_*\mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D),$$

resp:

$$f^*\mathcal{F} \longrightarrow \mathcal{O}_S(D).$$

Remark 2.1.

- If D is f-globally generated. Then:

  $$f^*f_*\mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D)$$

  is surjective and ψ is a morphism.

- If the map

  $$f^*\mathcal{F} \longrightarrow \mathcal{O}_S(D)$$

  is surjective, then $\psi_{\mathcal{F}}$ is a morphism.

Take A, a sufficiently very ample divisor such that $f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)$ is a very ample vector bundle. Then the rank of $f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)$ is

$$r = H^0(F, D|_F).$$

And

$$\deg(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) = \deg f_*\mathcal{O}_S(D) + r.\deg(A).$$

Remark 2.2. Since $f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)$ is globally generated, i.e., $\exists\, n > 0$ such that we have a surjective map

$$\mathcal{O}_C^{\oplus n} \longrightarrow f_*\mathcal{O}_S(D) \otimes\ \mathcal{O}(A),$$

or, equivalently, for any $y \in C$ we have a surjective map giving by evaluation of section:

$$H^0(C, f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)|_y,$$

we remark that each section in $H^0(F, D|F)$ comes from some section of $H^0(C, f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$. In other words, the following map is surjective

$$H^0(C, f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \simeq H^0(S, D + f^*A) \longrightarrow H^0(F, D|F).$$

Now, $\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ and $\mathbb{P}_C(f_*\mathcal{O}_S(D))$ are isomorphic by an isomorphism s, see [11, Lemma 7.9]. Since

$$\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{P}(\mathcal{O}_C^{\oplus n}) = \mathbb{CP}^n \times C,$$

we can identify $\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ with $X \times C$ such that X is a projective variety in $\mathbb{CP}^n$.

The rational map

$$\phi : S \dashrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)),$$

defined by:

$$f^*(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathcal{O}_S(D) \otimes f^*\mathcal{O}(A)$$

is the rational map given by the linear system $|D + f^*A|$, and for a general fibre F of f, $\phi|_F$ is the map defined by $|D|_F|$.

The line bundle $\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1)$ on $\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))$ is very ample. Then it gives an embedding of this last projective bundle to a projective space $\mathbb{CP}^N$. We have the following commutative diagram:

$$\begin{tikzcd} S \arrow[r, dashed, "\phi"] \arrow[dr, "f"'] & \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{CP}^N \arrow[d, "\pi_A"] \\ & C \end{tikzcd}$$

Where π_A is the projection map, again we have $\psi = s \circ \phi$, the rational map ϕ is defined by the complete linear system $|D + f^*A|$, if it has no nontrivial fixed part then its image is contained in no hyperplane.

We assume that there is a fixed part Z of $|D + f^*A|$, so the linear system $|D - Z + f^*A|$ factorize the map defined by $|D + f^*A|$ and it defines a rational map $\phi^{'}$ such that the following diagram is commutative:

$$\begin{tikzcd} S \arrow[r, dashed, "\phi^{'}"] \arrow[rrd, bend right, "f"'] & X^{'} \times C \arrow[r, "i"] & \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{CP}^N \arrow[d, "\pi_A"] \\ & & C \end{tikzcd}$$

Where $X^{'}$ is a closed sub-variety of X, i is an injection from $X^{'} \times C$ to $\mathbb{CP}^N$ and $i \circ \phi^{'} = \phi$.

- The fixed part Z of $|D+f^*A|$ restricted in F is just the fixed part of the complete linear system $|D|_F|$.
- The system $|D - Z + f^*A|$ has no fixed part, so it has only a finite number of base points.
- The fixed part Z is a divisor such that the homomorphism:

  $$f^*(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathcal{O}_S(D - Z) \otimes f^*\mathcal{O}(A),$$

  is surjective in codimension 1.
- If $D = K_{S/C}$ and $g(F) \geq 2$, then the fixed part Z has no horizontal components.

Theorem 2.3. *There exist a surface $\tilde{S}$ birational to S (i.e $\exists\, \epsilon : \tilde{S} \to S$ which is birational) and a morphism*

$$\lambda^{'} : \tilde{S} \longrightarrow X^{'} \times C$$

such that the following diagram is commutative:

$$\begin{tikzcd} \tilde{S} \arrow[d, bend right, "\epsilon"] \arrow[rd, bend left, "\lambda^{'}"] & & \\ S \arrow[r, dashed, "\phi^{'}"] \arrow[rrd, bend right, "f"'] & X^{'} \times C \arrow[r, "i"] & \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{CP}^N \arrow[d, "\pi_A"] \\ & & C \end{tikzcd}$$

$\phi^{'} \circ \epsilon = \lambda^{'}$ and:

$$(\lambda^{'})^* \mathcal{O}_{X^{'} \times C}(1) = \epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E).$$

Where E is the exceptional divisor of ϵ.

Remark 2.4. $\epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E)$ is globally generated.

