Traffic Modeling and Estimating Origin-Destination Matrix

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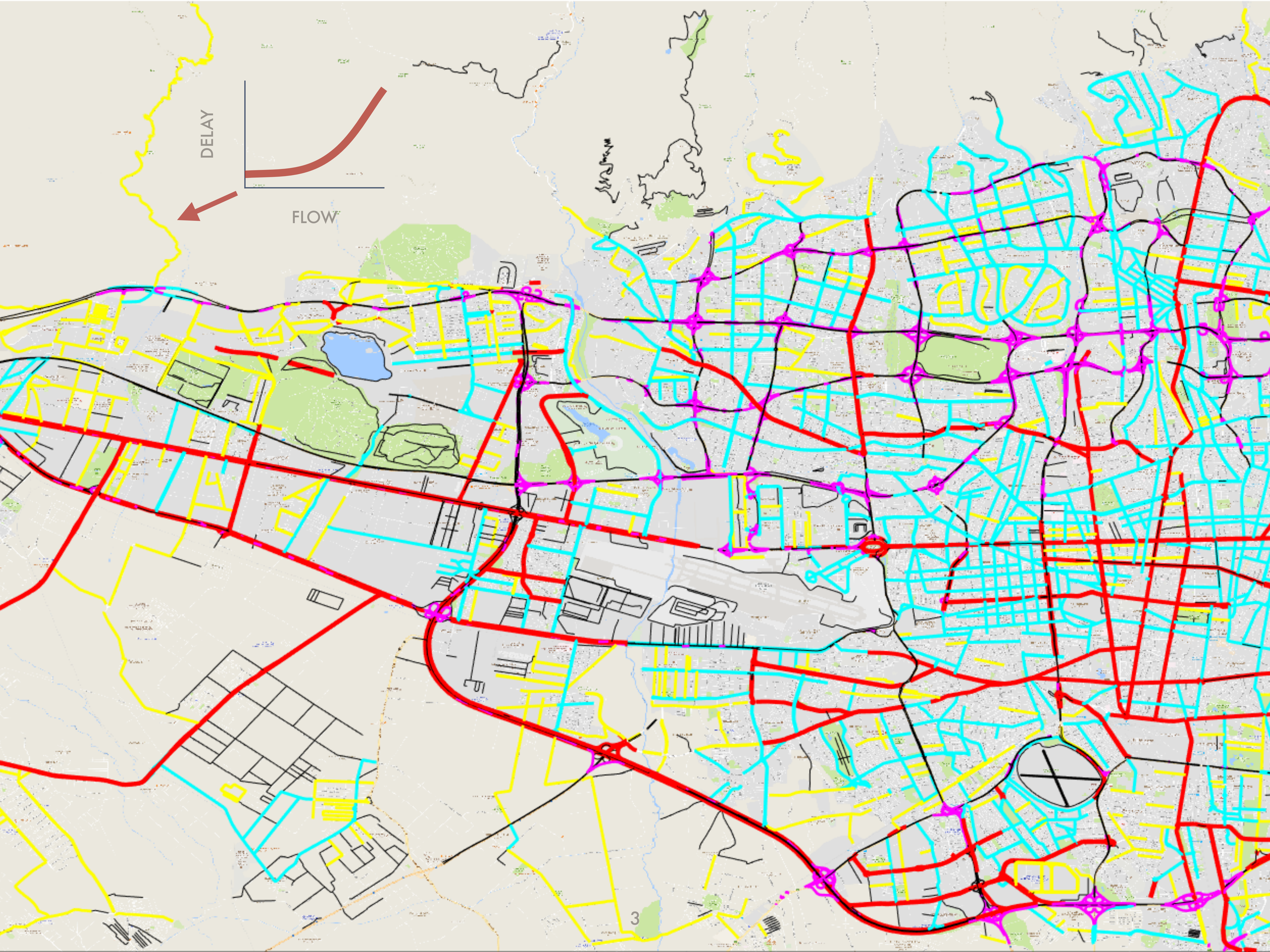
OUTLINE

- Network Structure
- Wardrop Equilibrium
- Traffic Assignments (Theory and Implementation)
- Price of Anarchy
- Problem Definition and Solution
- Results


DELAY
FLOW

- ## Network Structure:

$$J = \{(i,j) | (i,j) \in E\}, \quad R = \{(i_1, i_2, \dots, i_k) | (i_u, i_v) \in E\}$$

$$A := \text{link-route incident matrix} \in \mathbf{R}^{|J| \times |R|}$$

$$H := \text{source destination pair-route incident matrix} \in \mathbf{R}^{|V|^2 \times |R|}$$

