Plugging these expansions back into the Schrödinger equation (58) we obtain two new PDEs for the two components ψ^0 and ψ^1:

$$i\,\frac{\partial}{\partial\eta}\psi^0 = \frac{1}{2}\left(\pi_k^2 - \omega_k^2(\eta)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi^0, \tag{61}$$

$$i\,\frac{\partial}{\partial\eta}\psi^1 = \frac{1}{2}\left(\pi_k^2 - \omega_k^2(\eta)\,\frac{\partial^2}{\partial\pi_k^2}\right)\psi^1 + F, \tag{62}$$

$$F = F(\eta,\pi_k) = -\frac{\pi_k^4}{3}\psi^0(\eta,\pi_k), \tag{63}$$

where F indicates a source term for the μ^2-order equation that results to be dependent on the zero-order solution.

Now, the zero-order PDE (61) is the Schrödinger equation of a time-dependent harmonic oscillator with standard operators, but in the momentum polarization; therefore the solution $\psi^0(\eta,\pi_k)$ is just the Fourier transform of $\psi^0(\eta,\xi_k)$, properly rescaled to account for the normalization of Hermite polynomials:

$$\psi_n^0(\eta,\pi_k) = (-i)^n \frac{h_n\left(\frac{\pi_k f}{|R|}\right)}{\sqrt{2^n\,n!}}\sqrt{\frac{(R^*)^n\,f}{R^{n+1}\sqrt{\pi}}}\;e^{-\frac{\pi_k^2 f^2}{2R}}\,e^{i\alpha_n}, \tag{64}$$

where R has been defined in (44), and f and α_n have the same expressions as before.

On the other hand, the first order PDE (62) is the same of the zero order one but with the addition of the source term $F(\eta,\pi_k)$. In order to solve it, we consider that the eigenfunctions $\psi_n(\eta,\pi_k)$ form a complete orthonormal basis such that $\langle\psi_{n_1}|\psi_{n_2}\rangle = \delta_{n_1,n_2}$ and any function can be expressed as a linear combination of them. Therefore we can write ψ^1 and ψ^0 as

$$\psi^0 = \sum_n c_n(\eta)\,\psi_n(\eta,\pi_k), \quad \psi^1 = \sum_n d_n(\eta)\,\psi_n(\eta,\pi_k), \tag{65}$$

where $c_n(\eta)$, $d_n(\eta)$ are time-dependent coefficients; when we plug these expansions back into the first order Schrödinger equation (62) we are left with just a recurrence relation for the coefficients, since all the eigenfunctions ψ_n^0 satisfy the zero-order equation (61) that corresponds to the homogeneous part of the first order one:

$$i\sum_n \frac{\mathrm{d}\,d_n}{\mathrm{d}\eta}\,\psi_n^0(\eta,\pi_k) = -\frac{\pi_k^4}{3}\sum_n c_n(\eta)\,\psi_n^0(\eta,\pi_k). \tag{66}$$

Considering just the ground state and using the result (43) for π_k, we obtain

$$\pi_k^4\,\psi_0 = \frac{3}{4}\frac{R^2(R^*)^2}{f^4}\psi_0 - \frac{3}{\sqrt{2}}\frac{R^3R^*}{f^4}e^{2i\varphi}\psi_2 + \sqrt{\frac{3}{2}}\,\frac{R^4}{f^4}e^{4i\varphi}\psi_4; \tag{67}$$

it is thus clear that, when $c_n = \delta_{0,n}$, the only non-zero coefficients on the left hand side are d_0, d_2 and d_4. Therefore the relations for these coefficients are:

$$i\frac{\mathrm{d}\,d_0}{\mathrm{d}\eta} = -\frac{1}{4}\,\frac{(1+f^2f'^2)^2}{f^4}, \tag{68}$$

$$i\frac{\mathrm{d}\,d_2}{\mathrm{d}\eta} = +\frac{1}{\sqrt{2}}\,\frac{(1+f^2f'^2)(1-iff')^2}{f^4}e^{2i\varphi}, \tag{69}$$

$$i\frac{\mathrm{d}\,d_4}{\mathrm{d}\eta} = -\frac{1}{\sqrt{6}}\,\frac{(1-iff')^4}{f^4}e^{4i\varphi}. \tag{70}$$

Finally, the ground state of our system in the π_k representation is

$$\psi_0^{\mathrm{tot}}(\eta,\pi_k) = \psi_0^0(\eta,\pi_k) + \mu^2\sum_{n=0}^{2} d_{2n}(\eta)\,\psi_{2n}^0(\eta,\pi_k). \tag{71}$$

From here on we will omit the superscript indicating the order, since we expressed ψ^1 and ψ^0 as a linear combination of ψ_n.

Now, in order to find the final spectrum of perturbations we have to evaluate the expectation value $\left\langle\hat{\xi}_k^2\right\rangle$ on the ground state; we can use the expression (42) and therefore write

$$\begin{aligned}\left\langle\psi_0^{\mathrm{tot}}\middle|\hat{\xi}_k^2\middle|\psi_0^{\mathrm{tot}}\right\rangle &= \int d\pi_k\,\psi_0^{\mathrm{tot}\,*}\hat{\xi}_k^2\psi_0^{\mathrm{tot}} = \int d\pi_k\left|\hat{\xi}_k\psi_0^{\mathrm{tot}}\right|^2 \\ &= \int d\pi_k\,f^2\left|\frac{1+\mu^2 d_0}{\sqrt{2}}\,e^{i\varphi}\psi_1 + \mu^2 d_2 e^{-i\varphi}\psi_1 + ...\right|^2,\end{aligned} \tag{72}$$

where the dots stand for terms proportional to ψ_3 and ψ_5, whose square modulus would contribute with terms of order μ^4 which we would neglect. Given that the norm of ψ_0^{tot} is easily calculated to be $|N|^2 = 1 + 2\mu^2\,\mathrm{Re}(d_0)$, since $\int d\pi_k|\psi_n| = 1$, the normalized expectation value of $\hat{\xi}_k^2$ results to be

$$\begin{aligned}\frac{\left\langle\hat{\xi}_k^2\right\rangle}{|N|^2} &= \frac{f^2}{2|N|^2}\left(1 + 2\mu^2\,\mathrm{Re}(d_0) + 2\sqrt{2}\,\mu^2\,\mathrm{Re}(d_2e^{-2i\varphi})\right) = \\ &= \frac{f^2}{2}\left(1 + \frac{2\sqrt{2}\,\mu^2\,\mathrm{Re}(d_2e^{-2i\varphi})}{1+2\mu^2\,\mathrm{Re}(d_0)}\right).\end{aligned} \tag{73}$$

As expected, the zero-order term is the same as for the standard Spectrum (45); on the other hand, for the μ^2-order correction we see that we only need d_0 and d_2 among the coefficients of the expansion.

Now, looking at equation (68) we see that the right hand side is real; therefore d_0 has a purely imaginary time derivative, and its real time-independent part must be set through initial conditions; we will adopt the same prescription as in [33, 34] where we assume that the wavefunction is in the instantaneous ground state at the beginning of inflation: we therefore write $d_0(\eta_s) = 0$ and, since its real part is independent of time, it will remain zero throughout the evolution. Then we solve the integral (69) for $d_2(\eta)$, insert it into the Spectrum and find