# Minimizing the number of complete bipartite graphs in a $K_s$ -saturated graph

Beka Ergemlidze\* Abhishek Methuku<sup>†</sup> Michael Tait<sup>‡</sup> Craig Timmons<sup>§</sup>

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#### Abstract

A graph G is F-saturated if it contains no copy of F as a subgraph but the addition of any new edge to G creates a copy of F. We prove that for  $s \geq 3$  and  $t \geq 2$ , the minimum number of copies of  $K_{1,t}$  in a  $K_s$ -saturated graph is  $\Theta(n^{t/2})$ . More precise results are obtained when t=2 where the problem is related to Moore graphs with diameter 2 and girth 5. We prove that for  $s \geq 4$  and  $t \geq 3$ , the minimum number of copies of  $K_{2,t}$  in an n-vertex  $K_s$ -saturated graph is at least  $\Omega(n^{t/5+8/5})$  and at most  $O(n^{t/2+3/2})$ . These results answer a question of Chakraborti and Loh. General estimates on the number of copies of  $K_{a,b}$  in a  $K_s$ -saturated graph are also obtained, but finding an asymptotic formula remains open.

# 1 Introduction

Let F be a graph with at least one edge. A graph G is F-free if G does not contain F as a subgraph. The study of F-free graphs is central to extremal combinatorics. Turán's Theorem, widely considered to be a cornerstone result in graph theory, determines the maximum number of edges in an n-vertex  $K_s$ -free graph. An interesting class of F-free graphs are those that are maximal with respect to the addition of edges. We say that a graph G is F-saturated if G is F-free but the addition of an edge joining any pair of nonadjacent vertices of G creates a copy of F. The function  $\operatorname{sat}(n,F)$  is the saturation number of F, and is defined to be the minimum number of edges in an n-vertex F-saturated graph. In some sense, it is dual to the Turán function  $\operatorname{ex}(n,F)$  which is the maximum number of edges in an n-vertex F-saturated graph.

One of the first results on graph saturation is a theorem of Erdős, Hajnal, and Moon [10] which determines the saturation number of  $K_s$ . They proved that for  $2 \le s \le n$ , there is a unique n-vertex  $K_s$ -saturated graph with the minimum number of edges. This graph is the join of a complete graph on s-2 vertices and an independent set on n-s+2 vertices, denoted  $K_{s-2} + \overline{K_{n-s+2}}$ . The Erdős-Hajnal-Moon Theorem was proved in the 1960s and since then,

<sup>\*</sup>Department of Mathematics and Statistics, University of South Florida, Tampa, Florida, U.S.A. E-mail: beka.ergemlidze@gmail.com

<sup>&</sup>lt;sup>†</sup>School of Mathematics, University of Birmingham, United Kingdom. E-mail: abhishekmethuku@gmail.com. Research is supported by the EPSRC grant number EP/S00100X/1 (A. Methuku).

<sup>&</sup>lt;sup>‡</sup>Department of Mathematics & Statistics, Villanova University, U.S.A. E-mail: michael.tait@villanova.edu. Research is supported in part by National Science Foundation grant DMS-2011553.

<sup>§</sup>Department of Mathematics and Statistics, California State University Sacramento, U.S.A. E-mail: craig.timmons@csus.edu. Research is supported in part by Simons Foundation Grant #359419.

graph saturation has developed into its own area of extremal combinatorics. We recommend the survey of Faudree, Faudree, and Schmitt [12] as a reference for history and significant results in graph saturation.

The function  $\operatorname{sat}(n, F)$  concerns the minimum number of edges in an F-saturated graph. More generally, one can ask for the minimum number of copies of H in an n-vertex F-saturated graph. Let us write  $\operatorname{sat}(n, H, F)$  for this minimum. This function was introduced in [18] and was motivated by the well-studied generalized Turán function whose systematic study was initiated by Alon and Shikhelman [2]. Recalling that the Erdős-Hajnal-Moon Theorem determines  $\operatorname{sat}(n, K_s) = \operatorname{sat}(n, K_2, K_s)$ , it is quite natural to study the generalized function  $\operatorname{sat}(n, K_r, K_s)$ , where  $2 \leq r < s$ . Answering a question of Kritschgau, Methuku, Tait and Timmons [18], Chakraborti and Loh [5] proved that for every  $2 \leq r < s$ , there is a constant  $n_{r,s}$  such that for all  $n \geq n_{r,s}$ ,

$$sat(n, K_r, K_s) = (n - s + 2) {s - 2 \choose r - 1} + {s - 2 \choose r}.$$

Furthermore, they showed that  $K_{s-2} + \overline{K_{n-s+2}}$  is the unique graph that minimizes the number of copies of  $K_r$  among all n-vertex  $K_s$ -saturated graphs for  $n \ge n_{r,s}$ . They proved a similar result for cycles where the critical point is that  $K_{s-2} + \overline{K_{n-s+2}}$  is again the unique graph that minimizes the number of copies of  $C_r$  among all n-vertex  $K_s$ -saturated graphs for  $n \ge n_{r,s}$  under some assumptions on r in relation to s (see Theorem 1.4 below). Chakraborti and Loh then asked the following question (Problem 10.5 in [5]).