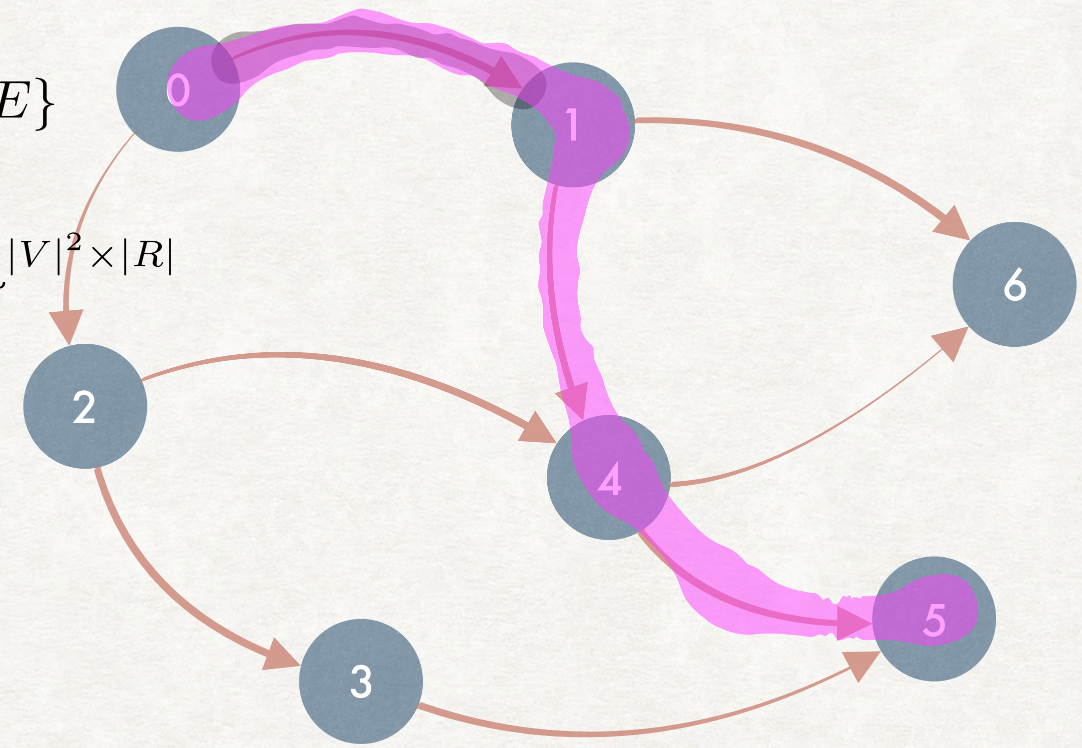
$$A_{j,r} = \begin{cases} 1 & \text{if link } j \text{ in route } r \\ 0 & \text{otherwise} \end{cases}$$

$$H_{p,r} = \begin{cases} 1 & \text{if route } r \text{ serve source destination pair } p \\ 0 & \text{otherwise} \end{cases}$$

$$x_r := \text{flow on route } r$$

$$y_j := \text{flow on link } j = \sum_{r \in R} x_r A_{j,r}$$

$$\text{conservation of flow} \rightarrow f_p = \sum_{r \in R} x_r H_{p,r}$$

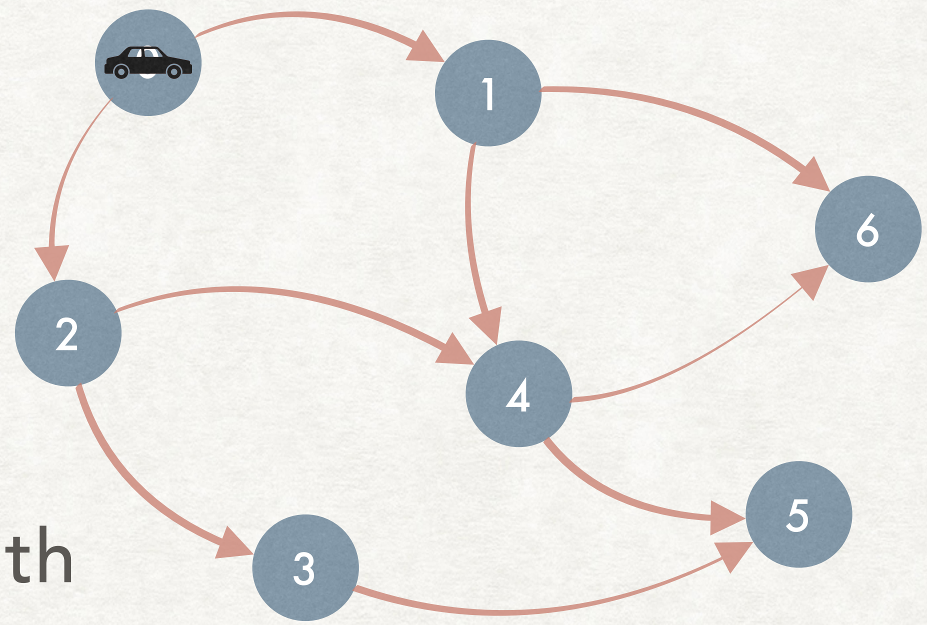


$$A = \begin{array}{c} \\ 01 \\ 02 \\ \vdots \\ 35 \\ 45 \end{array} \begin{array}{c} \begin{array}{ccc} 0145 & 0245 & \dots \end{array} \\ \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \dots \\ 1 & 1 & \dots \end{pmatrix} \end{array}$$

WARDROP EQUILIBRIUM

- Commuters Go Through Roads To Minimize Their Delay.

$$\text{if } x_r > 0 \Rightarrow \sum_{j \in J} D_j(y_j) A_{j,r} \leq \sum_{j \in J} D_j(y_j) A_{j,r'}$$



- Does a Non-Negative Vector x Exists With Specified Conditions? Yes!, and Fortunately (Under Mild Conditions) is Unique

- Equilibrium: Knowing Other Opponents Strategies You Have No Incentive To Change Yours

$$\phi(x) = \sum_{j\in J}\int_0^{y_j} D_j(u)du \longrightarrow \frac{\partial \phi}{\partial x_r} = \sum_{j\in J} D_j(y_j)A_{j,r}$$

$$\min \sum_{j\in J}\int_0^{y_j} D_j(u)du$$

$$\phi(x_r - \delta, x_{r'} + \delta, x_{-\{r,r'\}}) - \phi(x_r, x_{r'}, x_{-\{r,r'\}}) \approx \delta(\frac{\partial \phi}{\partial x_{r'}} - \frac{\partial \phi}{\partial x_r})$$

$$s.t \quad x \geq 0, y$$

$$Hx = f, Ax = y$$

Nobody Has an Incentive To Change Their Route!

