Supplementary Material to "Comparison of Bayesian and Frequentist Multiplicity Correction For Testing Mutually Exclusive Hypotheses Under Data Dependence"

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(Dated: October 28, 2019)

Gaussian distribution properties

Lemma S.1.

$$\boldsymbol{X} \sim multinorm\left(\begin{pmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{pmatrix}, \begin{pmatrix}1 & \rho & \cdots & \rho\\ \rho & 1 & \cdots & \rho\\ \vdots & \vdots & \ddots & \vdots\\ \rho & \rho & \cdots & 1\end{pmatrix}\right)$$

is equivalent to

$$X_i = \theta_i + \sqrt{\rho}Z + \sqrt{1-\rho}Z_i \ \forall\, i \in \{1, 2, ..., n\}\,, \tag{S.2}$$

where $Z, Z_1, ..., Z_n \sim iid\, N(0,1)$. *Furthermore, if* $\theta_j = 0\ \ \forall j$, *then, as* $n \to \infty$,

$$\begin{cases} \frac{\bar{\boldsymbol{x}}}{\sqrt{\rho}} = z + O\left(\frac{1}{\sqrt{n}}\right) \\ \frac{x_i - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}} = z_i + O\left(\frac{1}{\sqrt{n}}\right) \end{cases}. \tag{S.3}$$

Proof. It is straightforward to show that the expectation and covariance of (S.2) are as desired. (S.3) follows from the definitions and the central limit theorem. ☐

Fact S.4 (Normal tail probability)**.** *Letting* $\Phi(t)$ *denote the cumulative distribution function of the standard normal distribution,*

$$\frac{t\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t^2+1} \leq \ 1 - \Phi(t) \leq \frac{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t}$$

$$1 - \Phi(t) = \frac{\frac{1}{\sqrt{2\pi}}e^{-t^2/2}}{t} + O\left(\frac{e^{-t^2/2}}{t^3}\right)$$

The proof can be found in [1].

Fact S.5 (Weak law for triangular arrays (WLTA))**.** *For each* n, *let* $X_{n,i}$, $1 \leq k \leq n$ *be independent. Let* $\beta_n > 0$ *with* $\beta_n \to \infty$ *and let* $\bar{\boldsymbol{x}}_{n,k} = X_{n,k}1_{\{|X_{n,k}| \leq \beta_n\}}$. *Suppose that as* $n \to \infty$: $\sum\limits_{k=1}^{n} P(|X_{n,k}| > \beta_n) \to 0$ *and* $1/\beta_n^2 \sum\limits_{k=1}^{n} E\bar{X}_{n,k}^2 \to 0\,.$ *then*

$$\frac{(S_n - \alpha_n)}{\beta_n} \to 0 \ in\ probability$$

$$where\ S_n = X_{n,1} + ... + X_{n,n}\ \ and\ \alpha_n = \sum_{k=1}^{n} E\bar{X}_{n,k}.$$

See [1] for the proof.

An Ad hoc Procedure

Proof of Corollary ?? . If $\rho = 0$, $\alpha = 1 - (\Phi(c) - \Phi(-c))^n = 1 - (2\Phi(c) - 1)^n$, from which it follow that

$$\Phi(c) = \frac{1 + (1-\alpha)^{1/n}}{2} = \frac{1 + 1 + \frac{\log(1-\alpha)}{n} + O(1/n^2)}{2}\,.$$

If $\rho \to 1$, by Lemma S.1,

$$\begin{aligned}
&\lim_{\rho \to 1} P\left(\max_{1 \le j \le n} |X_j| > c \;\middle|\; \theta_i = 0 \;\forall i\right) \\
&= 1 - \lim_{\rho \to 1} \mathbb{E}^{Z_1,\ldots,Z_n}\left\{P\left(|\sqrt{\rho}Z + \sqrt{1-\rho}Z_j| < c \mid Z_1,..,Z_n\right)\right\} \\
&= 1 - \mathbb{E}^{Z_1,\ldots,Z_n}\left\{\lim_{\rho \to 1} P\left(|\sqrt{\rho}Z + \sqrt{1-\rho}Z_j| < c \mid Z_1,..,Z_n\right)\right\} \\
&= 1 - (\Phi(c) - \Phi(-c)) \\
&= 2(1 - \Phi(c))\,.
\end{aligned}$$

□

Likelihood Ratio Test

Proof of Theorem ??. Denote

$$\Sigma_0 = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}$$

and its inverse

$$\Sigma_0^{-1} = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix} \; where \; \begin{cases} a = a_n = \frac{1+(n-2)\rho}{(1+(n-1)\rho)(1-\rho)} \\ b = b_n = \frac{-\rho}{(1+(n-1)\rho)(1-\rho)} \end{cases}\,.$$

The likelihood ratio is then, letting $f(\cdot)$ denote the density of X and $\tilde{\mathbf{x}}_{\mathbf{i}} = (x_1, ..., x_{i-1}, x_i - \theta_i, x_{i+1}, ...x_n)'$,

$$LR = \frac{f(\boldsymbol{x} \mid \theta_i = 0, \forall\, i)}{\max\limits_{i,\theta_i} f(\boldsymbol{x} \mid \theta_i \neq 0, \theta_{-i} = 0)} = \frac{(\det \Sigma_0)^{-\frac{1}{2}} \exp\left\{\frac{-1}{2}\boldsymbol{x}^T \Sigma_0^{-1}\boldsymbol{x}\right\}}{\max\limits_{i} (\det \Sigma_0)^{-\frac{1}{2}} \exp\left\{\frac{-1}{2} \sup_{\theta_i} \tilde{\mathbf{x}}_{\mathbf{i}}^T \Sigma_0^{-1} \tilde{\mathbf{x}}_{\mathbf{i}}\right\}}\,. \tag{S.6}$$

Computation yields, defining $u_i = \sum\limits_{j \neq i}^{n} x_j$,

$$\hat{\theta}_i = \arg\max_{\theta_i} \frac{-1}{2} \tilde{\mathbf{x}}_{\mathbf{i}}^T \Sigma_0^{-1} \tilde{\mathbf{x}}_{\mathbf{i}} = x_i + \frac{b}{a} u_i\,,$$

from which it is immediate that

$$LR = \min_i \frac{\exp\left\{\frac{-1}{2}\left((a-b)(\sum_1^n x_j^2) + b(\sum_1^n x_j)^2\right)\right\}}{\exp\left\{\frac{-1}{2}\left((a-b)(\frac{b^2}{a^2}u_i^2 + \sum_{j\neq i}^n x_j^2) + b(-\frac{b}{a}u_i + \sum_{j\neq i}^n x_j)^2\right)\right\}}$$

$$= \min_i \exp\left\{\frac{-1}{2}\left[(a-b)(x_i^2 - \frac{b^2}{a^2}u_i^2) + b\left((u_i+x_i)^2 - u_i^2(\frac{b}{a}-1)^2\right)\right]\right\}$$

$$\text{(since } \sum_1^n x_j = u_i + x_i\text{)}$$

$$= \min_i \exp\left\{\frac{-1}{2}\left[ax_i^2 + 2bu_ix_i + \frac{b^2}{a}u_i^2\right]\right\}$$

$$= \min_i \exp\left\{\frac{-1}{2a}(ax_i + bu_i)^2\right\}.$$

Noting that

$$\frac{1}{a}(ax_j + bu_j)^2$$

$$= \frac{1}{(1+(n-1)\rho)(1+(n-2)\rho)(1-\rho)}\left((1+(n-2)\rho)x_j - \rho\sum_{k\neq j} x_k\right)^2$$

$$= \frac{1}{(1+(n-1)\rho)(1+(n-2)\rho)(1-\rho)}\left[(1-\rho)x_j + n\rho(x_j - \bar{\boldsymbol{x}})\right]^2$$

$$= \frac{1}{(1+(n-1)\rho)(1+(n-2)\rho)}\left[\sqrt{1-\rho}x_j + n\rho\left(\frac{x_j - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}}\right)\right]^2,$$

it is immediate that LR is equivalent to the test statistic T.

