Scott Clark
MPIPKS

1 Introduction

We want to examine the extreme statistics of "Gaussian seas" which are superpositions of plane waves with normally distributed amplitudes. We will look at both the real and complex case and numerically determine the density of maxima and the distribution of the values of the maxima for these two cases as well as other statistically significant information.

1.1 Real Case

The function that we are examining in the real case is

$$f(x,y) = \sum_{i=1}^{N} a_i \sin\left(\frac{\sin(\theta_i)x}{\lambda/2\pi} + \frac{cos(\theta_i)y}{\lambda/2\pi} + \delta_i\right)$$

where each a_i independent and identically normally distributed with mean 0 and variance 1 (N(0,1)). And each θ_i, δ_i is independent and identically uniformly distributed from 0 to 2π.

1.2 Complex Case

The function that we are examining in the complex case is

$$\Psi(x,y) = \sum_{i=1}^{N} a_i \cos\left(\frac{\sin(\theta_i^{(1)})x}{\lambda/2\pi} + \frac{cos(\theta_i^{(1)})y}{\lambda/2\pi} + \delta_i^{(1)}\right) + b_i i \sin\left(\frac{\sin(\theta_i^{(2)})x}{\lambda/2\pi} + \frac{cos(\theta_i^{(2)})y}{\lambda/2\pi} + \delta_i^{(2)}\right)$$

where each a_i and b_i is independent and identically normally distributed with mean 0 and variance 1 (N(0,1)). And each $\theta_i^{(1)}, \theta_i^{(2)}, \delta_i^{(1)}, \delta_i^{(2)}$ is independent and identically uniformly distributed from 0 to 2π.

2 Density of maxima

2.1 Real Case

2.1.1 Scaling with area

To test dependence we superimposed 1000 plane waves of varying amplitudes and heights and found how many local maximums (of Ψ^2) we could find in different areas ($A = 100, 400, 900, \ldots, 100^2$) with the wavelength λ set to 2π. The test was run for 100 different sets of 1000 waves to create some statistical stability. The results are as follows,
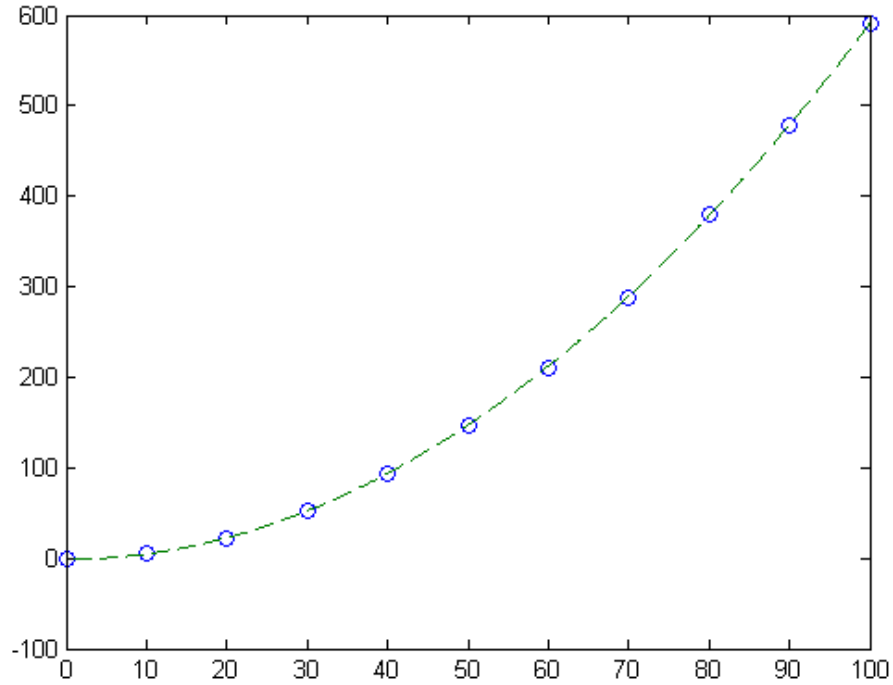

Figure 1: Local maximums found of Ψ^2 versus L, the length of a side of the square area examined.

As we can see the relation looks to be one of $n \propto L^2$

2.1.2 Scaling with wavelength

Now we want to test the wavelength (λ) dependence and so we set A = 400 (L = 20) and test the number of nodes observed for $\lambda = 2\pi, \frac{2\pi}{2}, \frac{2\pi}{3}, \dots, \frac{2\pi}{10}$. The results are as follows,
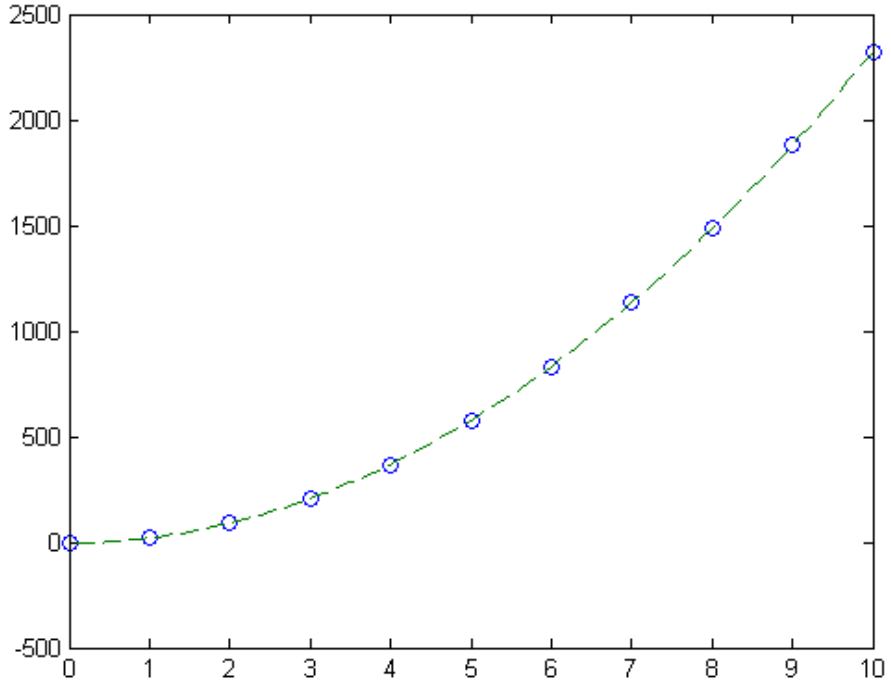

Figure 2: Local maximums found of Ψ^2 versus z, where $\lambda = \frac{2\pi}{z}$.

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Once again we see a quadratic proportionality, $n \propto z^2 = \frac{1}{\lambda^2}$

2.1.3 Scaling with both parameters

We put these two realizations together to conclude that the number of local maximums (n) of Ψ^2 observed for any given wavelength in a box of size A is approximately governed by the function,

$$n_{max} \propto \frac{A}{\lambda^2}$$

For the exact constant we turn to the previously derived analytics [1] and conclude

$$n_{max,f} = \frac{\pi}{2\sqrt{3}}\frac{A}{\lambda^2} = 0.9070\frac{A}{\lambda^2}$$

what we are concerned with is Ψ^2 which will have a coefficient of exactly twice this because all minima will become maxima and the quadratic function is monotonic so all maxima will be preserved. So we have

$$n_{max,f^2} = \frac{\pi}{\sqrt{3}}\frac{A}{\lambda^2} = 1.8138\frac{A}{\lambda^2}$$

and we test this against the number of average number of maxima found for a box of length 512, 128 and 1000 waves of wavelength 2π over 1000 realizations.

$$\frac{(11903.937)(2\pi)^2}{512^2} = 1.7927$$

$$\frac{(745.831)(2\pi)^2}{128^2} = 1.79713$$

both of these results are within about 1 percent of the analytic results. They are slightly lower because we use a finite mesh size to discretized space when we are looking for the maxima and this can cause two maxima to become aliased and only count for one on rare occasions.

2.2 Complex Case

For the complex case we find the same proportionality dependence between the number of maxima and the area and wavelength,

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$$n_{max,|\Psi|^2} \propto \frac{A}{\lambda^2}$$

and we turn to the previous numeric [2] (approximated from graph) and analytic [3] (weak argument) results to conclude

$$n_{max,|\Psi|^2} \approx 2\frac{A}{\lambda^2}$$

this agrees nicely with our numerics for the number of average number of maxima found for a box of length 512, 256 and 1000 waves of wavelength 2π over 1000 realizations.

$$\frac{(13368.45)(2\pi)^2}{512^2} = 2.01326$$

$$\frac{(3515.369)(2\pi)^2}{256^2} = 2.11763$$

2.3 Result

These values give us an effective dimensionality of the system. Each maximum represents a node that can be used in a discrete basis for the superposition. We have the two formula for this dimensionality,

$$n_{real} \approx \frac{\pi}{\sqrt{3}}\frac{A}{\lambda^2}$$

$$n_{complex} \approx 2\frac{A}{\lambda^2}$$

3 Mean value of extremes and their standard deviation

These values will be essential to transforming the distributions of the extremes. We now attempt to numerically derive formulas for the means and standard deviations of extreme points as they correspond to area and wavelength. Below are tables representing key statistical values.