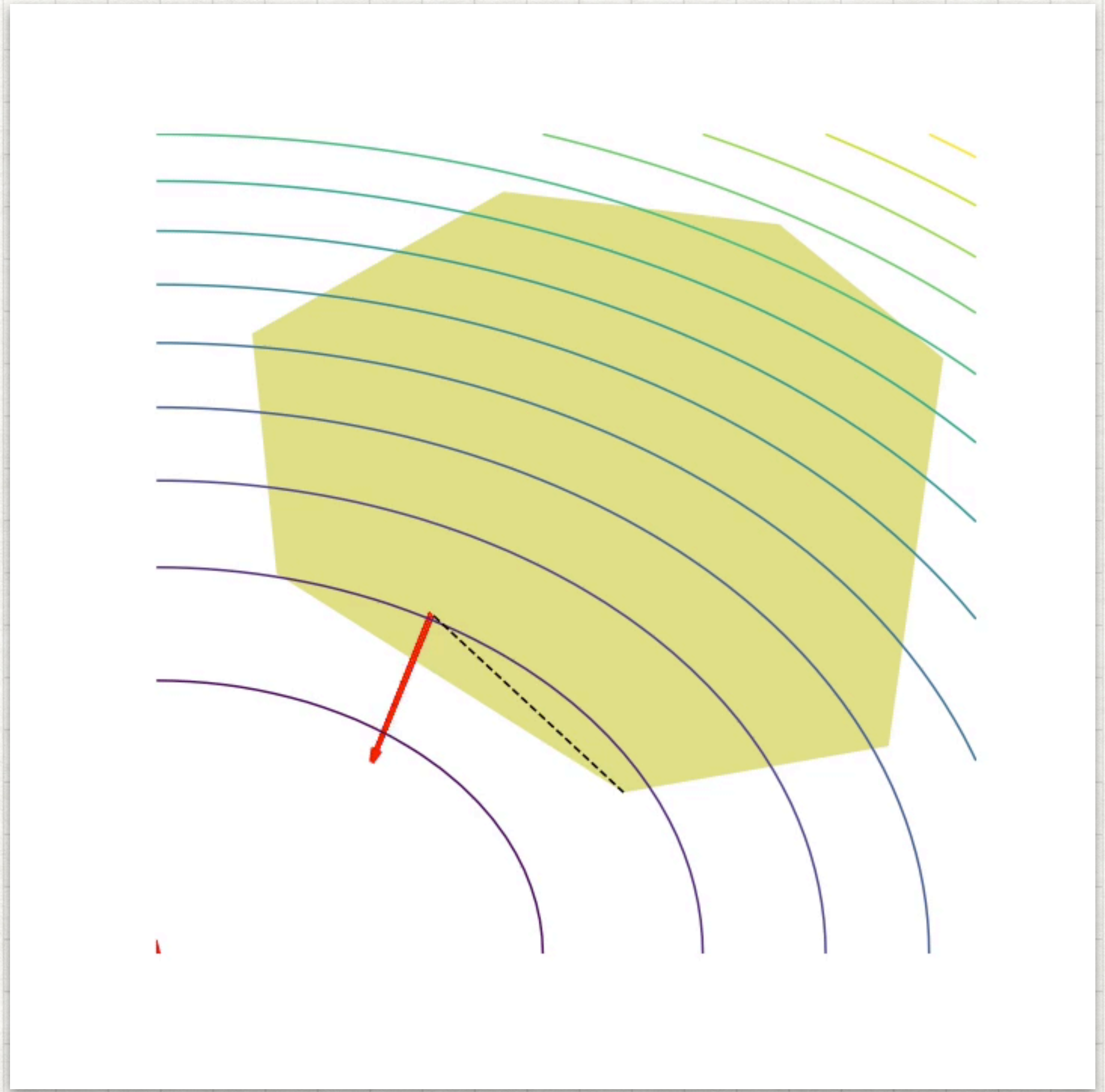
FRANK WOLFE METHOD

$$\begin{array}{ll} \min & f(x) \\ \text{s.t} & x \in \mathbf{C} \end{array}$$

$$f(x) \geq f(x_0) + < \nabla f(x_0), x - x_0 >$$

$$\begin{array}{ll} \max & < -\nabla f(x_0), x - x_0 > \\ \text{s.t} & x \in \mathbf{C} \end{array}$$

