Supplementary Material to "Comparison of Bayesian and Frequentist Multiplicity Correction For Testing Mutually Exclusive Hypotheses Under Data Dependence"

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Appendix

Normal Theory

Lemma 1.

$$\boldsymbol{X} \sim multinorm\left(\begin{pmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{pmatrix}, \begin{pmatrix}1 & \rho & \cdots & \rho\\ \rho & 1 & \cdots & \rho\\ \vdots & \vdots & \ddots & \vdots\\ \rho & \rho & \cdots & 1\end{pmatrix}\right)$$

is equivalent to

$$X_i = \theta_i + \sqrt{\rho} Z + \sqrt{1-\rho} Z_i \;\forall\, i \in \{1, 2, ..., n\}\,, \tag{1}$$

where $Z, Z_1, ..., Z_n \sim iid\, N(0,1)$. *Furthermore, if* $\theta_j = 0 \;\; \forall j$, *then, as* $n \to \infty$,

$$\begin{cases} \frac{\bar{\boldsymbol{x}}}{\sqrt{\rho}} = z + O\left(\frac{1}{\sqrt{n}}\right) \\ \frac{x_i - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}} = z_i + O\left(\frac{1}{\sqrt{n}}\right) \end{cases} . \tag{2}$$

Proof. It is straightforward to show that the expectation and covariance of (1) are as desired. (2) follows from the definitions and the central limit theorem.

□

Corollary 1.

- *When* $\rho = 0$, $\Phi(c) = 1 + \frac{\log(1-\alpha)}{2n} + O(1/n^2)$, *essentially calling for the Bonferroni correction.*
- *When* $\rho \to 1$, $\Phi(c) \to 1 - \frac{\alpha}{2}$, *so the critical region is the same as that for a single test.*

Proof. If $\rho = 0$, $\alpha = 1 - (\Phi(c) - \Phi(-c))^n = 1 - (2\Phi(c) - 1)^n$, from which it follow that

$$\Phi(c) = \frac{1 + (1-\alpha)^{1/n}}{2} = \frac{1 + 1 + \frac{\log(1-\alpha)}{n} + O(1/n^2)}{2} .$$

If $\rho \to 1$, by Lemma 1,

$$\begin{aligned} &\lim_{\rho \to 1} P\left(\max_{1 \le j \le n} |X_j| > c \;\middle|\; \theta_i = 0 \;\forall i\right) \\ &= 1 - \lim_{\rho \to 1} \mathbb{E}^{Z_1, \ldots, Z_n}\left\{P\left(|\sqrt{\rho} Z + \sqrt{1-\rho} Z_j| < c \mid Z_1, .., Z_n\right)\right\} \\ &= 1 - \mathbb{E}^{Z_1, \ldots, Z_n}\left\{\lim_{\rho \to 1} P\left(|\sqrt{\rho} Z + \sqrt{1-\rho} Z_j| < c \mid Z_1, .., Z_n\right)\right\} \\ &= 1 - (\Phi(c) - \Phi(-c)) \\ &= 2(1 - \Phi(c))\,. \end{aligned}$$

□

Fact 1 (Normal tail probability)**.** *Letting $\Phi(t)$ denote the cumulative distribution function of the standard normal distribution,*

$$\frac{t\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t^2+1} \leq\ 1-\Phi(t) \leq \frac{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t}$$

$$1-\Phi(t) = \frac{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t} + O\left(\frac{e^{-t^2/2}}{t^3}\right)$$

The proof can be found in [1].

By expanding a, b in Theorem 3.1, one obtains the following explicit form for the posterior probabilities:

Corollary 2. *The posterior of any non-null model M_i is:*

$$P(M_i \mid \boldsymbol{x}) = \left\{\begin{array}{l} \left(\sqrt{\frac{(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho}{(1+(n-1)\rho)(1-\rho)}}\left(\frac{n\,r}{1-r}\right)*\right. \\ \exp\left\{\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\left((x_i-\bar{\boldsymbol{x}})+\frac{(1-\rho)\bar{\boldsymbol{x}}}{1+(n-1)\rho}\right)^2\right\}+ \\ \sum_{k=1}^{n}\exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\right. \\ \qquad \left.\left((x_i+x_k-2\bar{\boldsymbol{x}})(x_i-x_k)+2\frac{\bar{\boldsymbol{x}}(x_i-x_k)(1-\rho)}{1+(n-1)\rho}\right)\right] \end{array}\right\}^{-1}. \tag{3}$$