L	Real $\#_{max}$	$\bar{x}_{ex}$	σ_{ex}
8	2.299	3942	1433.8
16	10.429	5127	1487.4
32	44.488	6933	1565.1
64	183.473	8217	1642.4
128	745.831	9568	1599.6
256	1850.677	11324	1528.5
512*	11903.937	10833	-

Table 1: box length, avg. number of maxima, avg height of extreme values, standard deviation of extreme values. Grid spacing was 0.5 except for those marked with *, in which case it was 1.0

L	Complex $\#_{max}$	$\bar{x}_{ex}$	σ_{ex}
8	2.673	4194.6	(1572.3)
16	12.199	5569.1	(1623.1)
32	52.072	7159.5	1759.5
64	214.751	8722.9	1634.3
128	871.744	10263	1564.1
256	3515.369	11807	1487.2
512*	13368.45	12778	1395.2

Table 2: box length, avg. number of maxima, avg height of extreme values, standard deviation of extreme values. Grid spacing was 0.5 except for those marked with *, in which case it was 1.0

3.1 Real Case

Using a modified least squares (where later points are treated with more emphasis) we are able to obtain a function for the mean value of the highest value of a given superposition (the extreme point) (1000 waves, $\lambda = 2\pi$, box length L, 1000 iterations).

$$\bar{x} \approx 2142.943 \log(0.38265L)$$
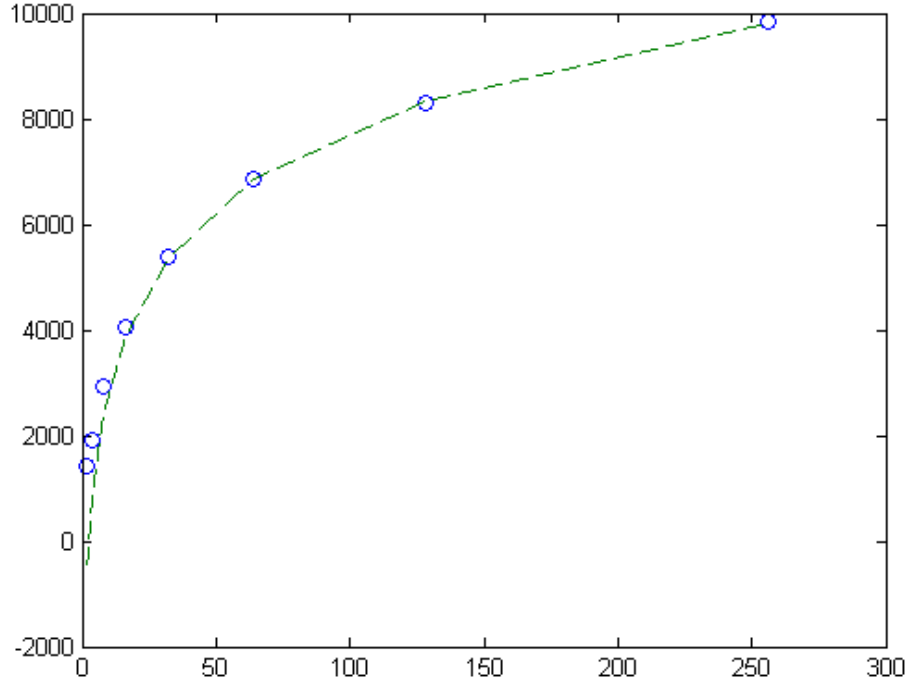
Which can be graphically compared to the data as follows,

Figure 3: mean extreme val vs L real

We can derive a formula similar to this using known values of the system namely the maxima density and the mean value of the field.

$$\bar{x} \approx 2\bar{x}_{field} \log(\pi \#_{max}) \approx 1000 \log(\pi \frac{\pi}{\sqrt{3}} \frac{A}{\lambda^2})$$

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The standard deviation looks to be nearly constant, although like the complex case it may have a negative logarithmic dependence on box length.

$$\sigma \approx 1584$$

3.2   Complex Case

Using a modified least squares (where later points are treated with more emphasis) we are able to obtain a function for the mean value of the highest value of a given superposition (1000 waves, $\lambda = 2\pi$, box length L, 1000 iterations).

$$\bar{x} \approx 2191.779 \log(0.8384L)$$

Which can be graphically compared to the data as follows,
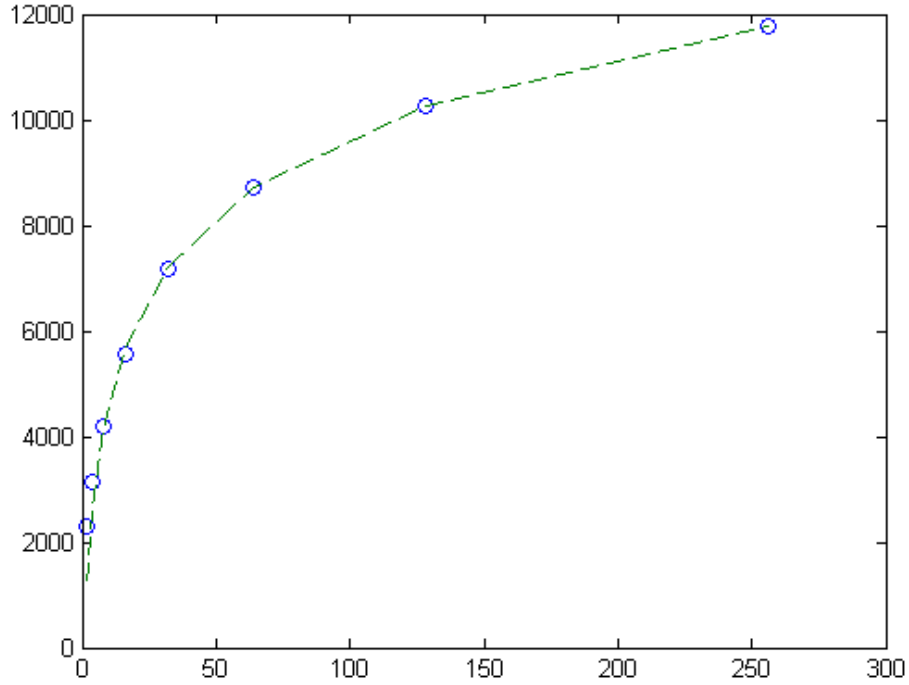
Figure 4: mean ex val vs L complex

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This can also be seen to fit more closely to the analytic results of [3] which state that the largest maximum should scale like

$$\bar{x} = |\psi|_\infty \propto c\sqrt{\log E_n}$$

Using a least squares fitting we obtain the results in the plot below which far outperforms the other model of

$$\bar{x} \approx 2\bar{x}_{max} \log \#_{max}$$
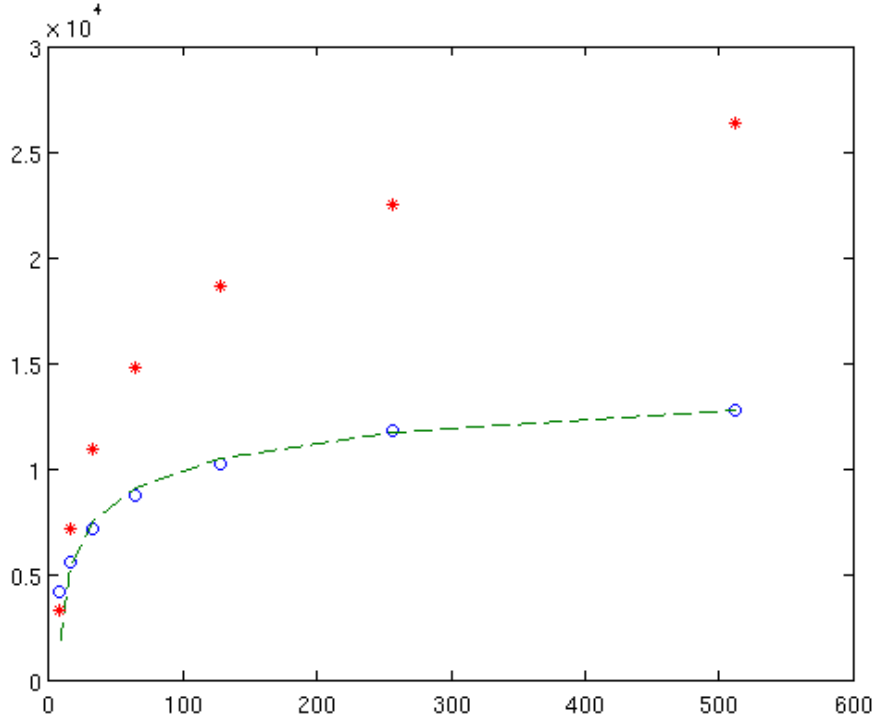
as seen by the plot below,

