THEORETICAL DERIVATION

LAPLACE TRANSFORM

Laplace transform is conversion from time domain to a complex valued frequency domain known as s-domain or s-plane. This derivation is founded by Pierre-Simon Laplace, hence the name Laplace Transform.

TRANSFORMING SYSTEM TIME DOMAIN EQUATION TO S-DOMAIN

Below is the transformation for a mass spring and damper:

First, convert the equation to a time domain

$$m\ddot{x} + c\dot{x} + kx = F_o \sin(\omega t)$$

$$\frac{1}{m\ddot{x} + c\dot{x} + kx} = \frac{1}{F_o \sin(\omega t)}$$

$$F_o = \frac{m\ddot{x} + c\dot{x} + kx}{\sin(\omega t)}$$

$$F(t) = \frac{m\ddot{x} + c\dot{x} + kx}{\sin(\omega t)} \text{, where } t=0$$

Then, use the Laplace replacement for sin and differential

$$\frac{1}{\omega} \sin \omega t = \frac{1}{s^2+\omega^2}$$

$$\sin \omega t = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}(f)$$

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Then, resolve the differential

$$\mathcal{L}(f) = x(s)$$

$$\mathcal{L}(f') = s\,x(s) - x(0)$$

$$\mathcal{L}(f'') = s^2 x(s) - s\,x(0) - \frac{dx(0)}{dt}$$

Then, plug into the time domain equation to become s-domain

$$F(s) = \frac{m\left[s^2 x(s) - s x(0) - \frac{dx(0)}{dt}\right] + c\left[s x(s) - x(0)\right] + k\left[x(s)\right]}{\frac{\omega}{s^2+\omega^2}}$$

$$= \frac{\left(m\left[s^2 x(s) - s x(0) - \frac{dx(0)}{dt}\right] + c\left[s x(s) - x(0)\right] + k\left[x(s)\right]\right)(s^2+\omega^2)}{\omega}$$

[Open]

Assume at initial state of the speed and position to be zero

$$x(0)=0,\ \frac{dx(0)}{dt}=0$$

Then, simplify the equation to give the X(s)/F(s)

$$F(s)=\frac{\left(m[s^2x(s)]+c[s\,x(s)]+k[x(s)]\right)(s^2+\omega^2)}{\omega} \Rightarrow \frac{X(s)}{F(s)}=\frac{\omega}{(ms^2+cs+k)(s^2+\omega^2)}$$

EXPRESSION OF STEADY STATE AND SPACE

The steady-state response is evaluated to understand how a system behaves after all temporary effects have died out, to see its true long-term performance. This helps us know whether the system will settle, oscillate, or become unstable.

FINDING THE STEADY STATE AND SPACE

Given the following G(x) :

$$\frac{X(s)}{F(s)}=\frac{\omega}{(ms^2+cs+k)(s^2+\omega^2)}$$

$$X(s)=\frac{\omega F(s)}{(ms^2+cs+k)(s^2+\omega^2)}$$

Resolve the frequency domain evaluation where s = jw :

$$(j\omega)^2=-\omega^2 \qquad H(j\omega)=\frac{\omega}{(m(j\omega)^2+cj\omega+k)((j\omega)^2+\omega^2)}$$

$$=\frac{\omega}{(m(-\omega^2)+cj\omega+k)(j\omega^2+\omega^2)}$$

$$=\frac{\omega}{(k-m\omega^2+cj\omega)(j\omega^2+\omega^2)}$$

first term

second term

$$=\frac{\omega}{(k-m\omega^2+cj\omega)(-\omega^2+\omega^2)}$$

whereby $c>0$ for amplitude to stay finite and avoid singularity

Using the sine wave steady state equation (as our input is sine wave) , derive the derivatives to find the amplitude:

$$x_{ss}(t)=X\sin(\omega t-\phi)$$

whereby X = amplitude , ϕ = phase lag

$$\dot{x}_{ss}=X\omega\cos(\omega t-\phi)$$

$$\ddot{x}_{ss}=-X\omega^2\sin(\omega t-\phi)$$

Substitute back into the original mass spring damper equation:

$$m\ddot{x} + c\dot{x} + kx = F_o \sin(\omega t)$$

$$m[-X\omega^2 \sin(\omega t - \phi)] + c[X\omega^2 \sin(\omega t - \phi)] + k[X \sin(\omega t - \phi)] = F_o \sin(\omega t)$$

$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \Rightarrow x_{ss}(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \sin\left(\omega t - \tan^{-1}\frac{c\omega}{k - m\omega^2}\right)$$

NATURAL FREQUENCY, DAMPING RATIO AND RESONANT FREQUENCY

Natural Frequency Calculation Theory

Natural Frequency is calculated by the following:

$f = \frac{\omega}{2\pi}$ where ω is the angular velocity

$\omega = \sqrt{\frac{k}{m}}$ where k is spring constant and m is mass

so the final equation will be

$$f = \frac{\sqrt{k/m}}{2\pi}$$

Damping Ratio Calculation Theory

Damping Ratio is calculated by the following:

$$\zeta = \frac{c}{2 * \sqrt{mk}}$$

where c is damping coefficient

m is mass

k is spring constant

Resonant Frequency Calculation Theory

Resonant Frequency are Frequency higher than the fundamental frequency (The lowest frequency that fits the string or space). Harmonics are measured based on the end conditions of the wave to determine whether it is fixed or free. The following describes the end conditions and the formula to apply:

End conditions	Same	Different
Fundamental Frequency	$f = \frac{x}{2L}$	$f = \frac{x}{4L}$
Harmonic multiples	Even ($2f, 4f, 6f, \ldots$)	Odd ($3f, 5f, 7f, \ldots$)

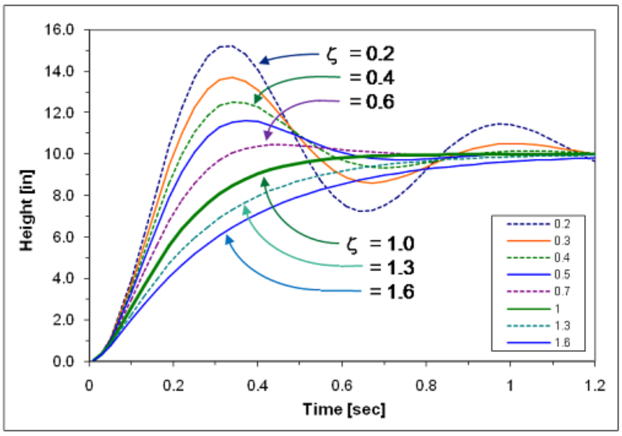
where x is the speed of wave in medium and L is length of string or pipe.

DAMPING VS SYSTEM RESPONSE

Zeta is calculated to see how system responds to disturbances. The range of zeta value indicates how quickly system oscillates in a system decay.

$$\tau = \sqrt{\frac{m}{g_c k}} \quad [=]\text{tau}$$

$$\zeta = \sqrt{\frac{g_c c^2}{4mk}} \quad [=]\text{zeta}$$



The range of zeta expresses different responses in system and can be referred to as following:

$\zeta = 0$ undamped (infinite continuous oscillations)

$0 < \zeta < 1$ undamped(gradual oscillations decay)

$\zeta = 1$ Critically damped (fast equilibrium, no oscillations)

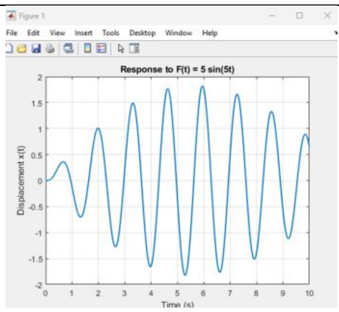
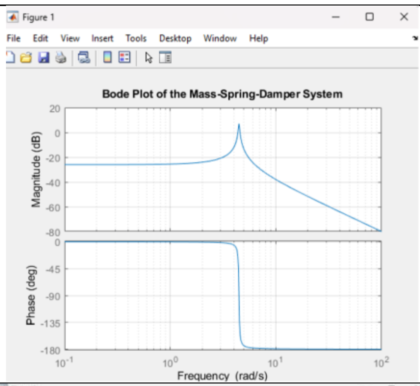
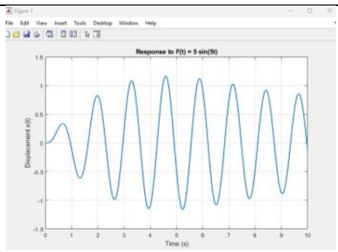
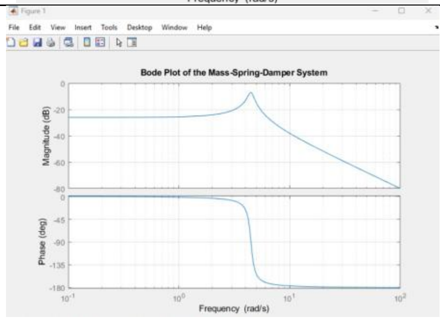
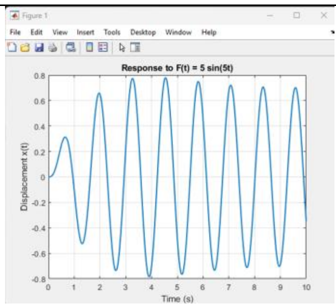
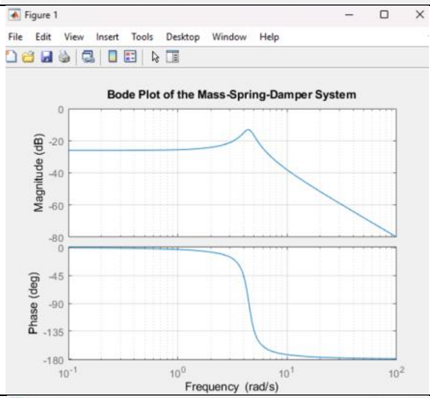
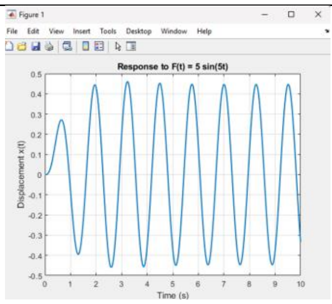
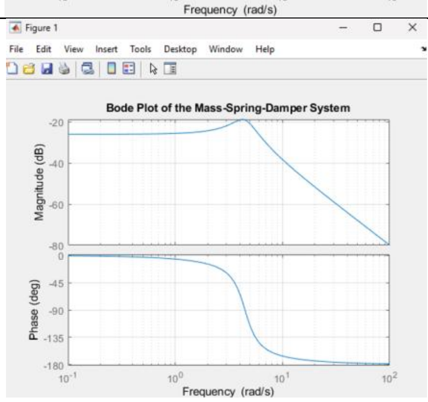
$\zeta > 1$ Overdamped (slow equilibrium, no oscillations)