Alternatively, in terms of $z(\boldsymbol{x})$, with $z_i = z_i(\boldsymbol{x})$:

$$P(M_i \mid \boldsymbol{x}) = \left\{\begin{array}{l} \left(\sqrt{\frac{(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho}{(1+(n-1)\rho)(1-\rho)}}\left(\frac{n\,r}{1-r}\right)*\right. \\ \exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\right. \\ \qquad \left.\left(\theta_i-\bar{\boldsymbol{\theta}}+\sqrt{1-\rho}(z_i-\bar{\boldsymbol{z}})+\frac{1-\rho}{1+(n-1)\rho}(\bar{\boldsymbol{\theta}}+\sqrt{\rho}z+\sqrt{1-\rho}\bar{\boldsymbol{z}})\right)^2\right] \\ +\sum_{k=1}^{n}\exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho]}\right. \\ \qquad \left(\frac{[\theta_i+\theta_k-2\bar{\boldsymbol{\theta}}+\sqrt{1-\rho}(z_i+z_k-2\bar{\boldsymbol{z}})][\theta_i-\theta_k+\sqrt{1-\rho}(z_i-z_k)]}{1-\rho}+\right. \\ \qquad\qquad \left.\left.2\frac{[\bar{\boldsymbol{\theta}}+\sqrt{\rho}z+\sqrt{1-\rho}\bar{\boldsymbol{z}}][\theta_i-\theta_k+\sqrt{1-\rho}(z_i-z_k)]}{1+(n-1)\rho}\right)\right] \end{array}\right\}^{-1}. \tag{4}$$

Lemma 2. *If Z_i, $i \in \{1,2,...,n\}$, are i.i.d. standard normal random variables, then*

$$|Z_i| \leq n^{1/2-\epsilon} \quad \forall\ i \qquad \textit{holds almost surely.}$$

Proof. By Fact 1:

$$\begin{aligned} & P(\text{ for all i}, |Z_i| \leq n^{1/2-\epsilon}) \\ = & \left(1-P(|Z_1| \geq n^{1/2-\epsilon})\right)^n \\ = & \left(1-2\frac{\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}n^{1-2\epsilon}\}}{n^{1/2-\epsilon}}+O\left(\exp\left\{\frac{-n^{1-2\epsilon}}{2}\right\}\right)\right)^n \\ = & \left(1-\frac{\frac{2n^{1/2-\epsilon}}{\sqrt{2\pi}}\exp\{-\frac{1}{2}n^{1-2\epsilon}\}}{n}+o(n^{-2})\right)^n \\ = & 1-O\left(2n^{1/2+\epsilon}\exp\{-\frac{1}{2}n^{1-2\epsilon}\}\right) \\ = & 1+o(1)\,. \end{aligned}$$

$\square$

Adaptive Choice of τ^2

Lemma 3.

$$\begin{aligned}&\arg\max_{\tau^2}\left[\left(1+\frac{1-\rho}{\tau^2}\right)\log\left(n\,\frac{p}{(1-p)(1-r)}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\right)+o(1)\right]\\&=(1-\rho)\left(2\log n+\log\log n+2\log\frac{p}{(1-p)(1-r)}+\log 2+o(1)\right).\end{aligned} \tag{5}$$

Proof. Letting $x=\frac{1-\rho}{\tau^2}$ and $c'=\frac{p}{(1-p)(1-r)}$, the expression in square brackets in (5) can be written

$$f(x)=(1+x)\big(\log(n\,c')+1/2\,log\,(1+1/x)\big)\,.$$

Clearly

$$f'(x)=\frac{1}{2}\big(2\log(n\,c')+\,\log(1+1/x)-1/x\big)\,,$$

so that, $f'(x)=0$ when $1/x=2\log\,n+\log\log n+2\log c'+\log 2+o(1)$, or

$$\tau^2=(1-\rho)(2\log\,n+\log\log n+2\log c'+\log 2)+o(1)\,.$$

. □

Fact 2 (Weak law for triangular arrays (WLTA))**.** *For each n, let $X_{n,i}$, $1\le k\le n$ be independent. Let $\beta_n>0$ with $\beta_n\to\infty$ and let $\bar{\boldsymbol{x}}_{n,k}=X_{n,k}1_{\{|X_{n,k}|\le\beta_n\}}$. Suppose that as $n\to\infty$: $\sum_{k=1}^{n}P(|X_{n,k}|>\beta_n)\to 0$ and $1/\beta_n^2\sum_{k=1}^{n}E\bar{X}_{n,k}^2\to 0$. then*

$$\frac{(S_n-\alpha_n)}{\beta_n}\to 0 \textit{ in probability}$$

$$\textit{where } S_n=X_{n,1}+...+X_{n,n} \textit{ and } \alpha_n=\sum_{k=1}^{n}E\bar{X}_{n,k}.$$

See [1] for the proof.

