The dual isogeny $\hat{\phi}$ is such that $\phi \circ \hat{\phi} = \times 3$, where $\times 3$ is the multiplication-by-3 map. The map defined by the dual isogeny $\hat{\phi}$ is given in the proof of Lemma 4.5.

Consider now the exact sequence ([15, Section X.4, Remark 4.7])

$$0 \to \hat{E}_D(\mathbb{Q})[\hat{\phi}]/\phi(E_{D'}(\mathbb{Q})[3]) \to \hat{E}_D(\mathbb{Q})/\phi(E_{D'}(\mathbb{Q})) \xrightarrow{\hat{\phi}} \tag{5}$$

$$\xrightarrow{\hat{\phi}} E_{D'}(\mathbb{Q})/3E_{D'}(\mathbb{Q}) \to E_{D'}(\mathbb{Q})/\hat{\phi}(\hat{E}_D(\mathbb{Q})) \to 0.$$

Since both $\ker_{\phi}(\overline{\mathbb{Q}}) = \mathcal{T}_{D'}$ and $\ker_{\hat{\phi}}(\overline{\mathbb{Q}}) = \hat{\mathcal{T}}_D$ contain no non-trivial rational point of order 3, the first quotient group of (5) vanishes and the rank $r(E_{D'})$ of $E_{D'}(\mathbb{Q})$ equals

$$r(E_{D'}) = \dim_{\mathbb{F}_3}(E_{D'}(\mathbb{Q})/3E_{D'}(\mathbb{Q})). \tag{6}$$

Consider the short exact sequence

$$0 \to \mathcal{T}_{D'} \to E_{D'}(\overline{\mathbb{Q}}) \xrightarrow{\phi} \hat{E}_D(\overline{\mathbb{Q}}) \to 0. \tag{7}$$

From this, we obtain the long exact cohomology sequence which gives in particular the following

$$0 \to \hat{E}_D(\mathbb{Q})/\phi(E_{D'}(\mathbb{Q})) \xrightarrow{\delta} H^1(G_{\mathbb{Q}}, \mathcal{T}_{D'}) \to H^1(G_{\mathbb{Q}}, E_{D'}(\overline{\mathbb{Q}}))[\phi] \to 0. \tag{8}$$

By localising at each prime p, we obtain the following commutative diagram, where res_p is the usual restriction map:

$$\begin{array}{ccccccccc}
0 & \to & \hat{E}_D(\mathbb{Q})/\phi(E_{D'}(\mathbb{Q})) & \longrightarrow & H^1(G_{\mathbb{Q}}, \mathcal{T}_{D'}) & \longrightarrow & H^1(G_{\mathbb{Q}}, E_{D'}(\overline{\mathbb{Q}}))[\phi] & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle res_p} & & \downarrow{\scriptstyle res_p} & & \\
0 & \to & \hat{E}_D(\mathbb{Q}_p)/\phi(E_{D'}(\mathbb{Q}_p)) & \to & H^1(G_{\mathbb{Q}_p}, \mathcal{T}_{D'}) & \to & H^1(G_{\mathbb{Q}_p}, E_{D'}(\overline{\mathbb{Q}}_p))[\phi] & \to & 0
\end{array}$$

Definition 2.1. *The Selmer group of $E_{D'}$ relative to the isogeny ϕ is*

$$\mathcal{S}_{\phi}(E_{D'}) = \{x \in H^1(G_{\mathbb{Q}}, \mathcal{T}_{D'}) \mid res_p(x) \in \mathrm{Im}(\hat{E}_D(\mathbb{Q}_p)/\phi(E_{D'}(\mathbb{Q}_p))) \text{ for all } p\}.$$

The Tate-Shafarevich group of $E_{D'}$ can now be defined as

$$Ш(E_{D'}) = \{x \in H^1(G_{\mathbb{Q}}, E_{D'}(\overline{\mathbb{Q}})) \mid res_p(x) = 0 \text{ for all } p\}.$$

These two groups are connected together as follows:

$$0 \to \hat{E}_D(\mathbb{Q})/\phi(E_{D'}(\mathbb{Q})) \to \mathcal{S}_{\phi}(E_{D'}) \to Ш(E_{D'})[\phi] \to 0. \tag{9}$$

Remark 2.2. *By considering the dual isogeny $\hat{\phi}$ instead, we get exact sequences analogous to (7), (8) and (9) which in turn give us the analogous definitions for $\mathcal{S}_{\hat{\phi}}(\hat{E}_D)$, $Ш(\hat{E}_D)$ and $Ш(E_D)[\hat{\phi}]$.*

Remark 2.3. *We also obtain exact sequences analogous to (7), (8) and (9) for $\times 3 = \phi \circ \hat{\phi}$ which in turn give us the analogous definitions for $\mathcal{S}_3(E_{D'})$, $Ш(E_{D'})$ and $Ш(E_{D'})[3]$, and similarly for $\hat{E}_D$.*