over $\mathbb{F}_q$. Let $\mathscr{C}_g$ be the affine plane curve defined by this equation. Note that $\mathscr{C}_g$ is non-singular and irreducible over $\overline{\mathbb{F}_q}$, an algebraic closure of $\mathbb{F}_q$. Denote the set of $\mathbb{F}_q$-rational points of $\mathscr{C}_g$ by

$$\mathscr{C}_g(\mathbb{F}_q) = \left\{(x,y) \in \mathbb{F}_q^2 \mid g^2x^\ell + gy^\ell + 1 = 0\right\}$$

and let $N_g = |\mathscr{C}_g(\mathbb{F}_q)|$ be its cardinality. Furthermore, let $\widetilde{\mathscr{C}_g}$ be the (Zariski) closure of $\mathscr{C}_g$ in the projective plane defined by the homogeneous equation

$$g^2X^\ell + gY^\ell + Z^\ell = 0. \tag{4}$$

Note that $\widetilde{\mathscr{C}_g}$ is also non-singular. We let $\widetilde{N_g}$ denote the cardinality of $\widetilde{\mathscr{C}_g}(\mathbb{F}_q)$. Having Conjecture B and Conjecture C in mind, we are especially concerned with whether or not a point $(x,y) \in \mathscr{C}_g(\mathbb{F}_q)$ satisfying $xy \neq 0$. The following lemma shows that this is always true except for very limited special cases.

Lemma 2.1. *Equation (4) has a nontrivial solution (x,y,z) with $xyz = 0$ if and only if one of the following situations holds*:

(i) $\ell = 1$ *or* 2;
(ii) $\ell = 4$ *and* $-1 \notin L$.

Moreover, if $\ell > 2$, then $xz \neq 0$.

Proof. Suppose that (x,y,z) is a nontrivial solution to Equation (4) with $xyz = 0$. Then only one of x,y,z is zero. Observe that if $x = 0$ or $z = 0$, then $-g \in L$ and gL is either of order 1 or 2 in $\mathbb{F}_q^\times/L$; if $y = 0$, then $-g^2 \in L$ and gL is of order 4 in $\mathbb{F}_q^\times/L$. In particular, we have $xz \neq 0$ provided that $\ell \neq 2$. In the case where $\ell = 4$, we see that $-L = g^2L \neq L$. It follows that $-1 \notin L$.

Conversely, if $\ell = 1$ then it's clear that Equation (4) has a nontrivial solution (x,y,z) with $xyz = 0$. Suppose that $\ell = 2$, then $\mathbb{F}_q^\times/L = \{L, gL\}$. If $-1 \notin L$, then $-L = gL$. In this case, $g = -a^2 \in L$ for some $a \in \mathbb{F}_q^\times$. Then, we clearly have solutions $(x,y,z) = (1,a,0)$ and $(0,1,a)$. Suppose $-1 \in L$, then $-g^2 = b^2 \in L$ for some $b \in \mathbb{F}_q^\times$ and we have the solution $(x,y,z) = (1,0,b)$ in this case.

Finally, suppose that $\ell = 4$ and $-1 \notin L$. Then both g^2L and $-L$ are of order 2 in the cyclic group $\mathbb{F}_q^\times/L$. Thus, $-L = g^2L$ and this gives a solution $(x,y,z) = (1,0,b)$ where $-g^2 = b^4 \in L$. □

Following [Wei48], the number N_g of solutions to Equation (3) can be expressed as a character sum which we now recall. As usual, by a multiplicative character of $\mathbb{F}_q$ we mean a character of the group $\mathbb{F}_q^\times$, i.e. a group homomorphism from $\mathbb{F}_q^\times$ to $\mathbb{C}^\times$. As we only deal with multiplicative characters of $\mathbb{F}_q$, we'll simply call them characters. The trivial character will be denoted by ε such that $\varepsilon(a) = 1$ for all $a \in \mathbb{F}_q^\times$. We extend the domain of a character χ such that $\chi(0) = 1$ if $\chi = \varepsilon$ and $\chi(0) = 0$ otherwise. We call the extension of χ an extended character and still denote the extension by χ if there is no