Proof of Theorem 2.3. If $|D - Z + f^*A|$ has no base point, then $\phi^{'}$ is a morphism and there is nothing to prove. We suppose that there is a base point x in $|D - Z + f^*A|$. We take the blow-up in x defined by ϵ^1, so $|(\epsilon^1)^*(D - Z + f^*A)|$ has a fixed part k_1E_1 with $k_1 \in \mathbb{Z}, k_1 \geq 1$ and $|D_1| = |(\epsilon^1)^*(D - Z + f^*A) - k_1E_1|$ has no fixed part. Hence it defines a rational map: $\lambda^1 : S_1 \dashrightarrow X^{'} \times C$ which is identical to $\phi^{'} \circ \epsilon^1$. If λ^1 is a morphism, then we are done; if not, we repeat the process. Thus, we get by induction a sequence $\epsilon^i : S_i \longrightarrow S_{i-1}$ of blow-ups and a linear system $|D_i|$ with no fixed part, where $D_i = (\epsilon^i)^*D_{i-1} - k_iE_i$ for $i \geq 1$. But we have:

$$D_i^2 = D_{i-1}^2 - k_i^2 < D_{i-1}^2.$$

Since D_i has no fixed part, $D_i^2 \geq 0$ and so this process must terminate. In other words, we arrive at a system D_n with no base points, which defines a morphism:

$$\epsilon = \epsilon^1 \circ ... \circ \epsilon^n : \tilde{S} \longrightarrow S.$$

We conclude that $|\epsilon^*(D - Z + f^*A) - E|$ define a morphism $\tilde{S} \xrightarrow{\lambda^{'}} X^{'} \times C$ such that:

$$\epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E) = (\lambda^{'})^* \mathcal{O}_{X^{'} \times C}(1).$$

Where $E = \sum_{i=1}^{i=n} K_iE_i$ is the exceptional divisor. $\square$

The last proof is inspired by the proof of [2, Theorem 2.7].

Corollary 2.5. *There exist a morphism $\lambda^A : \tilde{S} \to \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}_S(A))$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\tilde{S} \arrow[d, bend right, "\epsilon"'] \arrow[dr, bend left, "\lambda^{'}"] \arrow[drr, bend left, "\lambda^A"] & & \\
S \arrow[r, dashed, "\phi^{'}"] \arrow[drr, bend right, "f"'] & X^{'} \times C \arrow[r, "i"] & \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \subseteq \mathbb{CP}^N \arrow[d, "\pi_A"] \\
& & C
\end{tikzcd}$$

Proof. We take $\lambda^A = i \circ \lambda^{'}$ and it is a well-defined morphism defined from $\tilde{S}$ to $\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ and clearly verifies the property that:

$$(\lambda^A)^* \mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))}(1) = \epsilon^*(\mathcal{O}(D - Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E),$$

also:

$$\pi_A \circ \lambda^A = f \circ \epsilon.$$

$\square$

Corollary 2.6. *There exist a morphism $\lambda : \tilde{S} \to \mathbb{P}_C(f_*\mathcal{O}_S(D))$ such that the following diagram is commutative.*

$$\begin{tikzcd}
\tilde{S} \arrow[d, bend right, "\epsilon"'] \arrow[dr, bend left, "\lambda"] & \\
S \arrow[r, dashed, "\psi"] \arrow[dr, "f"'] & \mathbb{P}_C(f_*\mathcal{O}_S(D)) \arrow[d, "\pi"] \\
 & C
\end{tikzcd}$$

and we have:

$$(\lambda)^*(\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1)) = \epsilon^*(\mathcal{O}(D-Z)) \otimes \mathcal{O}(-E).$$

Proof. By Theorem 2.3 and Corollary 2.5, there exist a morphism $\lambda^A : \tilde{S} \longrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A))$ which has the property that:

$$(\lambda^A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1) = \epsilon^*(\mathcal{O}(D-Z) \otimes f^*\mathcal{O}(A)) \otimes \mathcal{O}(-E).$$

But $\exists s : \mathbb{P}_C(f_*\mathcal{O}_S(D) \otimes \mathcal{O}(A)) \longrightarrow \mathbb{P}_C(f_*\mathcal{O}_S(D))$ which is an isomorphism such that:

$$\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D)\otimes\mathcal{O}(A))}(1) = s^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) \otimes \pi_A^*\mathcal{O}(A)$$

$$\begin{aligned}
&\Longrightarrow (s \circ \lambda^A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) \otimes (\pi_A \circ \lambda^A)^*\mathcal{O}(A) \\
&\quad = \epsilon^*(\mathcal{O}(D-Z)) \otimes (f \circ \epsilon)^*\mathcal{O}(A) \otimes \mathcal{O}(-E) \\
&\Longrightarrow (s \circ \lambda^A)^*\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1) = \epsilon^*(\mathcal{O}(D-Z)) \otimes \mathcal{O}(-E).
\end{aligned}$$

