Nonlinear Systems and Control

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Winter 2025

Contents

Chapter 1

Introduction to nonlinear systems and nonlinear phenomena

1.1 Nonlinear models and nonlinear phenomena

In this course, we deal with dynamical systems modeled as a finite number of coupled differential equations:

$$\begin{aligned}\dot{x}_1 =& f_1(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_p)\\ &\vdots\\ \dot{x}_n =& f_n(t, x_1, x_2, \ldots, x_n, u_1, \ldots, u_p)\end{aligned}$$

We write this system more compactly using vector notation:

$$\dot{x}(t) = f(t, x, u),$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$, and $f(t, x, u) = \begin{pmatrix} f_1(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{pmatrix}$

We will also consider the output equation

$$y = h(t, x, u),$$

where $y \in \mathbb{R}^p$, $h \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$ with the output vector $y \in \mathbb{R}^p$ comprising variables of particular interest in the analysis of the dynamical system (e.g., variables that are physically measured of variables that are required to behave in a specific manner).

In a significant part of the analysis we will do in the course, the state equation will not have an explicit presence of an input u; that is, we will study the unforced state equation

$\dot{x} = f(t, x)$. This does not mean that the input u is necessarily zero. It can mean that u was already specified as a function of time or as a function of the state or both.

A special case of the unforced equation $\dot{x} = f(t, x)$ is when the function f does not depend explicitly on time. Such systems are called *time-invariant*.

1.2 An example of a nonlinear system

Consider the pendulum equation: a rod of length l and zero mass has a bob of mass m attached to one end, and it is fixed on the other end. The pendulum swings on the vertical plane. There is frictional force resisting the pendulum's motion with coefficient k. The equation of motion along the tangential direction is

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

We can write the equation in state-space form by setting $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{cases}$$

To find the equilibrium points, we set $\dot{x} = 0$ and solve for x_1 and x_2. We get

$$\begin{cases} 0 = x_2 \\ 0 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{cases}$$

which yields

$$\begin{cases} x_2 = 0 \\ x_1 = n\pi, \quad n = 0, \pm 1, \pm 2, \dots \end{cases}$$

From the physical description of the system, the pendulum has two equilibrium points: $(0, 0)$ and $(\pi, 0)$. Other equilibrium points are repetitions of these two positions.

1.3 Equilibrium of the state equation

One of the key concepts in the study of dynamical systems is equilibria. A point x^* in the state space is an equilibrium point of $\dot{x} = f(t, x)$ if it has the property that whenever the state of the system starts at x^*, the system will remain there for all future times. Equilibrium points can be isolated, or they can be in a continuum.

1.4 Nonlinear vs linear systems

We now compare some aspects of linear and nonlinear systems. Recall the structure of linear systems:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y &= C(t)x(t) + D(t)u(t)\end{aligned}$$

The trajectory of such a system can be obtained via the state transition matrix:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau,$$

where the state transition matrix is the solution to the system

$$\begin{aligned}\frac{\partial}{\partial t}\phi(t, \tau) &= A(t)\phi(t, \tau)\\ \phi(t, t) &= I\end{aligned}$$

In fact, for LTI systems, $\dot{x} = Ax$ with $A \in \mathbb{R}^{n\times n}$, the state transition matrix reads $\phi(t, t_0) = \exp(A(t - t_0))$, and the solution of the system is given by

$$x(t) = e^{A(t-t_0)}x(t_0)$$

This solution satisfies the following properties:

- It exists.
- It is unique.
- It is defined for all $t \geq t_0$.
- The set of the equilibrium points is the nullspace of the matrix A, which is a subspace. Hence, equilibrium points are connected and not isolated.
- If all trajectories converge to a given bounded set, then they converge to the origin.
- If all trajectories are bounded, they converge to a periodic solution.
- Solutions can be written down in closed form.

We also recall that the superposition principle being the fundamental property of linear systems. As we move to the nonlinear regime, the superposition principle does not hold. The analysis of nonlinear systems will require new techniques due to a few reasons:

- While we can always linearize a system about an equilibrium point and use linear techniques to analyze the linearized system, linearization provides an approximation to the original system that is only applicable in a neighborhood of the equilibrium point. Thus, linearization can only predict local properties of the system, not global ones.
- The dynamics of nonlinear systems are richer than those of linear systems. We proceed to discuss some.

1.5 Examples of nonlinear phenomena

1.5.1 Finite escape time

The state of an unstable linear system goes to infinity as time approaches infinity. A nonlinear system's state, however, can go to infinity in finite time.

Example.

> Consider the system $\dot{x} = 1+x^2$ with $x(0) = 0$. The solution to the system is $x(t) = \tan(t)$. The solution exists only during a bounded interval of time.

1.5.2 Non-uniqueness of solutions

Consider the system $\dot{x} = x^{2/3}$ with $x(0) = 0$. Then we can verify that $x(t) = 0$ and $x(t) = (t/3)^3$ are both solutions to the system.

1.5.3 Multiple isolated equilibria

A linear system can only have one isolated equilibrium point, that is, it can only have one steady-state operating point that attracts the state of the system irrespectively of the initial conditions. A nonlinear system, on the other hand, can have more than one isolated equilibria. The state may converge to one of several steady-state operating points depending on the initial state of the system.

1.5.4 Limit cycles

An LTI system oscillates if it has a pair of eigenvalues on the imaginary axis. This is a non-robust condition to maintain in the presence of perturbations. In addition, the amplitude of the oscillation depends on the initial conditions. In real life, a stable oscillation can be produced by a nonlinear system. There are nonlinear systems that can go into oscillations of fixed amplitude frequency, irrespective of the initial conditions. This type of oscillations is called a *limit cycle*.

Example.

> Consider the system $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$ for $\mu > 0$. Let $x_1 = x$ and $x_2 = \dot{x}$. We can write the system in state space form:
>
> $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\mu(x_1^2 - 1)x_2 - x_1 \end{cases}$$
>
> The solutions to this system converge to a bounded set, but do not converge to the origin. They converge to a limit cycle. It is an asymptotically stable oscillator. This can occur for systems with $n \geq 2$.

1.5.5 Strange attractor or chaos

A nonlinear system can have more complicated steady-state behavior, that is, not equilibrium, periodic, or almost periodic solution. Such behavior is called chaos.

Example.

> Lorentz's equations:
>
> $$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\beta x + xy \end{cases}$$
>
> where $\beta, \sigma > 0$. The solutions to this system are bounded, but they do not converge to anything. Graphical representations and animations are shown on the web. This phenomenon can occur for systems with $n \geq 3$.

1.5.6 Solutions cannot always be written in closed form

Example: Poincaré's three body problem. Can occur for systems with $n \geq 1$.

1.5.7 Sub-harmonic, harmonic, or almost periodic solutions

A stable linear system under a periodic input generates an output of the same frequency. A nonlinear system under a periodic input can oscillate with frequencies that are sub-multiples or multiples of the input frequency. It may even generate an almost-periodic oscillation (e.g., the sum of periodic oscillations that are not multiples of each other).

1.6 Conclusion

For a nonlinear system $\dot{x} = f(x)$, it does not automatically follow that solutions exist, are unique, etc. And when we can show that a solution exists, we will not be able to compute it in closed form. Hence, our analysis has to be indirect: instead of working with the solutions to the system, we will deduce properties of the solution by analyzing the ODE model instead of the solution itself. In the first part of the course, we develop the mathematical tools that allow us to do this analysis.

Chapter 2

Qualitative behavior of linear systems

2.1 Second order systems

Consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

Assume that the solution $x(t) = (x_1(t), x_2(t))$ with $x(t_0) = x_0$ exists and unique. Then the locus of $x(t)$ on the x_1-x_2 plane for all $t \geq t_0$ is a curve that passes through x_0. The right-hand side of the state equation expresses the tangent vector $\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to the solution curve. The solution curve is often called a trajectory orbit (from x_0). The family of all solutions is called the *phase portrait* of the system. We can qualitatively analyze the behavior of second order systems by using their phase portraits.

2.2 Solutions of second-order linear systems

We let

$$\dot{x} = Ax, \quad A \in \mathbb{R}^{2\times 2}.$$

The solution of the system from a given initial state x_0 is given by

$$x(t) = M \exp(J_r t) M^{-1} x_0,$$

where J_r is the real Jordan form of A and M is real and nonsingular such that $M^{-1}AM = J_r$.

Depending on the eigenvalues of A, J_r may have the following forms:

- 2 real and distinct eigenvalues: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- 2 real and equal eigenvalues: $\begin{pmatrix} \lambda & k \\ 0 & \lambda \end{pmatrix}$, where $k = 0, 1$

- complex eigenvalues: $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $\lambda_{1,2} = a \pm jb$

We now consider each case.

2.2.1 Case I: both eigenvalues are real

- Assume negative eigenvalues $\lambda_2 < \lambda_1 < 0$. λ_2 is the fast eigenvalue, and λ_1 the slow eigenvalue.

  The trajectories approach the origin tangent to the slow eigenvector, and are parallel to the fast eigenvector far from the origin. This is a stable node.

- If the eigenvalues are positive, we have the reversed trajectory direction compared with the previous situation. This is an unstable node.

- If the eigenvalues are of opposite signs, $\lambda_2 < 0 < \lambda_1$, we have a saddle point. λ_2 is the stable eigenvalue, and λ_1 the unstable one.

2.2.2 Case II: eigenvalues are complex

We can write $\lambda_{1,2} = a \pm jb$.

- If $a < 0$, we have a stable focus.

- If $a > 0$, we have an unstable focus.

- If $a = 0$, we have a center.

2.2.3 Case III: nonzero multiple eigenvalues

The eigenvalues can be expressed as $\lambda_1 = \lambda_2 = \lambda \neq 0$.

The phase portrait is similar to the portrait of a node, but not with the fast-slow asymptotic behavior we noticed earlier. We usually call the case $\lambda < 0$ a stable node and $\lambda > 0$ an unstable node.

2.2.4 Case IV: one of both eigenvalues are zero

In this case, A has a nontrivial nullspace. This means that any vector in the nullspace is an equilibrium point for the system. We also say that the system has an equilibrium space instead of an equilibrium point.

2.3 Qualitative behavior near equilibrium points

Except for some special cases, the qualitative behavior of a nonlinear system neat an equilibrium point can be determined via linearization with respect to that point.

Consider

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

and let $p = (p_1, p_2)$ be an equilibrium point. Assume f_1, f_2 are continuously differentiable. Letting $y_i = x_i - p_i$, the linearization of the system about p yields

$$\begin{cases} \dot{y} = Ay, \end{cases}$$

where A is the Jacobian of $f(x)$ evaluated at $x = p$.

Now, it is true that if the origin of the linearized system is a stable (resp. unstable) node with distinct eigenvalues, a stable (resp. unstable) focus, or a saddle point, then, in a small neighborhood of the equilibrium point, the trajectories of the nonlinear system will behave like a stable (resp. unstable) node, a stable (resp. unstable) focus, or a saddle point. However, if the linearized system has a center equilibrium, then the behavior of the nonlinear system around the equilibrium point could be quite distinct from that of the linearized system. In fact, in this case, the linearization method around the equilibrium point is inconclusive regarding the type of equilibrium in the nonlinear system.

Part I

Nonlinear System Analysis

Chapter 3

Existence and uniqueness of solutions

In Chapter 1, we highlighted via several examples why the fundamental properties of the solutions of ODEs should be studied. The *fundamental properties* we have in mind are

- existence,
- uniqueness,
- continuous dependence on initial conditions, and
- continuous dependence on parameters.

These properties are essential for the state equation $\dot{x} = f(t, x)$ to be a useful model for a physical system.

Example.

> Consider, for instance, the pendulum (Section 1.2). In principle, we expect that starting the pendulum from a given initial state at time t_0 will imply that the system will move and that its state will be defined at $t > t_0$. Moreover, for a deterministic system, we expect that, if we could repeat the experiment, we would get exactly the same motion and the same state at $t > t_0$. For the mathematical model to predict the state of the system, the problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$ must have a unique solution. These are the questions of existence and uniqueness.

The questions of existence and uniqueness of solutions can be treated by imposing constraints on f. The key constraint is the *Lipschitz condition*, namely

$$\|f(x,t) - f(t,y)\| \leq L \|x - y\| .$$

for all (x, t) and (y, t) in a neighborhood of (x_0, t_0) for $L < \infty$.

Our main objective is to determine conditions guaranteeing existence and uniqueness of solutions of a nonlinear ODE. We will need to cover some mathematical background first. Before proceeding, let's clarify what we mean by existence and uniqueness of the solution of a system of ordinary differential equations. Let

$$\begin{cases} \dot{x} = f(x,t) & \forall t \geq t_0 \\ x(t_0) = x_0 \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

For this system to be a useful representation of a physical system, we require the following characteristics:

- The system should have at least one solution. (Existence.)
- Even better, the system has exactly one solution for small enough values of t. This requires local existence and uniqueness of the solution.
- Even better, the system has exactly one solution for all $t \in [0, \infty)$. This requires global existence and uniqueness of the solution.

Remark.

Local and global existence and uniqueness are the topics of Theorems 3.1 and 3.2 in the textbook.

Recall that by the term *solution* of the initial value problem (IVP) 3.1 over the interval $[t_0, t_1]$, we mean a continuous function $x(t)\colon [t_0, t_1] \to \mathbb{R}^n$ such that $\dot{x}$ is defined and $\dot{x} = x(t, x)$ for all $t \in [t_0, t_1]$. We will assume that f is continuous in x, but only piecewise continuous in t.

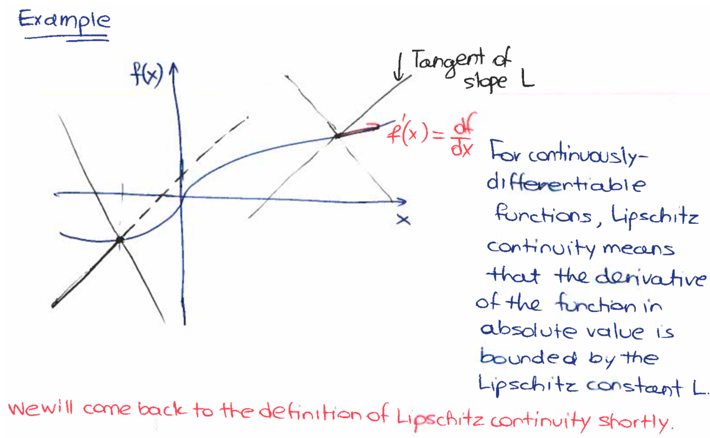
Example.

Recall the system $\dot{x} = x^{1/3}$ with $x(0) = 0$. We saw that $x(t) = \left(\frac{2}{3}t\right)^{3/2}$ and $x(t) = 0$ are both solutions to the system. Noting that the right-hand side of the equation is continuous in x tells us that continuity is not sufficient to guarantee uniqueness of the solution. Extra conditions must be imposed on f.

Remark.

We noted that this extra condition is the Lipschitz condition. To gain insight into the usefulness of this definition, consider a function $f\colon \mathbb{R} \to \mathbb{R}$. In this case the condition reads $\frac{|f(x)-f(y)|}{|x-y|} \leq L$. This means that on a plot of $f(x)$ versus x, a straight line joining any two points of f cannot have absolute value greater than L.

Example.



Let us summarize the main concepts of existence and uniqueness of solutions to the IVP 3.1:

Theorem 3.1 addresses the local existence and uniqueness of the solution $x(t)$ over a time interval $[t_0, t_0 + \delta]$, where $\delta > 0$. The theorem states that if the function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is such that

$$\forall x, y \in B_r(x_0).\ \|f(x) - f(y)\| \leq L \|x - y\|,$$

where $B_t(x_0) := \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$, then there exists a $\delta > 0$ for which we can guarantee that the solution $x(t)$ exists and is unique over the time interval $t \in [t_0, t_0 + \delta]$. In other words, if the function is locally Lipschitz, then the theorem can be used to assert that the solution exists and is unique over some time interval $[t_0, t_0 + \delta]$.

You may already have several questions, such as what is a locally Lipschitz function? How can we tell if a function is locally Lipschitz? What is the parameter δ that affects the existence and uniqueness interval? Can δ be made arbitrarily large? In other words, can we guarantee global existence and uniqueness?

To answer these questions in a rigorous manner, we will establish the proper mathematical framework. In the next part of the lecture, as well as in Appendices A and B of the textbook, we summarize fundamental concepts about Euclidean spaces, normed spaces, closed sets, convergence of sequences, etc., which are used towards defining Lipschitz continuity.

The proofs require the notion a contraction mapping. The quick notion of a contraction mapping is as follows:

Remark.

Let X be a normed space. A function $P \colon X \to X$ is called a contraction if there is a constant $0 \leq c < 1$ such that for all $x, y \in X$, we have

$$\|p(x) - p(y)\| \leq c \|x - y\|.$$

In other words, a contraction mapping maps points closer together. Note that a Lipschitz continuous function with constant $L < 1$ is a contraction mapping.

What makes contraction mappings particularly useful is the *contraction mapping principle*, which says that if $p \colon X \to X$ is a contraction, then there exists a unique fixed point $x^* \in X$ such that $p(x^*) = x^*$ and that for all $x_0 \in X$, the sequence $x_{n+1} = p(x_n)$ is Cauchy and $\lim_{n\to\infty} x_n = x^*$.

Now, the sequence means a collection of points $\{x_i\}_{i=0}^{\infty}$. A Cauchy sequence is one whose terms approach closer and closer to each other as i increases. So the contraction mapping principle says that from every initial point x_0, applying a contraction mapping iteratively creates a sequence that converges to a unique fixed point.

This is the main idea that the proof of Theorem 3.1 uses: it considers the solution of

$$\dot{x} = f(t, x(t)), \quad x(t_0) = x_0$$

given as

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

It views the right-hand side of this expression as a mapping from the state trajectory $x\colon [t_0, t_1] \to \mathbb{R}^n$ to itself and applies the contraction mapping principle to identify conditions under which the mapping $Px(t)$ is a contraction.

This way, we obtain that the solution to $x(t) = Px(t)$ exists and is unique—in fact, this is shown for a time interval $[t_0, t_0 + \delta] \subseteq [t_0, t_1]$, where

$$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr + h}, \frac{e}{L}\right\},$$

with L being the Lipschitz constant, r the radius of the ball around x_0, $h = \max_{s\in[t_0,t_1]} \|f(s, x_0)\|$, $e < 1$.

Theorem 3.1 does not provide a way to arbitrarily choose $\delta > 0$; the conditions on global existence and uniqueness are given by Theorem 3.2.

To continue with the main argument of Theorem 3.1, to prove that the right-hand side of the solution $x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds$ is a contraction mapping, we apply the contraction mapping principle, i.e., we consider

$$\begin{aligned}\|Px(t) - Py(t)\| &= \left\|\int_{t_0}^{t} (f(s, x(s)) - s, y(s))\, ds\right\| \leq \int_{t_0}^{t} \|f(s, x(s)) - s, y(s)\|\, ds \\ &\leq \int_{t_0}^{t} L\, \|x(s) - y(s)\|\, ds \leq \int_{t_0}^{t} L\, \|x - y\|_C\, ds,\end{aligned}$$

where

$$\|x\|_C = \sup_{t\in[t_0,t_0+\delta]} \|x(t)\|.$$

3.1 Mathematical review

We will define several topological notions. An esteemed reference in this subject is Kelley's General Topology [1].

3.1.1 Euclidean space

The set of all n dimensional vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, where the x_i are real numbers defines the n dimensional Euclidean space.

3.1.2 Extended real line

$\mathbb{R}_e = \mathbb{R} \cup \{\infty, -\infty\}$, where $x < \infty$ and $-\infty < x$ for all $x \in \mathbb{R}$.

3.1.3 Supremum = least upper bound

Let $S \subseteq \mathbb{R}$. We say that $a \in \mathbb{R}_e$ is the *supremum* of S if

- a is an upper bound, i.e., $s \leq a$ for all $s \in S$, and
- a is the least bound, i.e., if $b \in \mathbb{R}_e$ satisfies $s \leq b$ for all $s \in S$, then $a \leq b$.

We typically write $a = \sup S$.

Example.

> Let $S = (0, 1)$. Then $\sup S = 1$, while $\max S$ is not defined. Note that the difference is that the maximum element must be an element of S, while the sup is an element of $\mathbb{R}$.
>
> If $f \colon V \to \mathbb{R}$, then $\sup_{x \in V} f(x) \coloneqq \sup \{f(x) \mid x \in V\}$.

The notion of infimum, or inf, is obtained by reversing the inequalities in this discussion.

3.1.4 Norms

Let V be a vector space with field $\mathbb{R}$, then $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ is a norm if

- for all $x \in V$, $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
- for all $x \in V$ and $a \in \mathbb{R}$, $\|ax\| = |a| \, \|x\|$; and
- for all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

Definition 3.1.1:

$(V, \|\cdot\|)$ is called a normed space if V is a vector space with field $\mathbb{R}$ and $\|\cdot\|$ is a norm.

Example.

> Set $V = \mathbb{R}^n$ and
>
> - $\|x\|_2 = \sqrt{\sum_i x_i^2}$,
> - $\|x\|_1 = \sum_i |x_i|$,
> - $\|x\|_\infty = \max_i |x_i|$.
>
> Set $V = \{f \colon [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$ with the norm defined as $\|f\|_\infty = \max_{a \leq t \leq b} |f(t)|$.

3.1.5 Induced norms

Let $A \in \mathbb{R}^{n\times n}$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, then the induced norm on A is

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| .$$

Fact 3.1.2

For all $x \in \mathbb{R}^n$, $\|Ax\| \leq \|A\| \|x\|$.

Example.

The only induced norm we will use is the one coming from the 2-norm, $\|\cdot\|_2$. In this case, Appendix A in the textbook shows that

$$\|A\|_i = \sqrt{\lambda_{\max}(A^\top A)}.$$

Note this is the square root of the maximum eigenvalue of $A^\top A$.

3.1.6 Equivalent norms

Two norms $\|\cdot\|_a : V \to \mathbb{R}_{\geq 0}$ and $\|\cdot\|_b : V \to \mathbb{R}_{\geq 0}$ are equivalent if there exist positive constants k_1 and k_2 such that, for all $x \in V$,

$$k_1 \|x\|_a \leq \|x\|_b \leq k_2 \|x\|_a .$$

Remark.

It follows from the definition of equivalent norms that

$$1/k_2 \|x\|_b \leq \|x\|_a \leq 1/k_1 \|x\|_b .$$

Fact 3.1.3

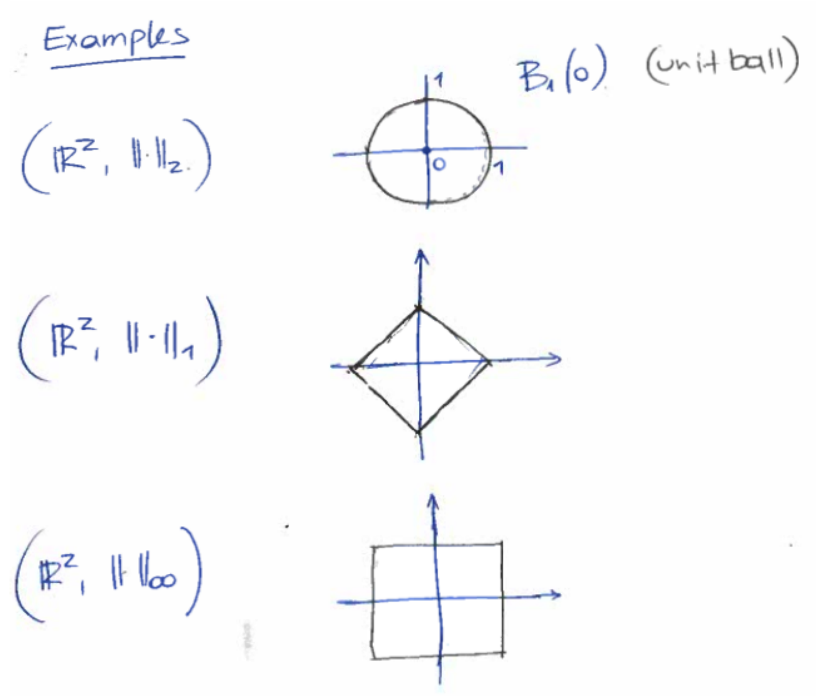
V is a finite-dimensional vector space if and only if all norms on V are equivalent.

3.1.7 Open balls, open sets, and closed sets

Let $x_0 \in V$ and $a \in \mathbb{R}$ with $a > 0$. We define the open ball of radius a centered at x_0:

$$B_a(x_0) = \{x \in V \mid \|x - x_0\| < a\} .$$

Example.



