Uniswap v4 - A Mathematical Primer

LVR & IL HEDGE HOOK FOR CONCENTRATED LIQUIDITY PROVIDERS,
DYNAMIC FEES & DELTA-GAMMA NEUTRALITY:
POWER PERPETUALS & LENDING PROTOCOLS

@mevquant, @seecoalba, @theblockchainer

Contents

1 Introduction

This is a draft for the *Hookathon*. It is still incomplete and some examples are missing.

This document provides an introduction into Uniswap Mathematics from the ground up. No prior knowledge by the reader beyond some basic algebra, calculus and some DeFi basics is required, although some familiarity with option contracts will be helpful. The explanation herein intends to be self-complete, meaning that the reader should not need any extra resources to understand any concept. All the formulas have their corresponding proves, corollaries and examples.

The author has tried to summarise to his best extent every resource he could find online, from whitepapers, articles, blogs and videos. Every single formula is derived in simple steps to facilitate comprehension and all of the plots have been re-done in Python to ensure the author had a complete understanding of every minor detail in the protocol.

Some of the examples and explanations contained in this article are directly extracted from the original sources. **All credit is due to their original authors, appropriate citations have been provided wherever necessary**.

This is an an opinionated summary of all the relevant information to fully understand Uniswap Mathematics. The first examples will be on *full range* positions to build intuition on the inner workings of the protocol and then move into *concentrated liquidity* positions.

The document intends to familiarise the reader with both protocol mathematics (`core`, `periphery`) and the necessary financial mathematics to be able to understand concepts like:

> A `Uniswap` liquidity provision position is a **perpetual short straddle** (no theta/time risk), earning a variable continuous premium from trading fees, and sold with strike prices equal to the current market price. In other words, a **swap of volatility or gamma risk** for a perpetual stream of payments from trading fees.

And be able to hedge a portfolio accordingly. Additionally, it will serve as a foundation to lay down all the necessary concepts for the hook the author is building for the *Hookathon* with the *Zealynx* team.

2 Swap Math

Uniswap is a Constant Function Automated Market Maker (CFMM), also know as Geometric Mean Market Maker (G3M). To determine how much token Y (dy) you get by swapping some amount of token X (dx) it uses the following formula:

$$x \cdot y = k; \quad k \; constant \tag{1}$$

k is formally known as the invariant. The invariant only remains constant during trades and only changes when liquidity is added or removed from the market. This is what makes LPs earn fees via their shares.
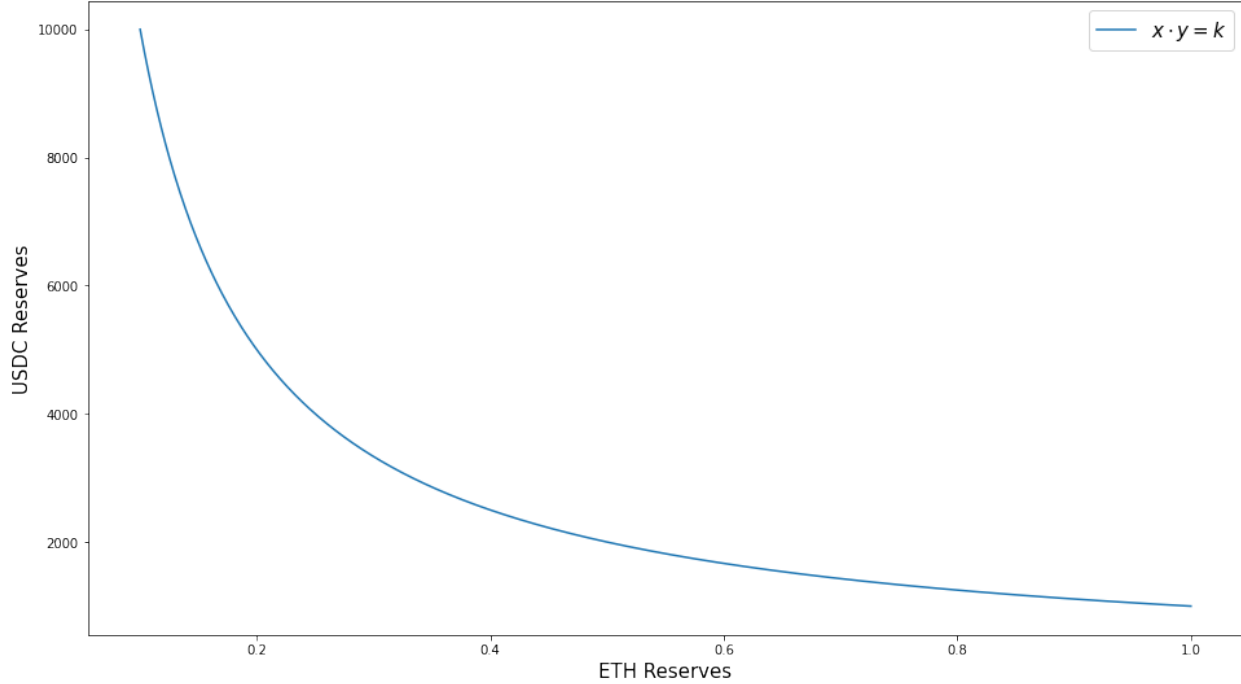
Visual representation of $y = \frac{k}{x}$, $k = 1_{ETH} \cdot 1000_{USDC}$