We take $\lambda = s \circ \lambda^A$. ∎

Remark 2.7. More generally, for $\mathcal{F} \subseteq f_*\mathcal{O}_S(D)$ a a locally free sub-sheaf, we take A a sufficiently very ample divisor such that $\mathcal{F} \otimes \mathcal{O}(A)$ is very ample. Let $L_{\mathcal{F}}$ be a sub-linear system of $|D + f^*A|$ wich correspond to a sections of $H^0(\mathcal{F} \otimes \mathcal{O}(A))$. Let $Z_{\mathcal{F}}$ a fixed part of $L_{\mathcal{F}}$, so $L_{\mathcal{F}} - Z_{\mathcal{F}}$ has no fixed part and it corresponds to a rational map from S to a projective sub-variety of $\mathbb{P}_C(\mathcal{F} \otimes \mathcal{O}(A))$ By the same arguments above $\exists\ \tilde{S}_{\mathcal{F}} \xrightarrow{\epsilon_{\mathcal{F}}} S$ which is birational and $\exists \lambda_{\mathcal{F}} : \tilde{S}_{\mathcal{F}} \longrightarrow \mathbb{P}_C(\mathcal{F})$ such that the following diagram is commutative:

$$\begin{tikzcd}
\tilde{S}_{\mathcal{F}} \arrow[d, bend right, "\epsilon_{\mathcal{F}}"'] \arrow[dr, bend left, "\lambda_{\mathcal{F}}"] & \\
S \arrow[r, dashed, "\psi_{\mathcal{F}}"] \arrow[dr, "f"'] & \mathbb{P}_C(\mathcal{F}) \arrow[d, "\pi_{\mathcal{F}}"] \\
 & C
\end{tikzcd}$$

and

$$(\lambda_{\mathcal{F}})^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F})}(1)) = \epsilon_{\mathcal{F}}^*(\mathcal{O}(D - Z_{\mathcal{F}})) \otimes \mathcal{O}(-E_{\mathcal{F}}),$$

where $E_{\mathcal{F}}$ is the exceptional divisor of $\epsilon_{\mathcal{F}}$.

3 Harder-Narasimhan filtration

In this section, we study Harder-Narasimhan filtration within the context of fibred surfaces.

Proposition 3.1 ([10]). *Let $\mathcal{E}$ a vector bundle over a smooth projective curve B. There exists a unique sequence of vector sub-bundles of $\mathcal{E}$:*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_{k-1} \subsetneq \mathcal{F}_k = \mathcal{E},$$

that satisfy the following conditions:

- *for $i = 1, \dots, k$, $\mathcal{F}_i/\mathcal{F}_{i-1}$ is a semi-stable vector bundles.*
- *for any $i = 1, \dots, k$, setting $\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) = \frac{\deg(\mathcal{F}_i/\mathcal{F}_{i-1})}{\mathrm{rk}(\mathcal{F}_i/\mathcal{F}_{i-1})}$, we have that:*

$$\mu_1 > \mu_2 > \dots > \mu_k.$$

In the context of Proposition 3.1 above, the filtration is called the *Harder-Narasimhan filtration* of $\mathcal{E}$. We set $\mu_f = \mu_k$ and call it the *final slope* of $\mathcal{E}$.

The following elementary lemma is important in what follows.

Lemma 3.2. *Let r_i be the rank of $\mathcal{F}_i$. Then:*

$$\deg \mathcal{E} = \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k \mu_k.$$

Proof. Indeed, we consider the exact sequence:

$$0 \longrightarrow \mathcal{F}_{k-1} \longrightarrow \mathcal{F}_k \longrightarrow \mathcal{F}_k/\mathcal{F}_{k-1} \longrightarrow 0.$$

From the additivity of degree, we have that:

$$\deg \mathcal{F}_k = \deg \mathcal{F}_{k-1} + \deg \mathcal{F}_k/\mathcal{F}_{k-1}.$$

Similarly, we have that

$$\deg \mathcal{F}_{k-1} = \deg \mathcal{F}_{k-2} + \deg \mathcal{F}_{k-1}/\mathcal{F}_{k-2}.$$

And so, by induction, we can conclude that:

$$\deg \mathcal{F}_k = \sum_{i=1}^{k} \deg \mathcal{F}_i/\mathcal{F}_{i-1}.$$

From the definition of slope, for every $i = 1, .., k$ we have: $\deg \mathcal{F}_i/\mathcal{F}_{i-1} = \mu_i(r_i - r_{i-1})$ and we obtain the desired formula. $\square$

Consider now a fibred surface $f : S \to C$ and let $(\mathcal{F}_i)$ be the Harder-Narasimhan filtration of $\mathcal{E} = f_*\mathcal{O}_S(D)$. By Corollary 2.6 and Remark 2.7, there exists a suitable smooth projective surface $\tilde{S}$ and a birational morphism $\epsilon : \tilde{S} \to S$ such that:

$$\lambda^*(\mathcal{O}_{\mathbb{P}_C(f_*\mathcal{O}_S(D))}(1)) = \epsilon^*(\mathcal{O}(D - Z)) \otimes \mathcal{O}(-E),$$

moreover, for any $\mathcal{F}_i$ in the filtration, we have:

$$\lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)) = \epsilon^*(\mathcal{O}(D - Z_{\mathcal{F}_i})) \otimes \mathcal{O}(-E).$$

Where $\lambda_i := \lambda_{\mathcal{F}_i}$, Z (resp. $Z_{\mathcal{F}_i}$) is a fixed part of $|D + f^*A|$ (resp. of $L_{\mathcal{F}_i} \subseteq |D + f^*A|$ which correspond to a sections of $H^0(\mathcal{F}_i \otimes \mathcal{O}(A))$). Here E, is the exceptional divisor of ϵ.