The rejection region is $LR \leq k$ for some k, which is clearly equivalent to $T \geq c$ for appropriate critical value c. □

A Bayesian test

Proof of Theorem ??. The posterior probability of M_i is

$$P(M_i \mid \boldsymbol{x}) = \frac{m_i(\boldsymbol{x})P(M_i)}{\sum_{j=0}^n m_j(\boldsymbol{x})P(M_j)}$$

$$= \frac{\frac{1-r}{n}|\Sigma_i|^{\frac{-1}{2}}\exp\left\{\frac{-1}{2}\boldsymbol{x}'\Sigma_i^{-1}\boldsymbol{x}\right\}}{r|\Sigma_0|^{\frac{-1}{2}} exp\left\{\frac{-1}{2}\boldsymbol{x}'\Sigma_0^{-1}\boldsymbol{x}\right\} + \sum_{j=1}^n \frac{1-r}{n}|\Sigma_j|^{\frac{-1}{2}}\exp\left\{\frac{-1}{2}\boldsymbol{x}'\Sigma_j^{-1}\boldsymbol{x}\right\}} \tag{S.7}$$

$$= \left\{\begin{array}{c}\left(\frac{n\,r}{1-r}\right)\left|\frac{\Sigma_0}{\Sigma_1}\right|^{\frac{-1}{2}}\exp\left\{\frac{-1}{2}\boldsymbol{x}'(\Sigma_0^{-1} - \Sigma_i^{-1})\boldsymbol{x}\right\} \\ +1 + \sum_{j\neq i}^n \exp\left\{\frac{-1}{2}\boldsymbol{x}'(\Sigma_j^{-1} - \Sigma_i^{-1})\boldsymbol{x}\right\}\end{array}\right\}^{-1}.$$

The expression can be simplified by further computing Σ_i^{-1},$(\Sigma_i^{-1} - \Sigma_k^{-1})$ and $det(\Sigma_i)$. First notice that by the Cholesky decomposition

$$\Sigma_0^{-1} = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix} = \boldsymbol{L}\boldsymbol{L}^T,$$

for some lower triangular matrix $\boldsymbol{L}$. Then by the Woodbury identity, the difference of two inverse matrices can be obtained:

$$\begin{aligned}\Sigma_i^{-1} = (\Sigma_0 + \boldsymbol{\tau}_i \boldsymbol{\tau}_i^T)^{-1} &= \Sigma_0^{-1} - \Sigma_0^{-1}\boldsymbol{\tau}_i(1+\boldsymbol{\tau}_i^T\Sigma_0^{-1}\boldsymbol{\tau}_i)^{-1}\boldsymbol{\tau}_i^T\Sigma_0^{-1} \\ &= \Sigma_0^{-1}\frac{-\tau^2}{1+\tau^2 a}\Sigma_0^{-1}\begin{pmatrix} & & & 0 & & & \\ b & \cdots & b & a & b & \cdots & b \\ & & & 0 & & & \end{pmatrix},\end{aligned}$$

where $\boldsymbol{\tau}_i = (0, \cdots, \tau, \cdots, 0)^T$ (the i^{th} element is τ^2). Therefore,

$$\begin{aligned}\boldsymbol{x}'(\Sigma_0^{-1} - \Sigma_i^{-1})\boldsymbol{x} &= \frac{\tau^2}{1+\tau^2 a}\left(x_i(a-b) + bn\bar{\boldsymbol{x}}\right)^2 = \frac{\tau^2}{1+\tau^2 a}\left(\frac{x_i}{1-\rho} + bn\bar{\boldsymbol{x}}\right)^2 \\ \boldsymbol{x}'(\Sigma_k^{-1} - \Sigma_i^{-1})\boldsymbol{x} &= \frac{\tau^2}{1+\tau^2 a}(x_i^2 - x_k^2)(a-b)^2 + 2b(a-b)(n\bar{\boldsymbol{x}})(x_i - x_k)\,.\end{aligned}$$

Also the ratio of two determinants is

$$\begin{aligned}\det(\Sigma_i)/\det(\Sigma_0) &= \det(\Sigma_1)/det(\Sigma_0) = \det(I + \Sigma_0^{-1}\boldsymbol{\tau_1}\boldsymbol{\tau_1}i^T) \\ &= \det(I + \boldsymbol{L}\boldsymbol{L}^T\boldsymbol{\tau_1}\boldsymbol{\tau_1}^T) = \det(I + \boldsymbol{\tau_1}^T\boldsymbol{L}^T\boldsymbol{L}\boldsymbol{\tau_1}) \\ &= (1+\tau^2 L_{11}^2) = (1+\tau^2 a)\,.\end{aligned}$$

By plugging back these quantities into (S.7), the proof is complete. □

Corollary S.8. *The posterior of any non-null model M_i is:*

$$P(M_i \mid \boldsymbol{x}) = \left\{\begin{array}{l} \left(\sqrt{\frac{(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho}{(1+(n-1)\rho)(1-\rho)}}\left(\frac{n\,r}{1-r}\right)*\right. \\ \exp\left\{\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\left((x_i-\bar{\boldsymbol{x}}) + \frac{(1-\rho)\bar{\boldsymbol{x}}}{1+(n-1)\rho}\right)^2\right\} + \\ \sum_{k=1}^{n}\exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\right. \\ \qquad\left.\left((x_i+x_k-2\bar{\boldsymbol{x}})(x_i-x_k) + 2\frac{\bar{\boldsymbol{x}}(x_i-x_k)(1-\rho)}{1+(n-1)\rho}\right)\right] \end{array}\right\}^{-1}. \tag{S.9}$$