MATHLAB SIMULINK SIMULATION

Given 4 Damping Coefficient value, the following is the G(s) output from MATHLAB

C Coefficient	0.1	0.5	1.0	2.0
Simulated G(s)	G = $\frac{1}{s^2 + 0.1 s + 20}$	G = $\frac{1}{s^2 + 0.5 s + 20}$	G = $\frac{1}{s^2 + s + 20}$	G = $\frac{1}{s^2 + 2 s + 20}$

The following graph are response to F(t) and bode plot with different damping coefficients

<table>
<tr><th>C Coefficient</th><th>Response to F(t)</th><th>Bode plot</th><th>Remarks</th></tr>
<tr><td>0.1</td><td>
</td><td>
</td><td>Wider phase and amplitude</td></tr>
<tr><td>0.5</td><td>
</td><td>
</td><td>Amplitude is less than 0.1 but has same phase</td></tr>
<tr><td>1.0</td><td>
</td><td>
</td><td>shorter amplitude than 0.1 and faster phase</td></tr>
<tr><td>2.0</td><td>
</td><td>
</td><td>Same phase as 1.0 but lesser amplitude than 1.0 (best damping as the phase reacts fastest) Here system vibrates the least</td></tr>
</table>

DAMPING AFFECT ON MAGNITUDE AND RESPONSE

Magnitude of response becomes smaller.
With more damping, the displacement amplitude decreases. The system does not vibrate as strongly when a force is applied.

Resonance peak becomes lower and wider.
At frequencies near the natural frequency, an undamped system can produce very large vibrations. Damping reduces this peak, preventing extreme amplitudes.

System responds more slowly.
Higher damping makes the system take longer to reach its steady-state motion. It settles smoothly instead of oscillating heavily.

DAMPING CHANGES VS RESONANT FREQUENCIES

Damping	**Amplitude**	**Resonance Peak**	**Phase Shift**
Low	Very high	Tall & sharp	Rapid change
Medium	Moderate	Lower & wider	Moderate change
High	Small	Low & broad	Slow, smooth change

THEORETICAL VS SIMULATION COMPARISON

Theoretical gives the equations to resolve the damping without any real value while simulation gives a more graphical view of the damping system

REFERENCE

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TrueGeometry. (n.d.). *Exploring HTML content* [Web blog post]. Retrieved November 29, 2025, from https://blog.truegeometry.com/api/exploreHTML/0437a2b859b8e8c6377d6a59649293ad.exploreHTML

APPENDIX (Code)

```
>> % System parameters
m = 1;        % mass (kg)
c = 0.5;      % damping constant (Ns/m)
k = 20;       % spring constant (N/m)
```

```
% Transfer function G(s) = 1 / (m*s^2 + c*s + k)
numerator = 1;
denominator = [m c k];

G = tf(numerator, denominator)

% Input force F(t) = 5*sin(5t)
% For simulation, define input and run lsim or use forced response

G =

      1
  ----------------
  s^2 + 0.5 s + 20

Continuous-time transfer function.

>> % System parameters
m = 1;       % mass (kg)
c = 0.5;     % damping (Ns/m)
k = 20;      % spring constant (N/m)

% Transfer function G(s) = 1 / (m s^2 + c s + k)
numerator = 1;
denominator = [m c k];
G = tf(numerator, denominator);

% Time vector
t = 0:0.001:10;    % simulate for 10 seconds

% Input force F(t) = 5 sin(5t)
F = 5 * sin(5*t);

% Simulate system response
x = lsim(G, F, t);

% Plot
figure;
plot(t, x, 'LineWidth', 1.5);
xlabel('Time (s)');
ylabel('Displacement x(t)');
title('Response to F(t) = 5 sin(5t)');
grid on;
>> % System parameters
m = 1;       % mass (kg)
c = 0.5;     % damping (Ns/m)
k = 20;      % spring constant (N/m)

% Transfer function G(s) = 1 / (m s^2 + c s + k)
numerator = 1;
denominator = [m c k];
G = tf(numerator, denominator);

% Bode plot
figure;
bode(G);
grid on;
title('Bode Plot of the Mass-Spring-Damper System');
>>
```