Definition 3.3 (See [23, Definition 3.11])**.** In this setting, just described above, we define:

- $Z(D, \mathcal{F}_i) = \epsilon^* Z_{\mathcal{F}_i} + E$ the *fixed part* of the vector sub-bundle $\mathcal{F}_i$.
- $M(D, \mathcal{F}_i) = \lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1))$ the *moving part* of the vector sub-bundle $\mathcal{F}_i$.
- Set $N(D, \mathcal{F}_i) := M(D, \mathcal{F}_i) - \mu_i F$. We call it the *Miyaoka divisor*.

Applying, [17], [19] and [15, Proposition 6.4.11] we prove the following result:

Lemma 3.4. *$N(D, \mathcal{F}_i)$ are nef divisors on $\tilde{S}$.*

Proof. Let's see that $\mathcal{E}\langle -\frac{c_1(\mathcal{E}/\mathcal{F}_{k-1})}{rk(\mathcal{E}/\mathcal{F}_{k-1})}\rangle$ is nef vector bundle.
Note:

$$\begin{cases} \mathcal{G}_i = \mathcal{F}_i/\mathcal{F}_{i-1}. \\ \delta_i = \frac{c_1(G_i)}{\mathrm{rk}(G_i)}. \end{cases}$$

$\mathcal{G}_i\langle -\delta_i\rangle$ is nef vector bundle, [15, Proposition 6.4.11], and $\deg \delta_i = \mu_i$. So:

$$-\deg \delta_1 < -\deg \delta_2 < ... < -\deg \delta_k.$$

This implies that $\mathcal{G}_i\langle -\delta_k\rangle$ is nef vector bundle. We have:

$$0 \longrightarrow \mathcal{F}_{k-1}\langle -\delta_k\rangle \longrightarrow \mathcal{E}\langle -\delta_k\rangle \longrightarrow \mathcal{G}_k\langle -\delta_k\rangle \longrightarrow 0.$$

$\mathcal{G}_k\langle -\delta_k\rangle$ is nef vector bundle and $\mathcal{F}_{k-1}\langle -\delta_k\rangle$ is nef by induction. $\Longrightarrow \mathcal{E}\langle -\delta_k\rangle$ is nef. $\Longrightarrow N(D, \mathcal{E})$ is nef. The proof is the same for $N(D, \mathcal{F}_i)$. ◻

Lemma 3.5. *$r_i = \mathrm{rk}\, \mathcal{F}_i \le h^0(F, N(D, \mathcal{F}_i)|_F)$.*

Proof. Let π_i be the projection form $\mathbb{P}_C(\mathcal{F}_i)$ to C, we have:

$$\begin{aligned} (\pi_i \circ \lambda_i)_*(M(D, \mathcal{F}_i)) &= (\pi_i)_*((\lambda_i)_* M(D, \mathcal{F}_i)) \\ = (\pi_i)_*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1) \otimes \lambda_{i*}\mathcal{O}_{\tilde{S}}) &\supseteq (\pi_i)_*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)) = \mathcal{F}_i \\ \Longrightarrow r_i &\le h^0(F, M(D, \mathcal{F}_i)|_F) \\ \Longrightarrow r_i &\le h^0(F, N(D, \mathcal{F}_i)|_F). \end{aligned}$$

◻

For simplification, set $N_i = N(D, \mathcal{F}_i)$ and $M_i = M(D, \mathcal{F}_i)$, $Z_i = Z(D, \mathcal{F}_i)$.

Proposition 3.6. *Let $d_i = \deg(N_i|_F) = N_i.F$, we have:*

$$d_k \geq d_{k-1} \geq ... \geq d_1 \geq 0.$$

Proof. Since F is a fibre, $F^2 = 0$. Thus:

$$d_i = N_i.F = (M_i - \mu_i F).F = M_i.F$$

$$= \lambda_i^*(\mathcal{O}_{\mathbb{P}_C(\mathcal{F}_i)}(1)).F = (\epsilon^*(D - Z_{\mathcal{F}_i}) - E).F$$

$$= \epsilon^*(D - Z_{\mathcal{F}_i}).F = (D - Z_{\mathcal{F}_i}).F \geq 0.$$

But : $Z_{\mathcal{F}_i} \geq Z_{\mathcal{F}_{i+1}}$. Thus, $D - Z_{\mathcal{F}_{i+1}} = D - Z_{\mathcal{F}_i} + (Z_{\mathcal{F}_i} - Z_{\mathcal{F}_{i+1}})$

$$\Longrightarrow d_{i+1} \geq d_i.$$

□

Proposition 3.7. *If $N_1|_F$ is nonspecial divisor on F. Then $N_j|_F$ is nonspecial divisor on F for any $j \geq 1$. Or more generally: if $\exists\, i \geq 1$ such that $N_i|_F$ is nonspecial. Then $N_j|_F$ is nonspecial divisor on F for $j \geq i$.*

Proof. Recall that

$$Z_{\mathcal{F}_1} \geq ... \geq Z_{\mathcal{F}_k}.$$

Thus

$$N_1|_F = \epsilon^*(D - Z_{\mathcal{F}_1})|_F = (D - Z_{\mathcal{F}_1})|_F \leq (D - Z_{\mathcal{F}_i})|_F.$$

So, if $h^1(F, D - Z_{\mathcal{F}_1}|_F) = 0$. Then $h^1(F, D - Z_{\mathcal{F}_i}|_F) = 0$. □