Theorem 3. *If $c_n\in(0,1)\,\forall n$ and $1-c_n=o(1)$, then*

$$\lim_{n\to\infty}\frac{1}{n}\sqrt{1-c_n}\sum_{i=1}^{n}\exp\left\{\frac{c_n}{2}z_i^2\right\}=\lim_{n\to\infty}2\Phi\left(\sqrt{\frac{2(1-c_n)}{c_n}\log\frac{n}{\sqrt{1-c_n}}}\right)-1$$

in probability.

Proof. Take $X_{n,i}=exp\left\{\frac{c_n}{2}z_i^2\right\}$; $\beta_n=\frac{n}{\sqrt{1-c_n}}$ in Fact 2.

Checking the first assumption of the WLTA:

$$\begin{aligned}P(|X_{n,i}|>\beta_n)&=P\left(|z_i|>\sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}\right)\\&=2\frac{\frac{1}{2\pi}exp\left\{\frac{-1}{2}\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}\right\}}{\sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}}+O\left(\frac{(\frac{n}{\sqrt{1-c_n}})^{-\frac{1}{c_n}}}{(\frac{1}{c_n}log\,\frac{n}{\sqrt{1-c_n}})^3}\right)\\&=\frac{1}{\sqrt{\pi}}\frac{\sqrt{1-c_n}^{\frac{1}{c_n}}}{n^{\frac{1}{c_n}}}\frac{1}{\sqrt{log\,\frac{n}{\sqrt{1-c_n}}}}(1+o(1))\\&<\frac{1}{\sqrt{\pi}}\frac{\sqrt{1-c_n}^{\frac{1}{c_n}}}{n^{\frac{1}{c_n}}}\frac{1}{\sqrt{log\,n}}(1+o(1))\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n} P(|X_{n,k}| > \beta_n) &= nP(|X_{n,k}| > \beta_n)\\ &< n^{1-\frac{1}{c_n}}(1-c_n)^{\frac{1}{2c_n}}\frac{1}{\sqrt{log\, n}}\\ &= n^{-\frac{1-c_n}{c_n}}(1-c_n)^{\frac{1}{2c_n}}\frac{1}{\sqrt{log\, n}} \to 0\,.\end{aligned}$$

Checking the second assumption of the WLTA:
Since $\lim_{n\to\infty} c_n \to 1$, without loss of generality, assume $c_n > 3/4$. Then

$$\begin{aligned}\frac{1}{\beta_n^2}\sum_{k=1}^{n} E\bar{X}_{n,k}^2 &= \frac{1-c_n}{n^2}n\int\limits_{|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} \exp\left\{c_n z^2\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-1}{2}z^2\right\}dz\\
&= \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\int\limits_{1<|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}}\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz + \int\limits_{|z|<1}\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz\right\}\\
&\le \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\int\limits_{1<|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} z\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz + d\right\}\\
&= \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\frac{1}{2(c_n-\frac{1}{2})}\exp\left\{(c_n-\frac{1}{2})(\frac{2}{c_n})\,\log\frac{n}{\sqrt{1-c_n}}\right\} + d'\right\}\\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2c_n-1}\right)\frac{1-c_n}{n}\left(\frac{n}{\sqrt{1-c_n}}\right)^{2-\frac{1}{c_n}} + o(1)\\
&= \frac{1}{\sqrt{2\pi}}\underbrace{\left(\frac{1}{2c_n-1}\right)}_{\le 2} n^{1-\frac{1}{c_n}}(1-c_n)^{\frac{1}{2c_n}} + o(1)\\
&\le \frac{2}{\sqrt{2\pi}}n^{-\frac{1-c_n}{c_n}}(1-c_n)^{\frac{1}{2c_n}} + o(1) = o(1)\,.\end{aligned}$$