**Question 1.1** Is there a graph H for which  $K_{s-2} + \overline{K_{n-s+2}}$  does not (uniquely) minimize the number of copies of H among all n-vertex  $K_s$ -saturated graphs for all large enough n?

Here we answer this question positively and show that there are graphs H for which  $K_{s-2} + \overline{K_{n-s+2}}$  is not the unique extremal graph.

We begin by stating our first two results, Theorems 1.2 and 1.3, where  $H = K_{1,t}$ . Together, they demonstrate a change in behaviour between the cases  $H = K_{1,2}$  and  $H = K_{1,t}$  with t > 2.

**Theorem 1.2** (i) For  $n \geq 3$ ,

$$\binom{n}{2} - \frac{n^{3/2}}{2} \le \operatorname{sat}(n, K_{1,2}, K_3) \le \binom{n-1}{2}.$$

(ii) For  $n \ge s \ge 4$ ,

$$sat(n, K_{1,2}, K_s) = (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2}.$$

Furthermore,  $K_{s-2} + \overline{K_{n-s+2}}$  is the unique *n*-vertex  $K_s$ -saturated with minimum number of copies of  $K_{1,2}$ .

**Theorem 1.3** For integers  $n \ge s \ge 3$  and  $t \ge 3$ ,

$$sat(n, K_{1,t}, K_s) = \Theta(n^{t/2+1}).$$

A consequence of Theorem 1.3 is that if  $s, t \geq 3$  and n is large enough in terms of  $t, K_{s-2} + \overline{K_{n-s+2}}$  does not minimize the number of copies of  $K_{1,t}$  among all n-vertex  $K_s$ -saturated graphs. Indeed,  $K_{s-2} + \overline{K_{n-s+2}}$  has  $\Theta(n^t)$  copies of  $K_{1,t}$ . Interestingly, the special case of determining  $\operatorname{sat}(n, K_{1,2}, K_3)$  is related to the existence of Moore graphs. This is discussed further in the Concluding Remarks section, but whenever a Moore graph of diameter 2 and girth 5 exists, this graph will have fewer copies of  $K_{1,2}$  than  $K_1 + \overline{K_{n-1}} = K_{1,n-1}$ . Thus, any potential result that determines  $\operatorname{sat}(n, K_{1,2}, K_3)$  exactly would have to take this into account.

The graph used to prove the upper bound of Theorem 1.3 is a  $K_s$ -saturated graph with maximum degree at most  $c_s n^{1/2}$ . This graph was constructed by Alon, Erdős, Holzman, and Krivelevich [1] and it is structurally very different from  $K_{s-2} + \overline{K_{n-s+2}}$ . Using this graph one can prove a more general upper bound that applies to any connected bipartite graph. This will be stated in Theorem 1.5 below.

Next we turn our attention to counting copies of  $K_{2,t}$  (for  $t \geq 2$ ) in  $K_s$ -saturated graphs. The graph  $K_1 + \overline{K_{n-1}}$  is  $K_3$ -saturated and  $K_{2,t}$ -free. Thus,  $\operatorname{sat}(n, K_{2,t}, K_3) = 0$  for all  $t \geq 2$ . For t = 2 and  $s \geq 4$  Chakraborti and Loh [5] proved that

$$sat(n, K_{2,2}, K_s) = (1 + o(1)) \binom{s-2}{2} \binom{n}{2}.$$
 (1)

Observe that the graph  $K_{s-2} + \overline{K_{n-s+2}}$  has

$$\binom{s-2}{2}\binom{n-s+2}{2} + \binom{s-2}{3}(n-s+2) + \binom{s-2}{4}$$

copies of  $K_{2,2}$  and this gives the upper bound in (1). Now the focus of [5] was on counting complete graphs and counting cycles, so here the above result is stated in terms of  $K_{2,2}$  but of course  $K_{2,2} = C_4$ . However, it is important and relevant to this work to mention the following theorem of Chakraborti and Loh which shows that  $K_{s-2} + \overline{K_{n-s+2}}$  minimizes the number of copies of  $C_r$  in certain cases.