Now, it is natural to ask about the sequence $(\frac{d_i}{h^0(F, N_i|_F) - 1})_{i \in \{1, ...k\}}$. For instance, is it increasing finite sequence? Is it decreasing? Is it bounded from below by strictly positive number? So we have the following results:

Lemma 3.8. *If $\exists\, t \in \{1, ..., k\}$ such that $h^0(F, N_t|_F) = 1$. Then $t = 1$.*

Proof. We have $h^0(F, N_t|_F) \geq \mathrm{rk}(\mathcal{F}_t)$. So, if $h^0(F, N_t|_F) = 1$ the only possibility is $t = 1$ and more than that the degree: $d_1 = g(F) - h^1(F, N_1|_F)$. □

Theorem 3.9. *Let $f : S \to C$ be a fibration as above, D be a divisor on S such that $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(f_*\mathcal{O}_S(D)) > 1$ and $h^0(F, N_1|_F) > 1$. Consider the Harder-Narasimhan filtration $(\mathcal{F}_i)$ of $\mathcal{E}$. Then, we have the following result:*

- *If $N_1|_F$ is nonspecial divisor and if:*
  - *$g(F) = 0$. Then:*

$$\frac{d_k}{h^0(N_k|_F) - 1} = ... = \frac{d_{i+1}}{h^0(N_{i+1}|_F) - 1} = \frac{d_i}{h^0(N_i|_F) - 1}$$

$$= ... = \frac{d_1}{h^0(N_1|_F) - 1} = 1.$$

– $g(F) \geq 1$. *Then:*

$$1 + \frac{g(F)}{h^0(D, D|_F) - 1} \leq \frac{d_k}{h^0(N_k|_F) - 1} \leq ... \leq \frac{d_{i+1}}{h^0(N_{i+1}|_F) - 1} \leq$$

$$\frac{d_i}{h^0(N_i|_F) - 1} \leq ... \leq \frac{d_1}{h^0(N_1|_F) - 1}.$$

- *if* $\exists t \in \{1, ..., k\}$ *such that* $N_t|_F$ *is special and* $N_{t+1}|_F$ *is nonspecial and if:*
  – $g(F) = 0$. *Then:*

$$\frac{d_k}{h^0(N_k|_F) - 1} = ... = \frac{d_{t+1}}{h^0(N_{t+1}|_F) - 1} = 1.$$

  – $g(F) \geq 1$. *Then:*

$$1 + \frac{g(F)}{h^0(D, D|_F) - 1} \leq \frac{d_k}{h^0(N_k|_F) - 1} \leq ... \leq \frac{d_{t+1}}{h^0(N_{t+1}|_F) - 1}.$$

  And we have: $\frac{d_i}{h^0(N_i|_F)-1} \geq 2, \quad \forall i \leq t$.

Proof. If $N_1|_F$ is nonspecial. Then by Riemann-Roch:

$$h^0(N_i|_F) = d_i + 1 - g(F).$$

- If $g(F) = 0$, then:

$$h^0(N_i|_F) = d_i + 1,$$

  and so evidently:

$$\frac{d_{i+1}}{h^0(N_{i+1}|_F) - 1} = \frac{d_i}{h^0(N_i|_F) - 1} = 1.$$

- If $g(F) \geq 1$, since:

$$h^0(N_{i+1}|_F) = h^0(N_i|_F) + d_{i+1} - d_i, \quad \forall i \in \{1, 2, ...k\}.$$

  It follows that:

$$\frac{d_{i+1}}{h^0(N_{i+1}|_F) - 1} = \frac{d_i + d_{i+1} - d_i}{h^0(N_i|_F) + d_{i+1} - d_i - 1} \leq \frac{d_i}{h^0(N_i|_F) - 1}.$$

Now we have $\forall i \in \{1, .., k\}$, $h^0(F, N_i|_F) \leq h^0(F, D|_F)$ because $N_i|_F = (D - Z_{\mathfrak{F}_i})|_F$ and $Z_{\mathfrak{F}_i}$ is a effective divisor. Using Riemann-Roch, we have:

$$\frac{d_i}{h^0(N_i|_F) - 1} \geq 1 + \frac{g(F)}{h^0(D, D|_F) - 1}.$$

Now, suppose that $\exists t \in \{1, ..., k\}$ such that $N_t|_F$ is special and $N_{t+1}|_F$ is nonspecial. So $N_i|F$ is special for $i \leq t$ and $N_i|_F$ is nonspecial for $i \geq t + 1$ and as above:

- If $g(F) = 0$ then:
$$\frac{d_k}{h^0(N_k|_F) - 1} = ... = \frac{d_{t+1}}{h^0(N_{t+1}|_F) - 1} = 1.$$
- If $g(F) \geq 1$ then:
$$1 + \frac{g(F)}{h^0(D, D|_F) - 1} \leq \frac{d_k}{h^0(N_k|_F) - 1} \leq ... \leq \frac{d_{t+1}}{h^0(N_{t+1}|_F) - 1}.$$

For $i \leq t$, by Clifford theorem, [1], we have:
$$d_i \geq 2(h^0(N_i|_F) - 1)$$
$$\implies \frac{d_i}{h^0(N_i|_F) - 1} \geq 2.$$

□

4 Slope inequality.

Now, we are ready to present the technical lemma to our method, we called it the **Modified Xiao Lemma**. Note that it is a more general form of Xiao [24, Lemma 2].