Noting that

$$\frac{\sqrt{1-c_n}}{n}\alpha_n = \frac{1-c_n}{n}\sum_{i=1}^{n} E\bar{X}_{n,i}$$

$$= (1-c_n)\int\limits_{|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} e^{\frac{c_n z^2}{2}}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz = 2\left(\Phi\left(\frac{\sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}}{\sqrt{(1-c_n)^{-1}}}\right)-\frac{1}{2}\right),$$

the WLTA yields

$$\frac{S_n-\alpha_n}{\beta_n} = \frac{\sum_{i=1}^{n} e^{c_n z_i^2}-\alpha_n}{\frac{n}{\sqrt{1-c_n}}} = \frac{\sqrt{1-c_n}\sum_{i=1}^{n} e^{c_n z_i^2}}{n} - \frac{\sqrt{1-c_n}}{n}\alpha_n \to 0\,.$$

in probability, and the result follows. □

Corollary 3. *Letting* $c_n = \frac{\hat{\tau}_n^2}{1-\rho+\hat{\tau}_n^2}$,

$$\frac{1}{n}\sqrt{1-c_n}\sum_{i=1}^{n}\exp\left\{\frac{c_n}{2}z_i^2\right\} \to \begin{cases} 1 & if\ \frac{log\, n}{\hat{\tau}_n^2} \to \infty\,,\\ 2\Phi\left(\sqrt{\frac{2}{k}}\right)-1 & if\ \frac{log\, n}{\hat{\tau}_n^2} \to \frac{1}{(1-\rho)k}\,,\\ 0 & if\ \frac{log\, n}{\hat{\tau}_n^2} \to 0\end{cases}$$

in probability.

Proof. By Theorem 3:

Case I: $\frac{log\, n}{\hat{\tau}_n^2} \to \infty$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 1\,.$$

Case II: $\frac{log\, n}{\hat{\tau}_n^2} \to \frac{1}{(1-\rho)k}$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 2\Phi\left(\sqrt{\frac{2}{k}}\right) - 1\,.$$

Case III: $\frac{log\, n}{\hat{\tau}_n^2} \to 0$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 0\,.$$

□

Lemma 4.

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n\sqrt{1+\tau_n^2 a}}\sum_{i=1}^{n}\exp\left\{\frac{\tau_n^2}{2(1+\tau_n^2 a)}\big(\frac{x_i}{1-\rho}+bn\bar{\boldsymbol{x}}\big)^2\right\}\\ =&\frac{1}{n}\sqrt{\frac{1-\rho}{1-\rho+\tau_n^2}}\sum_{i=1}^{n}\exp\left\{\frac{\tau_n^2 z_i^2}{2(1-\rho+\tau_n^2)}\right\}(1+o(1)) \quad a.s.\end{aligned} \tag{6}$$

Proof. Expanding the coefficients yields

$$\begin{aligned}\frac{1}{1+\tau_n^2 a} &= \left(1+\frac{\tau_n^2(1+(n-2)\rho)}{(1+(n-1)\rho)(1-\rho)}\right)^{-1}\\ &= \frac{1-\rho}{1-\rho+\tau_n^2\big(1+\frac{-\rho}{1+(n-1)\rho}\big)} = \frac{1-\rho}{1-\rho+\tau_n^2}(1+O(1/n))\,,\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{x_i}{1-\rho}+bn\bar{\boldsymbol{x}}\right)^2 = \frac{1}{(1-\rho)^2}\left(x_i+\frac{-\rho n\bar{\boldsymbol{x}}}{1+(n-1)\rho}\right)^2\\ &= \frac{1}{(1-\rho)^2}\left(x_i-\bar{\boldsymbol{x}}\big(1-\frac{1-\rho}{1-\rho+\rho n}\big)\right)^2\\ &= \frac{1}{(1-\rho)^2}\left(\sqrt{1-\rho}z_i+\sqrt{\rho}z\underbrace{\big(\frac{1-\rho}{1-\rho+\rho n}\big)}_{O(1/n)}+\sqrt{1-\rho}\underbrace{\bar{\boldsymbol{z}}}_{O(1/\sqrt{n})}\big(-1+\frac{1-\rho}{1-\rho+\rho n}\big)\right)^2\\ &= \frac{z_i^2}{1-\rho}+O\big((\log\, n)/\sqrt{n}\big)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{\sqrt{1+\tau_n^2 a}}1/n\sum_i\exp\left\{\frac{\tau_n^2}{2(1+\tau_n^2 a)}\big(\frac{x_i}{1-\rho}+bn\bar{\boldsymbol{x}}\big)^2\right\}\\ &=\sqrt{\frac{1-\rho}{1-\rho+\tau_n^2+o(1)}}1/n\sum_i\exp\left\{\frac{\tau_n^2}{2}\left[\frac{z_i^2}{1-\rho+\tau_n^2}+o(1)\right]\right\}\,.\end{aligned}$$