Convergence Rate $O(\frac{1}{\epsilon})$



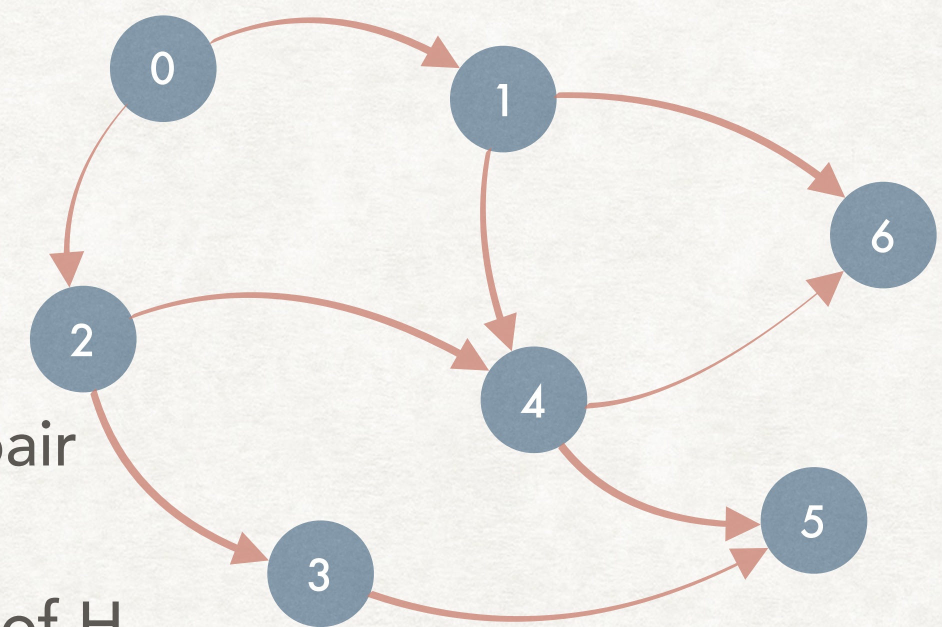
USER EQUILIBRIUM

SOLUTION

$$\nabla\phi(x) = [\ldots, \sum_{e\in r} D_e(y_e), \ldots]^T = [\ldots, w_r, \ldots]^T$$

$$\begin{array}{ll} \min & \sum_{r\in\mathbf{P}} w_r z_r \\ \text{s.t} & Hz = f \end{array}$$

Frank Wolfe

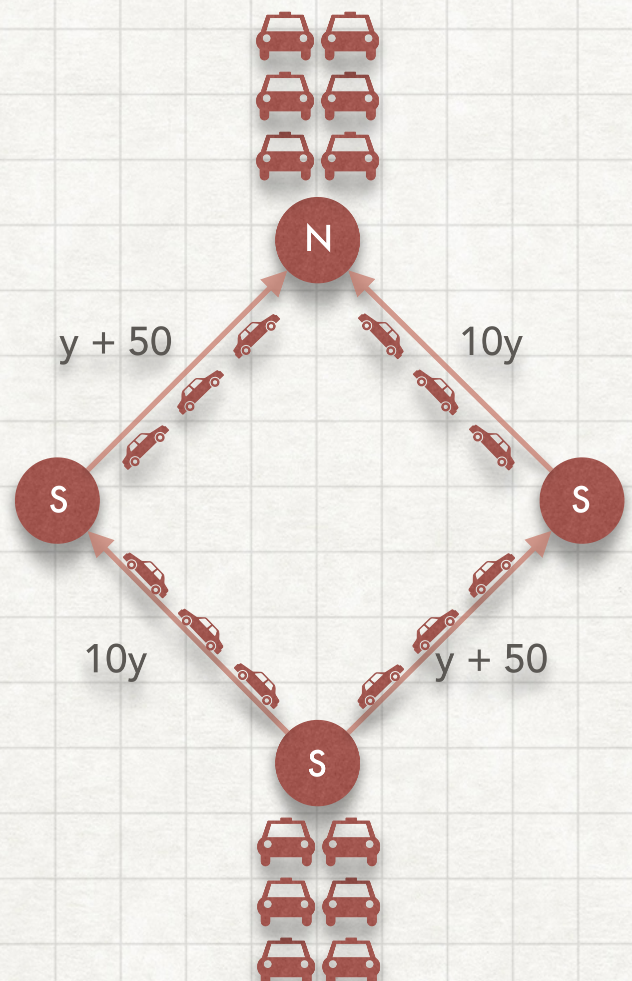


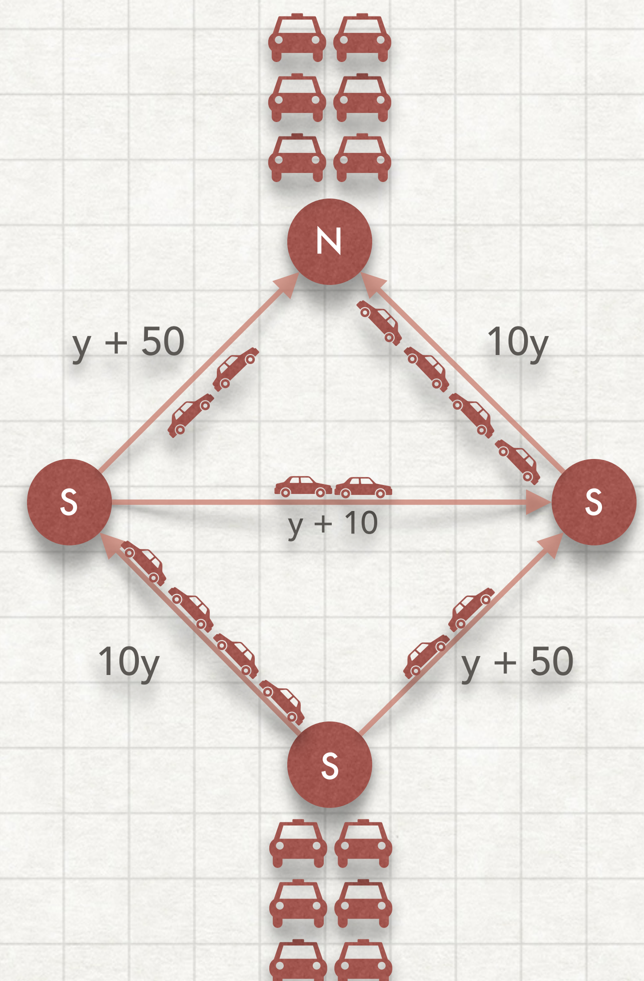
Assign demand flow on the shortest path in every pair