Alternatively, in terms of $z(\boldsymbol{x})$, with $z_i = z_i(\boldsymbol{x})$:

$$P(M_i \mid \boldsymbol{x}) = \left\{\begin{array}{l} \left(\sqrt{\frac{(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho}{(1+(n-1)\rho)(1-\rho)}}\left(\frac{n\,r}{1-r}\right)*\right. \\ \exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho](1-\rho)}\right. \\ \qquad\left.\left(\theta_i - \bar{\boldsymbol{\theta}} + \sqrt{1-\rho}(z_i-\bar{\boldsymbol{z}}) + \frac{1-\rho}{1+(n-1)\rho}(\bar{\boldsymbol{\theta}} + \sqrt{\rho}z + \sqrt{1-\rho}\bar{\boldsymbol{z}})\right)^2\right] \\ +\sum_{k=1}^{n}\exp\left[\frac{-\tau^2}{2}\frac{1+(n-1)\rho}{[(1-\rho+\tau^2)(1+(n-1)\rho)-\tau^2\rho]}\right. \\ \qquad\left(\frac{[\theta_i+\theta_k-2\bar{\boldsymbol{\theta}}+\sqrt{1-\rho}(z_i+z_k-2\bar{\boldsymbol{z}})][\theta_i-\theta_k+\sqrt{1-\rho}(z_i-z_k)]}{1-\rho}+\right. \\ \qquad\qquad\left.\left.2\frac{[\bar{\boldsymbol{\theta}}+\sqrt{\rho}z+\sqrt{1-\rho}\bar{\boldsymbol{z}}][\theta_i-\theta_k+\sqrt{1-\rho}(z_i-z_k)]}{1+(n-1)\rho}\right)\right] \end{array}\right\}^{-1}. \tag{S.10}$$

Proof. Expand a, b in Theorem **??** explicitly. □

Remark S.11. *Corollary S.8 gives the full expression for $P(M_i \mid \boldsymbol{x})$, without using a, b, and is utilized in the subsequent proofs.*

The situation as the correlation goes to 1

Proof of Theorem ??. By Lemma S.1, x_i can be written as $x_i = \theta_i + \sqrt{\rho} z + \sqrt{1-\rho} z_i$.
Case I: n=2:

$$\begin{aligned} \frac{x_i - \rho x_{(-i)}}{\sqrt{1-\rho^2}} &= \frac{(\theta_i - \theta_{(-i)}) + \sqrt{1-\rho}(z_i - \rho z_{(-i)}) + z\sqrt{\rho}(1-\rho)}{\sqrt{1-\rho^2}} \\ &= \frac{\theta_i - \theta_{(-i)}}{\sqrt{1-\rho^2}} + \frac{z_i - \rho z_{(-i)}}{\sqrt{1+\rho}} + \sqrt{1-\rho}\frac{\sqrt{\rho} z}{\sqrt{1+\rho}} . \end{aligned} \tag{S.12}$$

If M_0 is true, since both $\theta_i, \theta_{(-i)}$ are zero, the dominant term of (S.12) is $O(1)$, so the null posterior probability (Corollary **??**) becomes:

$$P(M_0 \mid \boldsymbol{x}) = (1 + O(\sqrt{1-\rho^2})) = 1 + o(1) .$$

If M_i is true, the dominant term of (S.12) is $O(1/\sqrt{1-\rho^2})$; hence as $\rho \to 1$:

$$\begin{aligned} P(M_i \mid \boldsymbol{x}) &= \left\{ \begin{array}{l} \sqrt{\frac{\tau^2}{2(1-\rho^2)}} \left(\frac{2r}{1-r}\right) \exp\left\{ \frac{-1}{2} \frac{\theta_i^2}{2(1-\rho^2)} + O(\frac{1}{\sqrt{1-\rho}}) \right\} \\ +1 + \exp\left\{ \frac{-1}{2} \theta_i(\theta_i + 2z) \right\} \end{array} \right\}^{-1} \\ &= \left(\frac{1}{1 + \exp\{\frac{-1}{2}\theta_i(\theta_i + 2z)\}} \right) (1 + o(1)) \\ &= \frac{1}{1 + \exp\left\{ \frac{-1}{2}\left(x_i^2 - x_{(-i)}^2\right) \right\}} (1 + o(1)) , \end{aligned}$$

using the fact that $e^{-1/\sqrt{1-\rho}}$ goes to zero faster than $1/\sqrt{1-\rho}$ goes to infinity.
If $M_{(-i)}$ is true: similar to the previous case, (S.12) is $O(1/\sqrt{1-\rho})$, so only the last term in (**??**) remains.
Case II: $n > 2$: denote $\boldsymbol{z} = (z_1, ..., z_n)$, $\bar{\boldsymbol{z}} = 1/n \sum_1^n z_i$;
If M_0 is true : take $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) = \mathbf{0}$ in (S.10), let $c = c(\rho) = \frac{1+(n-1)\rho}{(1-\rho+\tau^2)(1+(n-1)\rho) - \tau^2\rho}$, and note that $\lim_{\rho \to 1} c(\rho) = \frac{n}{\tau^2(n-1)}$. Then

$$\begin{aligned} &P(M_i \mid \boldsymbol{x}) \\ &= \left\{ \begin{array}{l} \sqrt{\frac{1}{c(1-\rho)}} \left(\frac{n\,r}{1-r}\right) * \\ \exp\left\{ \frac{-\tau^2}{2} c \left((z_i - \bar{\boldsymbol{z}}) + \underbrace{\frac{\sqrt{1-\rho}}{1+(n-1)\rho}(\sqrt{\rho}z + \sqrt{1-\rho}\bar{\boldsymbol{z}})}_{O(\sqrt{1-\rho})} \right)^2 \right\} + 1 + \\ \sum_{k \neq i}^n \exp\left\{ \frac{-\tau^2}{2} c \left\{ \begin{array}{l} \frac{(z_i + z_k - 2\bar{\boldsymbol{z}})(z_i - z_k)}{1-\rho} + \\ 2\, \underbrace{\frac{(1-\rho)\bar{\boldsymbol{z}}(z_i - z_k) + \sqrt{\rho(1-\rho)}z(z_i - z_k)}{1+(n-1)\rho}}_{O(\sqrt{1-\rho})} \end{array} \right\} \right\} \end{array} \right\}^{-1} \\ &= \left\{ \begin{array}{l} \sqrt{\frac{\tau^2(n-1)/n}{1-\rho}} \left(\frac{n\,r}{1-r}\right) \exp\left\{ \frac{-n}{2(n-1)} (z_i - \bar{\boldsymbol{z}})^2 \right\} + 1 + \\ \sum_{k \neq i}^n \exp\left\{ \frac{-n}{2(n-1)} \frac{(z_i + z_k - \bar{\boldsymbol{z}})(z_i - z_k)}{1-\rho} \right\} \end{array} \right\}^{-1} (1 + o(1)) \\ &\leq \left\{ \sqrt{\frac{\tau^2(n-1)/n}{1-\rho}} \left(\frac{n\,r}{1-r}\right) \exp\left\{ \frac{-n}{2(n-1)} (z_i - \bar{\boldsymbol{z}})^2 \right\} \right\}^{-1} (1 + o(1)) \\ &= \sqrt{1-\rho}\sqrt{\frac{1-r}{r\,\tau^2}} (1 + o(1)) \to 0 . \end{aligned}$$