Lemma 4.1 (Modified Xiao Lemma)**.** *Let $f : S \to C$ be a fibration, D be a divisor on S and suppose that there exist a sequence of effective divisors:*
$$Z_1 \geq Z_2 \geq ... \geq Z_k,$$
and a sequence of rational numbers:
$$\mu_1 > \mu_2 > ... > \mu_k,$$
such that for every $i \in \{1, ..., k\}$ we have:
$$\mathcal{N}_i := D - Z_i - \mu_i F$$
are nef $\mathbb{Q}$-divisors. Then:
$$D^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2D.Z_k - Z_k^2 + 2\mu_k d_k,$$
where $d_i = \mathcal{N}_i.F$.

Proof. First observe that $\mathcal{N}_1^2 \geq 0$ by nefness. And we have:
$$\mathcal{N}_i^2 = \mathcal{N}_i(\mathcal{N}_{i-1} + (Z_{i-1} - Z_i) + (\mu_{i-1} - \mu_i)F)$$
$$\geq \mathcal{N}_i(\mathcal{N}_{i-1} + (\mu_{i-1} - \mu_i)F)$$
$$\geq (\mathcal{N}_{i-1} + (Z_{i-1} - Z_i) + (\mu_{i-1} - \mu_i)F)(\mathcal{N}_{i-1} + (\mu_{i-1} - \mu_i)F)$$

$$\geq \mathcal{N}_{i-1}^2 + (\mu_{i-1} - \mu_i)(2\mathcal{N}_{i-1}F + (Z_{i-1} - Z_i)F)$$
$$= \mathcal{N}_{i-1}^2 + (\mu_{i-1} - \mu_i)(d_{i-1} + d_i).$$

So, by induction, we have:

$$\mathcal{N}_k^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

Hence

$$(D - Z_k - \mu_k F)^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

But

$$(D - Z_k - \mu_k F)^2 = (D - Z_k)^2 - 2\mu_k(D - Z_k)F$$
$$= D^2 - 2D.Z_k + Z_k^2 - 2\mu_k d_k.$$

So we have:

$$D^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2D.Z_k - Z_k^2 + 2\mu_k d_k.$$

□

Remark 4.2.

- When $\mu_k \geq 0$ and D is nef, we set $Z_{k+1} = 0$ and $\mu_{k+1} = 0$, $d_{k+1} = D.F$. With the techniques of the previous Lemma 4.1, we have:

$$D^2 \geq \sum_{i=1}^{k}(d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

  This is the original result of Xiao Gang [24, Lemma 2].

- The part: $2D.Z_k - Z_k^2 + 2\mu_k d_k$ describes the negativity of $f_*(\mathcal{O}_S(D))$ and D.

Example 4.3. If $D = K_{S/C} + L$ with L is nef and f-big (resp. trivial), then we have by [18], [13] (resp.[8]) $f_*(\omega_{S/C} \otimes L)$ is nef vector bundle. if f is relatively minimal fibration then $K_{S/C}$ is nef. So $K_{S/C} + L$ is also nef. By the discussion above $K_{S/C} + L$ is big if $f_*\mathcal{O}_S(D)$ is not semi stable.

Prior to stating our main result, we recall the following notation:

- N_1 is the Miyaoka divisor of the maximal destabilising sub-vector bundle in the Harder-Narasimhan filtration and :

$$\alpha = \begin{cases} \frac{g(F)}{h^0(D, D|_F) - 1} & \text{if } N_1|_F \text{is non special and } h^0(F, N_1|_F) > 1 \\ 2 & \text{else.} \end{cases}$$

- F is the general fiber of f.
- ϵ is a suitable birational morphism.
- Z_f : is the fixed part of $f_*\mathcal{O}_S(D)$.

- $N_D := \frac{2\epsilon^* D.Z_f - Z_f^2}{\deg f_* \mathcal{O}_S(D)}$ describes the negativity of D and $\mathcal{E}$ when $\deg f_*\mathcal{O}_S(D) \neq 0$.
- $N_D^{\mathcal{G}} := \frac{2\epsilon^* D.Z_f - Z_f^2}{\deg \mathcal{G}}$ for $\mathcal{G} \subseteq \mathcal{E}$ and $\deg \mathcal{G} \neq 0$.
- $d_f = N_f.F$ and $d_f^{\mathcal{G}} = N_f^{\mathcal{G}}.F$ where N_f (resp. $N_f^{\mathcal{G}}$) is the last Miyaoka divisor for $\mathcal{E}$ (resp. $\mathcal{G}$).