□

Lemma 5. *Under the null model, suppose*

$$\max_j \left(\frac{x_j - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}}\right)^2 = 2\log(n) + \log\log(n) + c\,.$$

Then

$$L_n(\tau^2) = \frac{1}{n}\sqrt{\frac{1-\rho}{1-\rho+\tau^2}}\sum_{i=1}^{n}\exp\left\{\frac{\tau^2 z_i^2}{2(1-\rho+\tau^2)}\right\}$$

is maximized at

$$\hat{\tau}_n^2 = (1-\rho)k(c)(\log\, n)(1+o(1))\,,$$

where

$$k(c) = (1 + 2/\sqrt{\pi}\exp\{-c/2\})^{-1}\,.$$

Proof. Without loss of generality, let $\max|z_i| = |z_1|$.

$$L_n(\hat{\tau}_n^2) = \Bigg(\underbrace{\sqrt{\frac{1-\rho}{1-\rho+\hat{\tau}_n^2}}1/n\exp\left\{\frac{\hat{\tau}_n^2 z_1^2}{2(1-\rho+\hat{\tau}_n^2)}\Bigg)\right\}}_{I} + \underbrace{\sqrt{\frac{1-\rho}{1-\rho+\hat{\tau}_n^2}}1/n\sum_{i=2}^{n}\exp\left\{\frac{\hat{\tau}_n^2 z_i^2}{2(1-\rho+\hat{\tau}_n^2)}\right\}}_{II}\Bigg)(1+o(1))\,.$$

First, note that $L_n(\hat{\tau}_n^2) \to 0$ when $\log n/\hat{\tau}_n^2 \to \infty$, since

$$\begin{aligned} I &= \frac{1}{\sqrt{\hat{\tau}_n^2}} n^{\frac{-(1-\rho)}{1-\rho+\hat{\tau}_n^2}} \sqrt{\log n}^{\frac{(\hat{\tau}_n^2)}{1-\rho+\hat{\tau}_n^2}} e^{\frac{c\hat{\tau}_n^2}{2(1-\rho+\hat{\tau}_n^2)}}(1+o(1)) \\ &= n^{\frac{-(1-\rho)}{1-\rho+\hat{\tau}_n^2}}\sqrt{\log n/\hat{\tau}_n^2}e^{c/2}(1+o(1)) = o(1)\,, \\ II &\to 0 \text{ by Corollary 3.} \end{aligned}$$

Similarly, one can show that $L_n(\hat{\tau}_n^2) \to 1$ when $\log n/\hat{\tau}_n^2 \to 0$, since

$$\begin{aligned} I &= n^{\frac{-(1-\rho)}{1-\rho+\hat{\tau}_n^2}}\sqrt{\log n/\hat{\tau}_n^2}e^{c/2}(1+o(1)) = o(1) \\ II &\to 1 \text{ by Corollary 3.} \end{aligned}$$

For the case in which $\log n/\hat{\tau}_n^2 \to k$, using Corollary 3, it follows that

$$L_n(\hat{\tau}_n^2) = [ve^{(\frac{c}{2}-v^2)} + 2\Phi(\sqrt{2}v) - 1](1+o(1)),$$

where $v = \sqrt{(1-\rho)/k}$. Differentiating $f(v) = [ve^{(\frac{c}{2}-v^2)} + 2\Phi(\sqrt{2}v)]$ and setting the derivative to 0, yields the solution $\hat{v} = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{\pi}}e^{-c/2}}$, which translates into $k(c)$ as in the statement of the lemma. It is straightforward to show that this extrema of $f(v)$ is the maximum, and

$$f(\hat{v}) > \max\{\lim_{v\to 0} f(v), \lim_{v\to\infty} f(v)\} = 1\,.$$

As this maximum thus exceeds the maximum over the domains $\log n/\hat{\tau}_n^2 \to \infty$ and $\log n/\hat{\tau}_n^2 \to 0$, the proof is complete. ∎

Lemma 6. *For the $k(c)$ defined above,*

$$\log(k(c)/2) + 2/k(c) - 1 > 0\,\forall\, c > 0$$

Proof. Note that $x = k/2 < 1$, so that we want to show that $f(x) = \log(x) + 1/x - 1 > 0$ over this region. Since $f'(x) = 1/x - 1/x^2 < 0$ over this region, $f(x)$ is minimized at $x = 1$, proving the result. $\square$

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[1] Durrett, R. (2010), *Probability: theory and examples*, vol. 3, Cambridge university press.