There is only one nonzero element in every column of H

Rows of H are disjoint partition of all paths in the network

BRAESS'S PARADOX



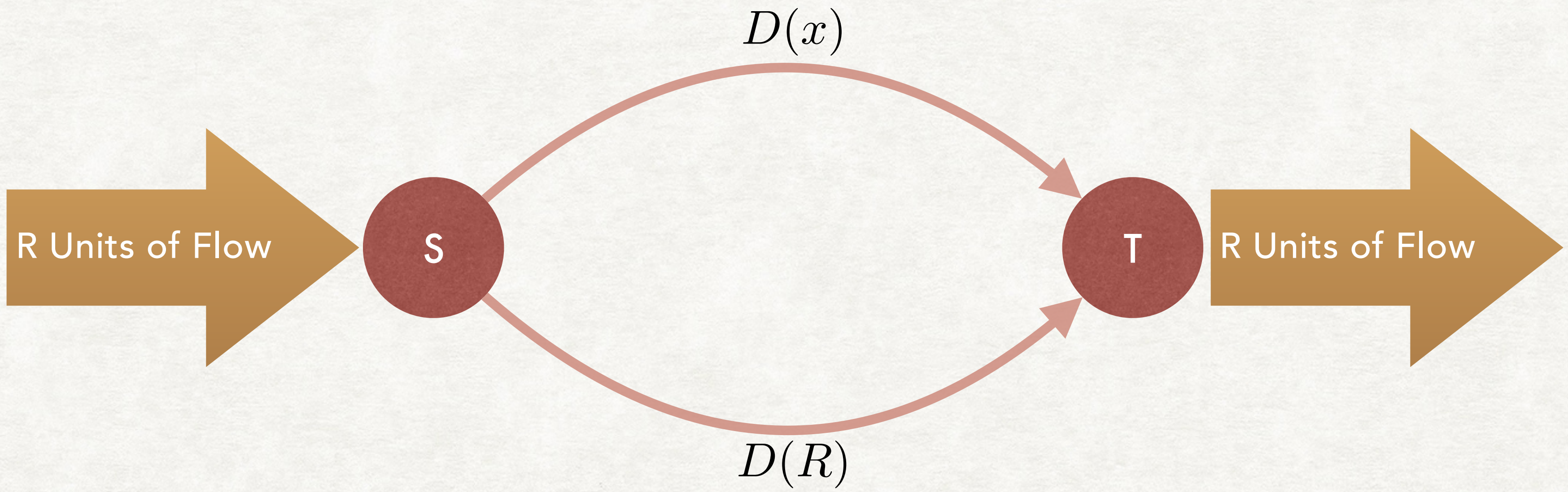


?

$$10 \times 3 + 3 + 50 = 83 \quad < \quad 10 \times 4 + 2 + 50 = 92$$

PRICE OF ANARCHY

PIGOU NETWORK



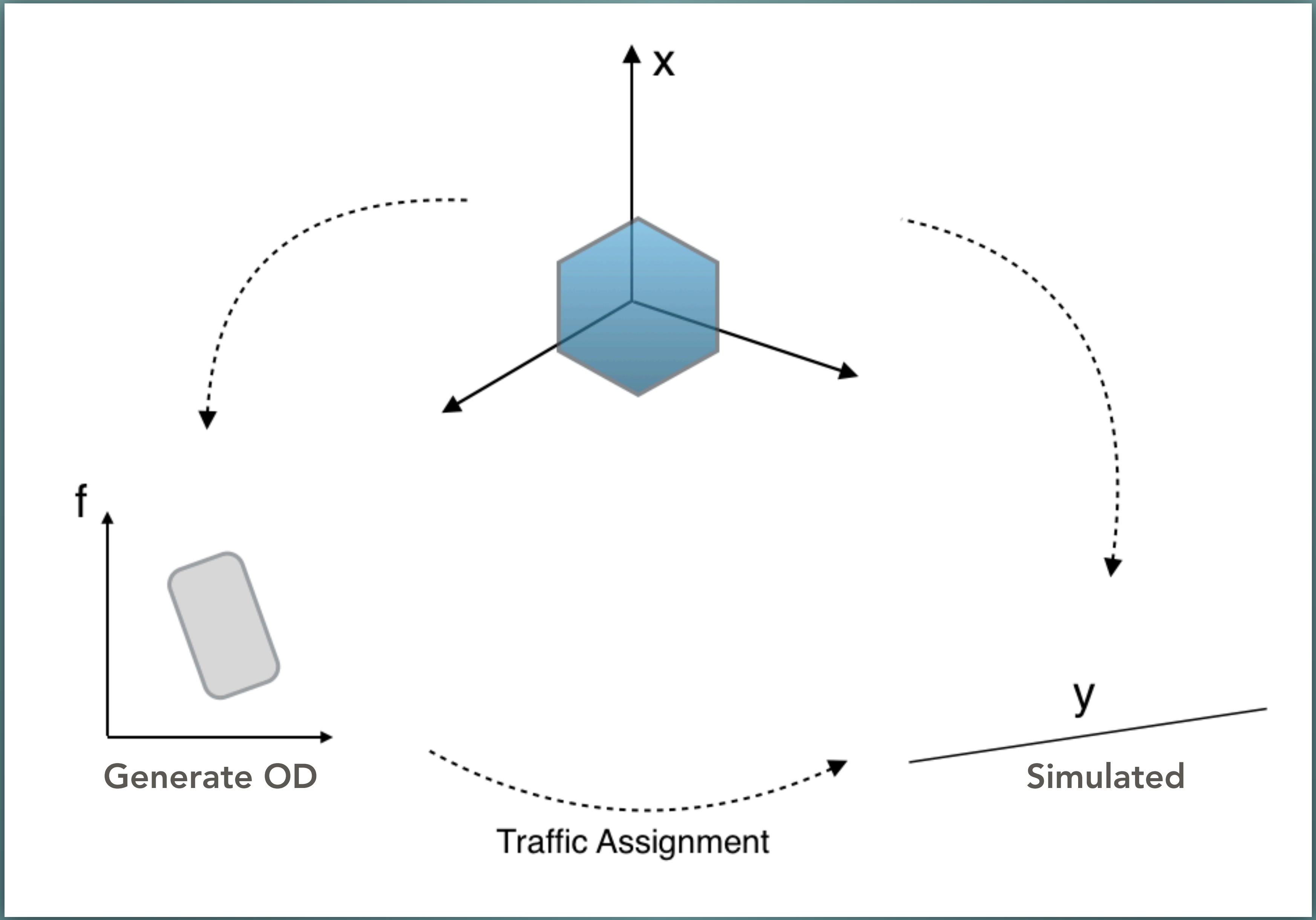
$$D \in \mathbf{L} = \{ax^2 + bx + c | a, b, c \geq 0\}$$