If M_j is true ($j > 0$): Take $\theta_k = 0 \,\forall\, k \neq j$ in (S.10):

$$
\begin{aligned}
&P(M_j \mid \boldsymbol{x}) \\
&= \left\{ \begin{array}{l} \sqrt{\frac{1}{c(1-\rho)}} \left(\frac{nr}{1-r} \right) \exp\left\{ \frac{-\tau^2 c}{2(1-\rho)} \left((1-\frac{2}{n})\theta_j + \sqrt{1-\rho}(z_i - \bar{\boldsymbol{z}}) + O(1-\rho) \right)^2 \right\} \\ +1 + \sum_{k \neq j}^{n} \exp\left\{ \frac{-\tau^2 c}{2} \left(\frac{(1-2/n)\theta_j + \sqrt{1-\rho}(z_j + z_k - 2\bar{\boldsymbol{z}})(\theta_j + \sqrt{1-\rho}(z_i - z_k))}{1-\rho} + O(1) \right) \right\} \end{array} \right\}^{-1} \\
&= \left\{ \begin{array}{l} \sqrt{\frac{\frac{n-1}{n}\tau^2}{1-\rho}} \left(\frac{n\,r}{1-r} \right) \exp\left\{ \frac{-n}{2(n-1)} \left(\frac{(1-2/n)^2\theta_j^2}{1-\rho} + O(\frac{1}{\sqrt{1-\rho}}) \right) \right\} + 1 \\ +(n-1)\exp\left\{ \frac{-n}{2(n-1)} \left(\frac{(1-2/n)\theta_j^2}{1-\rho} + O(\frac{1}{\sqrt{1-\rho}}) \right) \right\} \end{array} \right\}^{-1} (1 + o(1)) \\
&\to 1 \text{ since } \lim_{\rho \to 1} \sqrt{1/(1-\rho)} \exp\{-1/(1-\rho)\} = 0.
\end{aligned}
$$

□

Asymptotic frequentist properties of Bayesian procedures

Remark S.13. *Note that, by Lemma S.3,*

$$
z_i = z_i(\boldsymbol{x}) = \frac{x_i - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}} + O(1/\sqrt{n}), \tag{S.14}
$$

so that Lemma **??** *can be written, with respect to z_i:*

$$
P(M_i \mid \boldsymbol{x}) = \left(1 + \frac{n}{1-r} \sqrt{\frac{1-\rho+\tau^2}{1-\rho}} \exp\left\{ \frac{-\tau^2 z_i^2}{2(1-\rho+\tau^2)} \right\} (1+o(1)) \right)^{-1}.
$$

Lemma S.15. *If Z_i, $i \in \{1, 2, ..., n\}$, are i.i.d. standard normal random variables, then*

$$
|Z_i| \leq n^{1/2-\epsilon} \quad \forall\, i \qquad \textit{holds almost surely.}
$$

Proof. By Fact S.4:

$$
\begin{aligned}
&P(\text{ for all i}, |Z_i| \leq n^{1/2-\epsilon}) \\
=\;& \left(1 - P(|Z_1| \geq n^{1/2-\epsilon}) \right)^n \\
=\;& \left(1 - 2\frac{\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} n^{1-2\epsilon}\}}{n^{1/2-\epsilon}} + O\left(\exp\left\{ \frac{-n^{1-2\epsilon}}{2} \right\} \right) \right)^n \\
=\;& \left(1 - \frac{\frac{2n^{1/2-\epsilon}}{\sqrt{2\pi}} \exp\{-\frac{1}{2} n^{1-2\epsilon}\}}{n} + o(n^{-2}) \right)^n \\
=\;& 1 - O\left(2n^{1/2+\epsilon} \exp\{-\frac{1}{2} n^{1-2\epsilon}\} \right) \\
=\;& 1 + o(1).
\end{aligned}
$$

□

Remark S.16. *Figure 1 gives the estimated and true posterior probability of M_1 under the assumption that the null model is true, for fixed $r = \rho = 0.5$ and n, but varied τ. Notice that, for fixed n, the estimated probability is closer to the true probability when τ is small but is worse for larger τ, indicating that larger n is required for obtaining the same precision as τ grows.*

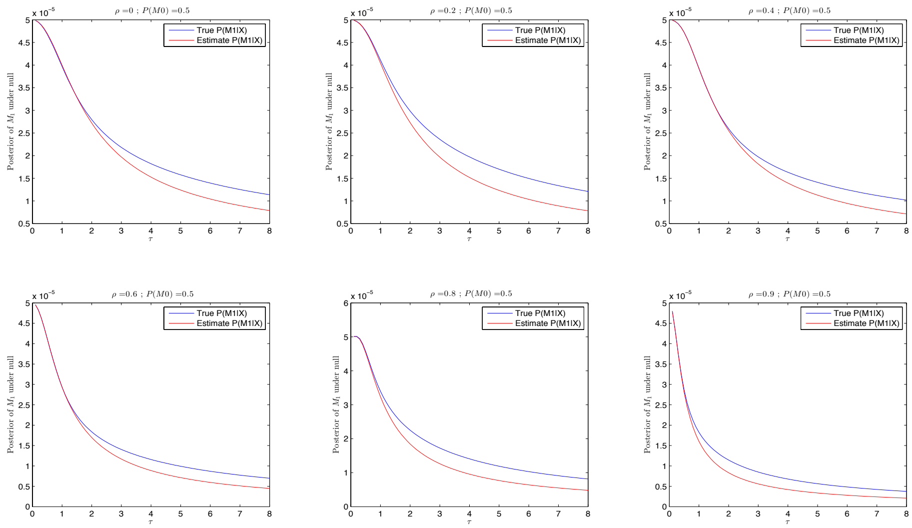

FIG. 1: Estimated (red line) and true posterior probability (blue line) of M_1 for different τ under the null model, for fixed $n = 2000, \rho = r = 0.5$.

Proof of Lemma ??. Take $\boldsymbol{\theta} = 0$ in (S.10):

$$
\begin{aligned}
&P(M_i \mid \boldsymbol{x}) = \\
&\left\{ \begin{array}{l} \sqrt{\frac{1}{c(1-\rho)}}\left(\frac{n\,r}{1-r}\right) \exp\left\{ \frac{-\tau^2}{2} c \Bigg(\underbrace{z_i - \bar{\boldsymbol{z}} + \frac{\sqrt{1-\rho}}{1+(n-1)\rho}(\sqrt{\rho} z + \sqrt{1-\rho}\bar{\boldsymbol{z}})}_{I} \Bigg)^2 \right\} + 1 + \\ \sum_{k \neq i}^{n} \exp\{ \frac{-\tau^2}{2} c \Bigg(\underbrace{(z_i + z_k - 2\bar{\boldsymbol{z}})(z_i - z_k) + 2\frac{(\sqrt{\rho} z + \sqrt{1-\rho}\bar{\boldsymbol{z}})(\sqrt{1-\rho}(z_i - z_k))}{1+(n-1)\rho}}_{II} \Bigg) \} \end{array} \right\}^{-1}
\end{aligned}
\tag{S.17}
$$