Definition 3.1.4

A set $S \subseteq V$ is open if for all $s_0 \in S$, there exists $\varepsilon(s_0) > 0$ such that $B_\varepsilon(s_0) \subseteq S$. The empty set, denoted $\emptyset$, is considered to be open.

Example.

$S = (0, 1) \subseteq \mathbb{R}$ is open.

$S = [0, 1) \subseteq \mathbb{R}$ is not open because for all $\varepsilon > 0$, $B_\varepsilon(0) \not\subseteq S$.

Fact 3.1.5

Equivalent norms define the same open sets.

A set S is closed if its complement, $\sim S$, is open.

Example.

$S = [0, 1] \subseteq \mathbb{R}$ is closed because $\sim S = (-\infty, 0) \cup (1, \infty)$ is open.

Fact 3.1.6

Arbitrary unions of open sets are open. Finite intersections of open sets are open. Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

Example.

> $S = [0, 1)$ is neither open nor closed.

3.1.8 Convergence of a sequence

Let $(V, \|\cdot\|)$ be a normed space.

Definition 3.1.7

A collection of points in V indexed by the natural numbers is called a sequence.

Definition 3.1.8:

A sequence $\{x_i\}_{i=0}^{\infty}$ converges to a point $\bar{x} \in V$ if for all $\epsilon > 0$, there exists $N(\epsilon) < \infty$ such that $\|x_i - \bar{x}\| < \epsilon$ for all $i \geq N$.
We use the notation $\lim x_i = \bar{x}$ or $x_i \to \bar{x}$.

Remark.

> We will soon use sequences as a means to iteratively write better and better approximations of solutions to an equation, such an ordinary differential equation, where we want to assert the existence of a solution. To apply the definition we just gave and test whether a sequence converges or not, we have to know the candidate limit $\bar{x}$. But if $\bar{x}$ is what we are trying to construct and prove that it exists, we seem to be trapped in a cycle: we want to prove that $\bar{x}$ exists by proving it is the limit of a convergent sequence. But to prove the sequence converges, we have to know the limit. To break this cycle, we use Cauchy sequences.

Definition 3.1.9

A sequence $\{x_i\}_i$ in V is a Cauchy sequence if for all $\varepsilon > 0$ there exists $N < \infty$ such that $\|x_i - x_j\| < \varepsilon$ for all $i, j \geq N$.

The intuition is that the father we go in the sequence, the closer the points get. Note that we can check this property without having access to the limit of the sequence.

Fact 3.1.10

Every convergent sequence is Cauchy, but the converse does not necessarily hold.

Definition 3.1.11

A normed space is complete is every Cauchy sequence converges to an element in that space.

Fact 3.1.12

Every finite-dimensional normed space is complete. The infinite-dimensional normed space $(C[a,b], \|\cdot\|_\infty)$ is complete.

3.1.9 Fixed points and contraction mappings

Let $(X, \|\cdot\|)$ be a normed space and let $P\colon X \to X$ be a mapping.

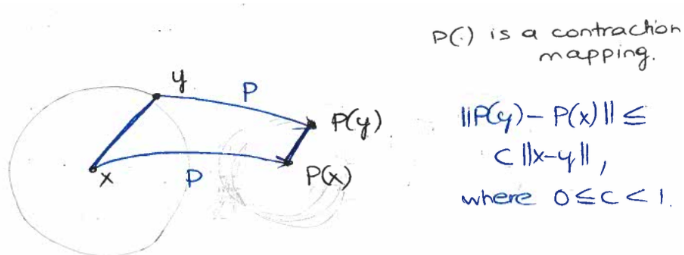
Definition 3.1.13

x^* is a fixed point of P if $P(x^*) = x^*$.

Definition 3.1.14

P is a contraction if there exists a constant $0 \leq x < 1$ such that for all $x, y \in X$, we have $\|P(x) - P(y)\| < c\,\|x - y\|$.

Note that a contraction map brings points closer together.



Theorem 3.1.15: Contraction mapping principle

Let $(X, \|\cdot\|)$ be a complete normed space, and let $P\colon X \to X$ be a contraction. Then there exists a unique $x^* \in X$ such that $P(x^*) = x^*$. Moreover, for all $x_0 \in X$, the sequence $x_{n+1} = P(x_n)$ is Cauchy and $\lim_{n\to\infty} x_n = x^*$.

Let us discuss more intuitive considerations about fixed points. Consider an equation of the form $x = P(x)$. A solution x^* to this equation is said to be a fixed point of the mapping P since P leaves x^* invariant.

A classical idea for finding a fixed point is the successive approximation method: we begin with an initial trial vector x_1 and compute $x_2 = P(x_1)$ and so on. The contraction mapping principle gives sufficient conditions under which there exists a fixed point x^* of $x = P(x)$, and $x_n \to x^*$. This is a powerful tool to prove the existence of the solution to an equation if the form $x = T(x)$. We will later use it to prove the existence of the solution of an equation of an ODE with locally Lipschitz right-hand side.

Let us summarize the discussion so far: suppose $P\colon X \to X$ is a contraction map.

- Fixed points are unique. To show this, suppose that x^* and y^* are both fixed points.

Then $x^* = P(x^*)$ and $y^* = P(y^*)$. We have

$$\|x^* - y^*\| = \|P(x^*) - P(y^*)\| \leq c \|x^* - y^*\|.$$

This is true if $c = 0$ or if $\|x^* - y^*\| = 0$, where $c = 0$ implies that $\|x^* - y^*\| = 0$, which means that $x^* = y^*$.

- $x_{n+1} = P(x_n)$ yields a Cauchy sequence. We have

$$\|x_{n+1} - x_n\| = \|Px_n - Px_{n-1}\| \leq \|x_n - x_{n-1}\| \leq c^n \|x_1 - x_0\|.$$

Note that this quantity goes to zero as n increases.

Example.

> Consider the normed space $(\mathbb{R}^n, \|\cdot\|_2)$ and seek to solve the linear equation $Ax = b$, where $A \in \mathbb{R}^{n\times n}$.
>
> Define $P(x) = x + (Ax - b)$. We note that
>
> $$x^* = P(x^*) \Leftrightarrow x^* = x^* + (Ax^* - b) \Leftrightarrow Ax^* = b.$$
>
> When is P a contraction?
>
> $$\begin{aligned} \|P(x) - P(y)\|_2 &= \|x + Ax - b - y - Ay + b\|_2 = \|(I + A)x - (I + A)y\|_2 \\ &= \|(I + A)(x - y)\|_2 \\ &\leq \|(I + A)\|_2 \|x - y\|_2. \end{aligned}$$
>
> Hence, P is a contraction if
>
> $$c \coloneqq \|I + A\|_2 = \left(\lambda_{\max}(I + A)^\top (I + A)\right)^{1/2} < 1.$$
>
> Later we will apply this method to prove the existence of solutions to differential equations.

3.1.10 Continuous functions

Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ be normed spaces.

Definition 3.1.16: Continuity

The function $h\colon V \to W$ is continuous at $x_0 \in V$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x_0\| < \delta \Rightarrow \|h(x) - h(x_0)\| < \varepsilon$.
We say that h is continuous if it is continuous at all $x_0 \in V$.

Definition 3.1.17: Lipschitz continuity

The function $h\colon V \to W$ is Lipschitz continuous at $x_0 \in V$ if there exists $r > 0$ and $L < \infty$ such that for all $x, y \in B_r(x_0)$,

$$\|h(x) - h(y)\| \leq L \|x - y\| .$$

L is called the Lipschitz constant.
h is called locally Lipschitz continuous if it is Lipschitz continuous at every point of V.
h is called globally Lipschitz continuous on V if there exists $L < \infty$ such that for all $x, y \in V$, $\|h(x) - h(y)\| \leq L \|x - y\|$.

Some remarks:

- The difference between local and global Lipschitz continuity is that with global Lipschitz continuity there is a single constant L that works everywhere, whereas with local Lipschitz continuity, as you vary x_0 and/or the radius of the open ball $B_r(x_0)$, the Lipschitz constant may have to change.
- Lipschitz continuous implies continuous at x_0; indeed, for $\varepsilon > 0$, we can select $\delta = \varepsilon/L$.
- Suppose that $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and that at some point $x_0 \in \mathbb{R}^n$, there exists $r > 0$ and $L < \infty$ such that $x \in B_r(x_0)$ implies that $\left\|\frac{\partial h}{\partial x}(x)\right\|_i \leq L$. Then for all $x, y \in B_r(x_0)$, we have $\|h(x) - h(y)\|_2 \leq L \|x - y\|_2$.
- If $h\colon V \to W$ is Lipschitz continuous at $x_0 \in V$ with norm $\|\cdot\|_V$ on V and norm $\|\cdot\|_W$ on W, then it is also Lipschitz continuous at x_0 for any equivalent norms, though the Lipschitz constant may be different.
- Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined as

  $$h(x) = \begin{cases} x, & |x| < 1 \\ 1, & x \geq 1 \\ -1, & x \leq -1. \end{cases}$$

  This function is not differentiable on $\mathbb{R}$, but it is globally Lipschitz continuous.

3.1.11 Piecewise continuity

Definition 3.1.18: Piecewise continuity

A function $h\colon \mathbb{R} \to V$ is piecewise continuous if

1. For every integer $k > 0$, $h\colon [-k, k] \to V$ is continuous, except possibly at a finite number of points.
2. At each point of discontinuity t_i, the limits from the left and from the right both exist and are finite.

Example.

The function

$$h(t) = \begin{cases} \sin(t), & t \neq k\pi \\ k^2\pi^2, & t = k\pi, \end{cases}$$

for $k \in \mathbb{Z}$ is piecewise continuous.

The function

$$h(t) = \begin{cases} \tan(t), & t \neq k\pi/2 \\ 0, & t = k\pi/2, \end{cases}$$

where $k \in \mathbb{Z}$ is not piecewise continuous because the limits from the left and right are not all bounded.

3.1.12 Negation of a statement

Recall the rule from logic that $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$, where the symbol $\neg$ denotes negation. The statement $\neg q \Rightarrow \neg p$ is called the contrapositive of $p \Rightarrow q$. To apply the contrapositive in a proof, we must be able to form the negation of a property, i.e., $\neg p$ and $\neg q$.

Example.

We let p be the property or statement $x_n \to \bar{x}$, i.e., the statement that a sequence $\{x_i\}$ converges to a point $\bar{x}$. Write out the formal expression for this statement:

$$p\colon \forall \varepsilon > 0.\ \exists N < \infty.\ (n \geq N) \Rightarrow \|x_n - \bar{x}\| < \varepsilon.$$

The negation of this statement is given by

$$p\colon \exists \varepsilon > 0.\ \forall N < \infty.\ (n \geq N) \wedge \|x_n - \bar{x}\| \geq \varepsilon.$$

General rules for negation:

- The existential quantifier is replaced with the universal quantifier of the negated formula.
- The universal quantifier is replaced with the existential quantifier of the negated formula.

Example.

We let p be the property: the function h is continuous at x_0. We write the formal statement:

$$p\colon \forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x \in B_\delta(x_0).h(x) \in B_\varepsilon(h(x_0)).$$

The negation of this statement is

$$\neg p\colon \exists \varepsilon > 0.\ \forall \delta > 0.\ \exists x \in B_\delta(x_0).\ h(x) \notin B_\varepsilon(h(x_0)).$$

Example.

When we are given a statement and are asked to prove another statement, it is often useful to write down formally the given statement and take it from there.

For example, suppose we have a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at $x_0 \in \mathbb{R}$ and also that we have a convergent sequence $\lim_{n\to\infty} x_n = x_0$. Suppose we want to show that the limit point of the sequence $f(x_n)$ will be $f(x_0)$. One way to approach these problems is to start with what we are given.

We know that f is continuous at x_0. This means that we can write the definition of continuity at x_0:

$$\forall \varepsilon > 0.\ \exists \delta > 0.\ \forall x \in B_\delta(x_0).\ f(x) \in B_\varepsilon(f(x_0)).$$

We also know that x_n is a convergent sequence to x_0, so we can write the definition of a convergent sequence:

$$\forall \varepsilon > 0.\ \exists N < \infty.\ \forall n \geq N.\ \|x_n - x_0\| < \varepsilon.$$

Since the definition holds for all $\varepsilon > 0$, it holds for some $\delta > 0$. Hence, we can write $\|x_n - x_0\| < \delta$ as $n \to \infty$.

From the continuity of $f\colon \mathbb{R} \to \mathbb{R}$, we have that there exists $\delta > 0$ such that

$$\|x_n - x_0\| < \delta \Rightarrow \|f(x_n) - f(x_0)\| < \varepsilon$$

for all $\varepsilon > 0$, i.e., what we wanted to prove.

Example.

Sometimes it is helpful to use proof by contradiction to prove an argument. Suppose we want to show that a statement is true. We start by assuming that the statement is false and try to reach a contradiction. Finding a contraction means that the original assumption about the falsehood of the statement was wrong, so the statement must be true.

For example, Problem 3 gives a closed set $S \subseteq V$, where V is a normed space, and a sequence of points $\{x_i\}_i$. The problem also states that sequence is convergent, i.e., $x_i \to \bar{x}$. Then we are asked to show that $\bar{x} \in S$.

We can show this by contraction. The idea is to assume the opposite of the conclusion (i.e., $\bar{x} \notin S$) and show that this leads to a falsehood.

Suppose that $\bar{x} \notin S$. Then $\bar{x}$ belongs to the complement of S, denoted $\sim S$, which must be open because S is closed. Since $\bar{x} \in \sim S$ and $\sim S$ is an open set, there exists an $\varepsilon > 0$ such that $B_\varepsilon(\bar{x}) \subseteq \sim S$. Then $B_\varepsilon(\bar{x}) \cap S = \emptyset$. This means that for all $x_n \in S$, $\|x_n - \bar{x}\| > \varepsilon$. We also know that x_n is also convergent to $\bar{x}$. This means that $\|x_n - \bar{x}\| < \varepsilon$ for all $\varepsilon > 0$ and for all $n > N$ for some $N < \infty$. These two statements are in contradiction since for all elements $x_1, x_2, \ldots$ the first statement says that the sequence $\bar{x}$ cannot be made smaller than or equal to $\varepsilon > 0$. Thus, $\bar{x} \notin S$ was a false statement. We conclude that $\bar{x} \in S$.

3.2 Existence and uniqueness of solutions to ODEs

With the math background in place, we proceed to discuss the existence and uniqueness to ODEs.

Theorem 3.2.1: Local existence and uniqueness (Theorem 3.1 in the book)

Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|$$

for all $x, y \in B_r(x_0)$ and all $t \in [t_0, t_1]$. Then there exists some $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Remark.

> Note that we extended the definition of the Lipschitz condition to functions $f(t, x)$ instead of $f(x)$ that we saw earlier. See page 89 of the textbook.

Proof. From the continuity of f, we have that $\dot{x} = f(t, x)$ has at least one solution $x(t)$. A solution $x(t)$ of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds. \tag{3.2}$$

Thus, we can proceed with investigating the existence and uniqueness of (3.2). We view the RHS of this equation as a mapping of the continuous function $x\colon [t_0, t_1] \to \mathbb{R}^n$:

$$x(t) = (Px)(t), \tag{3.3}$$

i.e., we just defined a mapping P. Note that Px is continuous in t, and we can view it as a contraction mapping. More specifically, we have that a solution to (3.3) is a fixed point of the mapping P that maps x to Px. Then we can use the contraction mapping principle to establish the existence of a fixed point of this equation.

To proceed with this approach, we need a complete normed linear space (aka Banach space) and a closed set $S \subseteq X$ such that the mapping $P\colon S \to S$ maps S into S and is a contraction over S.

Thus, in our proof, we proceed by defining a Banach space X, i.e., a normed vector space such that every Cauchy sequence in X converges to a point in X, and by constructing a closed set $S \subseteq X$ and showing that P maps S into S and is a contraction over S.

We consider the set of continuous functions $x\colon [t_0, t_0 + \delta] \to \mathbb{R}^n$, denoted as $C[t_0, t_0 + \delta]$. This set forms a vector space on $\mathbb{R}$. The justification is given in example B1. To obtain a normed vector space, we have to equip the vector space with a norm. We pick the norm

$$\|x\|_C = \max_{t \in [t_0, t_0 + \delta]} \|x(t)\|,$$

and thus we have a normed space, where δ is a constant to be chosen.

We proceed by defining the set $S = \{x \in X \mid \|x - x_0\| \le r\}$, where $r > 0$. We want to show that P maps S to S. Observe that

$$Px(t) - x_0 = \int_{t_0}^{t} f(s, x(s))ds = \int_{t_0}^{t} [f(s, x(s)) - f(s, x_0) + f(s, x_0)]ds.$$

Now we can make use of the following assumptions from the theorem:

- $f(t, x_0)$ is bounded on $[t_0, t_1]$ because f is piecewise continuous in its first argument. We let
$$h \coloneqq \max_{t \in [t_0, t_1]} \|f(t, x_0)\|.$$
- For all $x \in S$,
$$\|x(t) - x_0\| \le \max_{t \in [t_0, t_1]} \|x(t) - x_0\| \le r$$
for all $t \in [t_0, t_1]$.

We can thus write

$$\begin{aligned} \|(Px)(t) - x_0\| &\le \int_{t_0}^{t} \|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\| \, ds \le \int_{t_0}^{t} (L \|x(s) - x_0\| + h) \\ &\le \int_{t_0}^{t} (Lr + h) = (Lr + s)(t - t_0). \end{aligned}$$

We can now restrict the value of δ to satisfy $\delta \le t_1 - t_0$. This way, we would have $[t_0, t_0 + \delta] \subseteq [t_0, t_1]$. Then for $t = t_1$, we can write

$$\|(Px)(t) - x_0\| \le (Lr + h)\delta.$$

Note that $\|\cdot\|$ denotes the norm on $\mathbb{R}^n$, whereas $\|\cdot\|_C$ is the norm on X. We have

$$\|(Px) - x_0\|_C = \max_{t \in [t_0, t_1]} \|(Px)(t) - x_0\| \le (Lr + h)\delta.$$

Thus, choosing $\delta \le \frac{r}{Lr+h}$ ensures that P maps S into S since $\|(Px) - x_0\|_C \le r$.

Now we want to show that P is a contraction map over S. Let $x, y \in S$ and take

$$\begin{aligned} \|Px(t) - Py(t)\| &= \left\| \int_{t_0}^{t} (f(s, x(s)) - f(s, y(s))) \, ds \right\| \le \int_{t_0}^{t} \|(f(s, x(s)) - f(s, y(s)))\| \, ds \\ &\le \int_{t_0}^{t} L \|x(s) - y(s)\| \, ds \le L \|x(s) - y(s)\|_C \int_{t_0}^{t} ds \\ &\le L \|x(s) - y(s)\|_C \, \delta. \end{aligned}$$

We showed that
$$\|Px - Py\|_C \le L\delta \|x - y\|_C \le \rho \|x - y\|_C.$$

Under which condition is this a contraction? We see that, choosing $\rho < 1$ and $\delta \le \rho/L$ ensures that P is a contraction map over S. By the contraction mapping theorem, we conclude that if δ is chosen to satisfy

$$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lt+h}, \frac{\rho}{L}\right\}$$

for $\rho < 1$, then $x(t)$ is unique in S.

To complete the proof, we need to establish the uniqueness of the solution. To accomplish this, we start from the fact that since *(i)* $x(t_0) = x_0$, *(ii)* the solution starts in S, and *(iii)* it is continuous, it must lie in S for some interval of time. Suppose that $x(t)$ leaves the ball $B_r(x_0)$, and that the first time that $x(t)$ intersects the boundary of the ball is at $t_0 + \mu$. Then $\|x(t_0+\mu) - x_0\| = r$. In addition, for all $t \in [t_0, t_0+\mu]$, we can write

$$\begin{aligned}
\|x(t) - x(t_0)\| &\leq \int_{t_0}^{t} \left(\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|\right) ds \\
&\leq \int_{t_0}^{t} \left(L\,\|x(s) - x_0\| + h\right) ds \leq \int_{t_0}^{t} (Lr + h)\, ds.
\end{aligned}$$

Thus, $r = \|x(t_0+\mu) - x_0\| \leq (Lr+h)\mu$, which implies that $\mu \geq \frac{r}{Lr+h} \geq \delta$.

This implies that the time $t_0+\mu$ at which the solution $x(t)$ would leave the ball $B_r(x_0)$ is greater than $t_0+\delta$, or in other words, the solution $x(t)$ cannot leave the ball $B_r(x_0)$ within the time interval $[t_0, t_0+\delta]$, which implies that any solution in X over the time interval $[t_0, t_0+\delta]$ must lie in S. Consequently, uniqueness of the solution in S implies uniqueness in X. □

Corollary 3.2.2

Suppose that $f(t,x)$ is piecewise continuous in t and there exists $r > 0$, $T > t_0$ and $0 < L < \infty$ such that, for all $x, y \in B_r(x_0)$ and $t \in [t_0, T]$, we have $\left\|\frac{\partial f}{\partial x}(t,x)\right\| \leq L$. Then there exists $\delta > 0$ such that the differential equation $\dot{x} = f(t, x(t))$ has exactly one solution over $[t_0, t_0+\delta]$.

Remark.

> See also Lemmas 3.1, 3.2, 3.3.

Example.

> Consider the system $\dot{x} = f(x)$ with
>
> $$f(x) = \begin{pmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{pmatrix}.$$
>
> This is continuously differentiable on $\mathbb{R}^2$. Let us compute a Lipschitz constant over the convex set $W = \left\{x \in \mathbb{R}^2 \;\middle|\; |x_1| \leq a_1, |x_2| \leq a_2\right\}$. The Jacobian is
>
> $$\frac{\partial f}{\partial x} = \begin{pmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{pmatrix}.$$

Use $\|\cdot\|_\infty$ for vectors in $\mathbb{R}^2$ and the induced matrix norm for matrices. Then

$$\left\|\frac{\partial f}{\partial x}\right\| = \max\{|x_1| + |-1 + x_2|, |x_2| + |1 - x_1|\}.$$

For all points in W, we have $|-1+x_2|+|x_1| \leq 1+a_2+a_1$ and $|1-x_1|+|x_2| \leq a_2+1+a_1$. Hence, $\left\|\frac{\partial f}{\partial x}\right\|_\infty \leq 1 + a_1 + a_2$ and $L = 1 + a_1 + a_2$.

Lemma 3.2.3: (Lemma 3.1)

Let $f\colon [t_0, t_1] \times D \to \mathbb{R}^m$ be a continuous function for some open and connected set, also called a domain $D \in \mathbb{R}^r$. Suppose that $\frac{\partial f}{\partial x}$ exists and is continuous on $[t_0, t_1] \times D$. If for some convex subset $W \subseteq D$ there is a constant $L \geq 0$ such that $\left\|\frac{\partial f}{\partial x}(t, x)\right\| \leq L$ on $[t_0, t_1] \times W$, then

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|$$

for all $t \in [t_0, t_1]$ for all $x, y \in W$.

The lemma tells us that a continuously differentiable function with a bounded derivative on a convex set W satisfies the Lipschitz condition on the set $W \subseteq D$. In fact, the derivative bound is the Lipschitz constant. This leads to the following result:

Lemma 3.2.4: (Lemma 3.2)

If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times D$ for some domain $D \subseteq \mathbb{R}^n$, then f is locally Lipschitz in x on $[t_0, t_1] \times D$.

An extension to the global Lipschitz condition is given in Lemma 3.3.

Lemma 3.2.5: (Lemma 3.3)

If $f(t, x)$ and $\frac{\partial f}{\partial x}(t, x)$ are continuous on $[t_0, t_1] \times \mathbb{R}^n$, then f is globally Lipschitz in x on $[t_0, t_1] \times \mathbb{R}^n$ if and only if $\frac{\partial f}{\partial x}(t, x)$ is uniformly bounded[a] on $[t_0, t_1] \times \mathbb{R}^n$.

[a] The same bound applies for all $x \in \mathbb{R}^n$.

Theorem 3.2.6: Global existence and uniqueness

Suppose that $f(t, x)$ is piecewise continuous in t and satisfies $\|f(t, x) - f(t, y)\| \leq L \|x - y\|$ for all $x, y \in \mathbb{R}^n$ and for all $t \in [t_0, t_1]$. Then the state equation $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

Proof. See Appendix C. The key point is to show that the constant δ of Theorem 3.1 can be made independent of the initial state x_0. $\square$

Example.