Figure 5: Mean value of extremes using the 2 theoretical models, dashed from [3]

Using a modified least squares (where later points are treated with more emphasis) we are able to obtain a function for the standard deviation of the

highest value of a given superposition (1000 waves, $\lambda = 2\pi$, box length L, 1000 iterations).

$$\sigma \approx 2187.859 + 127.975 \log(\frac{1}{L}) \quad L \geq 32$$

$$\sigma \approx 127.975 \log(\frac{2.65885E7}{L}) \quad L \geq 32$$

Which was used as a prediction for the σ of $L = 512$ with the following results,

$$\sigma_{pred} = 1395.59$$

and

$$\sigma_{num} = 1395.2$$

so we can see that the standard deviation does drop like $\log(\frac{1}{L})$ which is what we would expect if

$$e^{1/b} \propto L^2$$

3.3 Mean value of all maxima vs. mean value of field

3.3.1 Real Case

The average mean value of a maximum in the system was found to be about 3.66 larger than the average value for the system in general (518.4105). This was done by taking 1000 waves of wavelength 2π and superimposing them on a grids from 8x8 to 512x512 with 1000 realizations and taking the average of the maximum heights and the average of every point in the field for each realization and comparing the ratio of the average of these quantities.

3.3.2 Complex Case

We can also find the average mean value of each extreme in the system which was found to be 2.779 times larger than the average of the field itself (1000). The test performed was identical to the one above.
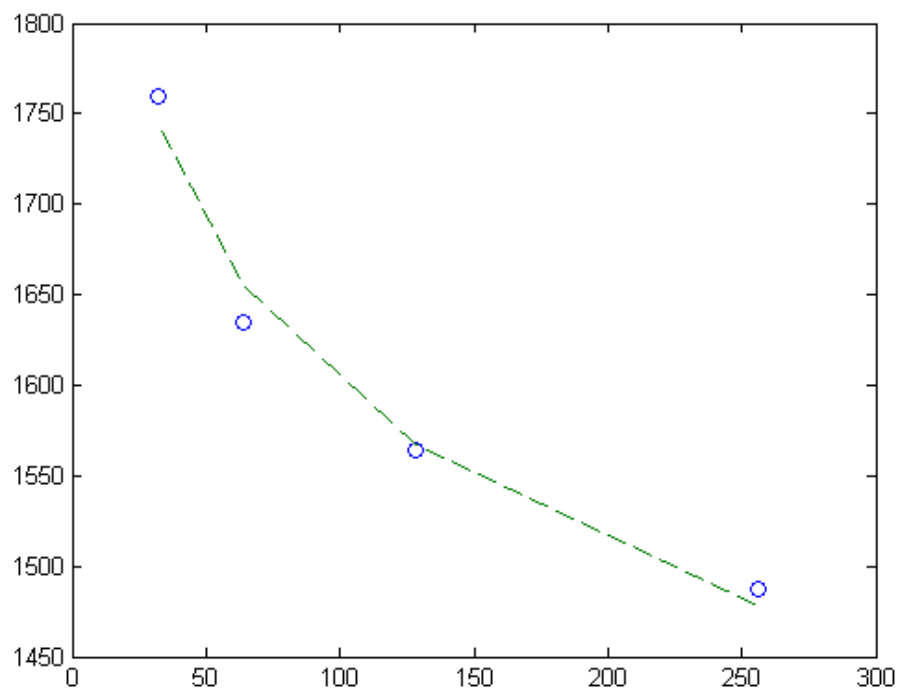
Figure 6: Standard deviation vs. box length for $L \geq 32$

4 Extreme Value Distribution

With the values calculated above we are now prepared to examine how the extreme values (greatest valued point) of our planar superpositions vary.

4.1 Real Case

4.1.1 Fitting to Gumbel by definition

We must transform the variable (32x32, 2pi, 1000 waves, 40,000 it);

$$Y = \frac{X - a}{b}$$

using the definition

$$b = \frac{\sqrt{6}}{\pi}\sigma = 1253.0$$

$$a = \bar{x} - \gamma b = 4707.1$$

4.1.2 Predictive fitting

from above we know

$$\bar{x} \approx 2142.943 log(0.38265 * L)$$

and

$$\sigma \approx 1584.0$$

from this we can obtain the linear transformation

$$b = \frac{\sqrt{6}}{\pi}1584.0 = 1235.0$$

$$a = 2142.943 log(0.38265 * 32) - 0.57721\frac{\sqrt{6}}{\pi}1584.0 = 4655.4$$

which looks like
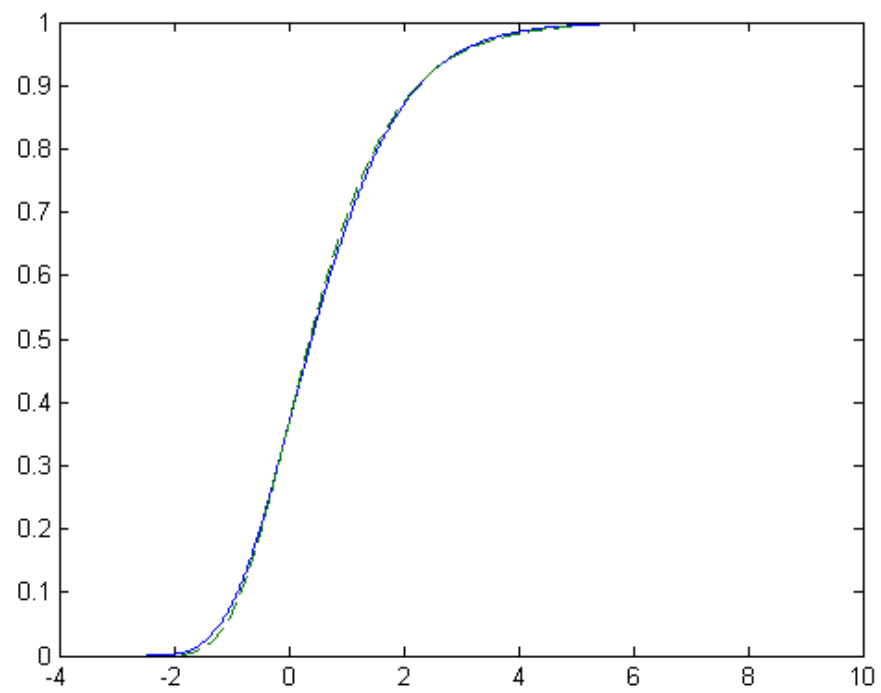
Figure 7: Distribution of extreme points as transformed by definition (real, 40000 iterations)
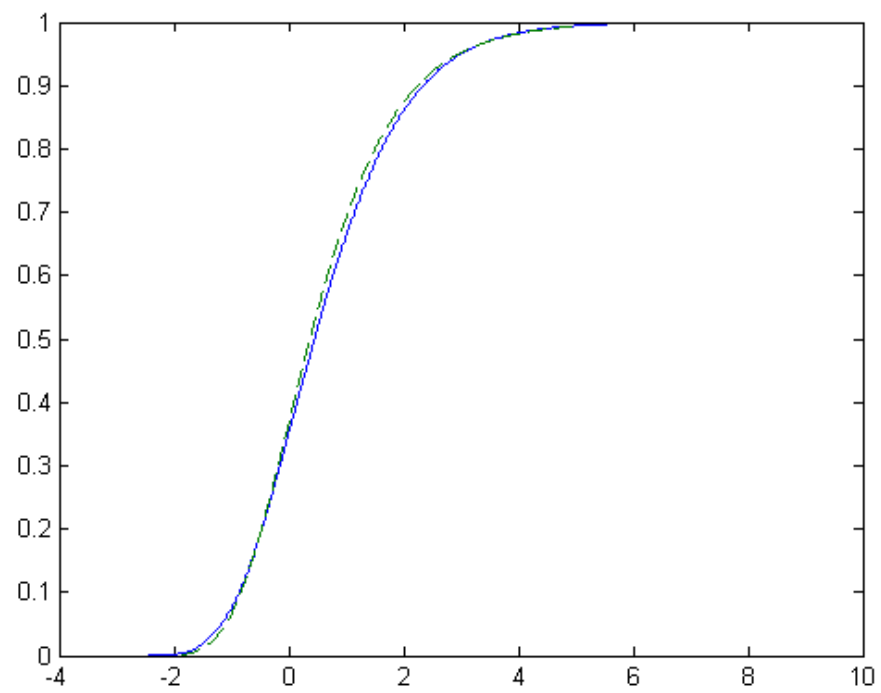
Figure 8: Distribution of extreme points as transformed by prediction (real, 40000 iterations)
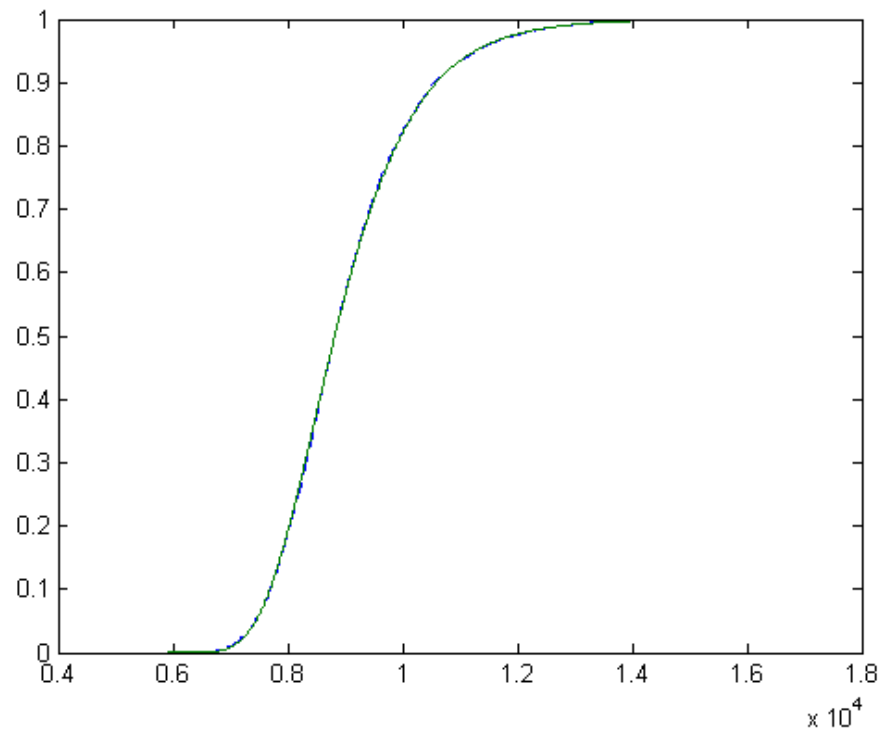
4.1.3 Distribution of 10 largest maxima for each realization