**Theorem 1.4 (Chakraborti and Loh [5])** Let  $s \geq 4$  and  $r \geq 7$  if r odd, and  $r \geq 4\sqrt{s-2}$  if r is even. There is an  $n_{r,s}$  such that for all  $n \geq n_{r,s}$ , the graph  $K_{s-2} + \overline{K_{n-s+2}}$  minimizes the number of copies of  $C_r$  over all n-vertex  $K_s$ -saturated graphs. Moreover, when  $r \leq 2s-4$ , this is the unique extremal graph.

It is conjectured in [5] that  $K_{s-2} + \overline{K_{n-s+2}}$  is the unique graph that minimizes the number of copies of  $C_r$  among all  $K_s$ -saturated graphs. Currently it is only known that  $K_{s-2} + \overline{K_{n-s+2}}$  minimizes the number of copies of  $K_r$  (Erdős-Hajnal-Moon for r=2 and [5] for r>2), and minimizes the number of copies of  $C_r$  under certain assumptions (stated in Theorem 1.4). Theorem 1.3 shows  $K_{s-2} + \overline{K_{n-s+2}}$  does not minimize the number of copies of  $K_{1,t}$ . We extend this to  $K_{a,b}$  with  $1 \le a+1 < b$  using the following theorem.

**Theorem 1.5** Let F be a connected bipartite graph with parts of size a and b with  $1 \le a+1 < b$ . If  $s \ge 3$  be an integer, then

$$sat(n, F, K_s) = \begin{cases} 0 & \text{if } a > s - 2, \\ O(n^{\frac{1}{2}(a+b+1)}) & \text{if } a \le s - 2 \end{cases}$$

where the implicit constant can depend on a, b, and s.

Theorem 1.5 naturally suggests the following question: how many copies of  $K_{2,t}$  must there be in a  $K_s$ -saturated graph? In this direction we prove the following.

**Theorem 1.6** Let  $s \ge 4$  and  $t \ge 3$  be integers. There is a positive constant C such that

$$sat(n, K_{2,t}, K_s) \ge C n^{t/5 + 8/5}.$$

By Theorem 1.5,  $\operatorname{sat}(n, K_{2,t}, K_s) \leq O_{s,t}(n^{t/2+3/2})$  for  $s \geq 4$  and  $t \geq 3$ , so that there is a gap in the exponent in the upper and lower bounds.

Saturation problems with restrictions on the degrees have also been well-studied. Duffus and Hanson [7] investigated triangle-saturated graphs with minimum degree 2 and 3. Day [8] resolved a 20 year old conjecture of Bollobás [15] which asked for a lower bound on the number of edges in  $K_s$ -saturated graphs with minimum degree t. Gould and Schmitt [14] studied  $K_2^t$ -saturated graphs (where  $K_2^t$  is the complete t-partite graph with parts of size 2) with a given minimum degree. Furthermore,  $K_s$ -saturated graphs with restrictions on the maximum degree were studied in [1, 13, 19]. Turning to generalized saturation numbers, as a step towards generalizing Day's Theorem, Curry et. al. [6] proved bounds on the number of triangles in a  $K_s$ -saturated graph with minimum degree t. Motivated by these results we prove a lower bound on the number of copies of  $K_{a,b}$  in  $K_s$ -saturated graphs in terms of its minimum degree.

**Theorem 1.7** Let  $s \ge 4$  and  $2 \le a < b$  be integers with  $a \le s - 2$ . If G is an n-vertex  $K_s$ -saturated graph with minimum degree  $\delta(G)$ , then G contains at least

$$c\left(\frac{n-\delta(G)-1}{\delta(G)^{a-1}}\right)^{b/2}$$

copies of  $K_{a,b}$  for some constant c = c(s, a, b) > 0.

Theorem 1.7 shows that if  $0 \le \alpha < \frac{1}{a-1}$  and  $\delta(G) \le \kappa n^{\alpha}$  for some  $\kappa > 0$ , then G contains at least  $cn^{b/2(1-\alpha(a-1))}$  copies of  $K_{a,b}$ . In particular, when  $\delta(G)$  is a constant, we obtain  $\Omega(n^{t/2})$  copies of  $K_{2,t}$ . This improves the lower bound of Theorem 1.6, but comes at the cost of a minimum degree assumption.

In the next subsection we give the notation that will be used in our proofs. Section 2 contains the proofs of Theorems 1.2 and 1.3. Section 3 contains the proofs of Theorems 1.5, 1.6, and 1.7.