Without loss of generality, assuming $|z_i| \le n^{1/2-\epsilon}$ for all i, which holds almost surely by Lemma S.15, asymptotic analysis of (S.17) yields:

$$
\begin{aligned}
I = &\Bigg((1 - 1/n)^2 z_i^2 + \underbrace{\bar{\boldsymbol{z}}^2_{(-i)}}_{\mathrm{O}(n^{-1})} - 2(1 - 1/n) \underbrace{z_i \bar{\boldsymbol{z}}_{(-i)}}_{O(n^{-\epsilon})} \Bigg) \text{ (where } \bar{\boldsymbol{z}}_{(-i)} = 1/n \sum_{k \neq i} z_k) \\
&+ 2 \underbrace{\left(\frac{\sqrt{1-\rho}}{1+(n-1)\rho} \right)}_{O(n^{-1})} \underbrace{\left((1 - 1/n) z_i - \bar{\boldsymbol{z}}_{(-i)} \right)}_{O(n^{1/2-\epsilon})} \underbrace{\left(\sqrt{\rho} z + \sqrt{1-\rho}\bar{\boldsymbol{z}} \right)}_{O(1)} \\
&+ \underbrace{\left(\frac{\sqrt{1-\rho}}{1+(n-1)\rho} \right)^2}_{O(n^{-2})} \underbrace{(\sqrt{\rho} z + \sqrt{1-\rho}\bar{\boldsymbol{z}})^2}_{O(1)} \\
= &(1 - 1/n)^2 z_i^2 + O(n^{-\epsilon}) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sqrt{\frac{1}{c(1-\rho)}}\left(\frac{n\,r}{1-r}\right)\exp\left\{\frac{-\tau^2}{2}c\,I\right\} \\
&=\sqrt{\frac{1}{c(1-\rho)}}\left(\frac{n\,r}{1-r}\right)\exp\left\{\frac{-\tau^2}{2}c\,(1-1/n)^2\,z_i^2\right\}\Big(1+O(n^{-\epsilon})\Big) \qquad \text{(S.18)}\\
&=\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\left(\frac{n\,r}{1-r}\right)\exp\left\{\frac{-\tau^2}{2(1-\rho+\tau^2)}z_i^2\right\}(1+o(1))\,.
\end{aligned}
$$

$$
\begin{aligned}
II &= \big(z_i^2-\underbrace{z_k^2}_{O(n^{1-2\epsilon})}\big)(1-2/n)-\underbrace{\big(2/n\sum_{l\notin\{i,k\}}z_l\big)}_{O(n^{-1/2})}\underbrace{(z_i-z_k)}_{O(n^{1/2-\epsilon})} \\
&\quad+\underbrace{\left(\frac{2}{1+(n-1)\rho}\right)}_{O(n^{-1})}\left\{\begin{array}{l}\sqrt{\rho(1-\rho)}\;\underbrace{z_i z}_{O(n^{1/2-\epsilon})}\;+(1-\rho)\,\underbrace{z_i/n(-z_k)}_{O(n^{-1/2-\epsilon})}\\ +(1-\rho)\;\underbrace{z_i^2/n}_{O(n^{-2\epsilon})}\;+\underbrace{(\sqrt{\rho}z+\sqrt{1-\rho}\bar{\boldsymbol{z}}_{(-i)})(-z_k)}_{O(n^{1/2-\epsilon})}\end{array}\right\} \\
&=(z_i^2-z_k^2)\,(1-2/n)+O(n^{-\epsilon})\,.
\end{aligned}
$$

The summation term in (S.17) becomes:

$$
\begin{aligned}
&\sum_{k\neq i}^{n}\exp\left\{\frac{-\tau^2}{2}c\,II\right\} \\
&=\exp\left\{-c\frac{\tau^2}{2}(1-2/n)z_i^2\right\}\big(1+O(n^{-\epsilon})\big)\sum_{k=1}^{n}\exp\left\{c\frac{\tau^2}{2}(1-2/n)z_k^2\right\} \\
&=\exp\left\{\frac{-\tau^2}{2(1-\rho+\tau^2)}z_i^2\right\}n\left(\mathbb{E}^Z\left[\exp\left\{\frac{\tau^2}{2(1-\rho+\tau^2)}Z^2\right\}\right]+o(1)\right)(1+o(1)) \qquad \text{(S.19)}\\
&\qquad\qquad \text{by the Law of Large Numbers and since } Z\sim N(0,1) \\
&=\exp\left\{\frac{-\tau^2}{2(1-\rho+\tau^2)}z_i^2\right\}n\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}(1+o(1))\,.
\end{aligned}
$$

The proof is completed by adding (S.18) and (S.19). □

Proof of Theorem ??. Under the null model, by (**??**), $P(M_i \mid \boldsymbol{x}) \geq p$ is equivalent to

$$
z_i^2 \geq 2\left(\frac{1-\rho+\tau^2}{\tau^2}\right)ln\left(\frac{n}{1-r}\frac{p}{1-p}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}(1+o(1))\right) \qquad \text{(S.20)}
$$

By Fact S.4:

$$
\begin{aligned}
&\mathbb{P}\left(|Z_i|\geq\underbrace{\sqrt{2\left(\frac{1-\rho+\tau^2}{\tau^2}\right)ln\left(\frac{n}{1-r}\frac{p}{1-p}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\right)+o(1)}}_{\gamma_n}\right) \\
&=1/n\underbrace{\left(\frac{\frac{2}{\sqrt{2\pi}}n^{-\frac{1-\rho}{\tau^2}}\left(\frac{1}{1-r}\frac{p}{1-p}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\right)^{-(1+\frac{1-\rho}{\tau^2})}(1+o(1))}{\sqrt{2\left(\frac{1-\rho+\tau^2}{\tau^2}\right)ln\left(\frac{n}{1-r}\frac{p}{1-p}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}})\right)+o(1)}}\right)}_{d_n}+O\left(\frac{1}{n\,(\log\,n)^2}\right).
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}(\text{any false positive} \mid M_0) = 1 - \prod_i^n \mathbb{P}(|Z_i| < \gamma_n) \\
&= 1 - (1 - \mathbb{P}(|Z_1| \geq \gamma_n))^n = 1 - \left(1 - \frac{d_n}{n}\right)^n = 1 - (1 - d_n) + O(d_n^2) \\
&= \left(\frac{1}{n^{\frac{1-\rho}{\tau^2}}}\right)\left(\frac{|\tau|}{\sqrt{\pi}(1-\rho+\tau^2)^{1+\frac{1-\rho}{2\tau^2}}}\right)\left(\frac{(1-r)(1-p)}{p}\right)^{1+\frac{1-\rho}{\tau^2}} \\
&\quad \left(\log\frac{n}{1-r} + \log\left(\frac{p}{1-p}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\right)\right)^{\frac{-1}{2}}(1+o(1)) \\
&= O(n^{-(\frac{1-\rho}{\tau^2})}(\log\, n)^{\frac{-1}{2}})\,.
\end{aligned}
$$

□

Adaptive Choice of τ^2

Lemma S.21.

$$
\begin{aligned}
&\arg\max_{\tau^2}\left[\left(1+\frac{1-\rho}{\tau^2}\right)\log\left(n\frac{p}{(1-p)(1-r)}\sqrt{\frac{1-\rho+\tau^2}{1-\rho}}\right)+o(1)\right] \\
&=(1-\rho)\left(2\log n+\log\log n+2\log\frac{p}{(1-p)(1-r)}+\log 2+o(1)\right).
\end{aligned} \tag{S.22}
$$

Proof. Letting $x = \frac{1-\rho}{\tau^2}$ and $c' = \frac{p}{(1-p)(1-r)}$, the expression in square brackets in (S.22) can be written

$$f(x) = (1+x)\big(\log(n\,c') + 1/2\,log\,(1+1/x)\big)\,.$$

Clearly

$$f'(x) = \frac{1}{2}\big(2\log(n\,c') + \log(1+1/x) - 1/x\big)\,,$$

so that, $f'(x) = 0$ when $1/x = 2\log\, n + \log\log n + 2\log c' + \log 2 + o(1)$, or

$$\tau^2 = (1-\rho)(2\log\, n + \log\log n + 2\log c' + \log 2) + o(1)\,.$$

. □

Theorem S.23. *If $c_n \in (0,1)\,\forall n$ and $1 - c_n = o(1)$, then*

$$\lim_{n\to\infty}\frac{1}{n}\sqrt{1-c_n}\sum_{i=1}^n \exp\left\{\frac{c_n}{2}z_i^2\right\} = \lim_{n\to\infty} 2\Phi\left(\sqrt{\frac{2(1-c_n)}{c_n}\log\frac{n}{\sqrt{1-c_n}}}\right) - 1$$

in probability.