We want to find the distribution of the 10 highest maxima of all of the realizations now, not just the very highest one. We immediately notice that this distribution is Gumbel with constants (by definition)

(This is done with 1000 waves of wavelength 2π on a grid of 512x512 and 1000 realizations, the following 3 sections will have an identical setup)

$$a = 8476.5$$

and

$$b = 942.3357$$

which is a perfect fit,

Figure 9: 1

4.1.4 Distribution of 10 smallest maxima for each realization

Now we want to find the distribution of the 10 smallest maxima of each of the distributions. We anticipate that the distribution should be Weibull which has a distribution function as follows,

$$F(x, k, \lambda) = 1 - e^{-(x/\lambda)^k}$$

We use a least squares minimization to find the constants

$$\lambda = 65.5$$

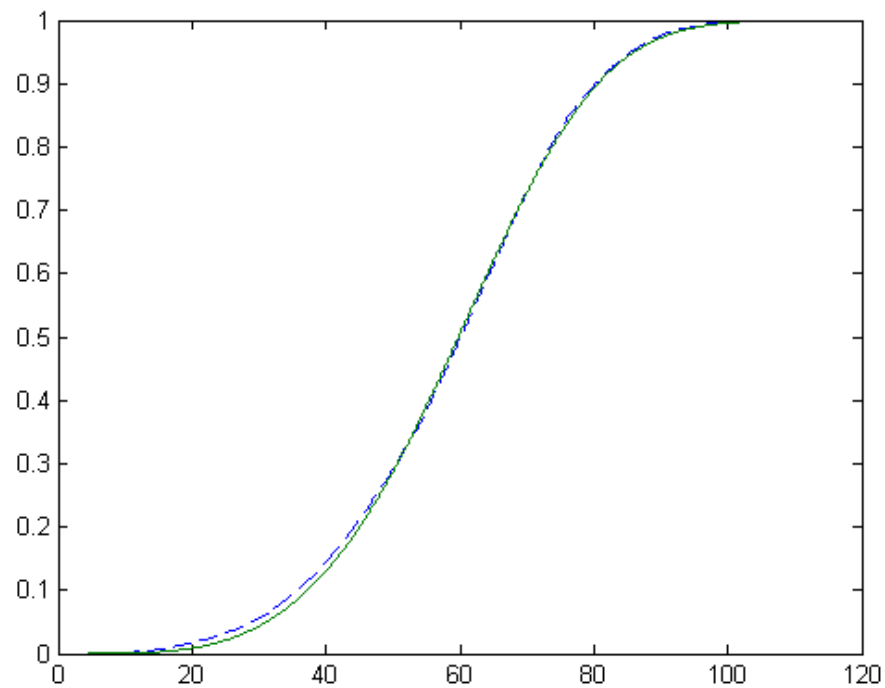$$k = 4$$

which comes to the following results,

Figure 10: 2

4.1.5 Distribution of the difference between the extreme value and 9 next highest maxima

Now we want to look at the distribution of the distance between the value of the highest point of each realization and the closest 9 maxima. We see that this resembles a Gumbel distribution with constants (by definition)

$$a = 1465.0$$

$$b = 951.8022$$
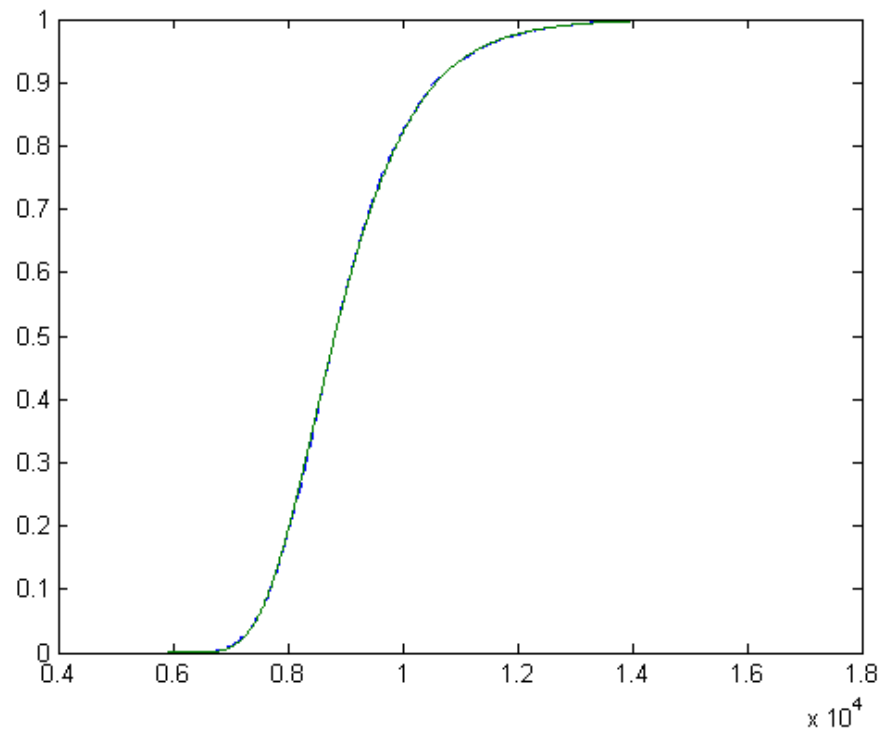
we see that this is a good fit,