Consider $\dot{x} = 1 + x^2$, $x(t_0) = 0 \in \mathbb{R}$. Check the Lipschitz condition for $-2 < x < 2$. We have $\left\|\frac{\partial f}{\partial x}\right\| = |2x| \leq 4$. Hence, there exists $\delta > 0$ such that the solution exists and is unique on $[t_0, t_0 + \delta]$.

Example.

Consider $\dot{x} = Ax(t) + Bu(t)$ where $t \in [t_0, \infty)$ and $x(t_0) = x_0 \in \mathbb{R}^m$. The input u is piecewise continuous and A and B are constants. We check the Lipschitz condition in some norm on $\mathbb{R}^n$:

$$\|f(t,x) - f(t,y)\| = \|A(x-y)\| \leq \|A\|_i \|x-y\| .$$

Hence, the solution exists globally and is unique.

Example.

Consider $\dot{x} = x^{2/3}$. Claim: $x^{2/3}$ is not Lipschitz continuous at 0. Observe that the derivative of this function is arbitrarily large as we approach 0. Suppose that this function is Lipschitz continuous at zero. Then there exists $r > 0$ and $0 < L < \infty$ such that for all $x, y \in B_r(0)$

$$|x^{2/3} - y^{2/3}| \leq L \|x-y\| .$$

Setting $y = 0$, we have that for all $x \in B_r(0)$, $|x^{2/3}| \leq L|x|$. Then, for $x \neq 0$, we can divide both sides by $|x|$:

$$\left|\frac{1}{x^{1/2}}\right| \leq L.$$

But the left-hand side is arbitrarily large as x approaches the origin from the right, which is a contradiction to the Lipschitz continuity.

Remark.

Since the RHS of the ODE is not locally Lipschitz continuous, our theorems on local existence and uniqueness do not apply.

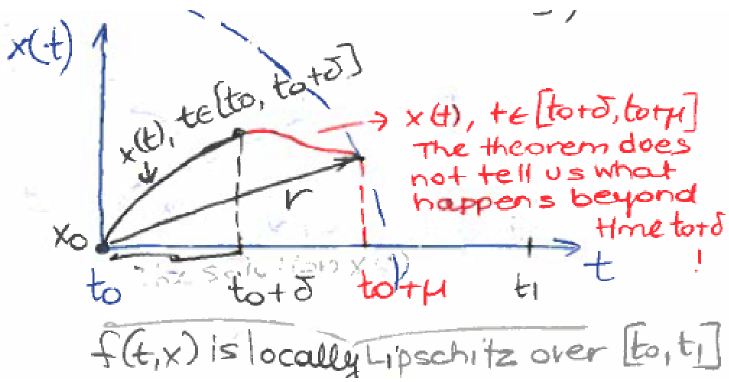
3.2.1 Recap of Theorems 3.1 and 3.2

What do the existence and uniqueness theorems tell us?

Let us start with the local existence and uniqueness Theorem (3.1). The theorem says that if the function $f(t,x)$ is piecewise continuous in t and satisfying the local Lipschitz property

$$\|f(t,x) - f(t,y)\| \leq L \|x-y\|$$

for all $x, y \in B_r(x_0)$ and all $t \in [t_0, t_1]$, then we can find a $\delta > 0$ such that the solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ exists and is unique on the time interval $[t_0, t_0 + \delta] \subseteq [t_0, t_1]$.



To better understand the physical meaning of the mathematical statement, it helps to go over the proof. The key points of the proof are

- We construct the set $S = \{x \in X \mid \|x - x_0\| \leq r\}$ and show that

$$x(t) = x_0 + \int_{t_0}^{t_0+\delta} f(s, x(s))ds$$

  is a mapping from S to S.

- That means that we prove that $x(t) = Px(t)$ exists in the set S and the time interval $[t_0, t_0 + \delta]$.

- Finally, we prove that the solution of $\dot{x}(t) = Px(t)$ is unique on $X = (C(t_0, t_0 + \delta), \|\cdot\|_C)$ over the time interval $[t_0, t_0 + \delta]$. To do this, we prove that the first time $t_0 + \mu$ that the trajectory would leave the set S is greater than $t_0 + \delta$. That means that for $t \in [t_0, t_0 + \delta]$, the unique solution in S is also a unique solution in X.

- In summary, if $f(t, x)$ is Lipschitz locally in X on a ball $B_r(x_0)$ over a time interval $[t_0, t_1]$, then we can guarantee that the ODE $\dot{x} = f(t, x)$, $x(t_0) = x_0$ will have a unique solution over $[t_0, t_0 + \delta] \subseteq [t_0, t_1]$.

Some additional remarks:

- We saw that Theorem 3.1 is a local theorem in the sense that it guarantees the existence and uniqueness of a solution over a time interval $[t_0, t_0 + \delta]$, where $\delta > 0$ might be very small. In other words, we have no control over δ, i.e., we cannot guarantee existence and uniqueness over a given interval $[t_0, t_1]$.

- How can we pursue a larger time interval for existence and uniqueness? We can try to extend the existence and uniqueness interval by repeated applications of the local theorem. That is, take $t_0 + \delta$ and out new initial time and $x(t_0 + \delta)$ as the new initial condition and try to apply Theorem 3.1 to ensure existence and uniqueness beyond $t_0 + \delta$. If the conditions of the theorem are satisfied, then there exists $\delta_2 > 0$ such that the original equation has a unique solution in $[t_0, t_0 + \delta + \delta_2]$.

- This idea can be applied repeatedly to keep extending the interval of the solution; however, in general the interval of solution may not be extended to infinity.

- In fact, there is a maximum interval $[t_0, T)$, where the unique solution starting at $(t_0, x(t_0))$ exists.
- Also, in general, T may be smaller than t_1, which means that as $t \to T$, the solution leaves the compact set over which f is locally Lipschitz in x over $[t_0, t_1]$.

Example.

> Consider $f(x) = -x^2$. We have $f'(x) = -2x$. This function is locally Lipschitz on every point of $\mathbb{R}$, but it is not globally Lipschitz since f' cannot be uniformly bounded on $\mathbb{R}$.
>
> However, we have that $f(x)$ is locally Lipschitz on any compact subset of $\mathbb{R}$.

Remark.

> The example above verifies that the condition of $f(t, x)$ being locally Lipschitz over $[t_0, t_1]$ does not guarantee that the solution $x(t)$ will exist and be unique over $[t_0, t_1]$, i.e., for arbitrary time t_1. The above problem is alleviated by having $f(t, x)$ satisfy the global Lipschitz condition as stated by the global existence and uniqueness theorem.

Remark.

> We should note that the global Lipschitz condition is restrictive. Many models of physical systems fail to satisfy it, still the models do have unique solutions. It is thus useful to have a global existence and uniqueness result that only requires the function to be locally Lipschitz.

Theorem 3.2.7: (Theorem 3.3)

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and for all $x \in D \subseteq \mathbb{R}^n$. Let W be a compact subset of D and suppose that it is known that every solution of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ lies entirely in W. Then there is a unique solution that is defined for all $t \geq t_0$.

Remark.

> The trick in applying Theorem 3.3 is in checking that every solution lies in W without actually solving the differential equation. The techniques of Chapter 4 will help us do that.

Chapter 4

Introduction to Lyapunov stability theory: a geometric picture

4.1 Mathematical background for stability

Definition 4.1.1

A subset $S \subseteq \mathbb{R}^m$ is bounded if there exists $k < \infty$ such that for all $x \in S$, $\|x\| \leq k$. Equivalently, one can say that $S \subseteq \mathbb{R}^n$ is bounded if there exists $k < \infty$ such that $S \subseteq B_k(0)$.

Lemma 4.1.2

Suppose that $V \colon \mathbb{R}^n \to \mathbb{R}$ is continuous. Then for all $x \in \mathbb{R}$, the set of points (i.e., the sublevel sets)

$$\mathcal{L}(c) = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$$

is closed.

Proof. We will use the sequence characterization of closed sets. Let $\{x_n\}_n$ be a sequence of points in $\mathcal{L}(c)$. This means that for all $n \geq 1$, $V(x_n) \leq c$ and such that $x_n \to \bar{x}$. We want to show that $\bar{x} \in \mathcal{L}(c)$.

Since V is continuous, $\lim_{n\to\infty} V(x_n) = V(\bar{x})$. Moreover, by definition, we have for all $n \geq 1$, $V(x_n) \leq c$. This means that $V(x_n)$ is a convergent sequence in $(-\infty, c] \subseteq \mathbb{R}$; by definition, $(\infty, c]$ is a closed set.

Now we have that the limit point $V(\bar{x})$ of $V(x_n)$ should belong to $(-\infty, c]$ because if a set S is closed, then it contains its limits points. Thus, we showed that $\bar{x} \in \mathcal{L}(c)$, i.e., $V(\bar{x}) \leq c$. $\square$

Definition 4.1.3

We often denote $\mathbb{R}_{\geq 0} := [0, \infty)$. A function is radially unbounded if for all $c < \infty$ there exists $k < \infty$ such that $\|x\| > k \Rightarrow V(x) > c$.

This definition essentially means that V is unbounded as $\|x\| \to \infty$.

Lemma 4.1.4

Suppose that $V\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and $V(x) \to \infty$ as $\|x\| \to \infty$. Then for all $c \geq 0$, $\mathcal{L}(x)$ are closed and bounded.

Proof. We already showed that this set is closed. Now we have to show that it is bounded. Fix $c \geq 0$. We will show that $\mathcal{L}(c)$ is not bounded, then V is not radially unbounded.

If $\mathcal{L}(c)$ is not bounded then for all $k < \infty$, there exists $x \in \mathcal{L}(c)$ with $\|x\| > k$. That means that for all $k < \infty$ there exists x such that $V(x) \leq c$ and at the same time $\|x\| \geq k$, which implies that V is not radially unbounded. This completes the proof. □

Definition 4.1.5

A subset $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

Fact 4.1.6

This definition is not valid for infinite-dimensional normed spaces!

Definition 4.1.7: Weierstrass Theorem

Suppose that $f\colon S \to \mathbb{R}$ is continuous and that S is compact. Then f achieves a minimum and a maximum on S; that is, there exists $s^* \in S$ and $s_* \in S$ such that $f(s^*) = \sup_{x \in S} f(x)$ and $f(s_*) = \inf_{x \in S} f(x)$.

Example.

- Observe that $S = [0, \infty)$. Then S is closed by not bounded. Consider the function $f(x) = \frac{1}{1+x^2} > 0$ for all $x \in S$. We have $\inf_{x \in S} f(x) = 0$. But there is no $s \in S$ such that $f(s) = 0$.
- Let $S = (0, 1) \subseteq \mathbb{R}$. This set is bounded but not closed. We have $\inf_{s \in S} f(s) = 0$, but the minimum is not achieved in S. Similarly, $\sup_{x \in S} f(s) = 1$, but the maximum is not achieved in S.

Definition 4.1.8

Let $S \subseteq \mathbb{R}^n$. A function $h\colon S \to \mathbb{R}^m$ is uniformly continuous of for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in S$,

$$\|x - y\| < \delta \Rightarrow \|h(x) - h(y)\| < \varepsilon \Rightarrow .$$

Key point: the same δ must work for all x, y. Of course, as ε gets smaller, δ may also get smaller, as it may depend on ε.

Example.

The function $h(x) = e^x$ is continuous at every $x \in \mathbb{R}$, but it is not uniformly continuous. To show this, we negate the statement of uniform continuity, i.e., we want to show that

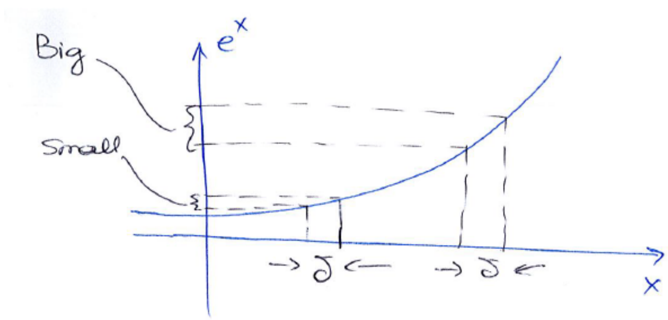
$$\exists \varepsilon > 0. \forall \delta > 0. \exists x, y \in \mathbb{R}.\ |x - y| < \delta \wedge |e^x - e^y| \geq \varepsilon,$$

where the symbol $\wedge$ means logical AND. Let $\varepsilon = 1$ and let $\delta > 0$ be fixed. Let

$$x = -\log(e^{0.9\delta} - 1) \text{ and } y = x + 0.9\delta.$$

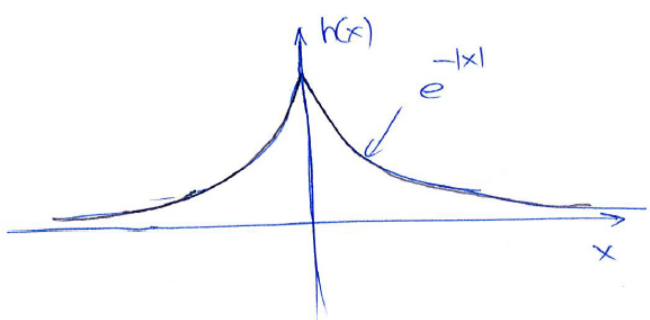
Then $|x - y| = 0.9\delta < \delta$ and

$$|e^x - e^y| = e^x |1 - e^{0.9\delta}| = 1.$$



Example.

$h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) = e^{-|x|}$ is uniformly continuous.



Theorem 4.1.9

If $h\colon S \to \mathbb{R}^m$ is continuous and S is compact, then h is uniformly continuous. In other words, continuity and compactness yields uniform continuity.

Example.

Consider $S = (0, 1]$ and $h(x) = 1/x$. Then h is not uniformly continuous.

Definition 4.1.10

A function $h\colon \mathbb{R} \to \mathbb{R}$ is bounded from below if there exists $m > \infty$ such that $h(x) \geq m$ for all $x \in \mathbb{R}$.

Definition 4.1.11

A function $h\colon \mathbb{R} \to \mathbb{R}$ is non-increasing if $y \geq x$ implies that $h(y) \leq h(x)$.

Remark.

The notions of bounded from above and non-decreasing are defined similarly.

Example.

Note that to be non-increasing, a function can be constant for a while, then decreasing for a while, and so on. It does not have to be *strictly decreasing* (i.e., $x < y \Rightarrow h(x) > h(y)$) in order to be non-increasing. Neither does it have to be continuous.

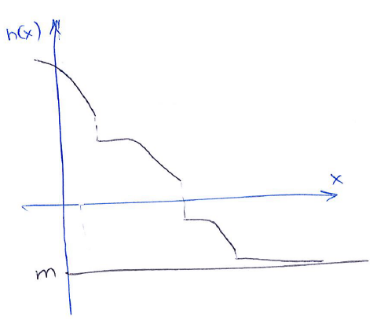


Fig ① Example of bounded from below and non-increasing.

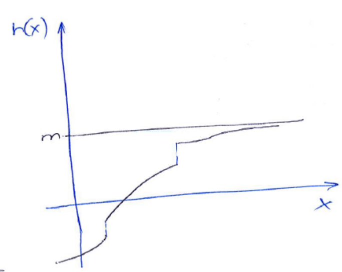


Fig ② Example of bounded from above and non-decreasing.

Theorem 4.1.12

If $h\colon \mathbb{R} \to \mathbb{R}$ is non-increasing and bounded from below, then there exists a unique $c \in \mathbb{R}$ such that $\lim_{x\to\infty} h(x) = c$. Similarly, if h is non-decreasing and bounded from above, then there exists a unique $c \in \mathbb{R}$ such that $\lim_{x\to\infty} h(x) = c$.

Definition 4.1.13

We say that $\lim_{x\to\infty} h(x) = c$ if for all $\varepsilon > 0$ there exists $k < \infty$ such that $x \geq k$ implies that $|h(x) - c| < \varepsilon$.

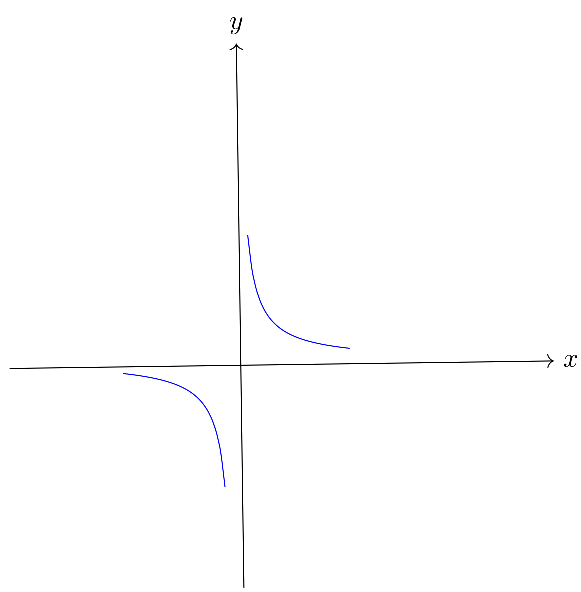
4.2 A geometric picture of stability

Up to now, we dealt with the problem of ensuring that our nonlinear model $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution over a time interval $[t_0, t_0 + \delta]$. We saw that if the function $f(t, x)$ ($t \in [t_0, t_1]$) is locally Lipschitz for all $x, y \in B_r(x_0)$ for all t then we can guarantee that the system has a unique solution over $[t_0, t_0 + \delta]$. Also, if the function $f(t, x), t \in [t_0, t_1]$ is globally Lipschitz for all $x, y \in \mathbb{R}^n$ then we can guarantee that the system has a unique solution over $[t_0, t_1]$.

The nature of these results is as sufficient but not necessary conditions. If the Lipschitz conditions of theorems 3.1 or 3.2 are not satisfied, the conclusions of the theorems do not apply, but this does not mean that the system fails to have (unique) solutions. In other words, we can have a system having unique solutions that fails to be Lipschitz (locally or globally). This is highlighted in example 3.5, where the uniqueness of the solution of $\dot{x} = -x^3, x(t_0) = x_0$ for all t does not imply that $f(x)$ should be globally Lipschitz. In fact, it is not.

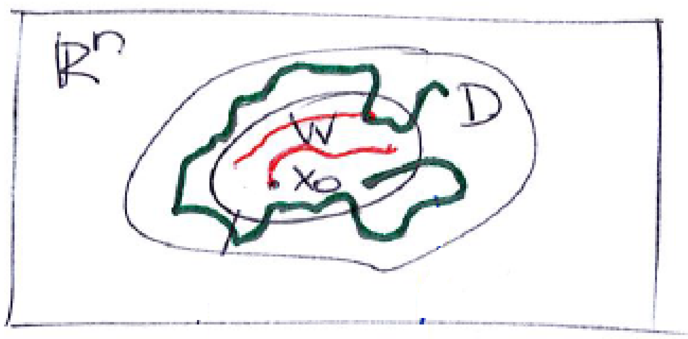
Example.

Consider $\dot{x} = x^{-2}, x(t_0) = x_0$. Obviously, $f(t, x)$ is not continuous at zero, hence it is not Lipschitz at $x = 0$. However, the system has a unique solution given as $x(t) = (3(t - t_0))^{1/3}$ for all $t \geq t_0$.



So, in summary, continuity of $f(t, x)$ implies at least one solution over a time interval. Lipschitz continuity implies a unique solution over a time interval (locally or globally depends on whether the Lipschitz property holds globally or locally).

We say that globally Lipschitz $f(t, x)$ implies existence and uniqueness over a time interval $[t_0, t_1]$, where t_1 can be chosen arbitrarily large. However, many of the models of physical systems do not possess the globally Lipschitz property, but still have unique solutions. Hence, it is useful to have theorems that guarantee global (for all $t \geq t_0$) existence and uniqueness of the solution, while requiring the locally Lipschitz property for $f(t, x)$. This comes at a price, the price being that we will have to know more about the system. Let us consider Theorem 3.3.



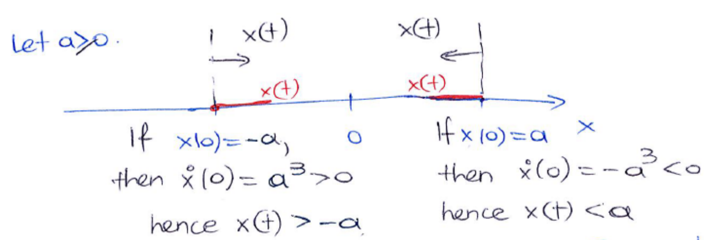
This theorem requires $f(t, x)$ locally Lipschitz on D, a compact subset $W \subseteq D$, $x_0 \in W$, and knowledge that all solutions starting in W remain in W. In other words, we are required to know that the solutions of the system that start from $x_0 \in W$ look like the red trajectories and that there are no solutions that look like the green trajectories. Then the theorem says that the solution from every $x_0 \in W$ is unique defined for all $t \geq t_0$.

While the theorem seems to be putting us in a vicious circle[1], there are ways out. Showing that the solution $x(t)$ remains in W may be non-trivial, but thankfully Lyapunov's methods addresses the challenge without solving the system of differential equations. The key is to show that the solution $x(t)$ is trapped in compact level sets of properly defined functions—called Lyapunov functions.

In the meantime, let us see an application of Theorem 3.3.

Example.

Consider $\dot{x} = -x^3 = f(x)$. This function is locally Lipschitz on $\mathbb{R}$. We want to assess that the solutions starting in a compact set $W \subseteq \mathbb{R}$ always remain in W so that we can apply Theorem 3.3.



[1] How can we guarantee that the solution $x(t)$ lies entirely in W for all $x_0 \in W$ and for all t? Wasn't the theorem supposed to tell us that whether the solution exists and is unique? The theorem requires f to be locally Lipschitz. In that case, we get confirmation about local existence and uniqueness. The theorem then requires that the solution cannot escape W. If those two conditions hold, then the theorem tells us that the solution exists and is unique for all $t \geq t_0$

That means that the solution is always in $W = \{x \in \mathbb{R} \mid |x| < a\}$. Hence, without calculating the solution, we can guarantee that the solution remains bounded in a closed set W, and the theorem assures us that it is unique and exists for all $t \geq 0$.

Now, the idea of ensuring that the solutions of the system evolve according to smaller and smaller closed and bounded level sets of what we call Lyapunov functions is fundamental in Lyapunov's stability theory, which studies the stability of the equilibrium points of $\dot{x} = f(t, x)$.

Example.

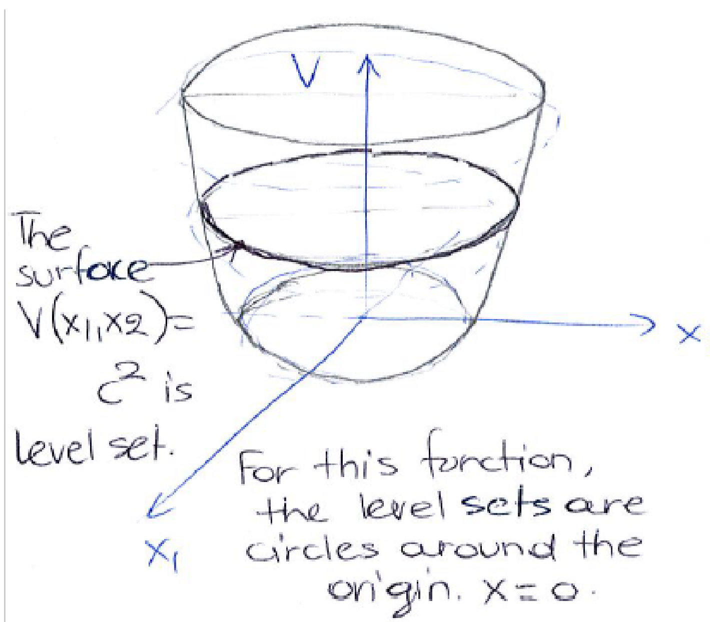
Le us see the application of Lyapunov's method on a linear system first, to connect stability for LTI systems with the geometric interpretation of a Lyapunov function.

Consider the harmonic oscillator $\ddot{y} + y = 0$ for $y \in \mathbb{R}$. For state space form, we define $x_1 = y$ and $x_2 = \dot{y}$. Thus,

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x = f(x).$$

We note that $\det(A) \neq 0$, hence $x_e = 0$ is the only equilibrium point. The eigenvalues of A are $\pm j$. Since they are imaginary, we know that the solutions will be periodic trajectories. Hence, we conclude that the system is stable, but not asymptotically stable.