$$\alpha_R(D) = \frac{RD(R)}{\min_{0 \leq x \leq R} D(x)x + (R - x)D(R)}$$

$$A_r(\mathbf{L}) = \max_{0 \leq \tilde{R} \leq R} \sup_{D \in \mathbf{L}} \alpha_{\tilde{R}}(D)$$

For an affine network upper bound is 4/3!

PROBLEM DEFINITION



GRAVITY MODEL

Attraction

Repulse

$$f_{i,j} = \frac{A_i R_j}{d_{i,j}}$$

Distance Between Regions

$$[f_{i,j}] \otimes [d_{i,j}] = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} \begin{bmatrix} R_1 & R_2 & \cdots & R_n \end{bmatrix}$$

This Is a Rank 1 Matrix!

- Modify Incident Matrix H

$$\tilde{H}_{p,r} = \begin{cases} d_{i,j} & \text{if } p = (i,j) \text{ and route } r \text{ is from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$$

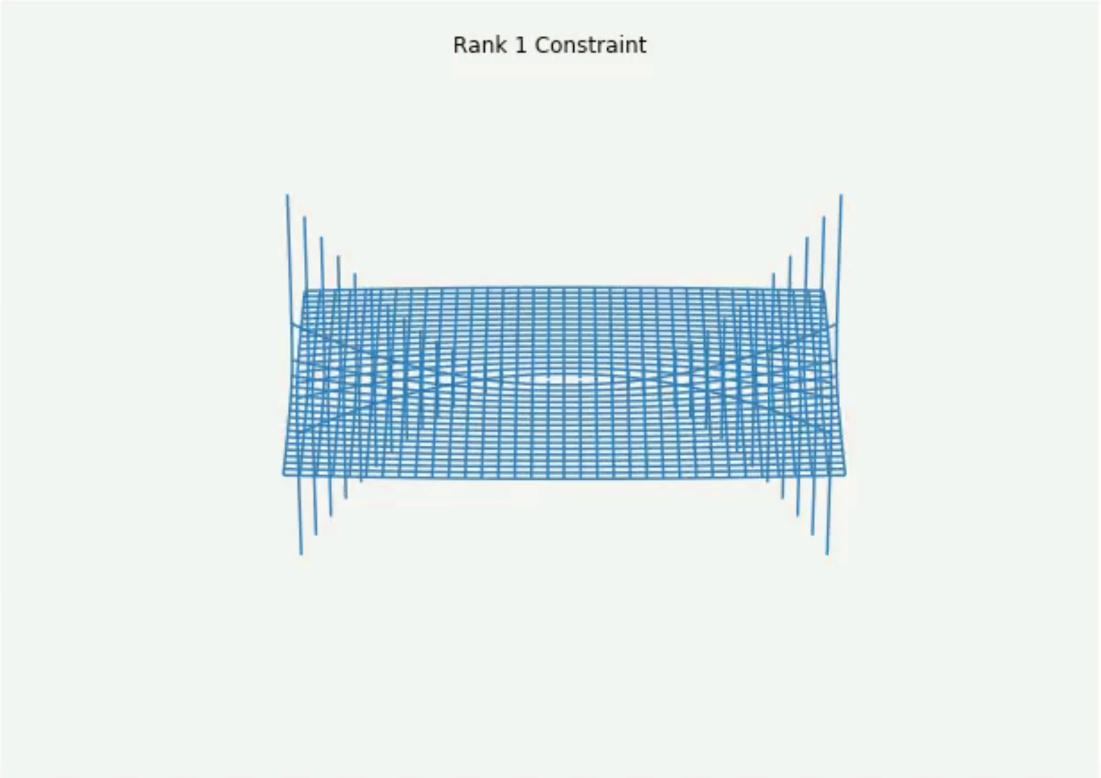
$$\square(\tilde{H}x) = [A_i][R_i]^T - diag([A_i R_i])$$