Figure 11: 3

4.1.6 Distribution of the difference between the smallest maxima and the 9 next smallest maxima

Now we want to find the same thing but in the opposite order. We want to find the distribution of the distance between the value of the lowest maxima and the 9 closest maxima to it. This is not immediately obvious what distribution it should be. The following plot is the Gumbel and the Weibul with constants found by definition and least squares respectively.
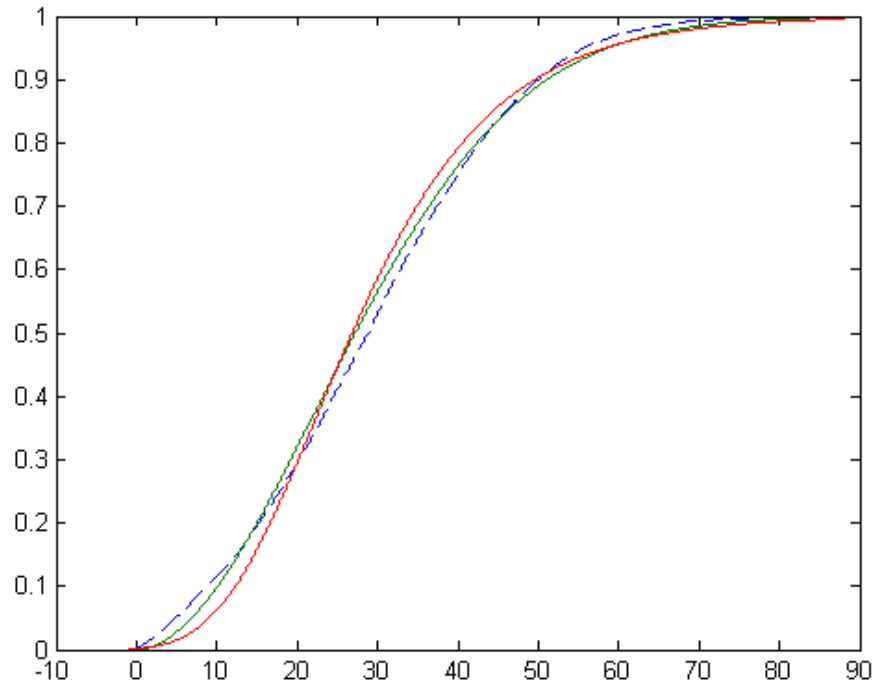

Figure 12: 4

We see that these two distributions seem to fail at representing the value we are interested in. The normal distribution with parameters $\mu = 29.38.78$ and $\sigma = 15.5360$ seems to fit pretty well though,
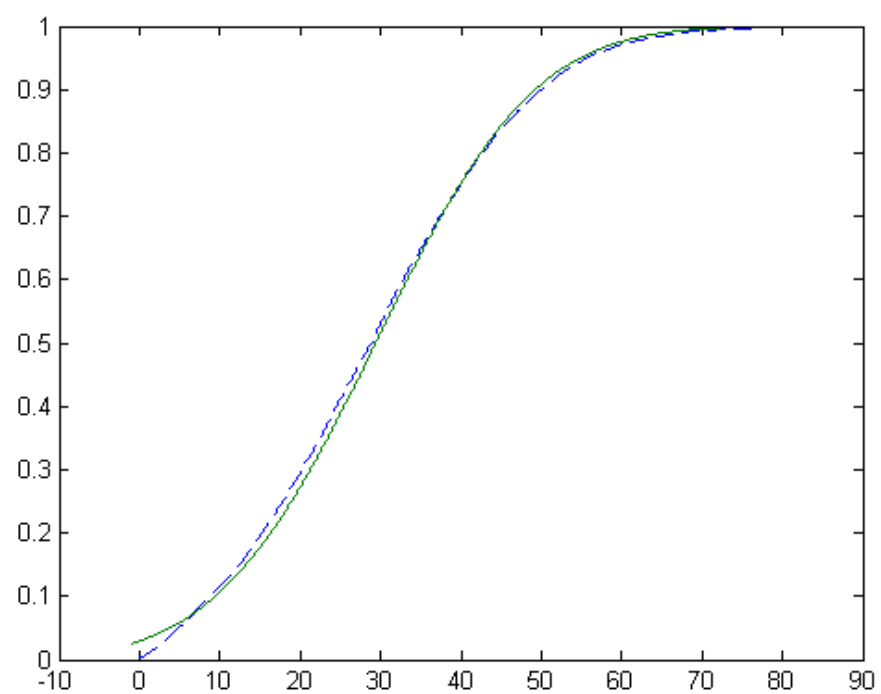
18

Figure 13: 5

4.2 Complex Case

4.2.1 Fitting to Gumbel by definition

We want to find if the distribution of the extreme values for different realizations of this superposition fit that of the Gumbel distribution. To do this we superimpose 1000 plane waves of wavelength 2π in a 100x100 grid (with discretization width .5) 1000 times and record the largest value for $|\psi|^2$ for each. We then make the distribution of these extremes. The idea is that by linearly transforming our random variable by

$$Y = \frac{X - a}{b}$$

We now use a transformation where we fix a and b by the equations

$$\bar{x} = a + \gamma b$$

$$\sigma = \frac{\pi}{\sqrt{6}} b$$

instead of the least squares data fitting. We have

$$\bar{x} = 9639.3$$

$$\sigma = 1610.5$$

which gives us

$$a = 8914.5$$

$$b = 1255.7$$

for the 100x100 case and for the 500x500 case we obtain,

$$\bar{x} = 12684$$

$$\sigma = 1404.7$$

which gives us

$$a = 12051$$

$$b = 1095.3$$

now we look at the distribution compared to that of the Gumbel,

We see that this is a good fit, and even when the area is increased to a box of 500 x 500, the distribution does not change (it is stationary at this point),
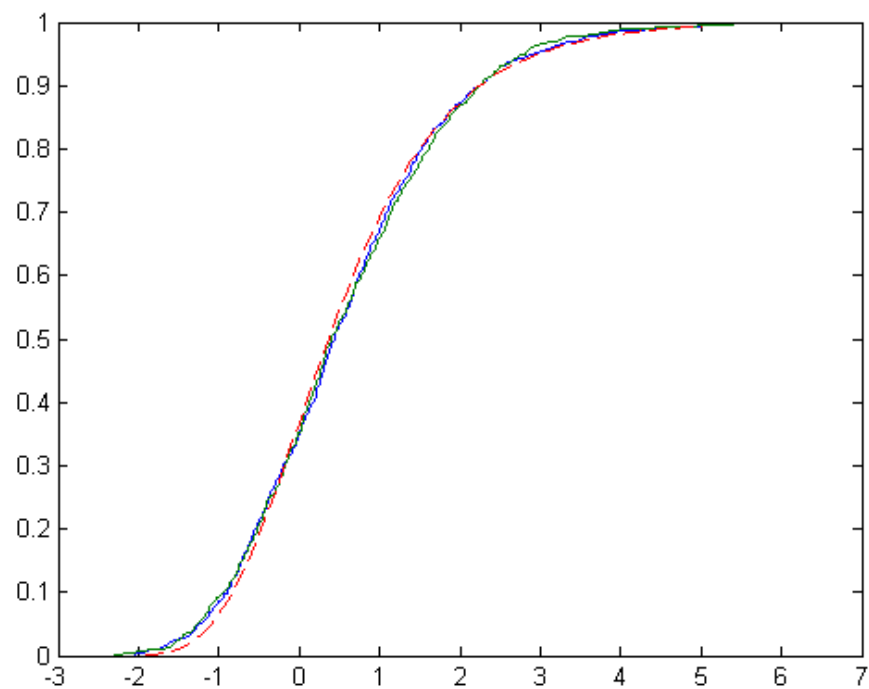
Figure 14: Transformed extreme distribution for 100x100 area (blue) vs. transformed extreme distribution for 500x500 area (green) vs. Gumbel (red, dashed)

4.2.2 Fitting from definition and large iterations

now we compute the extreme distribution for many more iterations in a 64x64 area. We preform 25000 iterations. With the following similar results,
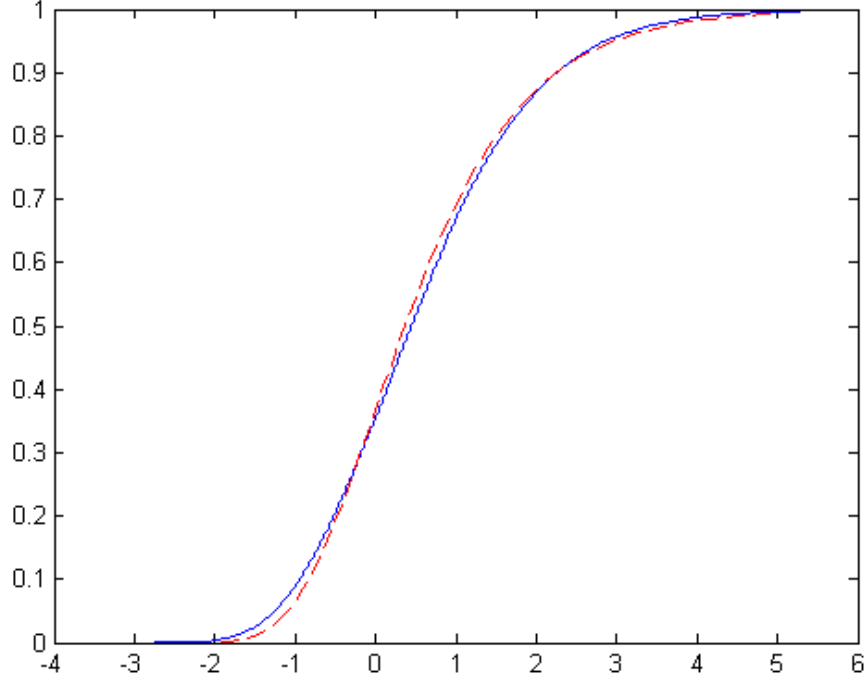