Let us build the elements of Lyapunov's theory for this simple example. The idea is to define an energy-like function $V(x) = x_1^2 + x_2^2$.



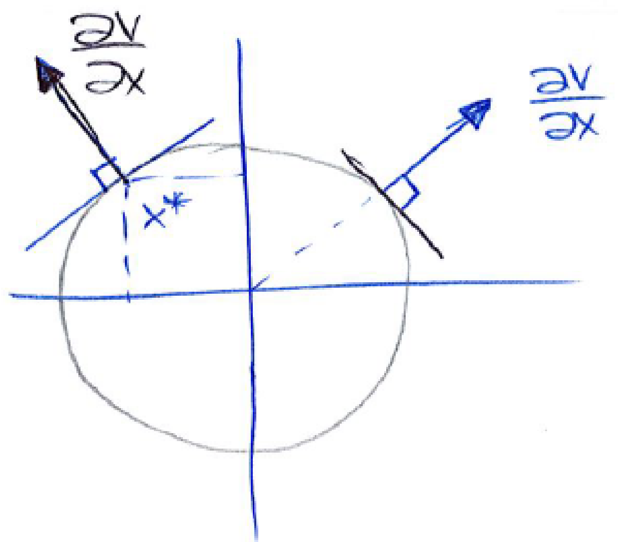
We call this energy-like because we can think of x_1 as the spring displacement and of x_2 as the mass velocity of the mass-spring system oscillating without friction.

Observe that given a constant $c > 0$, we have $V(x_1, .x_2) \leq c^2 \Rightarrow x_1^2 + x_2 \leq c^2$. This means that the sublevel sets of V are circular discs. In fact, they are closed and bounded (i.e., compact) sets for this particular choice of V.

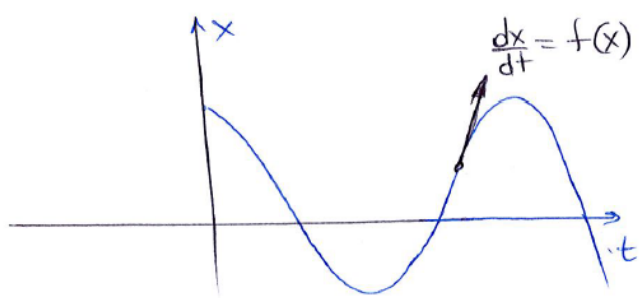
Let us define the gradient of the function $V(x)$:

$$\frac{\partial V}{\partial x} = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{pmatrix}.$$

The gradient vector at x^* is orthogonal to the tangent line of the level set $\{x \in \mathbb{R}^2 \mid V(x) = c^2\}$ at x^*.



Recall what our $\dot{x} = f(t, x)$ looks like:



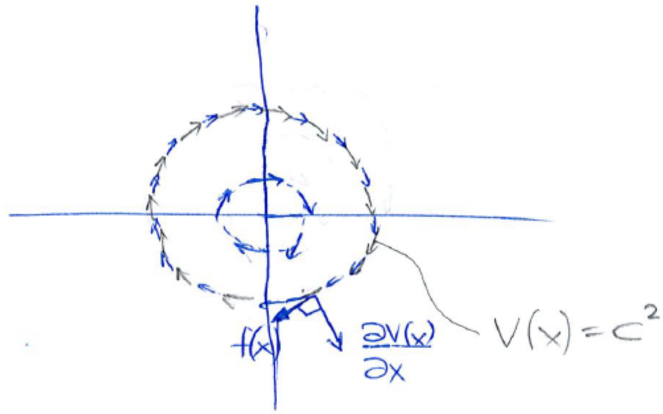
Recall also that the inner product $\langle \cdot, \cdot \rangle$ between two vectors is positive when both vectors point in the same direction, and negative when they point in opposite directions. Since $\dot{x} = f(t, x)$, we have the following situation:

- $\left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle = 0$ means that the trajectory evolves tangent to $V(x) = c^2$, i.e., along a direction of constant $V(x)$.
- $\left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle > 0$ means that the trajectory evolves in a direction of increasing $V(x)$.
- $\left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle < 0$ means that the trajectory evolves in a direction of decreasing $V(x)$.

Note that

$$\dot{V}(x(t)) = \frac{\partial V}{\partial x}(x(t)) \cdot \dot{x}(t) = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle.$$

Back to the example, $\dot{V}(x) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = 0$. This means that V remains constant along the trajectories of our system, i.e., the system evolves along circles. The phase portrait would look as follows:



Indeed, we expected for the equilibrium to be stable, i.e., trajectories remaining close to the equilibrium.

Example.

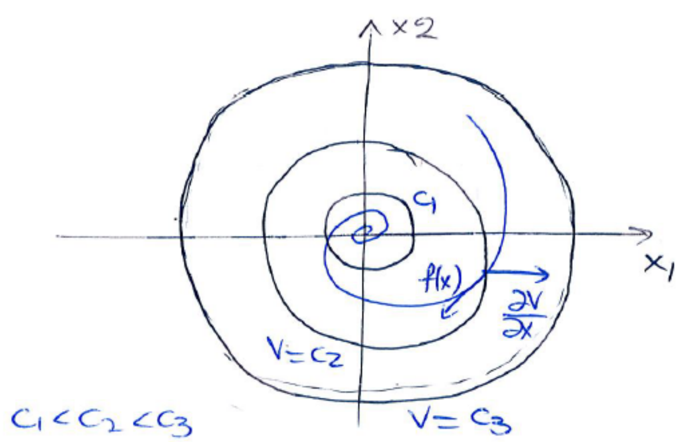
Consider the system

$$\dot{x} = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x = f(x).$$

We keep the same $V(x) = x_1^2 + x_2^2$. Compute

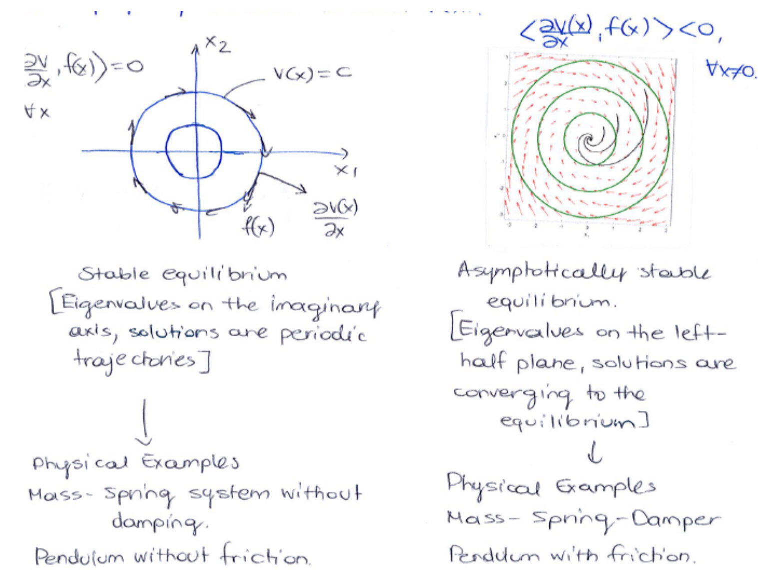
$$\dot{V}(x) = \begin{pmatrix} x_1 & 2x_2 \end{pmatrix} \begin{pmatrix} -1/2x_1 + x_2 \\ -x_1 - 1/2x_2 \end{pmatrix} = -x_1^2 - x_2^2 < 0.$$

Hence, we have $\left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle < 0$ for $x \neq 0$. This means that the solutions evolve according to lower and lower level sets of V.



4.3 Definitions and main stability theorem

We just considered a geometric approach to Lyapunov stability. More specifically, we considered the phase portraits of linear systems with stable and asymptotically stable equilibria, respectively, and studied their evolution relative to the level sets of a properly defined function $V(s)$.



Remark.

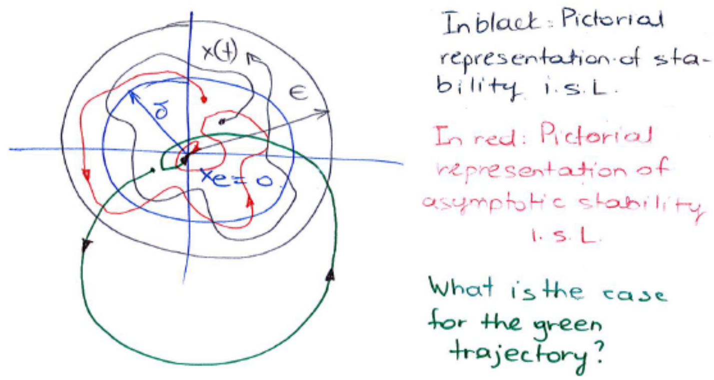
> The choice of the function $V(x)$ is not arbitrary. The physical motivation is that $V(x)$ expresses the total energy of the system. But for applying Lyapunov's method to investigate the stability of the equilibrium, we do not need V to represent a notion of energy. We will see guidelines of how to select a function V later on.

Now we will define formally the notion of stability of an equilibrium in the sense of Lyapunov.

Definition 4.3.1: Stability

Let $\dot{x} = f(x)$ be an ODE in $\mathbb{R}^n$ with equilibrium point x_e. Suppose there exists $\rho > 0$ such that for each initial condition $x_0 \in B_\rho(x_e)$, a solution $x(t, x_0)$ of the ODE exists on $[0, \infty)$ and is unique. Then the equilibrium x_e is

- *stable in the sense of Lyapunov* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0 - x_e\| < \delta \Rightarrow \|x(t, t_0) - x_e\| < \varepsilon$ for all $t \geq 0$;
- *unstable* if it is not stable;
- *asymptotically stable* if it is stable in the sense of Lyapunov and there exists $\eta > 0$ such that $\|x_0 - x_e\| < \eta \Rightarrow \lim_{t\to\infty} \|x(t, x_0) - x_e\| = 0$.



Remark.

> Define $\psi\colon \mathbb{R}^n \to C(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)$ as $\psi(x_0) = x(\cdot, x_0)$. That is ψ maps initial conditions to solutions of the ODE. Then, stability in the sense of Lyapunov is equivalent to continuity of ψ at x_e, where the sup norm is used on $C(\mathbb{R}_{\geq 0} \to \mathbb{R}^n)$.

Remark.

> The extra condition in the notion of asymptotic stability is often called *attractivity*. Thus, asymptotic stability can be understood as stability in the sense of Lyapunov plus attractivity.

Lyapunov's theorem provides us with the tools to investigate the stability properties of the equilibrium point x_e of $\dot{x} = f(x)$. We will also see how the fundamental existence and uniqueness theorems (Theorems 3.1, 3.2, and 3.3) are key in the investigations of stability via Lyapunov's method. In other words, our analysis will show that these theorems are not mathematical abstractions without application.

The best way to understand the physical interpretation of a theorem is to go through the proof[2]

[2] One of our professors used to say that proofs are what matters in theorems. The theorems themselves are like lampposts that remind us of statements we know to be correct.

Theorem 4.3.2: Lyapunov's direct method

Let $\dot{x} = f(x)$ be an ODE on $\mathbb{R}^n$ with equilibrium point $x_e = 0$. Assume there exists an open set D containing the origin such that:

- $f : D \to \mathbb{R}^n$ is locally Lipschitz; and
- there exists a function $V \in C^1(D, \mathbb{R})$ (i.e., V is a continuously differentiable function from D to $\mathbb{R}$) such that
  - $V(0) = 0$ and
  - $V(0) > 0$ for $x \in D \setminus \{0\}$ (i.e., V is positive everywhere outside of zero).

Then we may conclude the following:

- If $\dot{V}(x) \leq 0$ for $x \in D$, then the equilibrium point is stable.
- If $\dot{V}(x) < 0$ for $x \in D \setminus \{0\}$, then the equilibrium point x_e is asymptotically stable.

Proof. To show that the system is stable, we must establish that

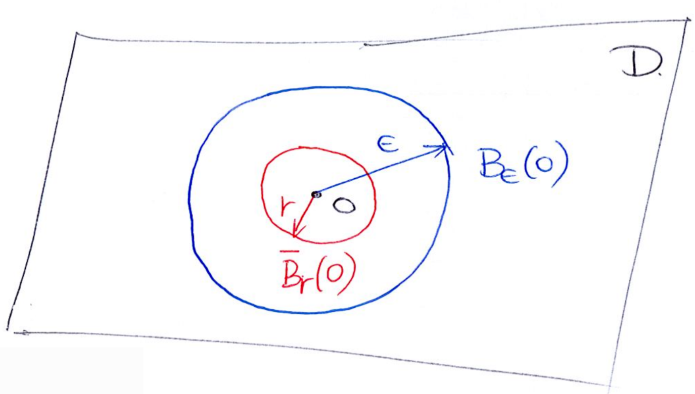
1. there exists $\rho > 0$ such that for all $x_0 \in B_\rho(0)$, a solution $x(t, x_0)$ exists and is unique on $[0, \infty)$; and
2. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \varepsilon$ for all $t \geq 0$.

We will show these two points at the same time by taking $\rho = \delta$:

Let $\varepsilon > 0$ be given. Since D is an open set containing zero, we can choose $0 < r < \varepsilon$ such that

$$\bar{B}_r(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq D.$$

Note we use the $\bar{(\cdot)}$ notation to mean the closure of a set (i.e., the smallest closed set that contains the given set). We have the following situation:



$B_\varepsilon(0)$ is an open set. Our objective is to find a ball of radius $\delta > 0$ satisfying the properties, namely, (i) for all $x_0 \in B_\delta(0)$, a solution $x(t, x_0)$ exists and is unique on $\mathbb{R}_{\geq 0}$, and (ii) if $x_0 \in B_\delta(0)$ then $x(t, x_0) \in B_\varepsilon(0)$ for all $t \in \mathbb{R}_{\geq 0}$.

Observe that we choose $\bar{B}_r(0)$ to be closed. The reason is to create a set that is closed and bounded, hence compact. The usefulness of compact sets in our analysis lies in the fact that a continuous function on a compact set attains its max and min values in the set.

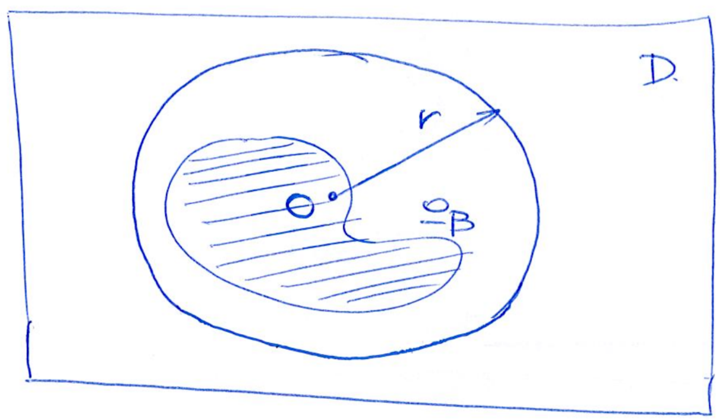
Let

$$a := \min_{\|x\|=r} V(x).$$

Note that we can define the minimum of V over $\|x\| = r$ since this set is closed and bounded (application of the Weierstrass theorem). From the definition of V, we know that $V(a) > 0$ because V is nonnegative and only zero at $x = 0$, which is not in the boundary of $B_r(0)$.

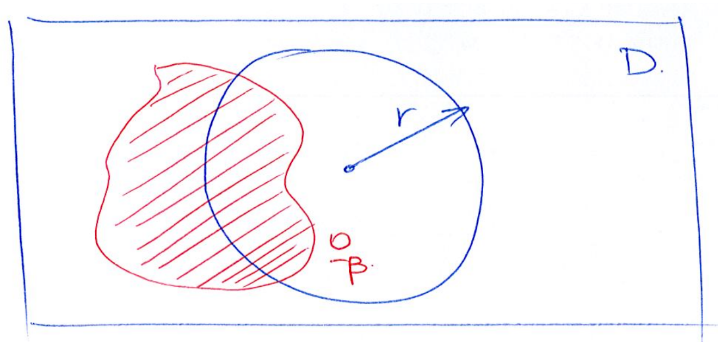
As a next step, we choose $0 < \beta < a$ and define the set

$$\Omega_\beta = \left\{x \in \bar{B}_r \;\middle|\; V(x) \leq \beta\right\}.$$



By definition, $\Omega_\beta \subseteq \bar{B}_r(0)$, but we actually have a stronger condition: $\Omega_\beta \subseteq B_r(0)$. Note that this means that Ω_β does not reach the boundary of $\bar{B}_r(0)$. To show this, consider $\|x\| = r$. For such x, we have $V(x) \geq a > \beta$, which means that $x \notin \Omega_\beta$.

By establishing that $\Omega_\beta \subseteq \bar{B}_r(0)$, we know that Ω_β looks like the diagram above and not like the diagram below.



The situation of this figure is not good for the following reason: if Ω_β spills out of $B_r(0)$, this would mean that a solution can start with $V(x_0) \leq \beta$, that is, $x_0 \in \Omega_\beta$, continue to satisfy $V(x(t, x_0)) < \beta$ and yet leave the set $B_r(0)$. However, we want to force the solution to be trapped inside $B_r(0)$. Note, in contrast, that the situation in the first figure is what we want because solutions that start with $x_0 \in \Omega_\beta$ will have to stay within Ω_β, which means they cannot leave the set $B_r(0)$.

Claim

Solutions of $\dot{x} = f(x)$ with $x_0 \in \Omega_\beta$ remain in Ω_β. Thus, by Theorem 3.3, we conclude that for all $x_0 \in \Omega_\beta$, solutions exist on $\mathbb{R}_{\geq 0}$ and are unique.

Proof for Claim.

Let $\phi(t)$, $t_0 \leq t \leq t_1$ be a solution of $\dot{x} = f(x)$ with $\phi(t_0) \in \Omega_\beta$. We want to show that $\phi(t) \in \Omega_\beta$ for all $t_0 \leq t \leq t_1$.

Since $\dot{V}(x) \leq 0$, we have $V(\phi(t)) \leq V(\phi(t_0))$. Also, since $\phi(t_0) \in \Omega_\beta$, we have $V(\phi(t_0)) \leq \beta$. Thus, $V(\phi(t)) \leq V(\phi(t_0)) \leq \beta$. Hence, we conclude that $\phi(t) \in \Omega_\beta$ for $t_0 \leq t \leq t_1$. ∎

Remark.

We have from Theorem 3.3 that a unique solution $x(t, t_0)$ exists on $\mathbb{R}_{\geq 0}$ for all $x_0 \in \Omega_\beta$. Also, since it never escapes Ω_β and since

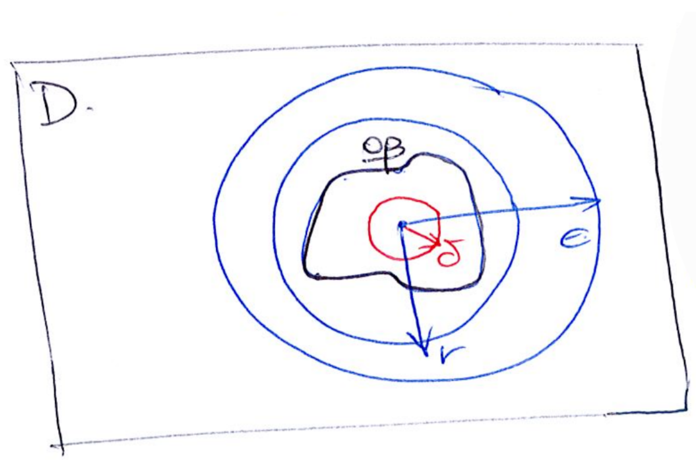
$$\Omega_\beta \subseteq B_r(0) \subseteq B_\varepsilon(0),$$

we have that the solution never escapes $B_\varepsilon(0)$.

Remark.

To conclude stability, we need to show that there exists $\delta > 0$ such that $B_\delta(0) \subseteq \Omega_\beta$.

Up to now, we have that trajectories starting in Ω_β stay in Ω_β, hence stay in $B_\varepsilon(0)$. We now need to show that there exists a $B_\delta(0)$ in Ω_β.



Claim

There exists $\delta > 0$ such that $B_\delta(0) \subseteq \Omega_\beta$.

Proof for Claim.

> Since V is continuous at 0, there exists $0 < \delta < r$ such that $\|x - 0\| < \delta \Rightarrow |V(x) - V(0)| < \beta$. Now, since $V(0) = 0$ and $V(x) \geq 0$, we can write $\|x\| < \delta \Rightarrow |V(x)| < \beta$. That is, $B_\delta(0) \subseteq \Omega_\beta$.

We have finished showing stability in the sense of Lyapunov. Now our objective is to address the asymptotic stability in the second part of the theorem.

We assume that $\dot{V}(x) < 0$ for $x \in D \setminus \{0\}$ and will show that there exists $\eta > 0$ such that $\|x_0\| < \eta \Rightarrow \lim_{t\to\infty} x(t, x_0) = 0$. We will in fact show that $\delta = \eta$ works, where δ was defined in the proof to the previous statement.

Claim

If $\|x_0\| < \delta$, then $\lim_{t\to\infty} V(x(t, x_0)) = 0$.

Proof for Claim.

> If $\|x_0\| < \delta$, then $x(t, x_0) \in \Omega_\beta \subseteq D$ for all $t \geq 0$. This implies that $\dot{V}(x(t, x_0)) \leq 0$ for all $t \geq 0$. Thus, $V(x(t, x_0))$ is a non-increasing function of t that is bounded below by zero. Hence, there exists a unique $c \in \mathbb{R}_{\geq 0}$ such that $\lim_{t\to\infty} V(x(t, x_0)) = c$.
>
> We know that $V(x(t, x_0))$ converges to a constant $c \geq 0$. Since $\dot{V}(x) \leq 0$, we know that $V(x(t, x_0))$ is non-increasing. Since $V(x) \geq 0$, we have that $V(x, t_0)$ is bounded from below. These two properties guarantee that $V(x(t, x_0))$ has a limit $c \geq 0$.

Now, to conclude that $x(t, x_0) \to 0$, we need to show that $c = 0$. We prove the argument by contradiction.

Assume that $c > 0$. We have that $V(x(t, x_0)) \geq 0$ for all $t \geq 0$. This is true because $x(t, x_0)$ is trapped in Ω_β, and we know that $V(x) \geq 0$ in D. We also know that, given the continuity of V at zero, there exists a $d > 0$ such that

$$\|x\| < d \Rightarrow V(x) < c/2.$$

Since $V(x(t, x_0))$ is a decreasing function of t with limit c, we have that $V(x(t, x_0)) \geq c$ for all $t \geq 0$. This implies that $x(t, x_0)$ never enters Ω_β and thus never enters $B_d(0)$.

Now, we have that on the compact set $\{x \in D \mid d \leq x \leq r\}$, $\dot{V}(x)$ is continuous. By the Weierstrass theorem, it attains a maximum value in the set. In addition, since $\dot{V}(x) < 0$ everywhere except for the origin, it follows that the maximum value of $\dot{V}$ on this set is negative. Let

$$-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x) < 0.$$

Using the fundamental theorem of calculus,

$$\begin{aligned} V(x(t, x_0)) &= V(x_0) + \int_0^t \dot{V}(x(\tau, x_0)) d\tau \\ &\leq V(x_0) - \gamma \int_0^t d\tau = V(x_0) - \gamma t, \end{aligned}$$

which tends to $-\infty$ as t tends to ∞. This implies that $V(x(t, x_0))$ becomes negative for sufficiently large t, which is a contradiction since $x(t, x_0) \in \Omega_\beta$ and $V(x) \geq 0$ in Ω_β. Therefore, it cannot be the case that $c > 0$, and we must have $c = 0$. This implies that $\lim_{t\to\infty} V(x(t, x_0)) = 0$.

□

Remark.

Note that we used the following three principles when analyzing $\dot{x} = f(x)$, $x \in D \subseteq \mathbb{R}^n$, where D is an open set and $V\colon D \to \mathbb{R}$ is continuously differentiable, and f is locally Lipschitz on D.

- Principle 1. Suppose $\dot{V}(x) \leq 0$ for $x \in D$. Then for all $t \geq 0$ for which a solution to the system exists and remains in D, we have

$$V(x(t, x_0)) \leq V(x_0).$$

- Principle 2. Let $\beta > 0$ and suppose the sublevel set Ω_β is compact and that $\dot{V}(x) \leq 0$ for $x \in D$. Then for all $x_0 \in \Omega_\beta$, a solution $x(t, x_0)$ to the system exists on the unbounded interval $\mathbb{R}_{\geq 0}$, is unique, and remains in Ω_β for all $t \geq 0$.
- Let $C \subseteq D$ be a compact set such that for all $x \in C$, $V(x) > 0$ and $\dot{V}(x) < 0$. Then any solution of the system starting in C must exit C. More precisely, for all $x_0 \in C$, there exists $0 < T < \infty$ such that $x(T, x_0) \notin C$.