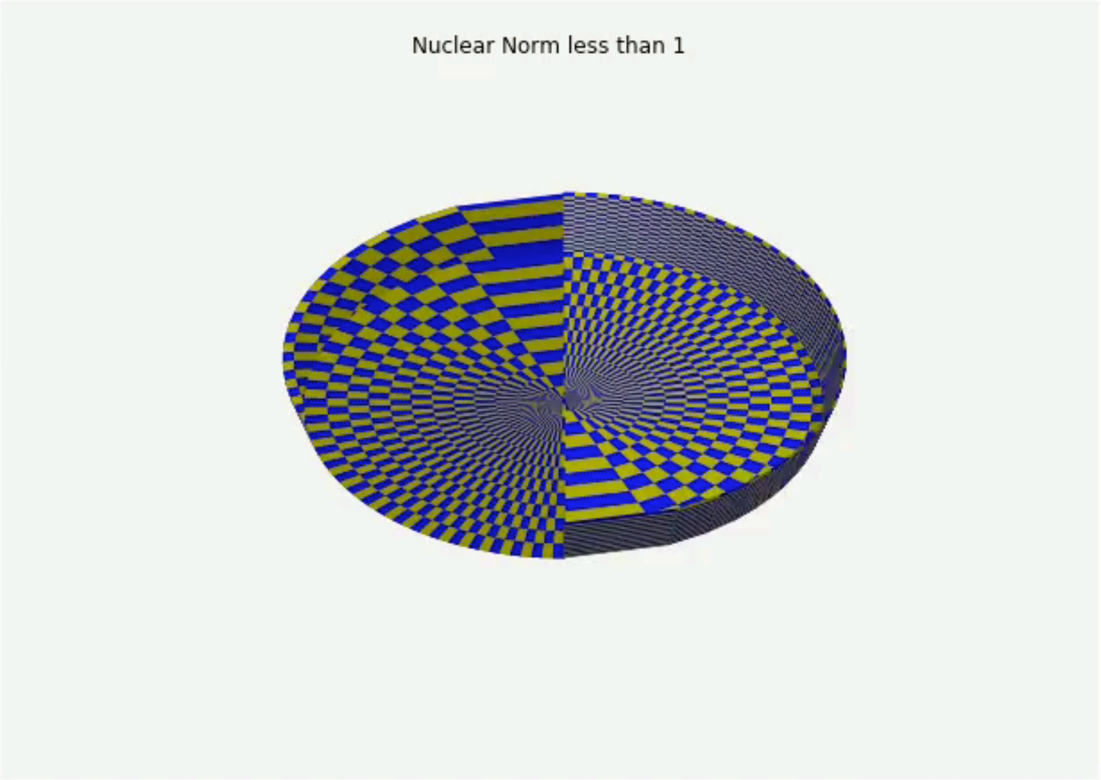
- We Need To Complete Diagonal In order To Have a Low Rank Matrix.

$$rank\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = 1$$

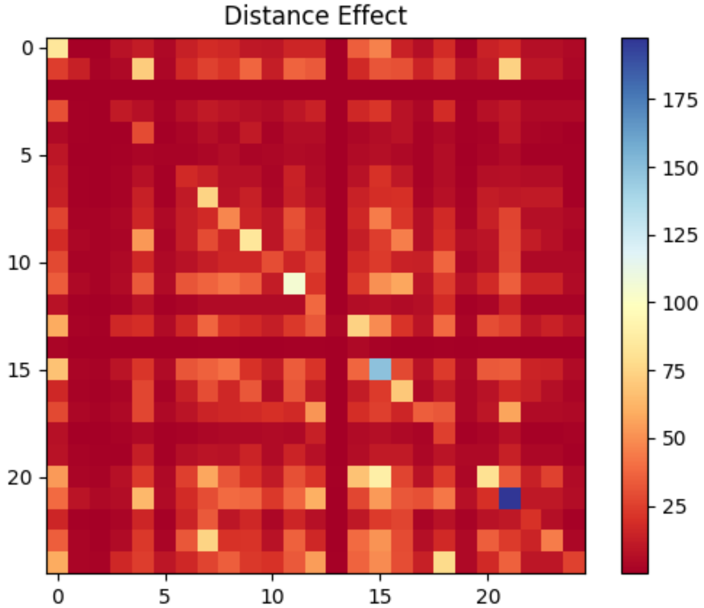
$$\begin{aligned} \min \quad & ||\square(\tilde{H}x) + D||_* + \lambda||Ax - y||^2 \\ s.t \quad & x \geq 0, D \geq 0, D \text{ is diagonal} \end{aligned}$$