#### 1.1 Notation

For graphs F and G, we write  $\mathcal{N}(F,G)$  for the number of copies of F in G. For a graph G and  $x,y\in V(G)$ , write N(x) for the neighborhood of x, and N(x,y) for  $N(x)\cap N(y)$ . More generally, if  $X\subseteq V(G)$  and  $v\in V(G)$ , then N(v,X) is the set of vertices adjacent to all of the vertices in  $\{v\}\cup X$ , and N(X) is the set of vertices adjacent to all vertices in X. We write  $d(v)=|N(v)|,\ d(X)=|N(X)|,\ \text{and}\ d(v,X)=|N(v,X)|.$  The set  $N[v]=N(v)\cup \{v\}$  is the closed neighborhood of v. For a graph G, let e(G) denote the number of edges in G.

For a hypergraph  $\mathcal{H}$ ,  $d_{\mathcal{H}}(v)$  is the number of edges in  $\mathcal{H}$  containing v. Similarly,  $d_{\mathcal{H}}(X)$  and  $d_{\mathcal{H}}(v,X)$  is the number of edges in  $\mathcal{H}$  containing X and  $\{v\} \cup X$ , respectively.

# 2 Bounds on $sat(n, K_{1,t}, K_s)$

#### 2.1 Proof of Theorem 1.2

Since the graph  $K_{s-2} + \overline{K_{n-s+2}}$  is  $K_s$ -saturated, by counting the number of copies of  $K_{1,2}$  in it, we have

$$sat(n, K_{1,2}, K_s) \le (s-2) \binom{n-1}{2} + (n-s+2) \binom{s-2}{2}.$$
(2)

In particular, if s=3 we have  $\operatorname{sat}(n,K_{1,2},K_3) \leq \binom{n-1}{2}$ . We now prove a matching lower bound up to an error term of order  $O(n^{3/2})$ . Let G be an n-vertex  $K_3$ -saturated graph. If  $e(G) \geq \frac{\sqrt{n-1}n}{2}$ , then for  $n \geq 3$ ,

$$\mathcal{N}(K_{1,2},G) = \sum_{v \in V(G)} {d(v) \choose 2} \ge n {2e(G)/n \choose 2} = \frac{2e(G)^2}{n} - e(G)$$
$$\ge {n \choose 2} - \frac{n^{3/2}}{2}.$$

Now assume that  $e(G) < \frac{\sqrt{n-1}n}{2}$ . If x and y are not adjacent, then since G is  $K_3$ -saturated, x and y must be joined by a path of length 2. Hence,

$$\mathcal{N}(K_{1,2},G) \ge e(\overline{G}) = \binom{n}{2} - e(G) \ge \binom{n}{2} - \frac{n^{3/2}}{2}.$$

This completes the proof of (i) of Theorem 1.2. To prove (ii) of Theorem 1.2, it suffices to show that for  $n \ge s \ge 4$ ,

$$sat(n, K_{1,2}, K_s) \ge (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2},$$

since (2) holds. Let G be an n-vertex  $K_s$ -saturated graph with  $n \ge s \ge 4$ . Kim, Kim, Kostochka and O [17, Theorem 2.1] proved that

$$\sum_{v \in V(G)} (d(v) + 1)(d(v) + 2 - s) \ge (s - 2)n(n - s + 1). \tag{3}$$

It is easy to check that

$$\sum_{v \in V(G)} (d(v) + 1)(d(v) + 2 - s) = \sum_{v \in V(G)} (d(v) - 1)d(v) + (4 - s) \sum_{v \in V(G)} d(v) + (2 - s)n.$$
 (4)

Therefore, combining (3) and (4), we have

$$\sum_{v \in V(G)} (d(v) - 1)d(v) \ge (s - 2)n(n - s + 1) + (s - 4)2e(G) + (s - 2)n.$$
 (5)

By the Erdős-Hajnal-Moon Theorem

$$sat(n, K_s) = (s-2)(n-s+2) + {s-2 \choose 2},$$

and  $K_{s-2} + \overline{K_{n-s+2}}$  is the unique n-vertex  $K_s$ -saturated with  $\operatorname{sat}(n, K_s)$  edges. Thus,

$$2e(G) \ge 2(s-2)(n-s+2) + 2\binom{s-2}{2} = (s-2)(2n-s+1).$$

Plugging this into (5) we get that if  $s \geq 4$ ,

$$\sum_{v \in V(G)} (d(v) - 1)d(v) \ge (s - 2)n(n - s + 1) + (s - 4)(s - 2)(2n - s + 1) + (s - 2)n.$$

Dividing through by 2 and simplifying the right-hand side yields

$$\sum_{v \in V(G)} {d(v) \choose 2} \ge (s-2) {n-1 \choose 2} + (n-s+2) {s-2 \choose 2},$$

where equality holds only if  $G = K_{s-2} + \overline{K_{n-s+2}}$ . This completes the proof of Theorem 1.2.

#### 2.2 Proof of Theorem 1.3

Now we prove a lower bound on the number of copies of  $K_{1,t}$  in a  $K_s$ -saturated graph that gives the correct order of magnitude for all  $t \geq 3$ .