We used the last principle in the proof of excluding that $c > 0$. In fact, we proved that the solution cannot stay in the compact set $\{x \in D \mid d \leq \|x\| \leq r\}$ as that leads to a contradiction (i.e., that $V(t, x_0) < 0$ for large t).

Definition 4.3.3

A continuously differentiable function $V\colon D \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0$ on $D \setminus \{0\}$, and $\dot{V}(x) \leq 0$ on D is called a *Lyapunov function.*
The surface $V(x) = c$ for $c > 0$ is called a level surface (or level set), or sometimes Lyapunov surface.

Remark.

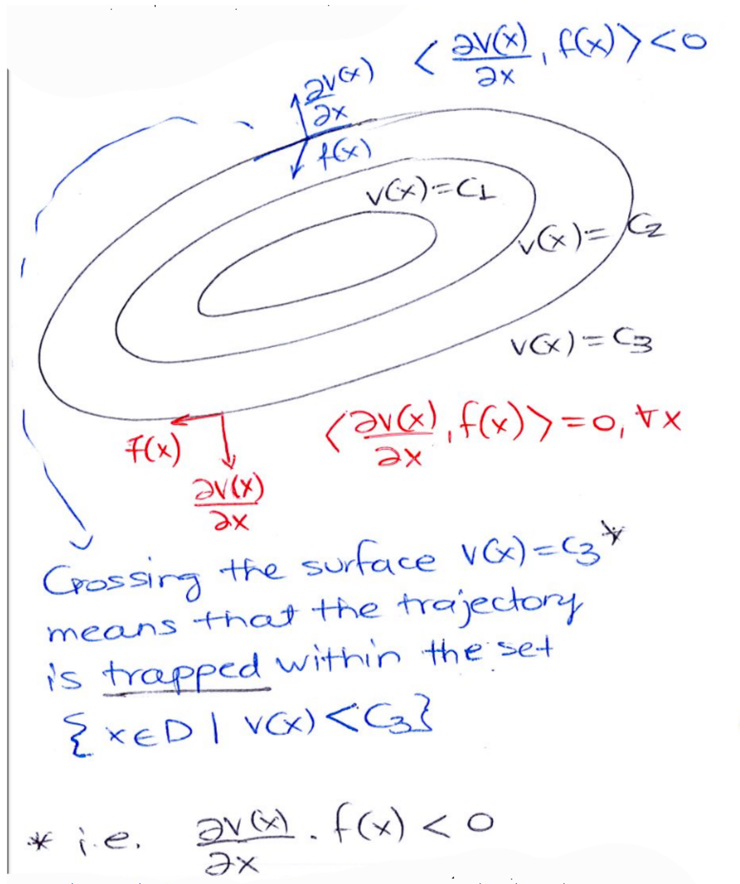
The condition

$$\dot{V}(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle \leq 0$$

implies that a when a trajectory crosses a Lyapunov surface $V(x) = c$, it moves inside the set

$$\Omega_c \coloneqq \{x \in D \mid V(x) \leq c\}$$

and cannot leave the set again.



In fact, when $\dot{V}(x) \leq 0$ (i.e. not strictly $\dot{V}(x) < 0$) then Theorem 4.1 does not tell us if the trajectory approaches zero, i.e., if the origin is asymptotically stable. For such cases, LaSalle's Theorem (to follow) can yield the solution.

Definition 4.3.4

A function $V : D \to \mathbb{R}$ such that $V(0) = 0$ and $V(x) \geq 0$ for $x \neq 0$ is called positive semi-definite.
A function $V : D \to \mathbb{R}$ is called negative (semi-) definite if $-V(x)$ is positive (semi-) definite.

Hence, Lyapunov's theorem can be rephrased as follows: the origin is stable if there is a continuously differentiable positive definite function $V(x)$, so that $\dot{V}(x)$ is negative semi-definite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.

Chapter 5

Main stability theorem—examples

We studied the main theorem for investigating the stability properties of an equilibrium point of $\dot{x} = f(x)$. The theorem tells us that if we can find a positive definite function $V : D \to \mathbb{R}$ whose derivative $\dot{V}(x) = \left\langle \frac{\partial V(x)}{\partial x}, f(x) \right\rangle$ is negative semi-definite, then we can conclude that the equilibrium point is stable in the sense of Lyapunov. If the derivative $\dot{V}(x)$ is negative definite, then the theorem tells us that the equilibrium point is asymptotically stable.

Remark.

> Recall that this theorem provides sufficient conditions for stability or asymptotic stability. If we fail to show stability or asymptotic stability with a candidate Lyapunov function $V(x)$, then the theorem does not conclude that the equilibrium point is not stable/asymptotically stable.

Example.

> (4.3 in the book.) Consider the pendulum without friction:
>
> $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a \sin x_1, \quad a > 0. \end{cases}$$
>
> Consider the equilibrium $(x_1, x_2) = (0, 0)$. We want to investigate its stability properties. Let us consider the positive definite function
>
> $$V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2.$$
>
> We check that the function is indeed positive definite. Note that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$.

We take time derivatives:

$$\dot{V}(x) = \left\langle \frac{\partial V}{\partial x}, f(x) \right\rangle = \begin{pmatrix} a\sin x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -a\sin x_1 \end{pmatrix} = 0.$$

From Theorem 4.1, we conclude that the origin is stable. In fact, since $\dot{V}(x) = 0$, we can conclude that the trajectories do not approach the origin. They get trapped on the level set $V(x(t_0)) = x$, i.e., on the level set of the function they start from.

Example.

(4.4 in the textbook.) Consider the pendulum with friction:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1 - bx_2, \quad a, b > 0. \end{cases}$$

Similarly, we want to investigate the stability properties of $(x_1, x_2) = (0, 0)$. Consider the following candidate Lyapunov function:

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2.$$

We take time derivatives...

$$\dot{V}(x) = \begin{pmatrix} a\sin x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_2 \\ -a\sin x_1 - bx_2 \end{pmatrix} = -bx_2^2 \leq 0.$$

The time derivative is non-positive. Observe that the time derivative is negative semi-definite, i.e., it is zero for all $(x_1, 0)$ with $x_1 \in \mathbb{R}$. From Theorem 4.1, we conclude that the origin is stable (but we cannot conclude it is asymptotically stable).

In this example, using the candidate Lyapunov function $V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$, we were able to conclude stability, but not asymptotic stability. However, we suspect that the origin is asymptotically stable. There are two ways in which we can do this:

- LaSalle's theorem, which we will study soon, treats the situations in which the derivative of the Lyapunov function is only negative semi-definite, and not negative definite.

- Try to find another Lyapunov function candidate.

We will pursue the second approach. Consider the candidate function

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x^\top Px,$$

where $P > 0$ is a positive definite matrix. For a 2×2 positive definite matrix $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}$, we have

$$P_{11} > 0 \text{ and } P_{11}P_{22} - P_{12}^2 > 0.$$

Take the time derivative of V:

$$\dot{V}(x) = a(1 - P_{22})x_2 \sin x_1 - aP_{12}x_1 \sin x_1 + (P_{11} - P_{12}b)x_1x_2 + (P_{12} - P_{22}b)x_2^2.$$

Now we can proceed as follows:

- We can attempt to pick P_{11}, P_{12}, P_{22} such that $\dot{V}(x)$ is negative definite and P is positive definite.
- We observe that the terms $x_1x_2, x_2 \sin x_1$ are sign indefinite, in the sense that their signs depend on the signs of x_1, x_2.

We can pick $P_{22} = 1$ and $P_{11} = bP_{12}$ to cancel out the corresponding terms from $\dot{V}(x)$. From the positive definiteness of P, we have $P_{11} > 0$ and $P_{11} > P_{12}^2$. This implies that $bP_{12} > P_{12}^2$ and $P_{12} > 0$, which means that $0 < P_{12} < b$. Let $P_{12} = b/2$. We obtain

$$\dot{V}(x) = -\frac{1}{2}abx_1 \sin x_1 - \frac{1}{2}bx_2^2.$$

Now we have $x_1 \sin x_1 > 0$ for all $0 < |x_1| < \pi$. Hence, we can define the domain $D = \left\{x \in \mathbb{R}^2 \;\middle|\; |x_1| < \pi\right\}$ and conclude that $\dot{V}(x)$ is negative definite over D. Thus, from Theorem 4.1, the origin is asymptotically stable.

Let us now focus our attention to the following question. First, we note that Theorem 4.1 assumes the existence of a Lyapunov function $V\colon D \to \mathbb{R}$, i.e., of a function V defined over a domain D in $\mathbb{R}^n$. Then, if the requirements of the theorem are met with $\dot{V}(x) < 0$ for $x \neq 0$, we conclude that the origin is asymptotically stable, i.e., that for all $\varepsilon > 0$, there exists $\delta > 0$, such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \varepsilon$ and $\lim_{t\to\infty} \|x(t, x_0)\| = 0$.

The question we can pose in this case is, how far from the origin can the trajectory be and still converge to the origin as $t \to \infty$? In other words, we are interested in estimating the region of attraction (or domain of attraction, basin, region of asymptotic stability)

$$R_A = \left\{x_0 \in \mathbb{R}^n \;\middle|\; \lim_{t\to\infty} x(t, x_0) = 0\right\}.$$

Finding the exact region of attraction might be exceedingly difficult to compute exactly. Lyapunov functions may be used to estimate the region of attraction. Essentially, we can try to find level sets of Lyapunov functions that are contained in the region of attraction. We will use this technique later in class.

Already from the proof of Theorem 4.1, we saw that if $V\colon D \to \mathbb{R}$ satisfies the conditions for asymptotic stability (V is positive definite and $\dot{V}$ is negative definite) and if $\Omega_c := \{x \in D \mid V(x) \leq c\}$ is bounded and contained in D[1], then Ω_c is an estimate of the region of attraction, as trajectories starting in Ω_c never escape Ω_c and eventually converge to the origin.

These observations tell us that Ω_c is an estimate of the region of attraction. However, it may be a very conservative estimate in the sense that it may be much smaller than the actual region of attraction. Recall that Ω_c contains points in $B_r(0)$ only. We will see techniques on estimating the region of attraction and enlarging the estimates later in class.

In the meantime, we will pose the following question: under what conditions will be

[1] In fact, Ω_c is by definition closed and bounded (i.e., compact).

region of attraction be the whole space $\mathbb{R}^n$. In other words, under what conditions the state trajectory $x(t, x_0)$ will approach the origin as $t \to \infty$ for any initial condition $x_0 \in \mathbb{R}^n$, no matter how large $\|x\|$ might be? If an asymptotically stable equilibrium at the origin possesses this property, then it is said to be globally asymptotically stable. We will seek an answer to this question.

Recall from the proof of Theorem 4.1 that we established asymptotic stability by showing that a ball of radius δ around the origin can be contained in a compact (i.e., closed and bounded) set Ω_c.

In order to be able to establish that any point $x \in \mathbb{R}^n$ can be contained in the interior or a compact set Ω_c, one obvious observation is that the conditions of the theorem hold globally, i.e., $D = \mathbb{R}^n$.

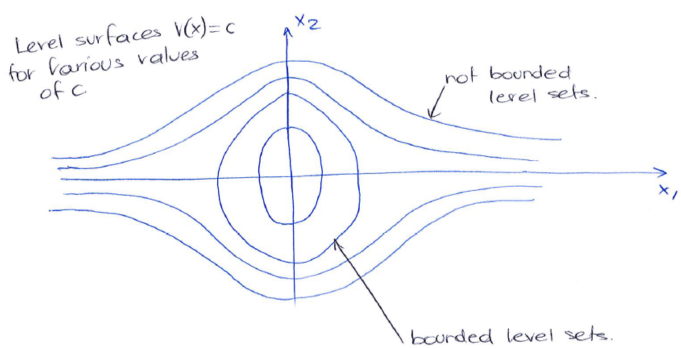
However, this condition is not sufficient. The problem is that it may be the case that for large c, the level sets of V may not be bounded sets.

Example.

Consider

$$V(x) = \frac{x_1}{1+x_1^2} + x_2^2.$$

This is a positive definite function



In this case, for small enough c, the level surfaces are closed and bounded. That is consequence of continuity and positive definiteness of V.

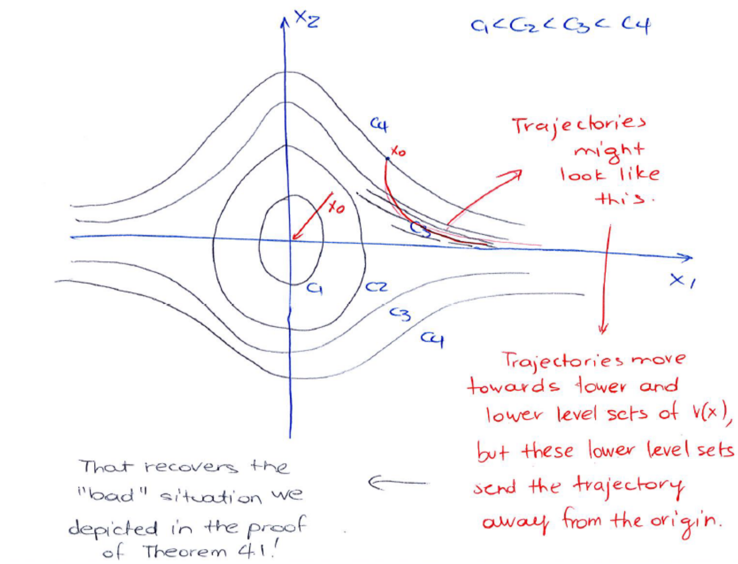
However, as c increases, the function $V(x)$ has unbounded level sets. Recall that a set $S \subseteq \mathbb{R}^n$ is bounded if there exists $k < \infty$ such that for all $x \in S$, $\|x\| < k$. Let us find the x for which the set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \le c\}$ is bounded. Let us check how the $\min_{\|x\|=r} V(x)$ varies as $r \to \infty$.

$$\begin{aligned}
l &= \lim_{r\to\infty} \inf_{\|x\|\ge r} V(x) = \lim_{r\to\infty} \min_{\|x\|=r} V(x) \\
&= \lim_{r\to\infty} \min_{\|x\|=r} \left(\frac{x_1^2}{1+x_1^2} + x_2^2 \right) = \lim_{\substack{|x_1|\to\infty \\ |x_2|\to 0}} \min_{\|x\|=r} \left(\frac{x_1^2}{1+x_1^2} + x_2^2 \right) \\
&= \lim_{|x_1|\to\infty} \min_{\|x\|=r} \left(\frac{x_1^2}{1+x_1^2} \right) = 1.
\end{aligned}$$

Hence, we conclude that the level set Ω_c is bounded for $c < l = 1$.

Hence, we verified that the level sets of the considered function V are bounded for $c < 1$

only. This means that we cannot verify global asymptotic stability with such a function. Intuitively, think that you applied the conditions of Theorem 4.1 with this function, and you indeed had $\dot{V} < 0$ for all $x \neq 0$. Then you would conclude that the origin is asymptotically stable for all initial conditions, i.e., globally. This would be false, as we can verify with the geometric representation of our Lyapunov stability theorem.



Thus, we have to impose an additional condition, apart from $D = \mathbb{R}^n$. This condition was introduced last time: $V(x)$ being radially unbounded.

Theorem 5.0.1: (Theorem 4.2.)

Let $x = 0$ be an equilibrium point for 4.1. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;
- $\|x\| \to \infty \Rightarrow V(x) \to \infty$;
- $\dot{V}(x) < 0$ for all $x \neq 0$.

Then $x = 0$ is asymptotically stable.

Proof. Given $p \in \mathbb{R}^n$. Let $x = V(p)$. Then from the second condition, we have that for all $c > 0$ there exists $r > 0$ such that $\|x\| > r \Rightarrow V(x) > c$. Hence, $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ is closed and bounded, and can be contained in a ball $B_r(0)$. The rest is similar to the proof of Theorem 4.1. $\square$

Remark.

This is known as the Barbashin-Krasovskii Theorem.

5.1 Lyapunov—instability

So far, we have studied two main theorems (4.1 and 4.2) that can be used to investigate and establish stability and asymptotic stability of an equilibrium point. There are also results that prove the instability of an equilibrium point.

Definition 5.1.1

Consider the system $\dot{x} = f(x)$, where f is locally Lipschitz with equilibrium point x_e. Then x_e is unstable if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x_0 \in B_\delta(x_e)$ and $T > 0$ resulting in

$$\|x(T, x_0) - x_e\| > \varepsilon.$$

The question is, how do we characterize the instability of the equilibrium? What are the theorems/tools that we can build to establish instability?

Claim

Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D bean open set containing $x_e = 0$, and suppose that f is locally Lipschitz continuous on D. Suppose there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that

1. $V(x) = 0$ and $V(x) > 0$ for all $x \in D \setminus \{0\}$ and
2. $\dot{V}(x) > 0$ for all $x \in D \setminus \{0\}$.

Then the equilibrium point is unstable.

Proof for Claim.

We approach this in steps:

1. Let $\varepsilon > 0$ be such that $U = \bar{B}_\varepsilon(0) \subseteq D$. Let $0 < \delta < \varepsilon$ be arbitrary and choose x_0 such that $\|x_0\| = \delta$. Further, let $\phi(t, x_0)$ be a solution starting at x_0.

   Claim

   There exists $T > 0$ such that $\|\phi(T, x_0)\| > \varepsilon$.

   Proof for Claim.

   We will prove this by contradiction. Assume there does not exist $T > 0$ such that $\|\phi(T, x_0)\| > \varepsilon$. That means that $\phi(t, x_0) \in U$ for all $t \geq 0$. As a result,

   $$V(\phi(t, x_0)) \leq \sup_{x \in U} V(x) < \infty,$$

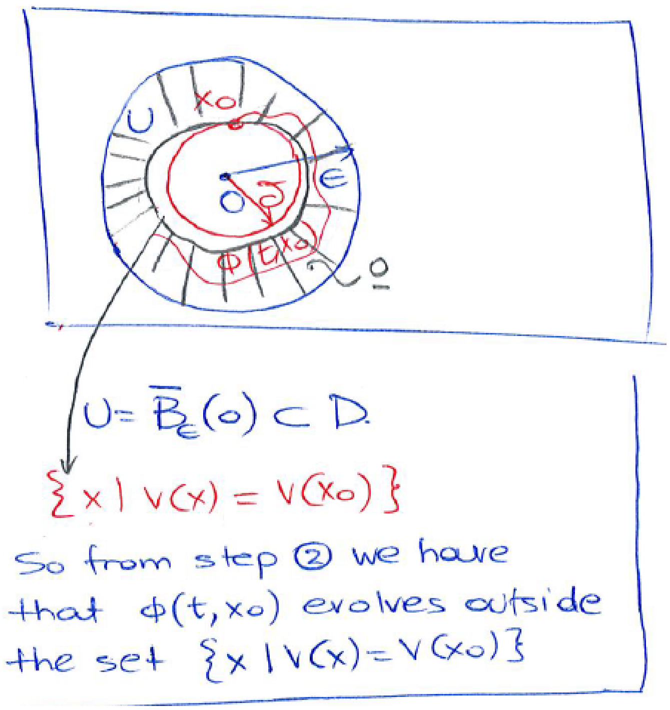
   i.e., $V(\phi(t, x_0))$ is bounded. ∎

2. Since we know that $\dot{V} > 0$, we know that $V(\phi(t, x_0))$ is non-decreasing and as thus, $V(\phi(t, x_0)) \geq V(x_0)$.

3. We furthermore have that the set (shaded in the figure)

$$\Omega_c = \{x \in U \mid V(x) \geq V(x_0)\}$$

is compact.



Define

$$\gamma \coloneqq \min_{x \in \Omega_c} \dot{V}(x).$$

Then $\gamma > 0$ by the positive definiteness of $\dot{V}$.

4. By the fundamental theorem of calculus:

$$V(\phi(t, x_0)) = V(x_0) + \int_0^T \dot{V}(\phi(t, x_0))dt \geq V(x_0) + \gamma t \to \infty$$

as $t \to \infty$. This is a contradiction to the assumption that $V(\phi(t, x_0))$ is bounded! This proves our initial claim.

The importance of our first instability result is twofold. First, it verifies our physical intuition that, if the time derivative of a candidate Lyapunov function is positive on a domain D that contains the equilibrium point, then this implies that the system trajectories must move along higher level sets of the function V, i.e., deviate from the equilibrium point and not be bounded within a ball of arbitrarily small radius ε. Second, the proof gives us nice practice on how to use δ-ε arguments. This formulation was similar to the proof of Theorem 4.1. Before we proceed with our discussion, we define some notation.

Definition 5.1.2

The closure of a set S, denoted $\bar{S}$, is the smallest closed set in $\mathbb{R}^n$ that contains S.

Example.

If $S = [0, 1)$, then $\bar{S} = [0, 1]$. If $S = B_r(0)$, then $\bar{S} = \bar{B}_r(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$.

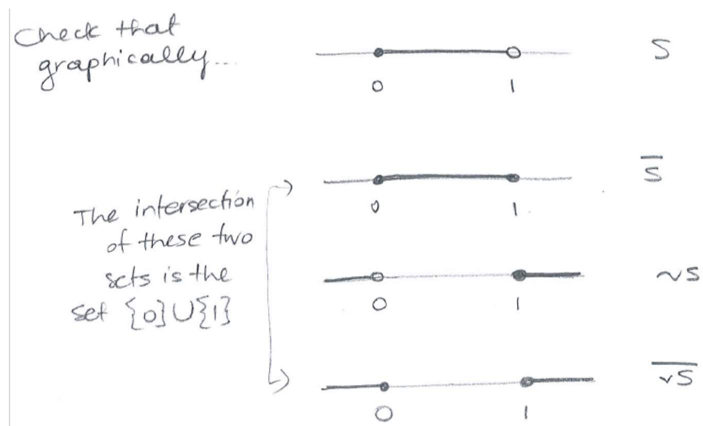
Definition 5.1.3: Boundary

The boundary of a set S, denoted ∂S, is defined as

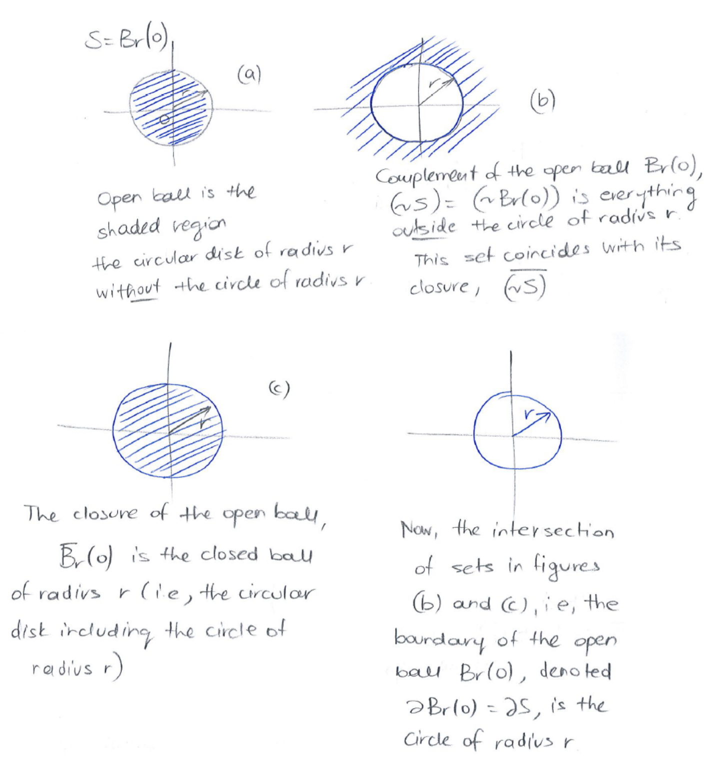
$$\partial S := \bar{S} \cap \overline{\sim S}.$$

Example.

If $S = [0, 1)$, we have $\partial S = \{0, 1\}$.



If $S = B_r(0)$, then $\partial S = \{x \in \mathbb{R}^n \mid \|x\| = r\}$.



If $S = \bar{B}_r(0)$, then $\partial S = \{x \in \mathbb{R}^n \mid \|x\| = r\}$.

Definition 5.1.4

Let $S \subseteq \mathbb{R}^n$. The interior of S, denoted S°, is the largest open set contained in S.