Proof. Take $X_{n,i} = exp\left\{\frac{c_n}{2}z_i^2\right\}$; $\beta_n = \frac{n}{\sqrt{1-c_n}}$ in Fact S.5.

Checking the first assumption of the WLTA:

$$
\begin{aligned}
P(|X_{n,i}| > \beta_n) &= P\Bigg(|z_i| > \sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}\Bigg) \\
&= 2\frac{\frac{1}{2\pi}exp\left\{\frac{-1}{2}\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}\right\}}{\sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}} + O\Bigg(\frac{(\frac{n}{\sqrt{1-c_n}})^{-\frac{1}{c_n}}}{(\frac{1}{c_n}log\frac{n}{\sqrt{1-c_n}})^3}\Bigg) \\
&= \frac{1}{\sqrt{\pi}}\frac{\sqrt{1-c_n}^{\frac{1}{c_n}}}{n^{\frac{1}{c_n}}}\frac{1}{\sqrt{log\frac{n}{\sqrt{1-c_n}}}}(1+o(1)) \\
&< \frac{1}{\sqrt{\pi}}\frac{\sqrt{1-c_n}^{\frac{1}{c_n}}}{n^{\frac{1}{c_n}}}\frac{1}{\sqrt{log\,n}}(1+o(1))\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{i=1}^{n} P(|X_{n,k}| > \beta_n) = nP(|X_{n,k}| > \beta_n) \\
&< n^{1-\frac{1}{c_n}}(1-c_n)^{\frac{1}{2c_n}}\frac{1}{\sqrt{log\,n}} \\
&= n^{-\frac{1-c_n}{c_n}}(1-c_n)^{\frac{1}{2c_n}}\frac{1}{\sqrt{log\,n}} \to 0\,.
\end{aligned}
$$

Checking the second assumption of the WLTA:
Since $\lim\limits_{n\to\infty} c_n \to 1$, without loss of generality, assume $c_n > 3/4$. Then

$$
\begin{aligned}
\frac{1}{\beta_n^2}\sum_{k=1}^{n} E\bar{X}_{n,k}^2 &= \frac{1-c_n}{n^2}n\int\limits_{|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} \exp\{c_n z^2\}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-1}{2}z^2\right\}dz \\
&= \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\int\limits_{1<|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}}\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz + \int\limits_{|z|<1}\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz\right\} \\
&\le \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\int\limits_{1<|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} z\exp\left\{(c_n-\frac{1}{2})z^2\right\}dz + d\right\} \\
&= \frac{1-c_n}{n}\frac{1}{\sqrt{2\pi}}\left\{\frac{1}{2(c_n-\frac{1}{2})}\exp\left\{(c_n-\frac{1}{2})(\frac{2}{c_n})\log\frac{n}{\sqrt{1-c_n}}\right\} + d'\right\} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2c_n-1}\right)\frac{1-c_n}{n}\left(\frac{n}{\sqrt{1-c_n}}\right)^{2-\frac{1}{c_n}} + o(1) \\
&= \frac{1}{\sqrt{2\pi}}\underbrace{\left(\frac{1}{2c_n-1}\right)}_{\le 2} n^{1-\frac{1}{c_n}}(1-c_n)^{\frac{1}{2c_n}} + o(1) \\
&\le \frac{2}{\sqrt{2\pi}}n^{-\frac{1-c_n}{c_n}}(1-c_n)^{\frac{1}{2c_n}} + o(1) = o(1)\,.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\frac{\sqrt{1-c_n}}{n}\alpha_n = \frac{1-c_n}{n}\sum_{i=1}^{n}E\bar{X}_{n,i} \\
= \ &(1-c_n)\int\limits_{|z|<\sqrt{\frac{2}{c_n}log\frac{n}{\sqrt{1-c_n}}}} e^{\frac{c_n z^2}{2}}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz = 2\Bigg(\Phi\Bigg(\frac{\sqrt{\frac{2}{c_n}\log\frac{n}{\sqrt{1-c_n}}}}{\sqrt{(1-c_n)^{-1}}}\Bigg) - \frac{1}{2}\Bigg),
\end{aligned}
$$

the WLTA yields

$$\frac{S_n - \alpha_n}{\beta_n} = \frac{\sum_{i=1}^{n} e^{c_n z_i^2} - \alpha_n}{\frac{n}{\sqrt{1-c_n}}} = \frac{\sqrt{1-c_n}\sum_{i=1}^{n} e^{c_n z_i^2}}{n} - \frac{\sqrt{1-c_n}}{n}\alpha_n \to 0\,.$$

in probability, and the result follows. □

Corollary S.24. *Letting* $c_n = \frac{\hat{\tau}_n^2}{1-\rho+\hat{\tau}_n^2}$,

$$\frac{1}{n}\sqrt{1-c_n}\sum_{i=1}^{n} \exp\left\{\frac{c_n}{2} z_i^2\right\} \to \begin{cases} 1 & \mathit{if}\ \frac{\log n}{\hat{\tau}_n^2} \to \infty\,, \\ 2\Phi\left(\sqrt{\frac{2}{k}}\right) - 1 & \mathit{if}\ \frac{\log n}{\hat{\tau}_n^2} \to \frac{1}{(1-\rho)k}\,, \\ 0 & \mathit{if}\ \frac{\log n}{\hat{\tau}_n^2} \to 0 \end{cases}$$

in probability.

Proof. By Theorem S.23:
Case I: $\frac{\log n}{\hat{\tau}_n^2} \to \infty$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} \log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 1\,.$$

Case II: $\frac{\log n}{\hat{\tau}_n^2} \to \frac{1}{(1-\rho)k}$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} \log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 2\Phi\left(\sqrt{\frac{2}{k}}\right) - 1\,.$$

Case III: $\frac{\log n}{\hat{\tau}_n^2} \to 0$. Clearly

$$\frac{\sqrt{1-c_n}}{n}\alpha_n \Rightarrow 2\Phi\left(\sqrt{\frac{2(1-\rho)}{\hat{\tau}_n^2} \log\left(n\sqrt{\frac{1-\rho+\hat{\tau}_n^2}{1-\rho}}\right)}\right) - 1 \to 0\,.$$

□

Lemma S.25.