Now, we are ready to prove our main results:

Theorem 4.4. *Let $f : S \to C$ be a fibration from a smooth complex projective surface to a smooth complex projective curve, let D be an arbitrary divisor on S such that $\operatorname{rk}\mathcal{E} = \operatorname{rk}(f_*\mathcal{O}_S(D)) > 1$ Then:*

1. *If D is nef and $\mu_f \geq 0$. Then:*
$$D^2 \geq \frac{2\alpha d_f}{d_f + \alpha} \deg f_*\mathcal{O}_S(D).$$
2. *If D is nef, $\mu_f < 0$ and $\exists\, \mathcal{G} \subsetneq \mathcal{E}$ nef locally free sub-sheaf, we have:*
$$D^2 \geq \frac{2\alpha d_f^{\mathcal{G}}}{d_f^{\mathcal{G}} + \alpha} \deg \mathcal{G}.$$
3. *If ($\mu_f < 0$ and there is no nef locally free sub-sheaf of $\mathcal{E}$) or (D is not nef and ($\mu_f > 0$ or ($\mu_f = 0$ and $\mathcal{E}$ is not semi-stable))), we have:*
$$D^2 \geq \left(\frac{2\alpha d_f}{d_f + \alpha} + N_D\right) \deg f_*\mathcal{O}_S(D).$$
4. *If D is not nef, $\mu_f < 0$ and $\exists\, \mathcal{G} \subsetneq \mathcal{E}$ nef locally free sub-sheaf such that: $\mu_f^{\mathcal{G}} > 0$ or ($\mu_f^{\mathcal{G}} = 0$ and $\mathcal{G}$ is not semi-stable). Then:*
$$D^2 \geq \left(\frac{2\alpha d_f^{\mathcal{G}}}{d_f^{\mathcal{G}} + \alpha} + N_D^{\mathcal{G}}\right) \deg \mathcal{G}.$$
5. *Otherwise, if $D^2 < 0$ and $\forall\, \mathcal{G} \subseteq \mathcal{E}$ nef locally free sub-sheaf $\Longrightarrow \deg \mathcal{G} = 0$. There is no sense to the slope inequality.*

Proof. For 1., Let D is nef and $\mu_f \geq 0$ such that:
$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_{k-1} \subsetneq \mathcal{F}_k = \mathcal{E}$$
be the Harder-Narasimhan filtration of $\mathcal{E}$. Following the previous discussion in section 2, we consider a suitable blow up $\epsilon : \tilde{S} \to S$ and over $\tilde{S}$, we define the fixed part $Z_i = Z(D, \mathcal{F}_i)$ and the moving part $M_i = M(D, \mathcal{F}_i)$ of $\mathcal{F}_i$. By (Lemma 4.1, see Remark 4.2) we have:
$$D^2 = (\epsilon^* D)^2 \geq \sum_{i=1}^{k} (d_i + d_{i+1})(\mu_i - \mu_{i+1}).$$

By the Theorem 3.9, if $N_1|_F$ is non special and $h^0(F, N_1|_F) > 1$. Then we take:

$$\alpha = 1 + \frac{g}{h^0(D, D|_F) - 1},$$

and else $\alpha = 2$ such that:

$$D^2 \geq \sum_{i=1}^{k-1} (\alpha(r_i - 1) + \alpha(r_{i+1} - 1))(\mu_i - \mu_{i+1}) + 2d_k\mu_k.$$

Because $d_i \geq \alpha(r_i - 1)$ and $d_{k+1} \geq d_k$. Since $r_{i+1} \geq r_i + 1$. Thus:

$$D^2 \geq 2\alpha \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) - \alpha(\mu_1 - \mu_k) + 2d_k\mu_k.$$

Hence

$$\begin{aligned} D^2 &\geq 2\alpha \left(\sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k\mu_k \right) - 2\alpha r_k\mu_k - \alpha(\mu_1 - \mu_k) + 2d_k\mu_k \\ &= 2\alpha \deg(f_*(\mathcal{O}_S(D))) - \alpha\mu_1 + (\alpha - 2\alpha r_k + 2d_k)\mu_k \\ &\implies D^2 \geq 2\alpha \deg(f_*(\mathcal{O}_S(D))) - \alpha(\mu_1 + \mu_k). \end{aligned}$$

Now, if:

$$\frac{d_k + \alpha}{2\alpha}(\mu_1 + \mu_k) \leq \deg f_*\mathcal{O}_S(D).$$

Then:

$$D^2 \geq \frac{2\alpha d_k}{d_k + \alpha} \deg f_*\mathcal{O}_S(D).$$

Else, we apply [24, Lemma 2] for:

$$Z_1 \geq Z_k,$$

and

$$\mu_1 > \mu_k.$$

So:

$$\begin{aligned} D^2 &\geq (\mu_1 - \mu_k)(d_1 + d_k) + \mu_k(d_k + d_{k+1}) \\ &\implies D^2 \geq d_k(\mu_1 + \mu_k) \\ D^2 &> \frac{2\alpha d_k}{d_k + \alpha} \deg f_*\mathcal{O}_S(D). \end{aligned}$$

For 2., if D is nef, $\mu_k < 0$ and if $\exists\ \mathcal{G} \subsetneq \mathcal{E}$ nef sub-vector bundle, using the same method for the first point with $\mathcal{G}$, we deduce:

$$D^2 \geq \frac{2\alpha d_k^{\mathcal{G}}}{d_k^{\mathcal{G}} + \alpha} \deg \mathcal{G}.$$