In a bit more detail, one can exactly determine how much dy they will get after exchanging dx accounting for fees with an expansion of 1. For simplicity, imagine that before the swap we subtract the fees from dx as $dx := dx - fees$ (this is actually how it was implemented on the code for V1, see *Appendix A*).

$$(x + dx) \cdot (y - dy) = k \tag{2}$$

Then with some algebra we see that:

$$(x + dx) \cdot (y - dy) = k$$

$$xy - xdy + ydx - dxdy = k$$

$$xy + ydx - xdy - dxdy = k$$

$$y(x + dx) - dy(x + dx) = k$$

$$y(x + dx) = k + dy(x + dx)$$

$$y(x + dx) - k = dy(x + dx)$$

$$dy = y - \frac{xy}{x + dx}$$

$$dy = y \cdot \underbrace{\left(\frac{x + dx}{x + dx}\right)}_{1} - \frac{xy}{x + dx}$$

$$dy = \frac{\cancel{yx} + ydx - \cancel{xy}}{x + dx}$$

$$\boxed{dy = \frac{ydx}{x + dx}} \tag{3}$$

If one would want to subtract the fee ($\phi \in [0, 1]$) directly and not at the beginning:

$$\boxed{dy = \frac{y \cdot dx(1 - \phi)}{x + dx(1 - \phi)}} \tag{4}$$

Similarly, if we wanted to know the amount of dx we would get if we supplied dy:

$$(x - dx) \cdot (y + dy(1 - \phi)) = k$$

$$xy + xdy(1 - \phi) - ydx - dxdy(1 - \phi) = k$$

$$x(y + dy(1 - \phi)) - dx(y + dy(1 - \phi)) = k$$

$$x(y + dy(1 - \phi)) - k = dx(y + dy(1 - \phi))$$

$$x - \frac{k}{y + dy(1 - \phi)} = dx$$

$$x \left(\frac{y + dy(1 - \phi)}{y + dy(1 - \phi)}\right) - \frac{k}{y + dy(1 - \phi)} = dx$$

$$\frac{\cancel{xy} + x \cdot dy(1 - \phi) - \overbrace{\cancel{xy}}^{k}}{y + dy(1 - \phi)} = dx$$

$$\boxed{dx = \frac{x \cdot dy(1 - \phi)}{y + dy(1 - \phi)}} \tag{5}$$

6

Additionally, one can obtain the final amount of tokens after some price movement α (15). Define

$$r = \frac{x}{y}$$

the exchange rate (or price) to convert X tokens into Y tokens. Using that

$$\begin{cases} x = r \cdot y \\ y = \frac{x}{r} \end{cases} \tag{6}$$

and 1:

$$x \cdot y = k \xrightarrow{6} \begin{cases} (r \cdot y) \cdot y = k & \longrightarrow & \boxed{y = \sqrt{k \cdot r}} \\ \\ x \cdot \frac{x}{r} = k & \longrightarrow & \boxed{x = \sqrt{\frac{k}{r}}} \end{cases} \tag{7}$$

2.1 Price Volatility on LP's Positions

In the following section we will explore what happens to our LP position holdings and value when there are price movements. Then, it will become evident to the reader in an intuitive way the concept of Impermanent (or Divergence) Loss.

2.1.1 LP Value without fees

Exchange rate r_0 (or price) is `$100 DAI/ETH`.

1. Alice stakes 1 `ETH` and $100 `DAI` into the `ETH/DAI` pool.

2. After her deposit there is a total of 10 `ETH` and `1000 DAI` in the pool. She has a 10% stake on the pool.

3. A week passes and 1 `ETH` is now worth `200 DAI` ($r_1 = 200$).

4. No liquidity is added to the pool during that period.

5. For simplicity, there are also no fees in the pool.

Then, how many assets does `Alice` hold at the end of the period? What is the new value of her LP position?