Example.

If $S = [0, 1)$, then $S^\circ = (0, 1)$.
If $S = \bar{B}_r(0)$, then $S^\circ = B_r(0)$.

Fact 5.1.5

$x_0 \in S^\circ$ if and only if there exists $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subseteq S$.

We can now proceed with examples on instability and more instability results.

Example.

Consider the system

$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2. \end{cases}$$

From linear systems tools, we can immediately verify that the system is unstable. Let us verify this using our new tool. Consider the positive definite function

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

Computing the time derivative, we get

$$\dot{V}(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2,$$

which is positive definite. This means the origin is unstable.

Remark.

We wonder: if we are to obtain a less restrictive test for instability, what should give? Should we drop V or $\dot{V}$ being positive definite? We investigate this via examples.

Example.

Consider the following system:

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2. \end{cases}$$

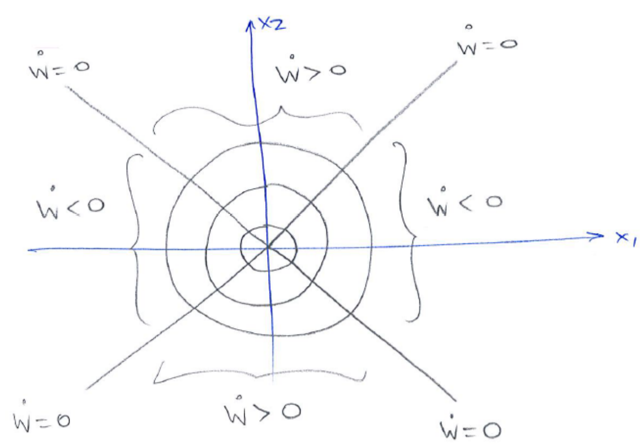
We can use linear systems tools to verify that the origin is unstable. We will use our new theorem. Consider the function

$$W(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

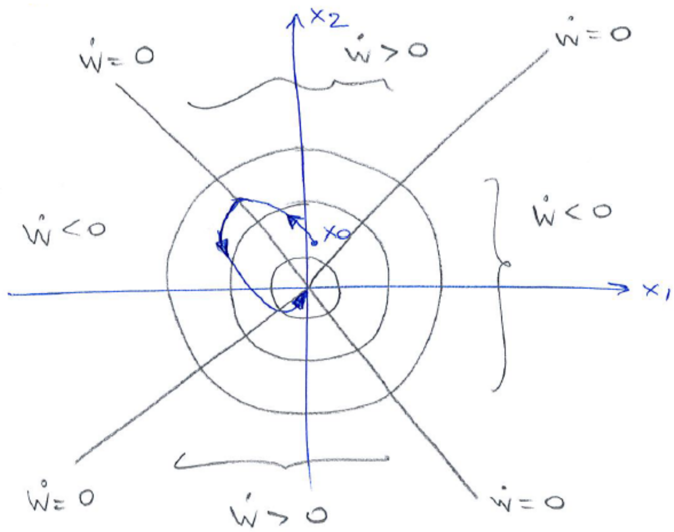
This is a positive definite function. We compute the time derivative and obtain

$$\dot{W}(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = -x_1^2 + x_2^2.$$

We note that $\dot{W}(x_1, x_2) = 0$ when $|x_1| = |x_2|$, and that $\dot{W}$ can take both positive and negative values (we call such functions indefinite).



This plot shows the sign of $\dot{W}(x_1, x_2)$ for the various values of x_1 and x_2. Then we could consider a trajectory that starts out in a region where $\dot{W} > 0$, moves through increasing level sets where $\dot{W} = 0$ and enters a region where $\dot{W} < 0$, at which point, it could move through decreasing level sets and approach the origin.



But we know that the origin is unstable, so there has to be a catch. Given any $\varepsilon > 0$, we cannot be sure that there is a $\delta > 0$ such that $\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \varepsilon$ for all $t \geq 0$.

Fact 5.1.6

If $W(x)$ is positive definite and $\dot{W}(x)$ is indefinite, then nothing can be concluded on the basis of $W(x)$.

Example.

Consider the same system as in the previous example:

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2. \end{cases}$$

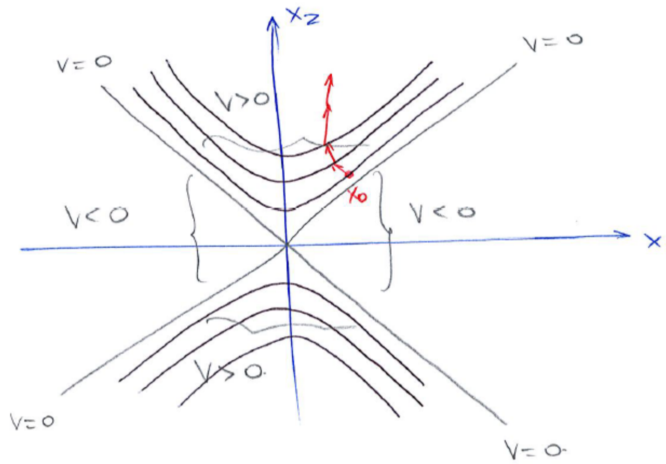
We build a different candidate function:

$$V(x_1, x_2) = -\frac{1}{2}x_1^2 + \frac{1}{2}x_2.$$

And let us take its time derivative:

$$\dot{V}(x_1, x_2) = \begin{pmatrix} -x_1 & x_2 \end{pmatrix} \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = x_1^2 + x_2^2,$$

which is positive definite. Noting that $V(x) = 0$ when $|x_1| = |x_2|$, we have the following situation:



We observe that if a trajectory starts in a region where both $V > 0$ and $\dot{V} > 0$, then it must move along increasing level sets. If the region is bounded by the sets where $V = 0$, then the trajectory must move outwards, away from the origin, showing instability.

Remark.

These ideas underlie the primary instability theorem from our textbook (given next) and an improved version (given after the book's version).

Remark.

Bottom line: if $V(x)$ is indefinite and $\dot{V}(x)$ is positive definite, then it may be possible to conclude something on the basis of V. We will later see that a weaker condition is even possible.

Theorem 5.1.7: (Theorem 4.3 in the textbook)

Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D be an open set containing $x_e = 0$ and suppose f is locally Lipschitz on D. Suppose also that there is a continuously differentiable function $V : D \to \mathbb{R}^n$ such that

- $V(0) = 0$ and
- For every $\delta > 0$ there exists $x_0 \in D$ with $\|x_0\| < \delta$ such that $V(x_0) > 0$. Choose $r > 0$ such that $B_r(0) \subseteq D$ and define

$$U = \{x \in B_r(0) \mid V(x) > 0\}$$

- $\dot{V}(x) > 0$ for all $x \in U$.

Then the equilibrium point is unstable.

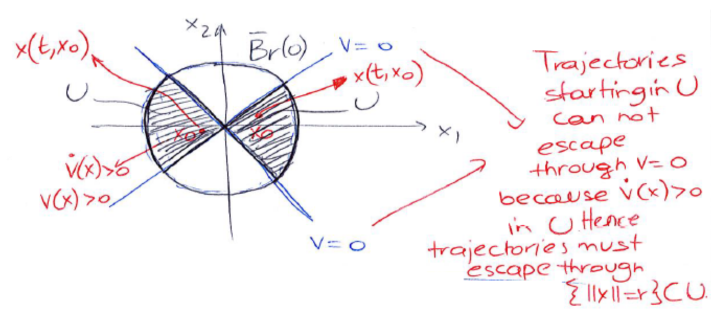
Remark.

The second condition is ensuring that the origin is on the boundary of the set U, that is, $0 \in \partial U$. That means that no matter how small $\delta > 0$ is selected, you can always find initial conditions x_0 which will be forced away from the origin.

Example.

Suppose that $V : D \to \mathbb{R}$ is continuously differentiable on a domain $D \subseteq \mathbb{R}^n$ that contains the origin. Suppose further that $V(0) = 0$. Assume also that there is a point x_0 arbitrarily close to the origin $x = 0$, i.e., contained within a ball $B_\delta(0)$ for any $\delta > 0$ such that $V(x_0) > 0$. Choose $r > 0$ such that the ball $\bar{B}_r(0)$ is contained in D and let $U = \{x \in \bar{B}_r(0) \mid V(x) > 0\}$.

Suppose, for example that we have the function $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ and that $\dot{V}(x) > 0$ in U.



In the region U, we have $V(x) > 0$. By construction, the boundary of the set U is the surface $V(x) = 0$ and the sphere $\|x\| = r$.

By Theorem 4.3, we conclude the origin is unstable.

Since we can find an x_0 arbitrarily close to the origin, $\|x_0\| < \delta$, from which trajectories cannot be confined within any $\varepsilon > 0$.

Theorem 5.1.8: Improvement over previous theorem

Let $x_e = 0$ be an equilibrium point of the ODE $\dot{x} = f(x)$ on $\mathbb{R}^n$. Let D be an open set containing $x_e = 0$ and suppose f is locally Lipschitz on D. Suppose also that there is a continuously differentiable function $V : D \to \mathbb{R}$ and an open set $U \subseteq D$ such that

- $V(0) = 0$
- $0 \in \partial U$
- $\dot{V}(x) > 0$ for all $x \in U$
- for all $\delta > 0$ there exists $x_0 \in U$ such that $V(x_0) > 0$ and $\|x_0\| < \delta$
- there exists $r > 0$ such that $x \in \partial U$ and $\|x\| < r \Rightarrow V(x) = 0$.

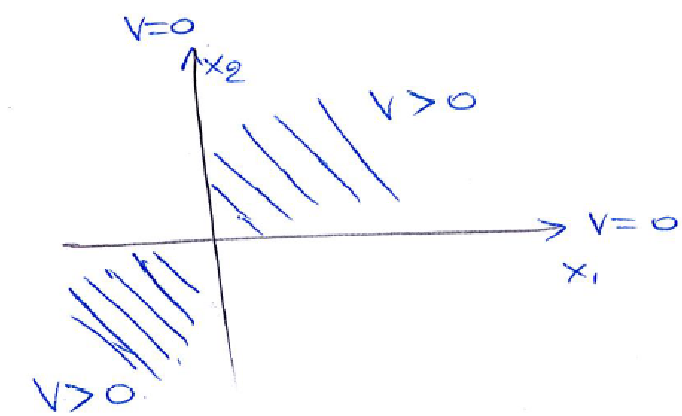
Then the equilibrium point is unstable.

Example.

Consider the system

$$\begin{cases} \dot{x}_1 = 3x_1 + x_2 + x_1^2 \\ \dot{x}_2 = x_1 - x_2 + x_1 x_2. \end{cases}$$

Consider the function $V(x_1, x_2) = x_1 x_2$.

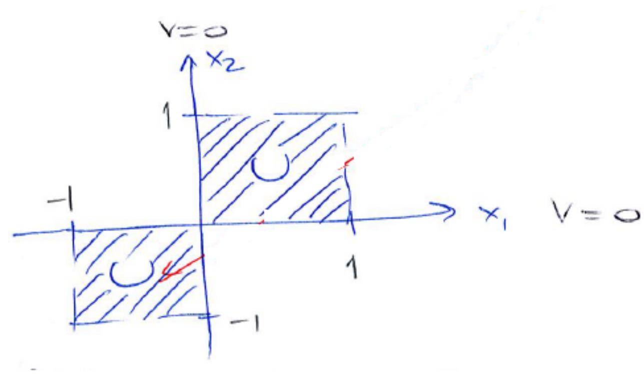


Its time derivative is

$$\dot{V}(x) = x_1^2 + x_2^2 + 2x_1 x_2 (x_1 + 1).$$

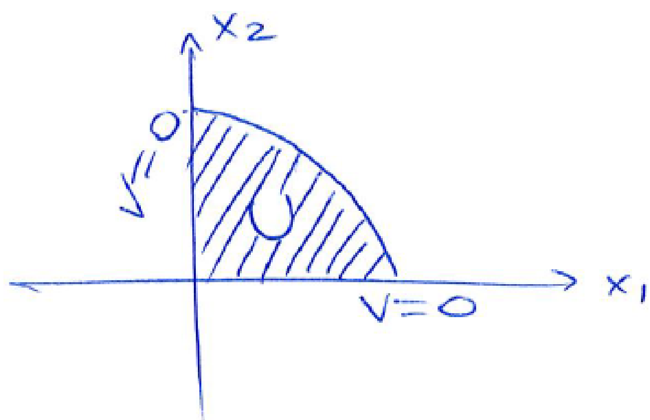
We consider two different solutions:

1. Choose $\|\cdot\| = \|\cdot\|_\infty$ and let $r = 1$. Define the set $U = \{x \in B_r(0) \mid V(x) > 0\}$.



We verify that $\dot{V}(x) > 0$ for all $x \in U$. From Theorem 4.3, the origin $x_e = 0$ is unstable.

2. Let $\|\cdot\|$ be any norm and pick $0 < r < \infty$. Define the set $U = \{x \in B_r(0) \mid x_1 > 0, x_2 > 0\}$. Then we have



- $0 \in \partial U$
- $\dot{V}(x) > 0$ for all $x \in U$
- for all $\delta > 0$ there exists x_0 with $\|x_0\| < \delta$ such that $V(x_0) > 0$
- $x \in \partial U$, $\|x\| < r \Rightarrow V(x) = 0$
- $V(0) = 0$

Per the improved version of Theorem 4.3, the origin is unstable.

We still have to address why the hypotheses of our second theorem are more relaxed than those of Theorem 4.3. We observe that the improved version does not require $\dot{V}(x)$ to be positive definite on $U = \{x \in \bar{B}_r(0) \mid V(x) > 0\}$ for any $r > 0$. The requirements of the theorem are the following: suppose there is a continuously differentiable function $V\colon D \to \mathbb{R}$ and an open set $U \subseteq D$ such that

- $V(0) = 0$

- $0 \in \partial U$
- $\dot{V}(x) > 0$ for all $x \in U$
- for all $\delta > 0$ there exists $x_0 \in U$ such that $V(x_0) > 0$ and $\|x_0\| < \delta$
- there exists $r > 0$ such that $x \in \partial U$ and $\|x\| < r \Rightarrow V(x) = 0$

Conclusion: if these conditions are met, then the origin is unstable.

Remark.

> Essentially, the "improvement theorem" constructs the set U in a more relaxed way that can still help us conclude instability.

Example.

Consider the function

$$V(x) = x_1^4 - x_2^2$$
$$\dot{V}(x) = -4x_1^5$$

Observe that $V(x)$ is not positive definite, thus cannot be used to assess stability. However, $V(x)$ takes on positive values near the origin, so it may be able to be used to show instability.

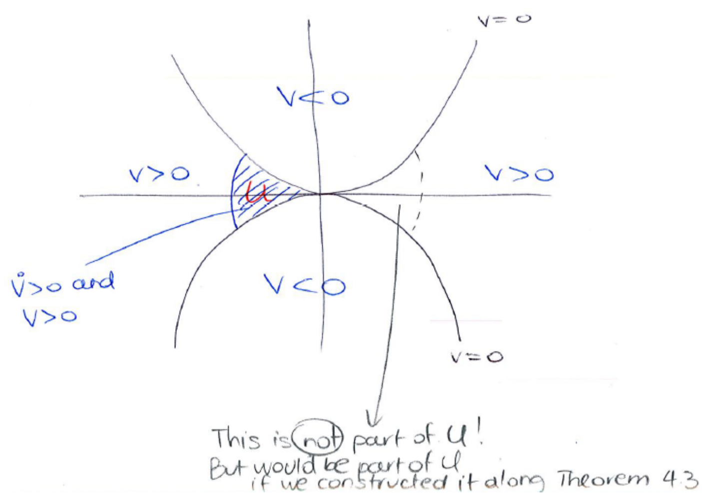
We have

$$\left\{x \in \mathbb{R}^n \;\middle|\; \dot{V}(x) > 0\right\} = \{x \in \mathbb{R}^n \mid x_1 < 0\}.$$

Next, we have to sketch where $V = 0$, $V > 0$, and $V < 0$. Note that

$$\{x \in \mathbb{R}^2 \mid V(x) = 0\} = \{x \in \mathbb{R}^2 \mid |x_2| = x_1^2\}.$$

Let $r = 1$ and consider the ball $B_r(0)$. Then we consider the set $U = \{x \in B_r(0) \mid V(x) > 0, x_1 < 0\}$, which is shown in the shaded region.



For this U, it is easy to verify that

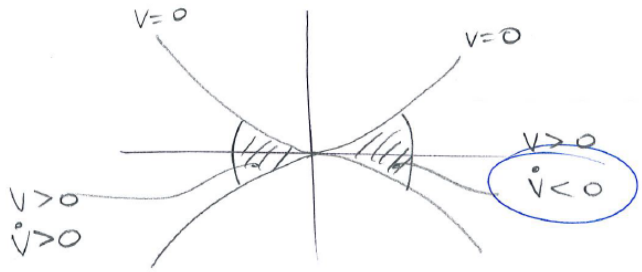
- $V(0) = 0$

- $0 \in \partial U$
- $\dot{V}(x) > 0$ for all $x \in U$
- for all $\delta > 0$ there exists $x_0 \in U$ such that $\|x_0\| < \delta$ and $V(x_0) > 0$
- for all $x \in \partial U$ with $\|x\| < 1$, we have $V(x) = 0$

Thus, by the improved theorem, the origin is unstable.

Remark.

If we constructed the set U per the requirements of Theorem 4.3, we would have



Note that $\dot{V}(x)$ is not positive definite in U because there are some parts of U for which $\dot{V}(x) < 0$. Hence, the hypotheses of Theorem 4.3 are not met in this case, and thus the theorem cannot be applied.

Chapter 6

Lyapunov's indirect method

We have already seen examples from our textbook where the use of positive definite matrices in the definition of our Lyapunov function candidates was useful in establishing the time derivative of the function negative definite (and thus in being able to conclude asymptotic stability).

Let us summarize/review the properties of positive definite matrices and see their usefulness in finding Lyapunov functions for linear systems.

Definition 6.0.1

Let P be a real, $n \times n$, symmetric ($P^\top = P$) matrix.

- P is positive semi-definite if $x^\top P x \geq 0$ for all $x \in \mathbb{R}^n$. We write $P \geq 0$.
- P is positive definite if $x^\top P x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$. We write $P > 0$.
- P is negative definite if $x^\top P x < 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$. We write $P < 0$.
- P is negative semi-definite if $x^\top P x \leq 0$ for all $x \in \mathbb{R}^n$. We write $P \leq 0$.

Remark.

> $P > 0$ does not mean that all the entries of P are positive.

Remark.

> Let M be an arbitrary, real, $n \times n$ matrix. Write M as
>
> $$M = \frac{M + M^\top}{2} + \frac{M - M^\top}{2}.$$

Claim

- $x^\top(M - M^\top)x = 0$ for all $x \in \mathbb{R}^n$.
- $\frac{M+M^\top}{2}$ is symmetric and is called the symmetric part of M.
- $\frac{M-M^\top}{2}$ is skew-symmetric and is called the skew-symmetric part of M.

Proof for Claim.

We show only the first statement. Note that, because $x^\top Mx$ is a scalar, it is equal to its transpose, which is $x^\top M^\top x$. This means that the first statement holds.

Remark.

From this result, we conclude that

$$x^\top Mx = x^\top \frac{M+M^\top}{2} x.$$

As thus, symmetry is always assumed as part of the definition of positive and negative (semi-)definitive matrices.

Fact 6.0.2

Let P be a real, $n \times n$ matrix, and let $\{\lambda_1, \dots, \lambda_n\}$ be its eigenvalues.

1. If $P = P^\top$, we have
   (a) all eigenvalues are real;
   (b) the Jordan canonical form is trivial (i.e., it only consists of size-1 blocks);
   (c) the eigenvectors are mutually orthogonal.
2. $P \geq 0$ if and only if $\lambda_i \geq 0$ for $i = 1, \dots, n$
3. $P > 0$ if and only if $\lambda_i > 0$ for $i = 1, \dots, n$
4. $P \leq 0$ if and only if $-P \geq 0$
5. $P < 0$ if and only if $-P > 0$
6. Let N be an $m \times n$ real matrix. Then $N^\top N \geq 0$ and moreover $N^\top N > 0$ if and only if $\operatorname{rank}(N) = n$.
7. $P \geq 0$ if and only if there exists N that is $m \times n$ such that $P = N^\top N$ and $m = \operatorname{rank}(P)$.
8. $P > 0$ if and only if there exists N that is $n \times n$ and invertible such that $P = N^\top N$.
9. $P > 0$ if and only if $P^{-1} > 0$.
10. Suppose that $P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{12} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \dots & \vdots \\ P_{1n} & P_{2n} & \dots & P_{nn} \end{pmatrix} = P^\top$. Then $P > 0$ if and only if its leading principal minors are positive, which means that

$$
\begin{aligned}
&P_{11} > 0 \\
&|P_{11} \quad P_{12} P_{12} \quad P_{22}| > 0 \\
&\vdots \\
&\begin{vmatrix} P_{11} & P_{12} & \dots & P_{1n} \\ P_{12} & P_{22} & \dots & P_{2n} \\ \vdots & \vdots & \dots & \vdots \\ P_{1n} & P_{2n} & \dots & P_{nn} \end{vmatrix} > 0
\end{aligned}
$$

Now we start building our Lyapunov functions.

6.1 Linear systems

Consider the following system:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n.$$

From linear systems theory, we know that the origin is asymptotically stable if and only if the matrix is Hurwitz, i.e., if all eigenvalues have negative real parts. We can investigate stability properties using the Lyapunov method, as well.

Let us consider the positive definite function

$$V(x) = x^\top P x,$$

where $P > 0$ and take its time derivative:

$$\begin{aligned} \dot{V}(x) &= \dot{x}^\top P x + x^\top P \dot{x} = x^\top A^\top P x + x^\top P A x \\ &= x^\top \underbrace{(A^\top P x + PA)}_{\text{-Q}} x = -x^\top Q x. \end{aligned}$$

We observe that if $A^\top P + PA < 0$, then we can conclude that the origin is asymptotically stable.

Remark.

> The equation $A^\top P + PA = -Q$ is called the *Lyapunov equation*.
>
> In fact, we can also derive the following: if we assume we started with a positive definite matrix Q, we can solve the Lyapunov equation to obtain a matrix P that is positive definite. Then we can conclude that the origin is asymptotically stable. We summarize this by saying that
>
> $\dot{x} = Ax$ is asymptotically stable if and only if there exists $V = x^\top P x$, $P > 0$ such that $\dot{V} = x^\top (A^\top P + PA) x$ is negative definite.

Remark.

> The same result with different wording is given in Theorem 4.6 of the textbook.

Theorem 6.1.1: (Theorem 4.6 in the textbook)

A matrix A is Hurwitz (i.e., $\dot{x} = Ax$ is globally asymptotically stable) if and only if for any given $Q > 0$ there exists $P > 0$ such that $A^\top P + PA = -Q$. Moreover, if A is Hurwitz, then P is the unique solution of $A^\top P + PA = -Q$.

Proof. First we prove sufficiency, meaning we want to show that if for all $Q > 0$, there exists $P > 0$ such that $A^\top P + PA = -Q$ then A is Hurwitz. To show this, we apply Theorem 4.1. Let us consider the function

$$V(x) = x^\top P x$$

and

$$\dot{V}(x) = -x^\top Q x < 0.$$

Hence, the origin is asymptotically stable, which means that A is Hurwitz.

Now we show necessity, which means that if the matrix is Hurwitz, then for all $Q > 0$ there exists $P > 0$ such that $A^\top P + PA = -Q$.