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n\sqrt{1+\tau_n^2 a}} \sum_{i=1}^{n} \exp\left\{\frac{\tau_n^2}{2(1+\tau_n^2 a)}\big(\frac{x_i}{1-\rho} + bn\bar{\boldsymbol{x}}\big)^2\right\} \\ &= \frac{1}{n}\sqrt{\frac{1-\rho}{1-\rho+\tau_n^2}} \sum_{i=1}^{n} \exp\left\{\frac{\tau_n^2 z_i^2}{2(1-\rho+\tau_n^2)}\right\}(1+o(1)) \quad a.s. \end{aligned} \tag{S.26}$$

Proof. Expanding the coefficients yields

$$\begin{aligned} \frac{1}{1+\tau_n^2 a} &= \left(1 + \frac{\tau_n^2(1+(n-2)\rho)}{(1+(n-1)\rho)(1-\rho)}\right)^{-1} \\ &= \frac{1-\rho}{1-\rho+\tau_n^2\big(1+\frac{-\rho}{1+(n-1)\rho}\big)} = \frac{1-\rho}{1-\rho+\tau_n^2}(1+O(1/n))\,, \end{aligned}$$

and

$$\left(\frac{x_i}{1-\rho} + bn\bar{\boldsymbol{x}}\right)^2 = \frac{1}{(1-\rho)^2}\left(x_i + \frac{-\rho n\bar{\boldsymbol{x}}}{1+(n-1)\rho}\right)^2$$
$$= \frac{1}{(1-\rho)^2}\left(x_i - \bar{\boldsymbol{x}}\big(1 - \frac{1-\rho}{1-\rho+\rho n}\big)\right)^2$$
$$= \frac{1}{(1-\rho)^2}\left(\sqrt{1-\rho}z_i + \sqrt{\rho}z\underbrace{\big(\frac{1-\rho}{1-\rho+\rho n}\big)}_{O(1/n)} + \sqrt{1-\rho}\underbrace{\bar{\boldsymbol{z}}}_{O(1/\sqrt{n})}\big(-1+\frac{1-\rho}{1-\rho+\rho n}\big)\right)^2$$
$$= \frac{z_i^2}{1-\rho} + O\big((\log\, n)/\sqrt{n}\big)\,.$$

Therefore,

$$\frac{1}{\sqrt{1+\tau_n^2 a}}1/n\sum_i \exp\left\{\frac{\tau_n^2}{2(1+\tau_n^2 a)}\big(\frac{x_i}{1-\rho} + bn\bar{\boldsymbol{x}}\big)^2\right\}$$
$$= \sqrt{\frac{1-\rho}{1-\rho+\tau_n^2+o(1)}}1/n\sum_i \exp\left\{\frac{\tau_n^2}{2}\left[\frac{z_i^2}{1-\rho+\tau_n^2} + o(1)\right]\right\}\,.$$

□

Lemma S.27. *Under the null model, suppose*

$$\max_j \left(\frac{x_j - \bar{\boldsymbol{x}}}{\sqrt{1-\rho}}\right)^2 = 2\log(n) + \log\log(n) + c\,.$$

Then

$$L_n(\tau^2) = \frac{1}{n}\sqrt{\frac{1-\rho}{1-\rho+\tau^2}}\sum_{i=1}^n \exp\left\{\frac{\tau^2 z_i^2}{2(1-\rho+\tau^2)}\right\}$$

is maximized at

$$\hat{\tau}_n^2 = (1-\rho)k(c)(\log\, n)(1+o(1))\,,$$

where

$$k(c) = (1 + 2/\sqrt{\pi}\exp\{-c/2\})^{-1}\,.$$

Proof. Without loss of generality, let $\max|z_i| = |z_1|$.

$$L_n(\hat{\tau}_n^2) = \Bigg(\underbrace{\sqrt{\frac{1-\rho}{1-\rho+\hat{\tau}_n^2}}1/n\exp\left\{\frac{\hat{\tau}_n^2 z_1^2}{2(1-\rho+\hat{\tau}_n^2)}\right\}}_{I} +$$
$$\underbrace{\sqrt{\frac{1-\rho}{1-\rho+\hat{\tau}_n^2}}1/n\sum_{i=2}^n \exp\left\{\frac{\hat{\tau}_n^2 z_i^2}{2(1-\rho+\hat{\tau}_n^2)}\right\}}_{II}\Bigg)(1+o(1))\,.$$

First, note that $L_n(\hat{\tau}_n^2) \to 0$ when $\log n/\hat{\tau}_n^2 \to \infty$, since

$$I = \frac{1}{\sqrt{\hat{\tau}_n^2}} n^{\frac{-(1-\rho)}{1-\rho+\hat{\tau}_n^2}} \sqrt{\log n}^{\frac{(\hat{\tau}_n^2)}{1-\rho+\hat{\tau}_n^2}} e^{\frac{c\hat{\tau}_n^2}{2(1-\rho+\hat{\tau}_n^2)}}(1+o(1))$$
$$= n^{\frac{-(1-\rho)}{1-\rho+\hat{\tau}_n^2}}\sqrt{\log n/\hat{\tau}_n^2}e^{c/2}(1+o(1)) = o(1)\,,$$

$II \to 0$ by Corollary $S.24$.

Similarly, one can show that $L_n(\hat\tau_n^2) \to 1$ when $\log n/\hat\tau_n^2 \to 0$, since

$$I = n^{\frac{-(1-\rho)}{1-\rho+\hat\tau_n^2}} \sqrt{\log n/\hat\tau_n^2}\, e^{c/2}(1+o(1)) = o(1)$$
$$II \to 1 \text{ by Corollary } S.24.$$

For the case in which $\log n/\hat\tau_n^2 \to k$, using Corollary S.24, it follows that

$$L_n(\hat\tau_n^2) = [ve^{(\frac{c}{2}-v^2)} + 2\Phi(\sqrt{2}v) - 1](1+o(1)),$$

where $v = \sqrt{(1-\rho)/k}$. Differentiating $f(v) = [ve^{(\frac{c}{2}-v^2)} + 2\Phi(\sqrt{2}v)]$ and setting the derivative to 0, yields the solution $\hat v = \sqrt{\frac{1}{2} - \frac{1}{\sqrt{\pi}} e^{-c/2}}$, which translates into $k(c)$ as in the statement of the lemma. It is straightforward to show that this extrema of $f(v)$ is the maximum, and

$$f(\hat v) > \max\{\lim_{v\to 0} f(v), \lim_{v\to\infty} f(v)\} = 1\,.$$

As this maximum thus exceeds the maximum over the domains $\log n/\hat\tau_n^2 \to \infty$ and $\log n/\hat\tau_n^2 \to 0$, the proof is complete. □

Lemma S.28. *For the $k(c)$ defined above,*

$$\log(k(c)/2) + 2/k(c) - 1 > 0\, \forall\, c > 0$$

Proof. Note that $x = k/2 < 1$, so that we want to show that $f(x) = \log(x) + 1/x - 1 > 0$ over this region. Since $f'(x) = 1/x - 1/x^2 < 0$ over this region, $f(x)$ is minimized at $x = 1$, proving the result. □