$$||\begin{bmatrix} x & y \\ y & z \end{bmatrix}||_* \leq 1$$

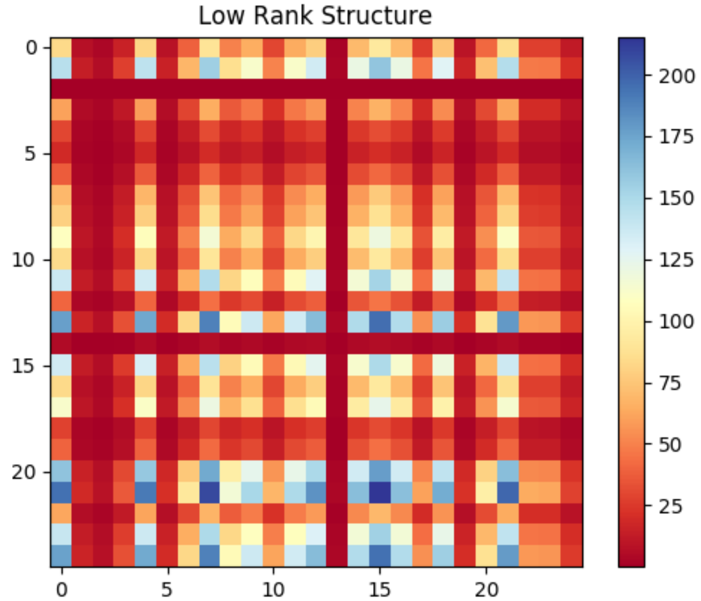


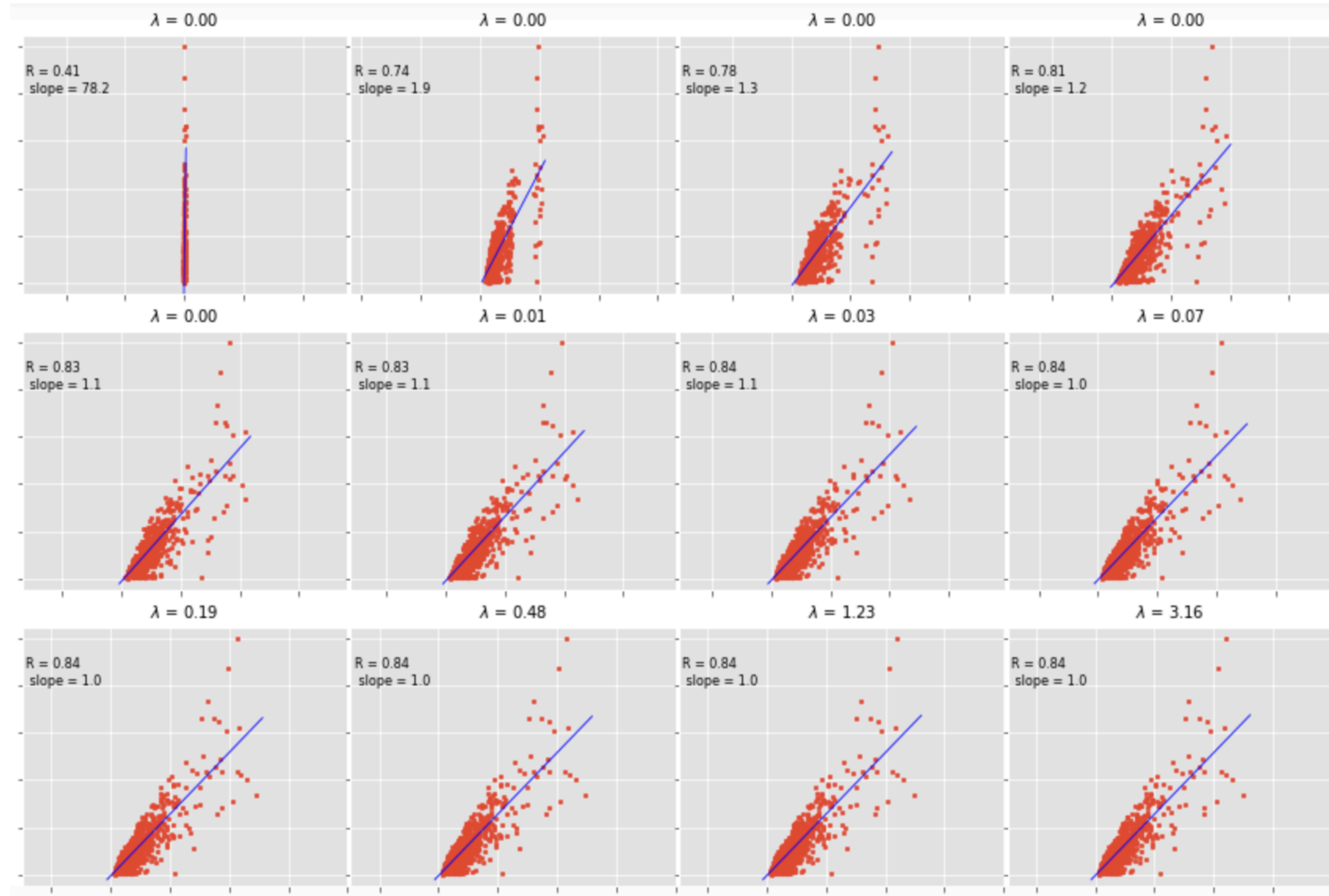


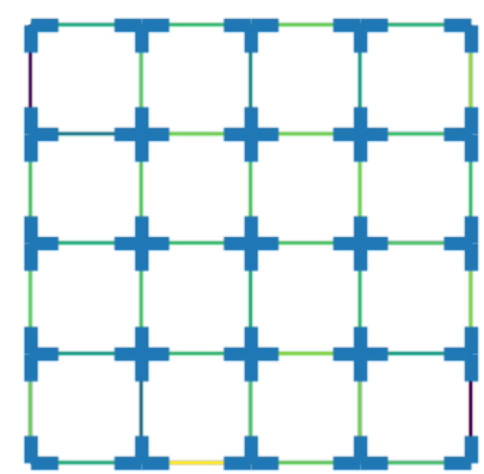
$$[f_{i,j}]$$



$$[f_{i,j}] \otimes [d_{i,j}]$$







Link Flows In Equilibrium

REFERENCES

- Game Theory, Alive by Anna J. Darlin and Yuval Peres
- The Princeton Companion To Mathematics
- Network Equilibrium and Pricing by Micheal Florian
- Braess's Paradox in Large Random Graphs by Gregory Valiant
- Convex Optimization by Stephan Boyd


Thank You!
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THERE WAS NEVER JUST ONE
Do you need a ride?
I’d rather walk !