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A NEW EXCLUDING CONDITION TOWARDS THE SOPRUNOV-ZVAVITCH CONJECTURE ON BEZOUT-TYPE INEQUALITIES

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ABSTRACT. In 2015, I. Soprunov and A. Zvavitch have shown how to use the Bernstein-Khovanskii-Kushnirenko theorem to derive non-negativity of a certain bilinear form F_Δ, defined on (pairs of) convex bodies. Together with C. Saroglou, they proved non-negativity of F_K characterizes simplices, among all polytopes. It is conjectured the characterization further holds among all convex bodies. Towards this conjecture, several necessary conditions on K (for non-negativity of F_K), were derived. We give a new necessary condition, expressed with isoperimetric ratios, which provides a further step towards a (conjectural) characterization of simplices among a certain subclass of convex bodies.

1. Introduction

The isoperimetric ratio of a convex body $K \subset \mathbb{R}^n$, is usually defined as $\frac{|\partial K|_{n-1}^n}{|K|_n^{n-1}}$, or as a power of this ratio. Here $|K|_n$ denotes the Lebesgue measure of K (its volume), and $|\partial K|_{n-1}$ its surface area (the $(n-1)$-Hausdorff measure of its topological boundary). A classical result in convex geometry, the isoperimetric inequality, states that Euclidean balls minimize this ratio, and are the only minimizers. On the other side, the isoperimetric ratio can be arbitrarily large. However, note this ratio is not affine invariant. Following K. Ball (see [Ba, Theorem 2]), consider instead the affine invariant quantity:

$$Iso(K) = \min_{T \in O(n)} \frac{|\partial(TK)|_{n-1}^n}{|TK|_n^{n-1}},$$

where the minimum runs over the orthogonal group $O(n)$. Then $Iso(K)$ is upper bounded. In fact, denoting by Δ_n an n-simplex, Ball has proved :

$$Iso(K) \le Iso(\Delta_n),$$

yielding a reverse isoperimetric inequality. Additionally, it was proved in [Ba] that n-simplices are the only maximizers of $Iso(K)$.

Denote $V_n(L_1, ..., L_n)$ the mixed volume of n convex bodies, and denote $V_n(K) = V(K[n])$ the volume of K. Since $|\partial K|_{n-1} = nV_n(K[n-1], B_2^n)$, where B_2^n denotes the (unit) l_2-ball in $\mathbb{R}^n$, Ball's result can be reformulated within the language of mixed volumes as follows : for any convex body $K \subset \mathbb{R}^n$, there exists an ellipsoid $\mathcal{E}$, such that

$$V_n(K[n-1], \mathcal{E})^n V_n(\Delta_n)^{n-1} \le V_n(\Delta_n[n-1], \mathcal{E})^{n-1} V_n(K)^{n-1}.$$

Another interesting inequality of mixed volumes involving Δ_n was derived by I. Soprunov and A. Zvavitch (see [SZ16]), using the Bernstein-Kushnirenko-Khovanskii theorem (see [Ber75], [Kho78], [Ku]), and an inequality from real algebraic geometry called Bezout inequality:

$$\forall L_1, L_2 \in \mathcal{K}^n : \quad V_n(L_1, L_2, \Delta_n[n-2]) V_n(\Delta_n) \le V_n(L_1, \Delta_n[n-1]) V_n(L_2, \Delta_n[n-1]).$$

This set of inequalities (called Bezout inequalities in [SZ16]) can be thought of as non-negativity of the bilinear form F_Δ, defined as

$$F_\Delta(A, B) = V_n(A, \Delta_n[n-1]) V_n(B, \Delta_n[n-1]) - V_n(A, B, \Delta_n[n-2]) V_n(\Delta_n).$$

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