Analysis as the information grows

Proof of Theorem??. Write $\sigma_n^2 = d_n/log\, n$, $X_i^* = X_i/\sigma_n$, and $\theta_i^* = \theta_i/\sigma_n$. Then a nonzero θ_i^* has prior $N(0, \tau^2/\sigma_n^2)$ and (**??**) becomes:

$$\boldsymbol{X}^* \sim multinorm\left(\begin{pmatrix}\theta_1^*\\ \theta_2^*\\ \vdots\\ \theta_n^*\end{pmatrix}, \begin{pmatrix}1 & \rho & \cdots & \rho\\ \rho & 1 & \cdots & \rho\\ \vdots & \vdots & \ddots & \vdots\\ \rho & \rho & \cdots & 1\end{pmatrix}\right).$$

Under the null model: by Theorem **??**,

$$\begin{aligned}
&P(M_0 \mid \boldsymbol{x})^{-1}\\
&= 1 + \left(\frac{1-r}{n\,r}\right)\sqrt{\frac{\sigma_n^2}{\sigma_n^2+\tau^2 a_n}} *\\
&\qquad \sum_{i=1}^n \exp\left\{\frac{\tau^2}{2(\sigma_n^2+\tau^2 a_n)}\left(\frac{z_i}{\sqrt{1-\rho}} + \frac{\sqrt{\rho}z}{1-\rho} + b_n n\sqrt{1-\rho}\bar{\boldsymbol{z}} + b_n n\sqrt{\rho}z\right)^2\right\}\\
&= 1 + \left(\frac{1-r}{nr}\right)\sqrt{\frac{\sigma_n^2}{\sigma_n^2+\tau^2/(1-\rho)}} \sum_{i=1}^n \exp\left\{\frac{\tau^2}{2(\sigma_n^2+\tau^2/(1-\rho))}\frac{z_i^2}{(1-\rho)}\right\}\left(1+O(\frac{1}{\sqrt{n}})\right)\\
&= 1 + (\frac{1-r}{nr})\sqrt{1-c_n}\sum_{i=1}^n \exp\left\{\frac{c_n}{2}z_i^2\right\}\left(1+O(1/\sqrt{n})\right),
\end{aligned} \tag{S.29}$$

where

$$\begin{cases} a_n = \frac{1}{1-\rho} + \frac{-\rho}{(1-\rho)(1+(n-1)\rho)} = \frac{1}{1-\rho} + O(1/n)\,,\\ nb_n = \frac{-1}{1-\rho} + \frac{1-\rho}{(1-\rho)(1+(n-1)\rho)} = \frac{-1}{1-\rho} + O(1/n)\,,\\ c_n = \frac{\tau^2}{(1-\rho)\sigma_n^2+\tau^2} = \frac{\tau^2}{(1-\rho)d_n/log\, n+\tau^2}\,. \end{cases}$$

Since $d_n = o(\log n)$, $1 - c_n = o(1)$, and $0 < c_n < 1$, one can apply Theorem S.23 to get the asymptotic analysis of (S.29):

$$
\begin{aligned}
&2\Phi\left(\frac{2(1-c_n)}{c_n}\log\frac{n}{\sqrt{1-c_n}}\right) - 1 = \frac{(1-\rho)\sigma_n^2}{\tau^2}\log\left(n\sqrt{\frac{(1-\rho)\sigma_n^2+\tau^2}{(1-\rho)\sigma_n^2}}\right) \\
&= \frac{(1-\rho)d_n}{\tau^2\log n}\left[\log\left(n\sqrt{\frac{log\, n}{d_n}}\right) + O(1)\right] \\
&= \frac{1-\rho}{\tau^2}\left[d_n + \frac{1}{2}\left(\frac{d_n}{log\, n}\right)\log\left(\frac{log\, n}{d_n}\right)\right](1+O(1)) \to \begin{cases} \infty & \text{if } d_n \to \infty\,, \\ \frac{1-\rho}{\tau^2}d & \text{if } d_n \to d\,, \\ 0 & \text{if } d_n \to 0\,. \end{cases}
\end{aligned}
$$

Hence, under the null hypothesis,

$$
P(M_0 \mid \boldsymbol{X}) \to \begin{cases} P(M_0) & \text{if } d_n \to \infty\,, \\ \left(1 + (\frac{1-r}{r})(2\Phi(\frac{1-\rho}{\tau^2}d) - 1)\right)^{-1} & \text{if } d_n \to d\,, \\ 1 & \text{if } d_n \to 0\,. \end{cases}
$$

Under the alternative model M_j: by Theorem **??**,

$$
\begin{aligned}
&P(M_j \mid \boldsymbol{x})^{-1} \\
&= \sqrt{1+\frac{a_n\tau^2}{\sigma_n^2}}\left(\frac{n\,r}{1-r}\right)\exp\left\{\frac{-\tau^2}{2(\sigma_n^2+a_n\tau^2)}\left[\frac{\theta_j^2}{\sigma_n^2(1-\rho)^2} + O(\frac{1}{\sigma_n})\right]\right\} \\
&+ 1 + \sum_{k\neq j}^{n}\exp\left\{\frac{-\tau^2}{2(\sigma_n^2+a_n\tau^2)}\left[\frac{\theta_j^2}{2(1-\rho)^2\sigma_n^2} + O(\frac{1}{\sigma_n})\right]\right\}\,.
\end{aligned}
$$

The first term is

$$
\begin{aligned}
&\sqrt{1+\frac{\tau^2/(1-\rho)}{\sigma_n^2}}\left(\frac{n\,r}{1-r}\right)\exp\left\{\frac{-\tau^2}{2(\sigma_n^2+\tau^2/(1-\rho))}\left[\frac{\theta_j^2}{(1-\rho)^2\sigma_n^2}\right]\right\}\left(1+O\left(\sqrt{\frac{log\, n}{d_n}}\right)\right) \\
&= \sqrt{\tau^2/(1-\rho)}\sqrt{\frac{log\, n}{d_n}}\left(\frac{r}{1-r}\right)n^{1-\frac{\theta_j^2}{2(1-\rho)d_n}}\left(1+O\left(\sqrt{\frac{log\, n}{d_n}}\right)\right)
\end{aligned}
$$

which $\to 0$ if and only if $\lim_{n\to\infty} d_n < \frac{\theta_j^2}{2(1-\rho)}$.

And the last term,

$$
\begin{aligned}
&\sum_{k\neq j}^{n}\exp\left\{\frac{-\tau^2}{2(\sigma_n^2+\tau^2/(1-\rho))}\left[\frac{\theta_j^2}{(1-\rho)^2\sigma_n^2}\right)\right]\right\}\left(1+O\left(\sqrt{\frac{log\, n}{d_n}}\right)\right) \\
&= n\exp\left\{-\frac{\theta_j^2}{2(1-\rho)}\frac{log\, n}{d_n}\right\}\left(1+O\left(\sqrt{\frac{log\, n}{d_n}}\right)\right) \\
&= n^{1-\frac{\theta_j^2}{2(1-\rho)d_n}}\left(1+O\left(\sqrt{\frac{log\, n}{d_n}}\right)\right), \\
&\to 0 \text{ when } \lim_{n\to\infty} d_n < \frac{\theta_j^2}{2(1-\rho)}\,.
\end{aligned}
$$

□

[1] Durrett, R. (2010), *Probability: theory and examples*, vol. 3, Cambridge university press.