Figure 15: Transformed distribution of extremes vs. Gumbel distribution (25000 iterations)

We can see that the fit improves a little and the resolution becomes much better.

$$a = 7728$$
$$b = 1285.5$$

4.2.3 Predictive fitting

Now we use the same definition, only this time we use the predictive fitting from the numerical equations derived in section 3,

$$\bar{x} \approx 2191.779 \log(0.8384L)$$

$$\sigma \approx 2187.859 - 127.975 \log(L) \quad L \geq 32$$

which gives

$$a \approx 2191.779 \log(0.8384L) - \frac{\gamma\sqrt{6}}{\pi}(2187.859 - 127.975 \log(L))$$

$$= -1370.9 + 1124.7 \log(A)$$

$$b \approx \frac{\sqrt{6}}{\pi}(2187.859 - 127.975 \log(L)) = 1705.9 - 49.89 \log(A)$$

This gives a very accurate prediction of the gumbel distribution that the extremes fit (256x256, 1000 waves, $\lambda = 2\pi$, 1000 iterations), $a = 11100, b = 1156.8$.

4.2.4   Distribution of 10 largest maxima for each realization

We want to find the distribution of the 10 highest maxima of all of the realizations now, not just the very highest one. We immediately notice that this distribution is Gumbel with constants (by definition)

(This is done with 1000 waves of wavelength 2π on a grid of 512x512 and 1000 realizations, the following 3 sections will have an identical setup)

$$a = 10310$$

and

$$b = 937.588$$

which is a perfect fit,

4.2.5   Distribution of 10 smallest maxima for each realization

Now we want to find the distribution of the 10 smallest maxima of each of the distributions. We anticipate that the distribution should be Weibull which has a distribution function as follows,
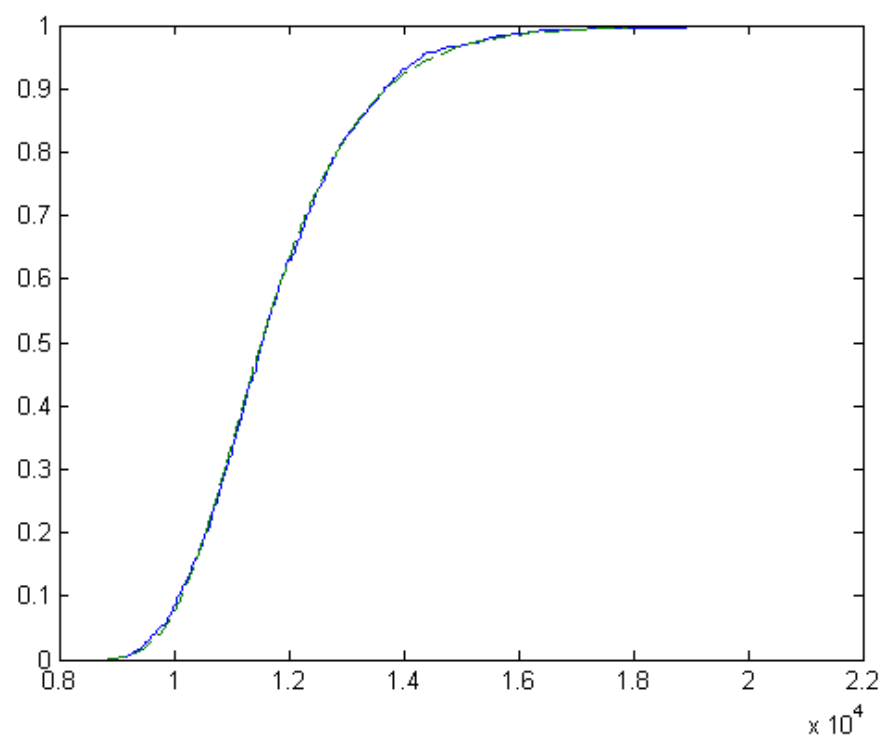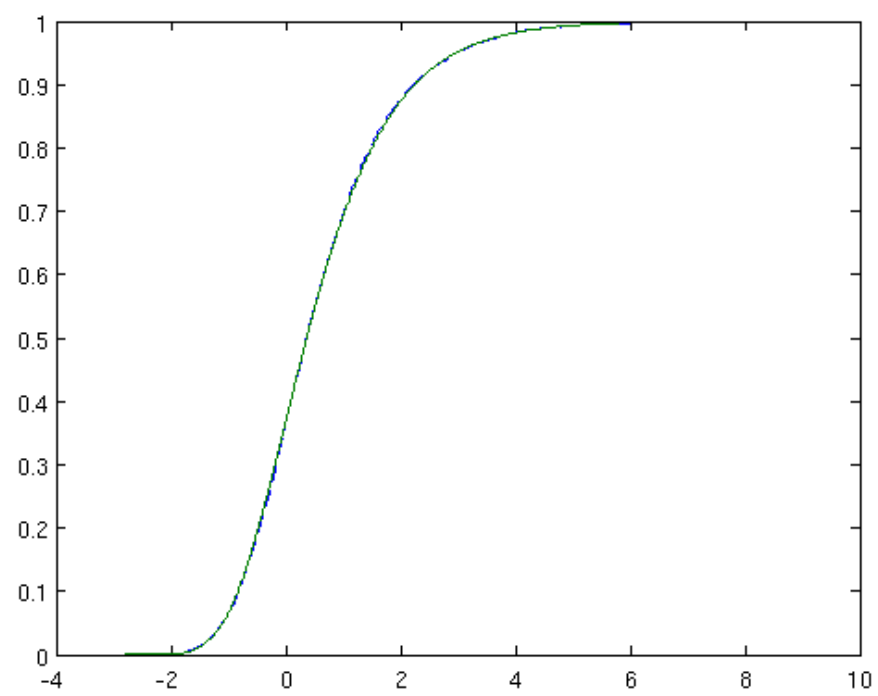
Figure 16: Predictive gumbel fit (complex)

Figure 17: 1

$$F(x, k, \lambda) = 1 - e^{-(x/\lambda)^k}$$

We use a least squares minimization to find the constants

$$\lambda = 273.992$$

$$k = 5.9129$$
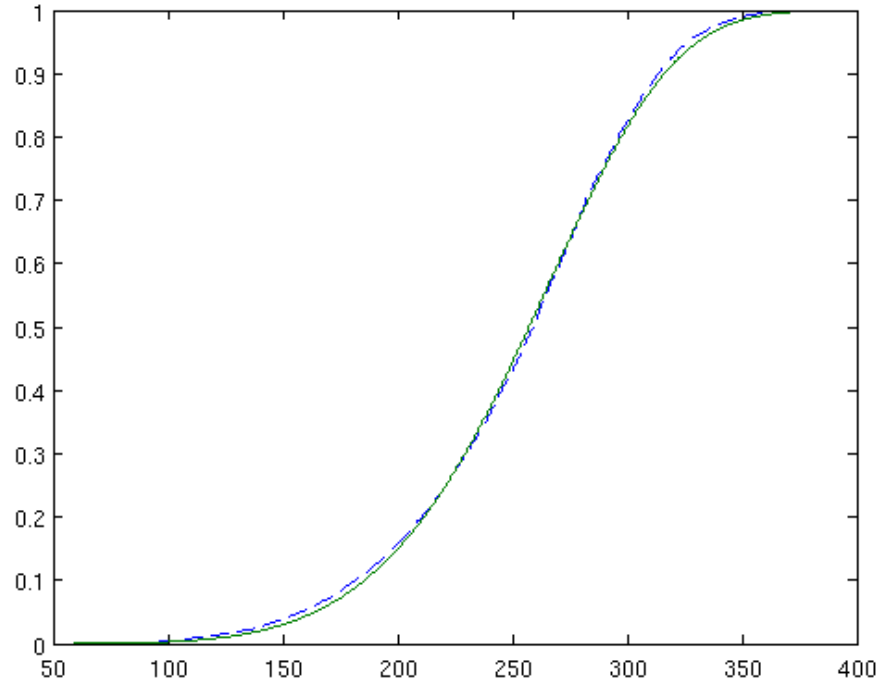
which comes to the following results,