For 3., if ($\mu_f < 0$ and there is no nef locally free sub-sheaf of $\mathcal{E}$) or (D is not nef and ($\mu_f > 0$ or ($\mu_f = 0$ and $\mathcal{E}$ is not semi-stable))). By the Lemma 4.1 we have:

$$D^2 \geq \sum_{i=1}^{k-1}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) + 2\epsilon^* D.Z_k - Z_k^2 + 2\mu_k d_k$$

$$\Longrightarrow D^2 \geq \sum_{i=1}^{k-1}(\alpha(r_i - 1) + \alpha(r_{i+1} - 1))(\mu_i - \mu_{i+1}) + 2d_k\mu_k + 2\epsilon^* D.Z_k - Z_k^2$$

$$\Longrightarrow D^2 \geq 2\alpha \sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) - \alpha(\mu_1 - \mu_k) + 2d_k\mu_k + 2\epsilon^* D.Z_k - Z_k^2$$

$$\Longrightarrow D^2 \geq 2\alpha \left(\sum_{i=1}^{k-1} r_i(\mu_i - \mu_{i+1}) + r_k\mu_k\right) - 2\alpha r_k\mu_k - \alpha(\mu_1 - \mu_k) + 2d_k\mu_k + 2\epsilon^* D.Z_k - Z_k^2$$

$$\Longrightarrow D^2 \geq 2\alpha \deg(f_*(\mathcal{O}_S(D))) - \alpha(\mu_1 + \mu_k) + 2\epsilon^* D.Z_k - Z_k^2.$$

If

$$\frac{d_k + \alpha}{2\alpha}(\mu_1 + \mu_k) \leq \deg f_*\mathcal{O}_S(D).$$

Then:

$$D^2 \geq \frac{2\alpha d_k}{d_k + \alpha} \deg f_*\mathcal{O}_S(D) + 2\epsilon^* D.Z_k - Z_k^2.$$

Else, we apply the Lemma 4.1 for:

$$Z_1 \geq Z_K,$$

and

$$\mu_1 > \mu_k.$$

So:

$$D^2 > \frac{2\alpha d_k}{d_k + \alpha} \deg f_*\mathcal{O}_S(D) + 2\epsilon^* D.Z_k - Z_k^2.$$

Put

$$N_D := \frac{2\epsilon^* D.Z_k - Z_k^2}{\deg f_*\mathcal{O}_S(D)}$$

$$\Longrightarrow D^2 \geq (\frac{2\alpha d_k}{d_k + \alpha} + N_D) \deg f_*\mathcal{O}_S(D).$$

For 4., if D is not nef, $\mu_k < 0$ and $\exists\, \mathcal{G} \subsetneq \mathcal{E}$ nef sub-vector bundle such that $\mu_k^{\mathcal{G}} > 0$ or ($\mu_k^{\mathcal{G}} \geq 0$ and $\mathcal{G}$ is not semi stable). Then using the same arguments above, we deduce:

$$D^2 \geq \left(\frac{2\alpha d_k^{\mathcal{G}}}{d_k^{\mathcal{G}} + \alpha} + N_D^{\mathcal{G}}\right) \deg \mathcal{G}.$$

$\square$

Remark 4.5. The result in the first point of the Theorem 4.4 is the same in [23, Theorem 3.20].

For the special case that $D = K_{S/C}$, Theorem 4.4 yields:

Corollary 4.6. *Let $f : S \to C$ be a relatively minimal fibration with $g(F) \geq 2$ and $D = K_{S/C}$. Then:*

$$K_{S/C}^2 \geq 4\frac{g-1}{g} \deg f_*\omega_{S/C}.$$

Proof. $K_{S/C}$ is a nef divisor and by [8], $f_*\omega_{S/C}$ is nef vector bundle so $\mu_k \geq 0$. Then by the first point of the Theorem 4.4 we have:

$$K_{S/C}^2 \geq \frac{2\alpha d_f}{d_f + \alpha} \deg f_*\omega_{S/C}.$$

Such that: $\alpha = \min\{2, 1 + \frac{g}{g-1}\} = 2$ and $d_f = 2g - 2 - Z.F$. But by construction Z is contained in the fiber. Thus: $Z.F = 0$ and we deduce the original result of Xiao [24]

$$K_{S/C}^2 \geq 4\frac{g-1}{g} \deg f_*\omega_{S/C}.$$

□

As a final observation, in the setting of Corollary 4.6, when the Miyaoka divisor N_1 restricted to the general fiber F is nonspecial and $h^0(F, N_1|_F) > 1$, then Corollary 4.6 takes the following more refined form.

Corollary 4.7. *Let $f : S \to C$ be a relatively minimal fibration with $g(F) \geq 2$ and $D = K_{S/C}$, if $N_1|_F$ is nonspecial and $h^0(F, N_1|_F) > 1$. Then:*

$$K_{S/C}^2 \geq 4\frac{(g-1)(2g-1)}{(g-1)(2g-1)+g} \deg f_*\omega_{S/C}.$$

Proof. In this case $\alpha = 1 + \frac{g}{g-1}$, so we deduce the result. □

Remark 4.8. The inequality in the Corollary 4.7 is more sharp than from the Corollary 4.6.

Remark 4.9. If $\mathrm{rk}(\mathcal{F}_1) > 1$ where $\mathcal{F}_1$ is the maximal destabilizing sub-vector bundle of $f_*\mathcal{O}_S(D)$. Then the condition $h^0(F, N_1|_F) > 1$ is verified by the Lemma 3.5.

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