In Case (b), where $b = 3c+2$ and $a = 2c+2$, equation (15) implies that $3|2^7yz^6D$ which is impossible. Hence, if Case (b) holds, then P cannot be the image of any point $Q \in \hat{E}_D(\mathbb{Q})$.

In Case (c) we have nothing to show since this case cannot happen.

In Case (d), $b = 3a/2$ and a must be even. We rewrite equation (15) as

$$3^{2+(3c-b+1)}(Bx^3z^3 + 3^{(3c-b+1)}2^7z^6yD) = yx^3. \tag{16}$$

Denote by ε the common exponent $\varepsilon = 3c - b + 1$.

If $\varepsilon \geq 0$, equation (16) implies that $3|yx^3$, a contradiction.

If $\varepsilon = -1$ then $b = 3c + 2$ and we are in Case a.

If $\varepsilon \leq -2$, equation (16) implies that $3|2^7z^6yD$, a contradiction.

Hence, if Case (d) holds, then P cannot be the image of any point $Q \in \hat{E}_D(\mathbb{Q})$.

Having exhausted all possible cases, we see that P cannot be the image of any point $Q \in \hat{E}_D(\mathbb{Q})$ and the lemma is proved. □

Before we move on to Proposition 4.8, we need the following two definitions.

Definition 4.6. *Given an order $\mathcal{O}$, we say that a proper $\mathcal{O}$-ideal $\mathfrak{a}$ is primitive if it is not of the form $k\mathfrak{a}$ for $1 < k \in \mathbb{Z}$ and $\mathfrak{a}$ a proper ideal ([8, §11.D, pg.214]). We will call an element π primitive if (π) is a primitive ideal.*

The second definition is a very brief definition of what we call a 3-virtual unit. The reader may find more on 3-virtual units in [1], and on l-virtual units in general in [6, Section 5.2.2].

Definition 4.7. *An element $\alpha \in K_{D'}^{\times}$ is a 3-virtual unit of $K_{D'}$ if there exists an ideal $\mathfrak{a}$ in the maximal order $\mathcal{O}_{D'}$ of $K_{D'}$ such that $\alpha\mathcal{O}_{D'} = \mathfrak{a}^3$.*

Proposition 4.8. *As before, D is a negative fundamental discriminant satisfying the congruence relations in (1). The subfamily of elliptic curves $E_{D'}$ for which $r_3(D) = r_3(D') + 1$ holds, have odd rank and no integral points.*

Proof. The fact that the whole family of elliptic curves $E_{D'}$ has odd rank was proved in Corollary 3.2. Assume now that we are in the escalatory case and assume by way of contradiction that an elliptic curve $E_{D'}$ in this subfamily has an *integral* point $P = (A, B) \in E_{D'}(\mathbb{Z})$. Plugging the point back in $E_{D'}$ and modifying the coefficients a little we obtain

$$3^4D = 4(\frac{3A}{4})^3 - 27(\frac{B}{4})^2.$$

Case (a): From the equation of $E_{D'}$, we see that if either A or B is even then so must be the other one, since $B^2 = A^3 - 3 \cdot 16D$. From the same equation we see that B must actually be divisible by 4 exactly, since $2 \nmid D$. Let $A = 2a$ and $B = 4b$. Cancelling out the 16 we end up with $2a^3 = b^2 + 3D$. Since $D \equiv 1 \bmod 4$ and b is odd, we have a contradiction unless $2|a$. Therefore, if both A and B are even then we must have that $4||B$ and $4|A$. This implies that the monic polynomial

$$g(x) = x^3 - \frac{3A}{4}x + \frac{B}{4} = x^3 - 3ax + b$$