**Proposition 2.1** Let  $n \ge s \ge 3$  and  $t \ge 3$  be integers. Then

$$\operatorname{sat}(n, K_{1,t}, K_s) \ge \left(\frac{\sqrt{s-2}}{t}\right)^t n^{t/2+1} + O_{s,t}(n^{t/2}).$$

**Proof.** Let G be an n-vertex  $K_s$ -saturated graph. Kim, Kim, Kostochka and O [17, Theorem 1.1] proved that

$$\sum_{v \in V(G)} d(v)^2 \ge (n-1)^2 (s-2) + (s-2)^2 (n-s+2) \tag{6}$$

and that equality holds if and only if G is  $K_{s-2} + \overline{K_{n-s+2}}$ , except for in the case that s=3 where equality holds if and and only if G is  $K_1 + \overline{K_{n-1}}$  or a Moore graph. By the Power Means Inequality,

$$\sum_{v \in V(G)} d(v)^2 \le n^{1-2/t} \left( \sum_{v \in V(G)} d(v)^t \right)^{2/t}.$$
 (7)

Combining (6) and (7) with the inequality  $\sum_{v \in V(G)} d(v)^t \leq t^t \sum_{v \in V(G)} {d(v) \choose t}$  and rearranging, we obtain that  $\mathcal{N}(K_{1,t},G)$  is equal to

$$\sum_{v \in V(G)} {d(v) \choose t} \ge \frac{((n-1)^2(s-2) + (s-2)^2(n-s+2))^{t/2}}{t^t n^{t/2-1}} = \left(\frac{\sqrt{s-2}}{t}\right)^t n^{t/2+1} + O_{s,t}(n^{t/2}).$$

This completes the proof of Proposition 2.1.

**Proposition 2.2** Let  $s \geq 3$  and  $t \geq 3$  be integers. For sufficiently large n,

$$sat(n, K_{1,t}, K_s) \le \frac{c_s^t n^{t/2+1}}{t!}$$

where  $c_s$  is a constant depending only on s.

**Proof.** By a result of Alon, Erdős, Holzman, and Krivelevich, for each  $s \geq 3$  and sufficiently large n, there is a  $K_s$ -saturated graph G with maximum degree  $c_s \sqrt{n}$  (the constant  $c_s$  satisfies  $c_s \to 2s$  as  $s \to \infty$ ). The number of copies of  $K_{1,t}$  in G is then

$$\sum_{v \in V(G)} \binom{d(v)}{t} \le n \binom{\Delta(G)}{t} \le \frac{c_s^t n^{t/2+1}}{t!}.$$

**Proof of Theorem 1.3.** Theorem 1.3 follows immediately from Propositions 2.1 and 2.2. ■

3 Bounds on  $sat(n, K_{2,t}, K_s)$  with  $s \ge 4$  and  $t \ge 3$ 

# 3.1 Upper bound on $sat(n, K_{2,t}, K_s)$

We begin this section with a basic lemma on counting copies of a graph F in a graph G with maximum degree  $\Delta$ . It is likely that this lemma, as well as Lemma 3.2, are known.

**Lemma 3.1** Let F be a connected bipartite graph with parts of size a and b. If G is an n-vertex graph with maximum degree  $\Delta$ , then

$$\mathcal{N}(F,G) \le n\Delta^{a+b-1}$$
.

**Proof.** We will prove the lemma by counting the number of possible embeddings of F in G. Let d be the diameter of F, and x be a vertex in F. For  $0 \le i \le d$ , let  $N_i(x)$  be the set of vertices at distance i from x in F. We count embeddings of F in G by starting with the vertex x, and then proceeding through  $N_1(x)$ , then  $N_2(x)$  and so on. There are n ways to choose a vertex in G that corresponds to x. Suppose that  $v_x$  is the chosen vertex in G. The vertices in G corresponding to those in  $N_1(x)$  must be neighbors of  $v_x$  in G and so there are at most  $\Delta^{|N_1(x)|}$  possibilities. This process is then repeated on  $N_2(x)$ ,  $N_3(x)$ , and so on. The crucial point is that each time a vertex of F is embedded in G, it is a neighbor (in G) of a previously embedded vertex (from F). Therefore, the number of possible embeddings of F in G is at most

$$n\Delta^{|N_1(x)|}\Delta^{|N_2(x)|}\cdots\Delta^{|N_d(x)|}=n\Delta^{a+b-1}.$$

Here we have used the assumption that since F is a connected graph with diameter d, we have the partition

$$\{x\} \cup N_1(x) \cup N_2(x) \cup \cdots \cup N_d(x) = V(F).$$

**Lemma 3.2** Let F be a connected bipartite graph with parts of size a and b. For any n-vertex graph G,

$$\mathcal{N}(K_{a,b},G) \leq \mathcal{N}(F,G).$$

**Proof.** If G has no  $K_{a,b}$ , then the lemma is trivial. Suppose K is a copy of  $K_{a,b}$  in G. Then, since F is a subgraph of  $K_{a,b}$ , we have that F is a subgraph of K so G has a copy of F. Moreover, since any two different copies of  $K_{a,b}$  have different vertex sets, they give rise to different copies of F. Thus, for each copy of  $K_{a,b}$  in G we obtain a copy of F, and no copy of F will be obtained twice in this way. This proves Lemma 3.2.

