Denote $F_{K,b}(A,B) := bV(A,K[n-1])V(B,K[n-1]) - V(A,B,K[n-2])V(K)$, then $b_2(K)$ can be equivalently defined as the least $b \geq 1$ such that $F_{K,b} \geq 0$. Notice that $b_2(TK) = b_2(K)$ for every affine transformation T.

Clearly, $F_K = F_{K,1}$. A Blaschke selection argument shows that the supremum is actually a maximum; in particular $b_2(K) < \infty$ for all $K \in \mathcal{K}^n$. In fact[6], $b_2(K) \leq 2$ for any K. Notice that $b_2(K) > 1$ is equivalent to not having $F_K \geq 0$ (i.e. to existence of a pair (A,B) such that $F_K(A,B) < 0$).

3. An excluding condition with isoperimetric ratios

If a property $\mathcal{P}$ (for instance, being decomposable) is such that when K has $\mathcal{P}$, then F_K cannot be non-negative (on all of $(\mathcal{K}^n)^2$), we shall say that $\mathcal{P}$ is an excluding condition (in the terminology of [SZ16], a convex body K cannot both satisfy $\mathcal{P}$ and satisfy *Bezout inequalities*). It was shown in [SSZ1, SSZ2] that being weakly decomposable (a property which in particular includes being a polytope other than an n-simplex, or being decomposable) is an excluding condition. Denote $\mathcal{K}_F$ the subclass of $\mathcal{K}^n$ consisting of convex bodies having at least one facet : $\mathcal{K}_F$ is closed under Minkowski addition, and contains the class of n-polytopes. In this section we give a new excluding condition, which concerns bodies $K \in \mathcal{K}_F$. We will work in $\mathbb{R}^n$ with $n \geq 3$.

Let $K \subset \mathbb{R}^n$ be a non-empty compact convex set. Recall there exists a unique affine subspace H of $\mathbb{R}^n$, such that $K \subset H$, and H has maximal (affine) co-dimension. The dimension of K is defined as the dimension of this subspace H. Alternatively $\dim(K)$ can be defined as the maximal $k \geq 1$, such that one may find $k+1$ affinely independent points, within K.

Let $k \geq 2$ and let $K \in \mathcal{K}^n$ be k-dimensional. Then denote

$$\mathrm{Isop}(K) = \frac{1}{k}\frac{|\partial K|_{k-1}}{|K|_k}.$$

Proposition 1. *Let K be a convex body such that K has a facet F satisfying : $Isop(F) > Isop(K)$. Then $b_2(K) > 1$.*

Proof. Assume K has a facet $F = K^{u_0}$ such that $\frac{|\partial F|_{n-2}}{(n-1)|F|_{n-1}} > \frac{|\partial K|_{n-1}}{n|K|_n}$. Set

$$c_0 := \frac{|\partial F|_{n-2}|K|_n}{n-1} - \frac{|\partial K|_{n-1}|F|_{n-1}}{n} > 0,$$

and $c = \frac{2}{n-1}|\partial F|_{n-2}|K|_n > 2c_0$. Fix $\epsilon > 0$ so that $c_0 > \epsilon c$.

Since $S_K(\{u_0\}) = |F|_{n-1} > 0$ and $S_K(\mathbb{S}^{n-1}) = |\partial K|_{n-1} < +\infty$, one may choose f a non-negative and continuous function on the sphere, such that $f(u_0) = \max_{\mathbb{S}^{n-1}} f = 1$ and $\int f(u)dS_K(u) < (1+\epsilon)S_K(u_0) = (1+\epsilon)|F|_{n-1}$. Fix such a positive function f, and define $L_t = W(h_K + tf)$, the Wulff shape with respect to function $h_K + tf$. One may think of L_t as a perturbed version of K, with most of the perturbation in direction u_0.

Set $M = H^-_{u_0} \cap B^n_2 = \{x \in B^n_2 : \langle x, u_0\rangle \leq 0\}$ to be a half-euclidean ball, such that its unique facet is the euclidean ball $M^{u_0} = \pi_{u_0^\perp}(M)$ with u_0 as an outer normal vector. We will show that $F_K(A,B) < 0$, for $A = L_t$, $B = M$, and $t > 0$ is small enough, proving that $b_2(K) > 1$.

Assume $t \geq 0$. Recall that the mixed surface area measure $\sigma := S(M, K[n-2], .)$ is a non-negative measure, and that, since $f \geq 0$, $h_K(u) \leq h_{L_t}(u)$, for all $u \in \mathbb{S}^{n-1}$. It follows that :

$$\begin{aligned} V_n(L_t, M, K[n-2]) - V_n(M, K[n-1]) &= \frac{1}{n}\int (h_{L_t} - h_K)(u)d\sigma(u) \\ &\geq \frac{1}{n}(h_{L_t} - h_K)(u_0)\sigma(\{u_0\}). \end{aligned}$$

[6] this is known as Fenchel inequality, see [FGM]