Figure 18: 2

4.2.6 Distribution of the difference between the extreme value and 9 next highest maxima

Now we want to look at the distribution of the distance between the value of the highest point of each realization and the closest 9 maxima. We see that

this resembles a Gumbel distribution with constants (by definition)

$$a = 1567.4$$
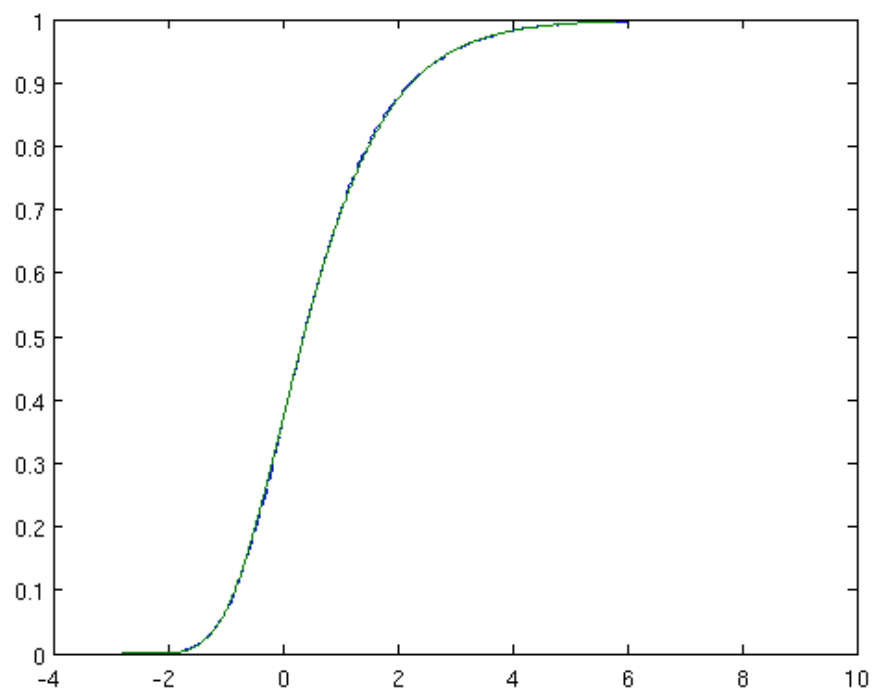
$$b = 992.6887$$

we see that this is a good fit,

Figure 19: 3

4.2.7 Distribution of the difference between the smallest maxima and the 9 next smallest maxima

Now we want to find the same thing but in the opposite order. We want to find the distribution of the distance between the value of the lowest maxima

and the 9 closest maxima to it. This is not immediately obvious what distribution it should be. The following plot is the Gumbel and the Weibul with constants found by definition and least squares respectively.
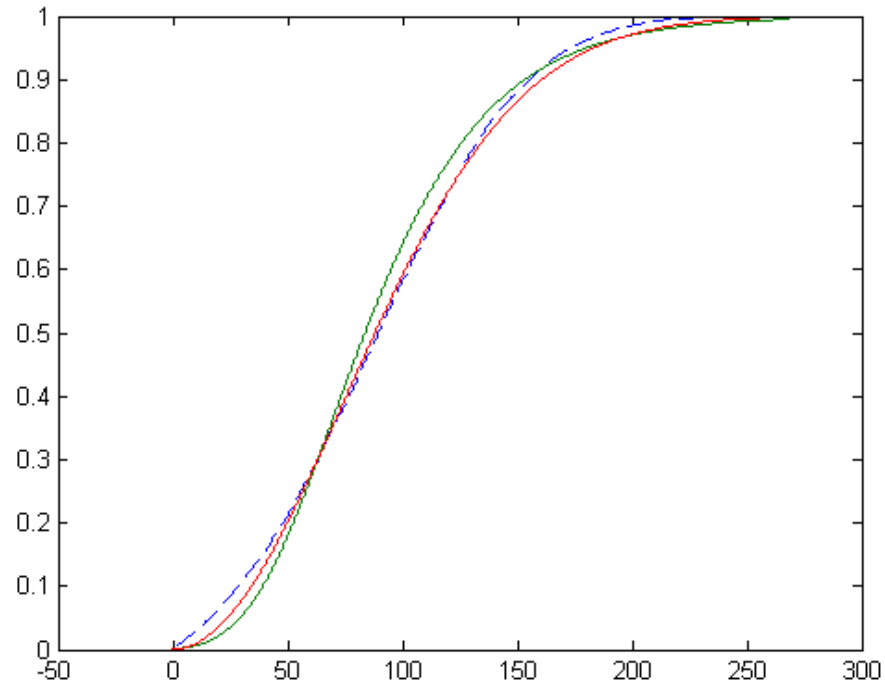

Figure 20: 4

We see that these two distributions seem to fail at representing the value we are interested in. The normal distribution with parameters $\mu = 91.4217$ and $\sigma = 47.6459$ seems to fit pretty well though,
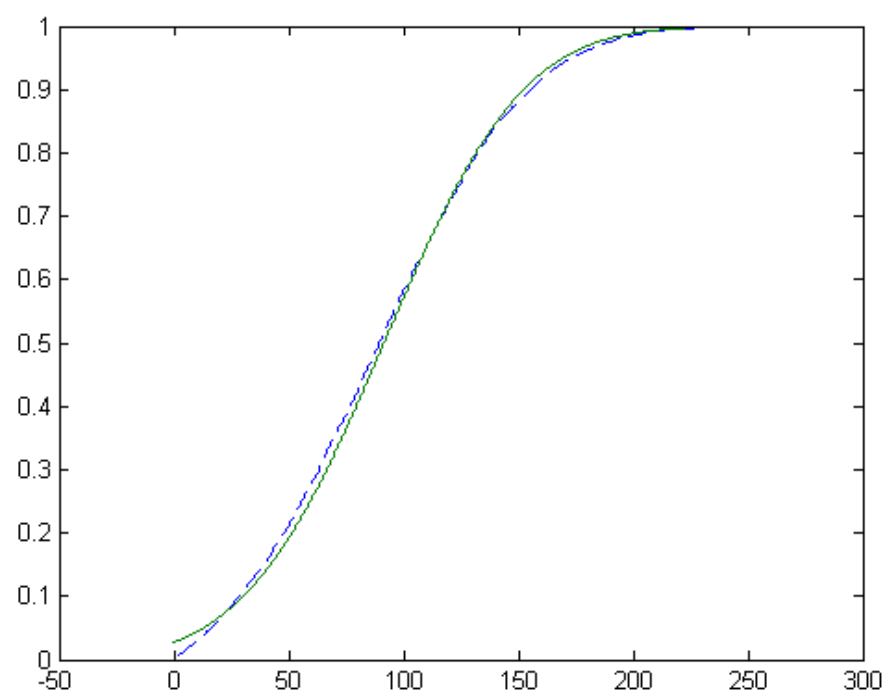
Figure 21: 5

5 Randomness Tests

5.1 Real

5.1.1 Point wise correlation

To ensure that we indeed have a random "gaussian sea" we implement 2 "randomness" tests. The first test takes 1000 random points from a Ψ function that was generated with 1000 plane waves in a 50x50 grid with wavelength 2π. It then plots the distribution of these points, the idea being that one would want this value to be normally distributed by the central limit theorem. We do just this with the following results,
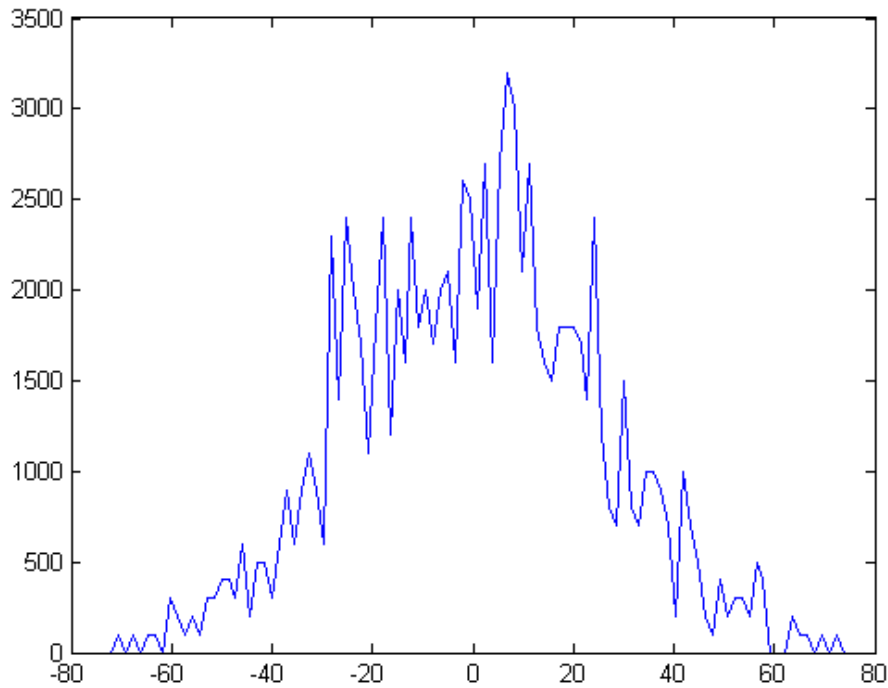

Figure 22: Distribution of 1000 values of random points of a sample Ψ.