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** If a > s - 2, then  $K_{s-2} + \overline{K_{n-s+2}}$  is  $K_s$ -saturated with no copies of F. Indeed, a copy of F would need at least a vertices from the  $K_{s-2}$ , but a > s - 2.

Now assume  $a \leq s-2$ . Let  $G_q^s$  be the  $K_s$ -saturated graph constructed in [1] where n (and thus q) is chosen large enough so that  $b < \frac{q+1}{2}$ . There is a constant  $c_s > 0$  such that  $\Delta(G_q^s) \leq c_s \sqrt{n}$ . By Lemma 3.1, the number of copies of F in  $G_q^s$  is at most  $nc_s^{a+b-1}n^{(1/2)(a+b-1)} = c_s^{a+b+1}n^{(1/2)(a+b+1)}$ .

We conclude this subsection by showing that the graph  $G_q^s$  used in the proof of Theorem 1.5 cannot be used to further improve upon the upper bound of  $O(n^{\frac{1}{2}(a+b+1)})$  when  $F = K_{a,b}$ . Since we are showing that  $G_q^s$  cannot be used to improve the upper bound, we will be brief in our argument. We will use the same terminology as in [1], but one point at which we differ is the notation we use for a vertex. A vertex in  $G_q^s$  is determined by its level, place, type, and copy. A vertex at level i, place j, type t, and copy c will be written as

$$((i-1)q+j,t,c).$$

First, take n large enough so that  $b < \frac{q+1}{2}$ . Choose a sequence  $i_1, i_2, \ldots, i_a$  of levels with  $1 \le i_1 < i_2 < \cdots < i_a \le \frac{q+1}{2}$ . Likewise, choose a sequence of b levels  $\frac{q+1}{2} \le i_{a+1} < i_{a+2} < \cdots < i_{a+b} \le q+1$ . This can be done in  $\binom{q+1}{2}\binom{q+1}{2}$  ways. Next, choose a place  $j_1 \in [q]$  which can be done in q ways, and a type  $t_1 \in [s-2]$  which can be done in s-2 ways. Finally, choose a sequence of copies  $1 \le c_1, c_2, \ldots, c_{a+b} \le s-1$  arbitrarily. This can be done in  $(s-1)^{a+b}$  ways. Using the definition of  $G_q^s$ , one finds that the a vertices in the set

$$\{((i_z-1)q+j_1,t_1,c_z):1\leq z\leq a\}$$

are all adjacent to the b vertices in the set

$$\{((i_z-1)q+(j_1+1)_q,(t_1+1)_{s-2},c_z):a+1\leq z\leq a+b\}$$

(here  $(j_1+1)_q$  is the unique integer  $\zeta$  in  $\{1,2,\ldots,q\}$  for which  $j_1+1\equiv \zeta \pmod q$ , and  $(t_1+1)_{s-2}$  is the unique integer  $\zeta'$  in  $\{1,2,\ldots,s-2\}$  for which  $t_1+1\equiv \zeta' \pmod {s-2}$ . This gives a  $K_{a,b}$  in  $G_q^s$  and so the number of  $K_{a,b}$  in  $G_q^s$  is at least

$$\binom{(1/2)(q+1)}{a}\binom{(1/2)(q+1)}{b}q(s-2)(s-1)^{a+b} \ge C_{s,a,b}q^{a+b+1} \ge Cn^{(1/2)(a+b+1)}.$$

By Lemmas 3.2 and 3.1,  $G_q^s$  is a  $K_s$ -saturated n-vertex graph with  $\Theta_{s,a,b}(n^{(1/2)(a+b+1)})$  copies of  $K_{a,b}$ .

## 3.2 Lower bound on $sat(n, K_{2,t}, K_s)$

First we prove Theorem 1.6.

**Proof of Theorem 1.6.** Let G be a  $K_s$ -saturated graph on n vertices. Note that we can assume

$$e(G) \le \frac{n^{\frac{7}{4}}}{10}.\tag{8}$$

Otherwise, a theorem of Erdős and Simonovits [9] implies that there is a positive constant  $\gamma$  such that

$$\mathcal{N}(K_{2,t},G) \ge \gamma \frac{e(G)^{2t}}{n^{3t-2}} = \Omega(n^{\frac{t}{2}+2}),$$
 (9)

proving Theorem 1.6.