To prove this statement, the idea is to start with the fact that A is Hurwitz. Then we construct/find a $P > 0$ that is a solution of the Lyapunov equation. The fact that A is Hurwitz means that the real parts of its eigenvalues are negative. We define the matrix

$$P = \int_0^\infty e^{A^\top t} Q e^{At} dt.$$

We know that the integrand exists (it's not infinite) because the terms in the integrand are the sum of terms of the form $t^{k-1} \exp(\lambda_i t)$, where $\mathrm{Re}(\lambda_i) < 0$. So we know that the matrix P just defined exists. The next step is to show that it is positive definite. We can see this as follows:

$$x^\top P x = \int_0^\infty (e^{At} x)^\top Q (e^{At} x) dt.$$

Since the integrand is positive for all t, so is the integral. This means that $P > 0$. Now that we know that P is positive definite, we have to show that it is the solution to the Lyapunov equation:

$$PA + A^\top P = \int_0^\infty e^{A^\top t} Q e^{At} A dt + \int_0^\infty A^\top e^{A^\top t} Q e^{At} dt = \int_0^\infty \frac{d}{dt}\left(e^{A^\top t} Q e^{At}\right) dt = -Q.$$

We have shown that P is a solution of the Lyapunov equation. This concludes the proof about necessity. Finally, we show that P is the unique solution. We assume that there is another matrix $\tilde{P}$ which is also a solution of the equation. We want to show that this solution must be equal to P. Since both P and $\tilde{P}$ are solutions to the Lyapunov equation, we have

$$0 = (P - \tilde{P})A + A^\top (P - \tilde{P}).$$

Multiplying on the left by $e^{A^\top t}$ and on the right by e^{At} yields

$$\begin{aligned} 0 &= e^{A^\top t}\left((P - \tilde{P})A + A^\top (P - \tilde{P})\right) e^{At} \\ &= \frac{d}{dt}\left(e^{A^\top t}(P - \tilde{P}) e^{At}\right). \end{aligned}$$

This means that $e^{A^\top t}(P - \tilde{P}) e^{At}$ is constant for all t. By setting $t = 0$, we have

$$P - \tilde{P} = e^{A^\top t}(P - \tilde{P}) e^{At}$$

for all t. This can only hold when $\tilde{P} = P$. □

Remark.

> The result above only applies to linear systems. So what is its use, if any, for nonlinear systems?

The Lyapunov equation provides a procedure for finding a Lyapunov function for any system $\dot{x} = Ax$, when A is Hurwitz. Now, the existence of such a Lyapunov function will allow us to draw conclusions about the system when the right-hand side of the equation is perturbed, i.e., if the coefficients of A were linearly perturbed of if the perturbation is nonlinear.

This becomes clearer through Theorem 4.7 of the textbook, which provides conditions under which we can draw conclusions about the stability of the origin as an equilibrium of a nonlinear system by investigating its stability as an equilibrium point of a linear system.

Theorem 6.1.2: Lyapunov's indirect method (Theorem 4.7)

Let $x = 0$ be the equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f\colon D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

be the Jacobian matrix evaluated at the equilibrium. Then

1. The origin is asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for all eigenvalues of A.
2. The origin is unstable if $\mathrm{Re}(\lambda_i) > 0$ for at least one eigenvalue of A.

Proof. We will sketch the main steps. The full proof is available in the book. First, we notice from the Mean Value Theorem (see the Appendices in the textbook) that

$$f_i(x) = f_i(0) + \left.\frac{\partial f}{\partial x}\right|_{z_i} (x - 0),$$

where z_i is a point on the line segment connecting x to the origin. Since $f(0) = 0$, we have

$$f_i(x) = \left.\frac{\partial f_i}{\partial x}\right|_{z_i} x = \left.\frac{\partial f_i}{\partial x}\right|_{0} x + \underbrace{\left.\frac{\partial f_i}{\partial x}\right|_{z_i} x - \left.\frac{\partial f_i}{\partial x}\right|_{0} x}_{g_i(x)},$$

or in vector form

$$f_i(x) = Ax + g(x),$$

where $A = \left.\frac{\partial f_i}{\partial x}\right|_0$. We have

$$|g_i(x)| \le \left\| \left.\frac{\partial f_i}{\partial x}\right|_{z_i} - \left.\frac{\partial f_i}{\partial x}\right|_{0} \right\| \|x\| .$$

From the continuity of $\frac{\partial f}{\partial x}$, we know that $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$[1]

[1] To see why, consider that the first term of the right hand side is a continuous function; then as $\|x\| \to 0$, $\left\| \frac{\partial f}{\partial x}|_x \right\| \to 0$. The left hand side being $\frac{\|g(x)\|}{\|x\|}$ is then bounded by something that goes to zero, hence it has

Now, $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ means that in a small neighborhood around the origin, we can approximate the nonlinear system as

$$\dot{x} = Ax,$$

where $A = \frac{\partial f}{\partial x}|_0$.

Note that the above analysis offers theoretical justification as to why the linearized system $\dot{x} = Ax$, where $A = \frac{\partial f}{\partial x}|_0$ is considered a valid approximation of the nonlinear system $\dot{x} = f(x)$ in a neighborhood of the origin.

So we have the system

$$\dot{x} = f(x) = Ax + g(x).$$

To proceed with proving the first claim of the theorem, let A be Hurwitz. By Theorem 4.6, we know that for any $Q > 0$, the solution P of the Lyapunov equation

$$A^\top P + PA = -Q$$

is positive definite. We will use $V(x) = x^\top Px$ as the Lyapunov function candidate *for the nonlinear system*. The time derivative is

$$\begin{aligned}\dot{V}(x) &= x^\top Pf(x) + f^\top(x)Px = x^\top P(Ax + g(x)) + (x^\top A^\top + g^\top(x))Px \\ &= x^\top(PA + A^\top P)x + \left(x^\top Pg(x) + g^\top(x)Px\right) = -x^\top Qx + 2x^\top Pg(x).\end{aligned}$$

The first term on the right-hand side is negative definite, but the second is in general indefinite. However, we showed that $\frac{\|g(x)\|}{\|x\|} \to 0$ as $x \to 0$. That means that for any $\gamma > 0$, there exists $r > 0$ such that $\|g(x)\| < \gamma \|x\|$ for all $\|x\| < r$.

Hence,

$$\dot{V}(x) \leq -x^\top Qx + 2\gamma \|P\| \|x\|^2$$

for all $\|x\| < r$. We also have $Q > 0$, so its eigenvalues are real and positive and also $x^\top Qx \leq \lambda_{\min}(Q) \|x\|^2$. This means that

$$\dot{V}(x) \leq -(\lambda_{\min}(Q) - 2\gamma \|P\|) \|x\|^2$$

for all $\|x\| < r$. It follows that if we choose $\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|}$, then $\dot{V}$ is negative definite. By Theorem 4.1, we complete the proof of the first claim.

Remark.

> We also showed that the Lyapunov function $V(x) = x^\top Px$, where P is the solution of the Lyapunov equation $A^\top P + PA = -Q$ for any $Q > 0$ is a local Lyapunov function for the nonlinear system (i.e., valid on a domain D).

Regarding the second part of the theorem, we defer the proof to the textbook. The key idea is that we can find a singular matrix T such that

$$TAT^{-1} = \begin{pmatrix} -A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

to go to zero as well.

where A_1 and A_2 are Hurwitz. Then we can define the two Lyapunov functions corresponding to each of the matrices A_1 and A_2 and define the function

$$V(z) = z_1^\top P_1 z_1 - z_2^\top P_2 z_2.$$

In general, this is indefinite. Our goal is to apply the instability theorem (4.3). In fact, we show that on a set U where $V(x) > 0$, we can also have $\dot{V}(x) > 0$, hence the origin is unstable. □

Remark.

> Wrapping up with Theorem 4.7, this is what we learned:
>
> - If $A = \frac{\partial f}{\partial x}|_0$ has all its eigenvalues with negative real parts, then the origin is asymptotically stable.
> - If $A = \frac{\partial f}{\partial x}|_0$ has at least one eigenvalue with a positive real part, then the origin is unstable.
> - If $A = \frac{\partial f}{\partial x}|_0$ has all its eigenvalues with non-positive real parts (i.e., negative or zero real parts), then the theorem is inconclusive. That is, linearization fails to determine the stability of the equilibrium point in this case.

Chapter 7

LaSalle's method

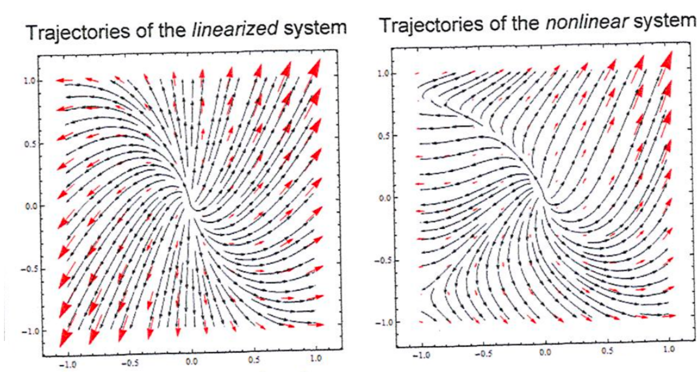
Last time, we studied the linearization method for drawing conclusions about the stability properties of the equilibrium point of a nonlinear system.

Our textbook has examples (4.14 and 4.15) on the application of the method. Let's consider one more:

Example.

Consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 = -x_1 + x_2^2 \end{cases}$$



Let us investigate the stability properties of the origin. We have

$$A = \left.\frac{\partial f}{\partial x}\right|_0 = \left.\begin{pmatrix} -1 + x_2 & 1 + x_1 \\ -1 & 2x_2 \end{pmatrix}\right|_{x=0} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of the matrix are $\lambda = \frac{-1 \pm i\sqrt{3}}{2}$. Thus, the origin is asymptotically stable. As an exercise, is this the only equilibrium?

Remark.

Theorem 4.7 is known as the indirect method, and Theorems 4.1 and 4.2 as the direct method of Lyapunov's stability theory.

We have already experienced the limitations of Lyapunov's theorem 4.1, which requires $\dot{V}(x) < 0$ for $x \neq 0$ in order to conclude asymptotic stability.

Fortunately, there are other tools that may help us establish asymptotic stability in cases when $\dot{V}(x) \leq 0$, i.e., when $\dot{V}$ is negative semi-definite.

One of the most popular is LaSalle's invariance principle. The core idea is to establish that the only trajectory that can stay identically on the set $\left\{x \in D \;\middle|\; \dot{V}(x) = 0\right\}$ is the zero trajectory.

In the sequel, we will see a motivating example, and how LaSalle's Theorem is formally stated.

Remark.

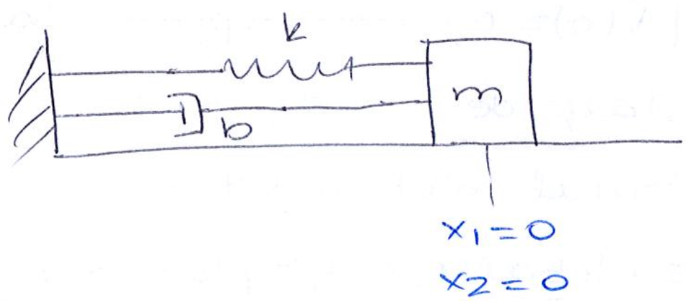
The key concepts under investigation are *invariant sets* and *positive invariant sets*.

7.1 LaSalle's invariance principle or LaSalle's method (both are used)

We consider the need for this result via an example:

Example.

Consider a nonlinear mass-spring damper.



The second order dynamics are given by

$$m\ddot{y} + \underbrace{b|\dot{y}|\dot{y}}_{\text{Nonlinear damping}} + \underbrace{k_0 y + k_1 y^3}_{\text{nonlinear spring}} = 0,$$

where y is the displacement, and $\dot{y}$ the velocity of the mass. Letting $x_1 = y$ and $x_2 = \dot{y}$, we have the following state-space representation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{m}|x_2|x_2 - \frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3. \end{cases}$$

Let us take the total energy function as a Lyapunov function candidate: $V(x) = \frac{1}{2}mx_2^2 +$

$\int_0^{x_1}(k_0\sigma + k_1\sigma^3)d\sigma$. We thus have

$$V(x) = \frac{1}{2}mx_2^2 + \frac{1}{2}k_0x_1^2 + \frac{1}{4}k_1x_1^4.$$

The time derivative can be computed to be

$$\dot{V}(x) = mx_2\dot{x}_2 + k_0x_1\dot{x}_1 + k_1x_1^3\dot{x}_1 = -bx_2^2|x_2|.$$

This function is negative semi-definite.

Remark.

> Note that this example is reminiscent of our pendulum example studied earlier.

The time derivative $\dot{V}(x)$ is negative semi-definite. Hence, we may only conclude that the origin is stable, but not asymptotically stable.

However, we expect from intuition that the origin is asymptotically stable, too, since we have losses. LaSalle's Theorem helps us to establish this result. Let us observe the following:

We have

$$\dot{V}(x) = 0 \text{ if and only if } \{x_2 = 0, x_1 \in \mathbb{R}\}.$$

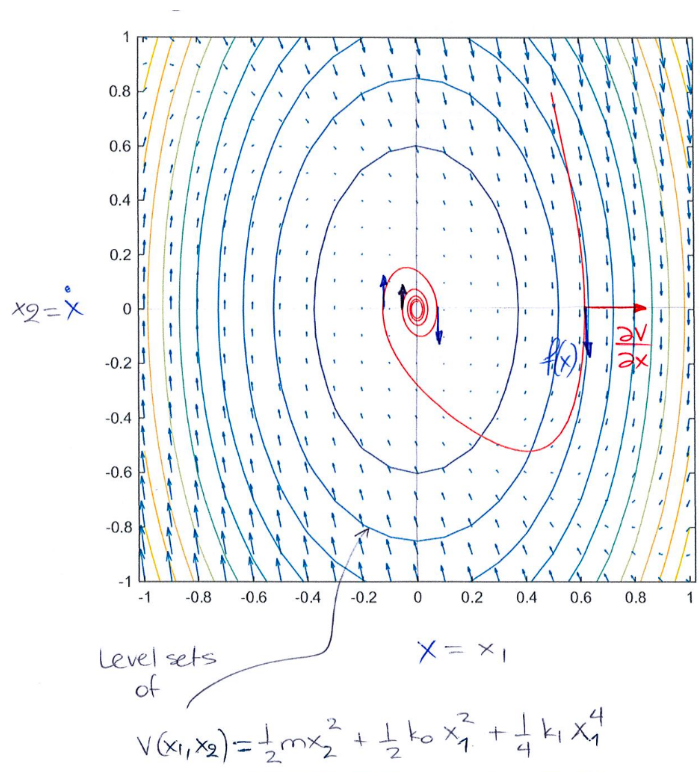
For $(x_1, x_2) = (x_1, 0)$, the system equation reads

$$m\dot{x}_2 + k_0x_1 + k_1x_1^3 = 0$$
$$m\dot{x}_2 = -(k_0x_1 + k_1x_1^3),$$

so $\dot{x}_2 \neq 0$ if $x_1 \neq 0$. But this means that x_2 does not remain constant, unless $x_1 = 0$. In other words, if $x_1 \neq 0$, at the next time we will have $x_2 \neq 0$. Then, from $\dot{x}_1 = x_2 \neq 0$, we know that x_1 will change, too. And since, $\dot{V}(x) = -b|x_2|x_2^2 < 0$ for $x_2 \neq 0$, this would mean that the system trajectories would move along lower level set of $V(x)$.

Geometrically, that means that when $\dot{V}(x) = \dot{V}(x_1, 0) = 0$ with $x_1 \neq 0$, the vector field $f(x)$ is tangent to the level set $V(x_1, 0)$.

This is the phase portrait of our system, with level surfaces of the Lyapunov function overlaid on it:



The hand-drawn arrows represent the vector field of the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{m}|x_2|x_2 - \frac{k_0}{m}x_1 - \frac{k_1}{m}x_1^3. \end{cases}$$

When $x_1 \neq 0$ and $x_2 = 0$, we observe that $\dot{V}(x) = 0$. From the system dynamics, we have $\dot{x}_1 = 0$ and $\dot{x}_2 \neq 0$. This implies that $x_2 = 0$ will not stay identically zero. From $\dot{x}_1 = x_2$, we have that x_1 will be forced to change as well. After this, $\dot{V}(x) = -b|x_2|x_2^2$ will take some negative value, implying that the system trajectories will keep evolving along lower level sets of $V(x_1, x_2)$.

This example shows the core concept used in LaSalle's invariance principle, i.e., the concept of an invariant set.

Definition 7.1.1: Invariant set

Consider a nonlinear system $\dot{x} = f(x)$, where $f\colon D \to \mathbb{R}^n$ is locally Lipschitz. A set M is an invariant set with respect to this system if $x(0) \in M \Rightarrow x(t) \in M$ for all $t \in \mathbb{R}$.

Note that this definition means that if the system is in the invariant set at any time, it is in the invariant set for all future and past times.

Remark.

Note that the invariance concept for a set is defined with respect to a system $\dot{x} = f(x)$, in the sense that $x(t)$ is the solution of the given ODE. In the sequel, we drop this notation for the sake of brevity.

Definition 7.1.2: Positively invariant set

A set M is called *positively invariant* if $x(0) \in M \Rightarrow x(t) \in M$ for all $t \geq 0$.

Remark.

We have already seen quite many examples of invariant and positively invariant sets.

Example.

The equilibrium point and the limit cycle are invariant sets since any solution starting in either set remains in the set for all $t \in \mathbb{R}$.

The set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq x\}$ with $\dot{V}(x) \leq 0$ for all $x \in \Omega_c$ is a positively invariant set, since as we saw in the proof of Theorem 4.1, any solution starting in Ω_c remains in Ω_c for all $t \geq 0$.

We are now ready to state LaSalle's Theorem (or the Invariance Principle, as it is often called.)

Theorem 7.1.3: LaSalle's Theorem

Let $\Omega_c \subseteq D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$, where $f\colon D \to \mathbb{R}^n$ is locally Lipschitz. Also let $V\colon D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω_c. Let E be the points in Ω where $\dot{V}$ is zero. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

Remark.

In this definition, the word "largest" is understood in the sense of set theory, i.e., M is the union of all invariant sets in $E = \left\{x \in \Omega \;\middle|\; \dot{V}(x) = 0\right\}$.

Remark.

In the requirements of Theorem 4.4, the function V does not have to be positive definite. This is one of the key extensions of LaSalle's relative to Lyapunov's Theorem.

Example.

Let us formally apply this result to the mass-spring-damper system, so that we illustrate the construction of the sets in the Theorem's assumptions.

We had the following function

$$V(x_1, x_2) = \frac{1}{2} m x_2^2 + \frac{1}{2} k_0 x_1^2 + \frac{1}{4} k_1 x_1^4.$$

Note that thus function is positive definite (but LaSalle's Theorem does not require it to be so). Nevertheless, the fact that V is positive definite tells us that the level set $\Omega_c = \{x \in D \mid V(x) \leq c\}$ is compact for sufficiently small c (as in the proof of Theorem 4.1). In fact, this happens to be a radially unbounded function. Thus, its level sets are compact sets for any $c > 0$. Let $\Omega = \Omega_c$. Since $\dot{V}(x) \leq 0$ on Ω, we have that any solution starting in Ω_c remains in Ω_c per the proof of Theorem 4.1. We conclude that Ω_c is a positively invariant set (with respect to the system) and $\dot{V}(x) = -b|x_2|x_2^2 \leq 0$ negative semi-definite on Ω_c.

Denote $E = \left\{x \in \Omega \;\middle|\; \dot{V}(x) = 0\right\}$. In our case,

$$E = \{x \in \Omega \mid x_1 \in \mathbb{R}, x_2 = 0\} = \{x \in \Omega \mid (x_1 \neq 0 \wedge x_2 = 0) \vee (x_1 = 0 \wedge x_2 = 0)\}.$$

Now, LaSalle's Theorem says that every solution starting in Ω will approach the largest invariant set M in E. Hence, for concluding convergence of the solutions to the origin, we need to show that the origin is the largest invariant set M in E. That means it suffices to show that

$$E_1 = \{x \in \Omega \mid x_1 \neq 0, x_2 = 0\}$$

is not an invariant set, while

$$E_2 = \{x \in \Omega \mid x_1 = 0, x_2 = 0\}$$

is an invariant set.

From the system's equations, we immediately verify that E_2 is indeed an invariant set. Now, we claim that E_1 is not invariant. We prove this claim by contradiction. Assume E_1 is invariant. Then for any trajectory with $x(0) \in E_1$, we will have $x(t) \in E_1$ for all $t \in \mathbb{R}$. However, from the system dynamics, we have $\dot{x}_2(t) \neq 0$. Thus, $x_2(t) \neq x_2(0)$ for $t > 0$. That means that trajectories in E_1 do not stay in E_1 for all $t \geq 0$, which contradicts the assumption about E_1 being invariant. This, in turn, implies that the largest invariant set of M is $\{0\}$. Hence, from Theorem 4.4, we have that the system trajectories approach the origin.

Remark.

Note our wording: we say "approach the origin" in Theorem 4.4. Why don't we say asymptotically stable?

It has to do with the math in the proof of Theorem 4.4. The proof shows that the solution approaches the positive limit set of Ω. No ε-δ balls were constructed.

In the example we studied, we had a positive definite function. In that case, LaSalle's Theorem is specialized into the following corollaries.

Corollary 7.1.4

Let $x = 0$ be the equilibrium point of $\dot{x} = f(x)$ for $f : D \to \mathbb{R}^n$ locally Lipschitz. Let $V : D \to \mathbb{R}$ be continuously differentiable, positive definite on D ($V > 0$ on $D \setminus \{0\}$ and $V(0) = 0$), and negative semi-definite on D.
Suppose that $S = \left\{ x \in D \;\middle|\; \dot{V}(x) = 0 \right\}$ and that no solution can stay identically at S other than the trivial solution $x(t) = 0$. Then the origin is asymptotically stable.

Corollary 7.1.5

Let $x = 0$ be the equilibrium point for $\dot{x} = f(x)$, $f : D \to \mathbb{R}^n$ locally Lipschitz. Let $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, radially unbounded, positive definite, and such that $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$.
Suppose that $S = \left\{ c \in \mathbb{R}^n \;\middle|\; \dot{V}(x) = 0 \right\}$ and that no other solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then the origin is globally asymptotically stable.

Remark.

> Corollaries 4.1 and 4.2 are known as the theorems of Barbashin and Krasovskii, who proved them before the introduction of LaSalle's Theorem.

Remark.

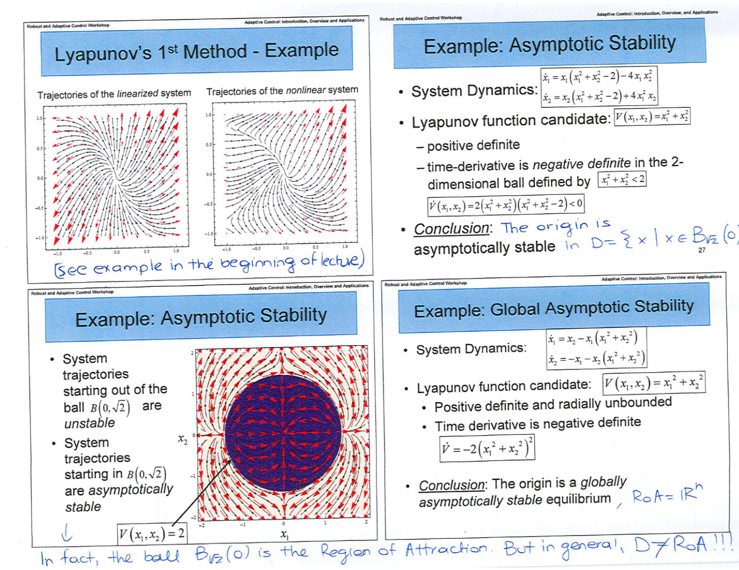
> When $\dot{V}(x)$ is negative definite, $S = \{0\}$, then the corollaries 4.1 and 4.2 coincide with theorems 4.1 and 4.2, respectively.

Example.

> See 4.8–4.11 in our textbook.

In summary, what do we gain with LaSalle's Theorem?

1. It relaxes the negative definiteness requirement from Lyapunov's theorem.
2. It does not require V to be positive definite.
3. It can be used when the system has an equilibrium set.
4. It gives an estimate of the region of attraction, as Ω in Theorem 4.4 can be any compact positively invariant set. Note that Ω does not have to be of the form $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq x\}$.



7.2 Applications of the linearization method, stability theorems, and LaSalle's Principle

Let us consider the following system:

$$\begin{cases} \dot{x}_1 = -x_2 - x_1^3 \\ \dot{x}_2 = x_1^5. \end{cases}$$

Investigate the stability properties of the origin.

We will use several methods:

1. We begin with linearization. We consider the system

$$A = \begin{pmatrix} -3x_1^2 & -1 \\ 5x_1^4 & 0 \end{pmatrix}\Bigg|_{x=0} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = 0$. Since the matrix A has eigenvalues on the imaginary axis, nothing can be concluded via the linearization method.

2. We consider the simplest possible Lyapunov candidate:

$$V = x_1^2 + x_2^2.$$

Let us take the time derivative as

$$\dot{V} = -2x_1x_2 - 2x_1^4 + 2x_2x_1^5.$$

The function does not seem to be useful. The function is indefinite for points near the origin.