We know that k must remain constant during swaps (1), that is:

$$k_0 = k_1 = x_0 \cdot y_0 = 10 \cdot 1000 = 10000 = x_1 \cdot y_1$$

We can first indeed check that 7 returns the correct number assets given a certain k and current exchange rate r_t. We have:

$$\begin{cases} x_0 = \sqrt{\frac{k_0}{r_0}} = \sqrt{\frac{10000}{100}} = \sqrt{100} = 10_{ETH} \\ y_0 = \sqrt{k_0 \cdot r_0} = \sqrt{10000 \cdot 100} = 1000_{DAI} \end{cases} \qquad \square \tag{8}$$

When $r_1 = 200$:

$$\begin{cases} x_1 = \sqrt{\frac{k_0}{r_1}} = \sqrt{\frac{10000}{200}} = \sqrt{50} \approx 7.07_{ETH} \\ y_1 = \sqrt{k_0 \cdot r_1} = \sqrt{10000 \cdot 200} \approx 1414.21_{DAI} \end{cases}$$

We know `Alice` has a 10% of stake in the pool so we can compute the total holding and position value:

$$V_{LP}(r_t = 200) = x_1 \cdot 10\% \cdot r_1 + y_1 \cdot 10\%$$
$$= 0.707_{ETH} \cdot 200 + 141.421_{DAI}$$
$$\approx 282.82_{DAI}$$

We can easily compute what would have happened if she just simply held the assets:

$$V_{HODL}^{50\%} = x_0 \cdot 10\% \cdot r_1 + y_0 = 1 \cdot 200 + 100 = 300_{DAI}$$

As no fees have been generated, by providing liquidity `Alice` has lost:

$$L = V_{LP} - V_{HODL}^{50\%}$$

$$\approx 282.82 - 300$$

$$\approx -17.18_{DAI}$$

$$L_\% = \frac{V_{LP} - V_{HODL}^{50\%}}{V_{HODL}^{50\%}}$$

$$= \frac{V_{LP}}{V_{HODL}^{50\%}} - 1$$

$$= \frac{282.82}{300} - 1$$

$$\approx -5.73\%$$

$$\boxed{L \approx -17.18_{DAI} \; ; \quad L_\% \approx -5.73\%}$$

However, notice that this is an *unrealised loss*, as if `Alice` does not withdraw and the price goes back to $r = 100$ again, her holdings (8) and therefore position value will be the same and $L = 0$; $L_\% = 0\%$.

This percentage difference on her holdings' value caused by price movement is indeed the so called:

<div align="center">

Impermanent Loss

</div>

It will be covered in detail in Section 4.

2.1.2 LP Value with fees

Wins money example :)

2.2 Spot Price, Execution Price & Slippage Math

Section :)

2.3 Understanding dx & dy

A numerical example follows explaining the logic behind 1. Define the current price of `ETH`=$1000. We create a new `ETH/USDC` pool with a fee of 0.3%, starting with `1 ETH` and `1000 USDC`. Therefore,

$$k = 1000_{USDC} \cdot 1_{ETH} = 1000$$

Now an `ETH` buyer sends 100_{USDC} to the pool. The 0.3% fee is extracted, therefore only 99.7_{USDC} is added to the pool. Next, the invariant is divided by the amount of `USDC` there currently is to determine the `ETH` the *pool* will have *after* the swap.

$$USDC_{pool}^{after} = USDC_{pool}^{before} + dx - \phi = 1000 + 100 - 0.3 = 1099.7_{USDC}$$

$$ETH_{pool}^{after} = \frac{k_{old}}{USDC_{pool}^{after}} = \frac{1000}{1099.7} \approx 0.9093_{ETH}$$

$$ETH_{buyer} = ETH_{pool}^{before} - ETH_{pool}^{after} = 1_{ETH} - \left(\frac{1000}{1099.7}\right)_{ETH} \approx 0.0907_{ETH}$$

With `100 USDC` *buyer* bought `0.0907 ETH`.

Alternatively, using 4:

$$dy = \frac{y \cdot dx(1-\phi)}{x + dx(1-\phi)}$$

$$= \frac{1 \cdot 100(1 - 0.003)}{1000 + 100 \cdot (1 - 0.003)}$$

$$= \frac{99.7}{1099.7}$$

$$\approx 0.0907_{ETH} \quad \square$$

After the swap has been completed the fee will be added to the pool.