Let  $K_4^-$  be the graph consisting of 4 vertices and 5 edges obtained by removing an edge from  $K_4$ . For a copy of  $K_4^-$  with vertices x, y, u, v, where  $uv \notin E(G)$ , let xy be called the *base edge* of this  $K_4^-$ . We estimate the number of copies of  $K_4^-$  in a  $K_s$ -saturated graph G.

For every u, v with  $uv \notin E(G)$  there is a set S such that  $S \subseteq N(u, v)$  and S induces a  $K_{s-2}$  in G. Therefore, there are at least  $\binom{s-2}{2}$  pairs  $x, y \in S$  such that u, v, x, y form a copy of  $K_4^-$ . On the other hand, every  $xy \in E(G)$  is the base edge of at most  $\binom{d(x,y)}{2}$  copies of  $K_4^-$  in G. Therefore,

$$\sum_{xy \in E(G)} {d(x,y) \choose 2} \ge \mathcal{N}(K_4^-, G) \ge \sum_{uv \in E(\overline{G})} {s-2 \choose 2} \ge e(\overline{G}) \stackrel{(8)}{\ge} \frac{n^2}{4}.$$

Thus, there is a constant  $c_t = c(t)$  such that the following holds:

$$\mathcal{N}(K_{2,t},G) \ge \sum_{xy \in E(G)} \binom{d(x,y)}{t} \ge \frac{1}{t^t} \sum_{xy \in E(G)} \left( \binom{d(x,y)}{2}^{\frac{t}{2}} - t^t \right) \ge$$

$$\frac{e(G)}{t^t} \left( \frac{\sum_{xy \in E(G)} \binom{d(x,y)}{2}}{e(G)} \right)^{\frac{t}{2}} - e(G) \ge \frac{(n^2/4)^{\frac{t}{2}}}{t^t e(G)^{\frac{t}{2}-1}} - e(G) = \frac{(n^2/4)^{\frac{t}{2}} - t^t e(G)^{t/2}}{t^t e(G)^{\frac{t}{2}-1}} \stackrel{(8)}{\ge} \frac{c_t n^t}{e(G)^{\frac{t}{2}-1}}.$$

Combining this with (9) we get

$$\mathcal{N}(K_{2,t},G) \ge \min\{\gamma \frac{e(G)^{2t}}{n^{3t-2}}, \frac{c_t n^t}{e(G)^{\frac{t}{2}-1}}\}.$$

Let  $e(G) = n^{\alpha}$ , then

$$\mathcal{N}(K_{2,t},G) \ge \min\left\{\gamma n^{2\alpha t - 3t + 2}, c_t n^{t - \alpha t/2 + \alpha}\right\}.$$

Choosing  $\alpha = \frac{8t-4}{5t-2}$  and  $C = \min\{\gamma, c_t\}$ , we get the desired lower bound  $Cn^{\frac{t}{5} - \frac{16}{125t-10} + \frac{41}{25}} > Cn^{\frac{t}{5} + \frac{8}{5}}$ .

Next we turn to the proof of Theorem 1.7. We need the following lemma.

**Lemma 3.3** Let  $s \ge 4$  and  $2 \le a \le b$  be integers with  $a \le s - 2$ . Suppose that G is an n-vertex  $K_s$ -saturated graph with vertex set V. There is a constant c = c(s, a, b) such that for any  $v \in V$ , there are at least

$$c\left(\frac{n-d(v)-1}{d(v)^{a-1}}\right)^{b/2}$$

copies of  $K_{a,b}$  containing v.

**Proof.** Let  $v \in V$ . For each  $u \in V \setminus N[v]$ , there is a set  $S_u \subset N(v)$  such that  $S_u$  induces a  $K_{s-2}$  in G. Fix such an  $S_u$  and define an (s-1)-uniform hypergraph  $\mathcal{H}$  to have vertex set  $V \setminus \{v\}$ , and edge set  $E(\mathcal{H}) = \{\{u\} \cup S_u : u \in V \setminus N[v]\}$ . By construction,  $\mathcal{H}$  has n - d(v) - 1 edges, each of which contains exactly one vertex from  $V \setminus N[v]$  and s-2 vertices from N(v). Also, no two edges of  $\mathcal{H}$  contain the same vertex from  $V \setminus N[v]$ . In what follows, we will add the subscript  $\mathcal{H}$  if we are referring to degrees in  $\mathcal{H}$ , and no subscript will be included if we are referring to degrees or neighborhoods in G.