5.1.2 Autocorrelation matrix

The next tests involves preforming an autocorrelation of the function (Ψ) with itself. The autocorrelation function for 2 dimensional arrays (like our 2D function is) is defined by the matrix,

$$\rho_{ij} = \sum_{m=0}^{M} \sum_{n=0}^{N} \Psi(x_m, y_n) \Psi(x_{m+i(modM)}, y_{n+j(modN)})$$

What we expect to see is a radially symmetric J_0 Bessel function which would imply no correlation. In a surface plot of our correlation matrix this would look like a peak at the center rippling out into smaller "waves" to be eventually met by 8 other peaks which are artifacts of the modulus in the definition. The results we obtained are as follows,
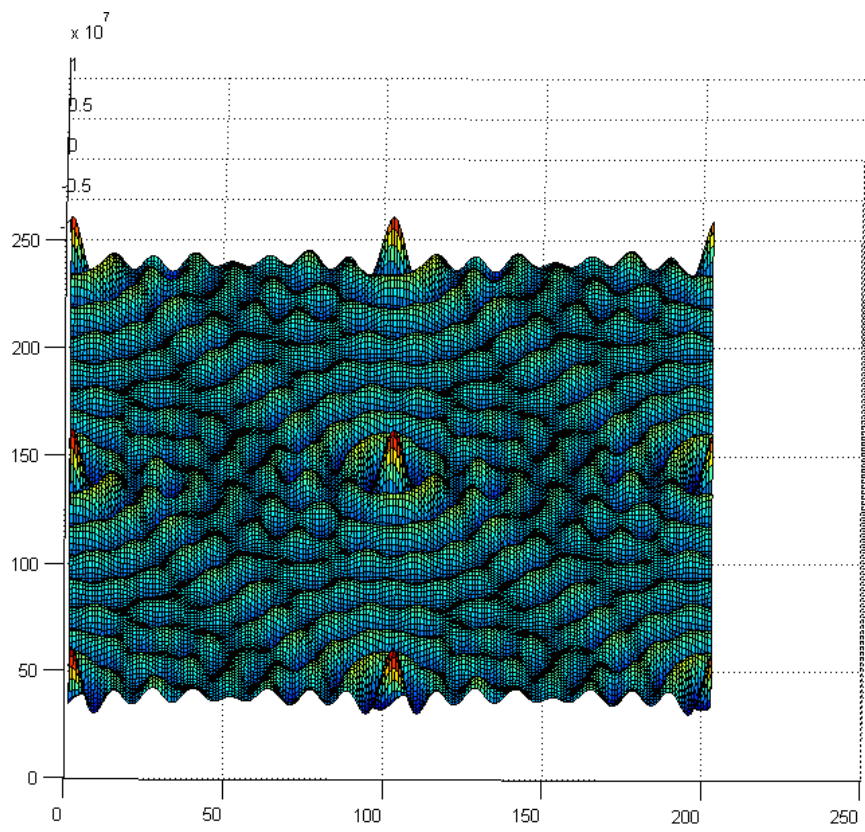
Figure 23: Autocorrelation matrix for a sample Ψ.

5.2 Complex

We then preform the same "randomness tests" as before obtaining the expected exponential distribution for $|\Psi|^2$,

5.2.1 Gaussian point wise correlation

We note that because the individual components of the waves were generated in the exact same way as with the real case (just separately) that individually they will follow a gaussian distribution.
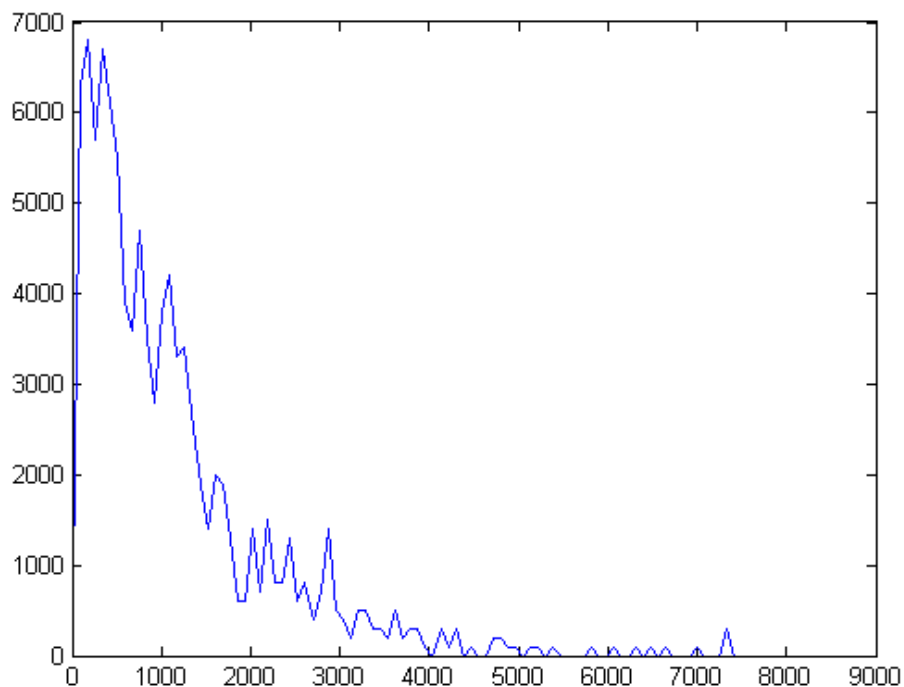

Figure 24: Distribution of 1000 values of random points of a sample $|\Psi|^2$.

5.2.2 Autocorrelation matrix

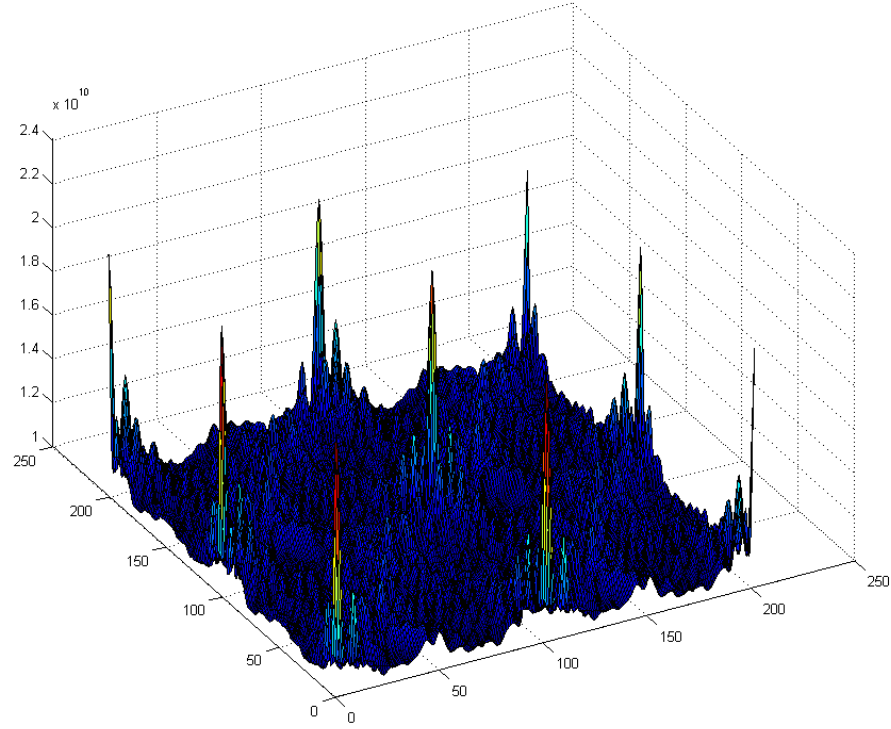and the expected J_0 bessel like function for the autocorrelation matrix,

Figure 25: Autocorrelation matrix for a sample $|\Psi|^2$.

References

[1] M.S. Longuet-Higgins, Phil. Trans. R. Soc. A **250**, 157 (1957)

[2] A. Weinrib, B.I. Halperin, Phys. Rev. B **26**, 1362 (1982)

[3] R. Aurich, A Bäcker, R. Schubert, M. Taglieber, Physica D **129**, 1 (1998)