> Sometimes when the square of the Euclidean norm does not work, some other quadratic function may do. Suppose we try
>
> $$V = x_1^2 + \alpha x_1 x_2 + \beta x_2^2,$$
>
> where α and β are values to be determined. The time derivative along system trajectories reads
>
> $$\dot{V} = -2x_1^2 + \alpha x_1^6 - 2x_1 x_2 - a x_1^3 x_2 + 2\beta x_1^5 x_2 - \alpha x_2^2.$$
>
> Again, the expression seems difficult to work with. In fact, if we set $(x_1, x_2) = (\delta_1, -\delta^2,$ we can verify that $\dot{V}$ is not negative, no matter how we choose α and β.

Other ideas at this point include customizing the choice of the Lyapunov function. This is similar to what we did with the pendulum example. Note that the right-hand side of the equation seems to suggest that x_1^3, x_2 should be treated with an order of magnitude.

We could try $V(x) = x_1^6 + \alpha x_2^2$ with $a > 0$ to be determined. Clearly, V is positive definite. The time derivative is

$$\dot{V} = (2a - 6)x_1^6 x_2 - 6x_1^8.$$

Then if $a = 3$, we have $\dot{V} = -6x_1^8 \leq 0$, i.e., a negative semi-definite function. From Theorem 4.1, the system is stable.

This reasoning shows that the system is stable, but is it asymptotically stable? We can resort to LaSalle to address this point.

Remark.

> Note that we could have tried to modify our Lyapunov function again to get a negative definitive time derivative for V. A bit of experimentation shows that the function $V = x^6 + xy^3 + 3y^2$ is positive definite in a neighborhood of the origin, while its derivative is negative definite. To establish the result, we use Young's inequality. We won't pursue the analysis here, but you should recall that if you have a negative semi-definite function, then tweaking it carefully (without destroying its positive definiteness) may yield a negative definite derivative. However, how to tweak the function may be an art...

Example.

> Consider the system
>
> $$\begin{cases} \dot{x}_1 = -x_2 - x_1^3 \\ \dot{x}_2 = x_1^5 \end{cases}$$
>
> We investigate the stability properties of the origin. Let the candidate Lyapunov function be
>
> $$V(x) = x_1^6 + 3x_2^2.$$
>
> The time derivative along the system trajectories is
>
> $$\dot{V}(x) = -6x_1^8 \leq 0.$$
>
> As $\dot{V}$ is negative semi-definite, Theorem 4.1 allows us to conclude that the system is

stable, but not that it is asymptotically stable. To investigate this point, we turn to LaSalle's invariance principle.

We have

$$S = \left\{ x \in \mathbb{R}^2 \;\middle|\; \dot{V}(x) = 0 \right\} = \left\{ x \in \mathbb{R}^2 \mid (x_2 \neq 0 \wedge x_1 = 0) \vee (x_2 = 0 \wedge x_1 = 0) \right\}.$$

For the set $\left\{ x \in \mathbb{R}^2 \mid x_2 \neq 0 \wedge x_1 = 0 \right\}$, we have $\dot{x}_1 \neq 0 \Rightarrow x_1 \neq 0$, and also $\dot{x}_2 \neq 0 \Rightarrow x_2 \neq 0$. This implies that the system trajectories cannot identically start in the set $\left\{ x \in \mathbb{R}^2 \mid x_2 \neq 0 \wedge x_1 = 0 \right\}$, i.e., that the set $\left\{ x \in \mathbb{R}^2 \mid x_2 \neq 0 \wedge x_1 = 0 \right\}$ is not invariant.

Then the largest invariant set M is the singleton $(x_1, x_2) = (0, 0)$. From Corollary 4.2, the origin is globally asymptotically stable.

Chapter 8

Region of attraction

It is often insufficient to establish that a system has an asymptotically stable equilibrium point. It is also important to know how far away from the equilibrium point the system trajectories can start and still converge to the equilibrium.

Definition 8.0.1: Region of attraction

Let $\dot{x} = f(x)$, where $f\colon D \to \mathbb{R}^n$, is locally Lipschitz, and let the origin be the equilibrium. D is a domain $D \subseteq \mathbb{R}^n$ that contains the origin. We are interested in finding the following set:

$$R_A = \left\{ x_0 \in D \;\middle|\; x(t, x_0) \text{ is defined for all } t \geq 0 \text{ and } \lim_{t\to\infty} x(t, x_0) = 0 \right\}.$$

R_A is called *the region of attraction*.

Remark.

> We already know that if the conditions of Theorem 4.2 are met, then the origin is globally asymptotically stable. That means the region of attraction is the entire $\mathbb{R}^n$.

We want to employ Lyapunov's Theorem 4.1 to obtain estimates of the region of attraction. By estimates, we mean that we want a set $\Omega \subseteq R_A$ such that any trajectory starting in Ω approaches the origin as $t \to \infty$.

Remark.

> Recall the assumptions of Theorem 4.1:
>
> 1. $D \subseteq \mathbb{R}^n$ is an open set containing the origin;
> 2. $\dot{x} = f(x)$, where f is locally Lipschitz on D;
> 3. $f(0) = 0$, i.e., the origin is the equilibrium;
> 4. $V\colon D \to \mathbb{R}$ is continuously differentiable and positive definite on D; and
> 5. $\dot{V}\colon D \to \mathbb{R}$ is negative definite on D.

Given $V(x)$ is positive definite on D and $\dot{V}(x)$ negative definite on D, we might want to jump to the conclusion that D is an estimate of R_A. This conjecture is not true! The textbook gives a counterexample.

Example.

(Example 8.8 in the book.)

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2. \end{cases}$$

We choose the Lyapunov function

$$V(x) = \frac{1}{2}x^\top \underbrace{\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{pmatrix}}_{P>0} x + \int_0^{x_1} \left(y - \frac{1}{3}y^3\right) dy.$$

Remark.

The Lyapunov candidate function and the system generalizations of the pendulum and mass-spring-damper examples; the first term corresponds to the kinetic energy, and the second term to potential energy.

The time derivative reads

$$\dot{V}(x) = -\frac{1}{2}x_1^2\left(1 - \frac{1}{3}x_1^2\right) - \frac{1}{2}x_2^2.$$

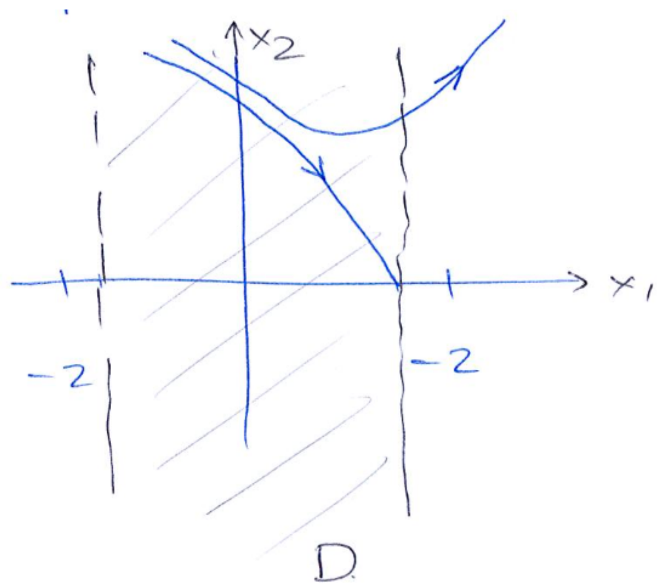
Then, if we define the domain

$$D = \left\{x \in \mathbb{R}^2 \;\middle|\; -\sqrt{3} < x_1 < \sqrt{3}\right\},$$

we obtain that $V(x)$ is positive definite on D, and $\dot{V}(x)$ is negative definite on D. Theorem 4.1 concludes that the origin is asymptotically stable.

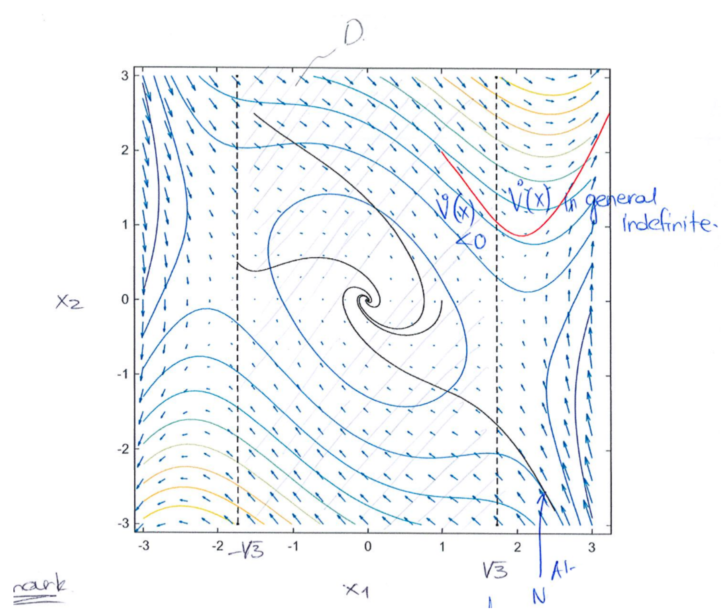
Remark.

Note, however, that the domain D is not a subset of the region of attraction R_A. This can be verified through inspection of the phase portrait.



There are trajectories starting in D that will not stay in D, and hence will not approach the origin. As we have mentioned earlier in class, the catch lies in the fact that the level sets of V are not necessarily compact for any $c \in \mathbb{R}_{\geq 0}$.

The phase portrait of the system in Example 8.8 with the level surfaces of the Lyapunov function used. We verify that the domain $D = \left\{x \in \mathbb{R}^2 \mid |x_1| < \sqrt{3}\right\}$ is not a subset of the region of attraction. For instance, the red trajectory starts in D but escapes D. Beyond that point, there is no guarantee that $\dot{V}(x) < 0$.



Note that the trajectory labeled in the bottom right does approach the origin, despite starting outside D.

Remark.

In general, the problem of computing the region of attraction is addressed via compact level sets of $V(x)$ that, in addition, are positively invariant, so that every trajectory starting in the set stays forever in the set.

Remark.

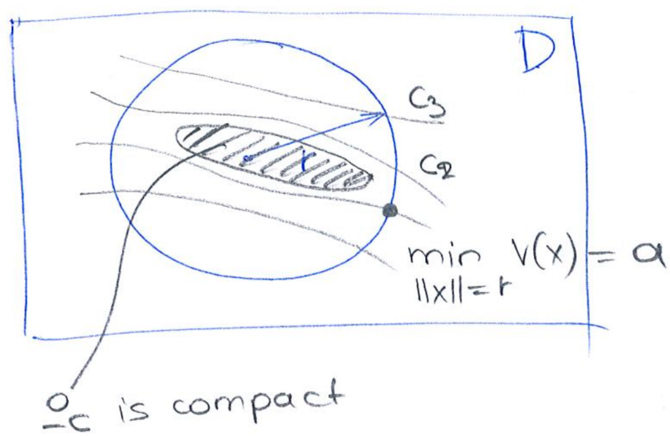
Problem objective: we seek $c > 0$ such that

$$\Omega_c = \{x \in D \mid V(x) \leq c\}$$

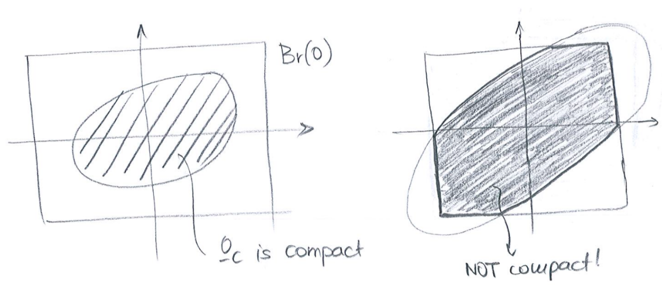
is compact. Then from Theorem 4.1, we have that

$$\forall x_0 \in \Omega_c. \lim_{t\to\infty} x(t, x_0) = 0.$$

In other words, $\Omega_c \subseteq R_A$. How do we pick the compact level sets? We construct Ω_c where $c < a = \min_{\|x\|=r} V(x)$. This set is closed because it contains all its limit points, and bounded because it is contained in $B_r(0)$.



In general, given any norm on $\mathbb{R}^n$, you can think of compact sets as those that can be contained in $B_r(0)$ such that $\partial\Omega \cap \partial B_r(0) = \emptyset$.



Remark.

We will investigate the estimation of the region of attraction using quadratic Lyapunov functions, i.e., functions of the form

$$V(x) = x^\top P x, \quad P = P^\top.$$

Recall that if P is symmetric and real, its eigenvalues are real. Moreover,

$$\lambda_{\min} x^\top x \leq x^\top P x \leq \lambda_{\max} x^\top x,$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of P, respectively.

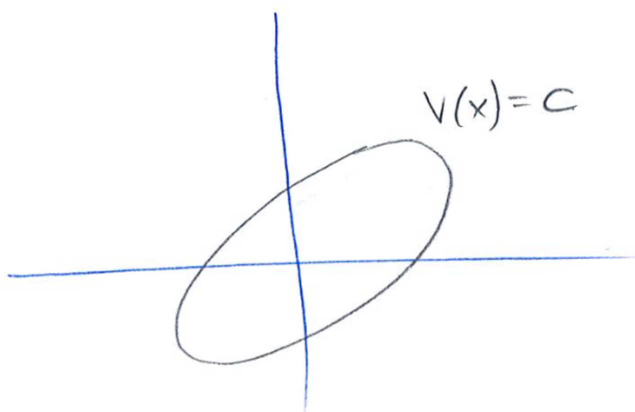
Remark.

Assumption: P is positive definite. This means that $\lambda_{\min} > 0$. Take $c > 0$ and let us investigate the geometry of the level sets of $V(x) = x^\top P x$.

- The level surfaces

$$\Omega_c = \{x \mid V(x) = c\} = \left\{x \;\middle|\; x^\top P x = x\right\}$$

are ellipses.



- Now let us consider the level sets

$$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\} = \left\{x \in \mathbb{R}^n \;\middle|\; x^\top P x \leq c\right\}.$$

For this set to be contained in a ball $B_r(0) := D$, it suffices to pick

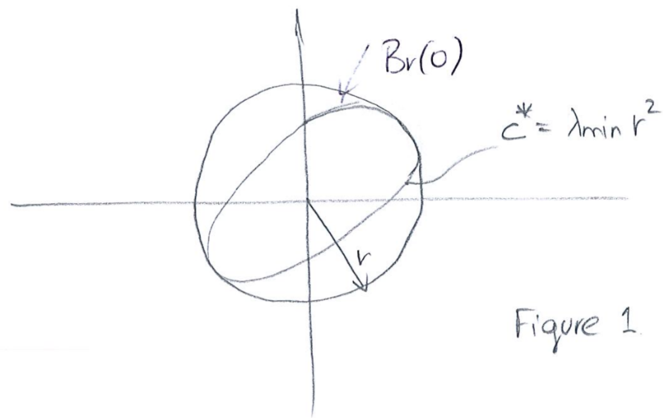
$$c < \min_{\|x\|=r} (x^\top P x) = \lambda_{\min} r^2.$$

This means that for a given r, the largest $c > 0$ such that

$$\Omega_c \subseteq \bar{B}_r(0)$$

is $c^* = \lambda_{\min} r^2$.

- Geometrically, this characterizes the largest ellipse that can be contained in a given circle.



Equivalently, for given $c > 0$, the smallest $r > 0$ such that $\Omega_c \subseteq \bar{B}_r(0)$ is $r^* = \sqrt{\frac{c}{\lambda_{\min}}}$. Geometrically, this characterizes the smallest circle that can contain a given ellipse.

- We may also want to define a ball $\bar{B}_r(0)$ contained in the level set

$$\Omega_c = \left\{ x \in \mathbb{R}^n \;\middle|\; x^\top P x \leq c \right\}.$$

For this, it suffices to pick $c > \max_{\|x\|=r} x^\top P x = \lambda_{\max} r^2$.

This means that, for given $r > 0$, the smallest $c > 0$ such that $\bar{B}_r(0) \subseteq \Omega_c$ is $c_* = \lambda_{\max} r^2$. Geometrically, this characterizes the smallest ellipse that contains a given circle.

We can also view it as, for given $c > 0$ the largest $r > 0$ such that $\bar{B}_r(0) \subseteq \Omega_c$ is $r^* = \sqrt{\frac{c}{\lambda_{\max}}}$. Geometrically, this is the largest circle contained in a given ellipse.

We will see how we can use these bounds to estimate the region of attraction through an example.

Remark.

Recall the Lyapunov equation $A^\top P + PA = -Q$, $P, Q > 0$ and what it implies for a system whose A matrix is Hurwitz.

Example.

Consider the system

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 - (x_1^2 + 1)x_2. \end{cases}$$

We are asked to determine if the origin is asymptotically stable, and if so, to give an estimate of R_A.

First, we apply the linearization method to draw conclusions about the stability of the origin.

$$A = \frac{\partial f}{\partial x}\Big|_0 = \begin{pmatrix} 0 & -1 \\ 1 - 2x_1x_2 & -(x_1^2+1) \end{pmatrix}\Bigg|_{x=0} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Compute the eigenvalues of these matrices to get the polynomial $\lambda^2 + \lambda + 1 = 0$. This means we have asymptotic stability. Hence, the origin of the nonlinear system is asymptotically stable. We need a Lyapunov function for the nonlinear system. For this, we consider the Lyapunov equation for $Q = I$. We obtain

$$A^\top P + PA = -I \Rightarrow P = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} > 0.$$

We take the Lyapunov function candidate for the nonlinear system as

$$V(x) = \frac{3}{2}x_1^2 - x_1x_2 + x_2^2 > 0.$$

The time derivative reads

$$\dot{V}(x) = -x_1^2(1 - x_1x_2) - x_2^2(1 + 2x_1^2).$$

We have $\dot{V}(x) < 0$ if $1 - x_1x_2 > 0$ and $x \neq 0$. We notice that $x_1x_2 < 1$ implies that $|x_1x_2| < 1$. Moreover, $|x_1x_2| \leq \frac{1}{2}\|x\|_2^2$. Hence, if $\frac{1}{2}\|x\|_2^2 < 1$, we have $\|x\|_2 < \sqrt{2}$. We also have $|x_1x_2| < 1$. Thus, we conclude that $\dot{V}(x)$ is negative definite on $D = B_{\sqrt{2}}(0)$ in the Euclidean norm. So the problem now reads: find $c > 0$ such that

$$\Omega_c = \left\{x \in \bar{B}_{\sqrt{2}}(0) \;\middle|\; V(x) \leq c\right\} \subseteq B_{\sqrt{2}(0)}.$$

In other words, we want to find the largest ellipse contained in

$$\left\{x \;\middle|\; x^\top x \leq 2\right\}.$$

We know that the major axis of the selected ellipse is aligned with the eigenvector corresponding to $\lambda_{\min}(P)$. Matlab can give the eigenvalues and eigenvectors. In this example, we get

$$V = \begin{pmatrix} -0.85 & -0.53 \\ 0.53 & -0.85 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1.81 & 0 \\ 0 & 0.691 \end{pmatrix}.$$

1. So the largest ellipse contained in $\left\{x \;\middle|\; x^\top x \leq (\sqrt{2})^2\right\}$ is the level surface

$$\left\{x \in B_{\sqrt{2}}(0) \;\middle|\; x^\top Px = c^*\right\},$$

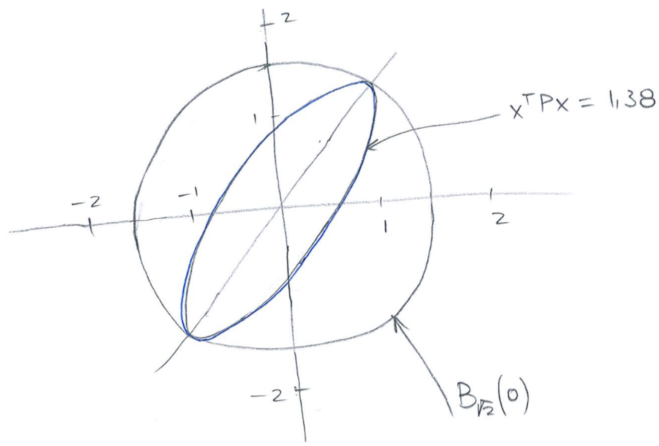
where

$$c^* = r^2\lambda_{\min} = (\sqrt{2})^2 0.691 = 1.38.$$

2. That means that for all $0 < c < c^*$,

$$\Omega_c = \left\{x \;\middle|\; x^\top Px = c\right\} \subseteq B_r(0).$$

3. Hence, Ω_c is compact and $x \in \Omega_c$ with $x \neq 0$ implies that $V(x) > 0$ and $\dot{V}(x) < 0$.
4. Thus, Ω_c is an estimate of the region of attraction.



Remark.

> A more conservative estimate is the largest ball contained in Ω_c. The radius of this ball is
> $$r^* = \sqrt{\frac{c^*}{\lambda_{\max}}} = \sqrt{\frac{1.38}{1.81}} \approxeq 0.873.$$

Alternatively, we can attempt to enlarge the region of attraction. Notice that we had

$$\dot{V}(x) = -x_1^2 + x_1^2 x_1 x_2 - x_2^2(1 + 2x_1^2) = -\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -x_1^2/2 \\ -x_1^2/2 & 1 + 2x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then if $\mu(x) > 0$, we have $\dot{V}(x) < 0$. When $\mu(x) > 0$, we have

$$1 + 2x_1^2 - \frac{x_1^2}{4} > 0 \Leftrightarrow -x_1^4 + 8x_1^2 + 4 > 0.$$

We consider $\mu = x_1^2$, then the solutions of $-\mu^2 + 8x_1^2 + 4 = 0$ are $\mu_1 \approxeq -0.4721$ and $\mu_2 = 8.4721$. Hence, we take $x_1^2 = 8.4721$, which implies that $x_1 = \pm\sqrt{8.4721}$ are points where $\mu(x)$ is no longer positive definite.

We conclude that $\dot{V}(x) < 0$ for $x_1^2 < 8.4721$ for $x \neq 0$. We now note that $\|x\|_2 < \frac{\sqrt{8.472}}{2.91}$ implies that $x_1^2 < 8.472$. Hence, we pick the ball $D = B_{2.91}(0)$ in Euclidean norm. Then we have

$$\begin{cases} V : D \to \mathbb{R} \text{ such that } V(x) > 0, x \neq 0, x \in D \\ \dot{V} : D \to \mathbb{R} \text{ such that } \dot{V}(x) < 0, x \neq 0, x \in D. \end{cases}$$

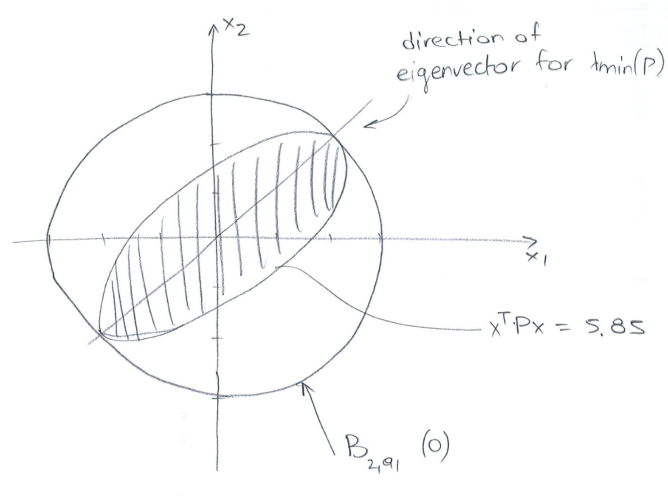
We now can approximate the region of attraction as the largest ellipse contained in

$$\bar{B}_{2.91}(0) = \{x \mid x^\top P x \leq c\} \subseteq B_{2.91}(0).$$

This means that Ω_c is compact. In addition, for all $x \in \Omega_c$, we have $\dot{V}(x) < 0$, $V(x) > 0$, $x \neq 0$. We conclude that Ω_c is an estimate of the region of attraction.

Remark.

This estimate is much bugger than the previous estimate. The improvement came from finding a larger set where $\dot{V}(x) < 0$. What does this look like?



Remark.

The reason that we obtained different results in the two approaches is that we only obtain *estimates* of the region of attraction. The RoA is the same in both cases, but in the first case we obtained a smaller region where $\dot{V} < 0$ compared with the second one. If we had tried a different Lyapunov function, we will likely get a different estimate.

Bibliography

[1] Kelley, J. L. *General topology*, vol. No. 27 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.].