By averaging, there is a set  $X \in \binom{N(v)}{a-1}$  such that

$$d_{\mathcal{H}}(X) \ge \frac{\binom{s-2}{a-1}(n-d(v)-1)}{\binom{d(v)}{a-1}}.$$

We then have

$$\sum_{y \in N(v,X)} d_{\mathcal{H}}(y,X) \ge \frac{d_{\mathcal{H}}(X)}{(s-2) - |X|} \ge c_1 \frac{n - d(v) - 1}{d(v)^{a-1}}$$
(10)

for some constant  $c_1 = c_1(s, a) > 0$ . The number of  $K_{a,b}$  with  $X \cup \{y\}$  forming the part of size a (y is an arbitrary vertex from N(v, X)) and v in the part of size b is at least

$$\sum_{y \in N(v,X)} {d_{\mathcal{H}}(y,X) \choose b-1} \ge d(v,X) {c_1(n-d(v)-1) \choose d(v,X)d(v)^{a-1} \choose b-1} \ge \frac{c_2(n-d(v)-1)^{b-1}}{d(v,X)^{b-2}d(v)^{(a-1)(b-1)}}.$$

Here we have used convexity, (10), and  $c_2 = c_2(s, a, b)$  is some positive constant.

Recalling that |X| = a - 1, there are  $\binom{d(v,X)}{b}$  copies of  $K_{a,b}$  where  $\{v\} \cup X$  is the part of size a and the part of size b is contained in  $N(v) \setminus X$ . Thus, for some constant  $c_3 = c_3(s,a,b) > 0$ , the number of  $K_{a,b}$  that contain v is at least

$$\frac{c_3(n-d(v)-1)^{b-1}}{d(v,X)^{b-2}d(v)^{(a-1)(b-1)}} + c_3d(v,X)^b.$$

By considering cases as to which is this the bigger term in this sum, we find that in both cases, there are at least

$$c_3 \left(\frac{n - d(v) - 1}{d(v)^{a - 1}}\right)^{b/2}$$

copies of  $K_{a,b}$  containing v.

Applying Lemma 3.3 to a vertex v with  $d(v) = \delta(G)$  proves Theorem 1.7.

# 4 Concluding Remarks

An interesting open problem is determining the minimum number of copies of  $K_{1,2}$  in a  $K_3$ -saturated graph. There is a connection between this problem and Moore graphs with diameter 2 and girth 5. It is easy to check that an n-vertex Moore graph with diameter 2 and girth 5 is  $K_3$ -saturated, and it is regular with degree  $d = \sqrt{n-1}$  [21] so it contains  $n\binom{d}{2} = n\binom{\sqrt{n-1}}{2}$  copies of  $K_{1,2}$ , and for all  $n \geq 3$ , this value is less than  $\binom{n-1}{2}$  which is the number of copies of  $K_{1,2}$  in  $K_1 + \overline{K_{n-1}} = K_{1,n-1}$ . Furthermore, one can duplicate vertices of a Moore graph and preserve the  $K_3$ -saturated property (where each duplicated vertex has the same neighborhood as the original vertex). Duplicating a vertex of the Petersen graph will lead to an 11-vertex  $K_3$ -saturated graph with 42 copies of  $K_{1,2}$ , but  $K_{1,10}$  has 45 copies of  $K_{1,2}$ . Starting from the Hoffman-Singleton graph, one can duplicate a vertex up to 4 times and we can still have fewer copies of  $K_{1,2}$  compared to the number of copies of  $K_{1,2}$  in  $K_{1,n-1}$ . Duplicating a single vertex is not necessarily the optimal way to minimize the number of copies of  $K_{1,2}$ , but the point is that there are other graphs besides the Moore graphs that have fewer copies of  $K_{1,2}$  than the number of copies of  $K_{1,2}$  in  $K_{1,n-1}$ .

It would also be interesting to determine the order of magnitude of  $sat(n, K_{2,t}, K_s)$ . There is a gap in the exponents (which is discussed in the introduction) and it would be nice to close this gap. It is not clear if our lower or upper bound is closer to the correct answer.

Another potential approach to studying  $\operatorname{sat}(n, H, F)$  is via the random F-free process. This random process orders the pairs of vertices uniformly and then adds them one by one subject to the condition that adding an edge does not create a copy of F. The resulting graph is then F-saturated. This process was first considered in [4, 11, 20, 22] and has since been studied extensively. If  $X_{H,F}$  is the random variable that counts the number of copies of H in the output of this process, then we have that  $\operatorname{sat}(n, H, F) \leq \mathbb{E}(X_{H,F})$ . It would be interesting to determine for which graphs H and F that this approach gives better bounds than the explicit constructions that are currently known.

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