The new invariant will be:

$$k_{new} = \overbrace{(USDC_{pool}^{after} + \phi)}^{x_{new}} \cdot \overbrace{ETH_{pool}^{after}}^{y_{new}}$$

$$= (USDC_{pool}^{after} + \phi)_{USDC} \cdot \left(\frac{k_{old}}{USDC_{pool}^{before} + dx^{+} - \phi}\right)_{ETH} \quad (9)$$

$$= (1099.7 + 0.3)_{USDC} \cdot \left(\frac{1000}{1099.7}\right)_{ETH}$$

$$\approx 1000.27280167$$

$k_{new} > k_{old}$. The increase is due to the new liquidity ($\phi = $ `fee`) added to the pool.

How does this affect the pool's price (or exchange rate)?

If now someone trades in the same direction `USDC`→`ETH` (deposits `USDC` in exchange for `ETH`, or "buys `ETH` with `USDC`") the amount of `ETH` (dy) for the same dx (`$100 USDC`) will be lower. We can again use 5:

$$dy = \frac{y_{new} \cdot dx(1 - \phi)}{x_{new} + dx(1 - \phi)}$$

$$= \frac{\frac{1000}{1099.7} \cdot 100(1 - 0.003)}{1100 + 100(1 - 0.003)}$$

$$\approx 0.07557_{ETH}$$

Now with `100 USDC` *buyer* only bought `0.07557 ETH`.

Clearly $dy_{new} < dy_{old}$, (-16.68%).

The change is quite large due to the low initial liquidity of the pool and the amount we are trading on it (see 2.4).

If instead one swaps `ETH` for `USDC`, the amount will be higher than the initial amount of `USDC` that would have been returned right after the pool was deployed. One can see that this mechanism benefits some kind of *mean reversion* towards the initial ratio the assets were deployed in. Arbitragers will ensure the price converges with that of other exchanges. Users can even use `flash swaps` (see 10.1) if there is an arbitrage on-chain, hacing to commit almost zero capital.

2.4   Liquidity Draining: ETH/USDC Example

We can further extend our understanding of this simple formula graphically. Imagine just after deploying the initial 1 `ETH` and 1000_{USDC} we want to swap $dx^+ = \$1000_{USDC}$ (current market price for `ETH`) for some amount dy^- of `ETH` (hopefully 1 `ETH`, right?). Running the above calculation we obtain:

$$dy \approx 0.49999925_{ETH}$$

Exactly $dy = 0.5_{ETH}$ if there was no fee. Did we just lose half our money? What happened?

The pool's constant product formula is designed in order to prevent liquidity draining. This also prices the tokens according to supply-demand dynamics, the more a token is demanded, the more expensive it gets. We can observe the actual initial *price curve* in terms of each asset with respect to the other.
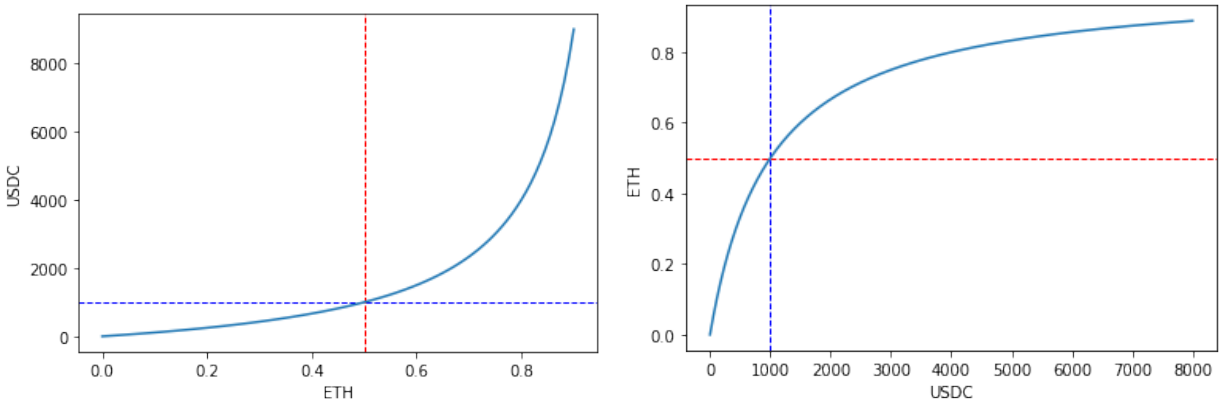
We can see that, indeed, if one tries to swap now 1000_{USDC} (or 0.5_{ETH} for `USDC`) the pool will return:

$$\$1000_{USDC} \rightleftarrows 0.5_{ETH}$$

If we had initially deployed, at the same exchange rate of $1_{ETH} = \$1000_{USDC}$, higher amounts of liquidity, for example:

$$1000_{ETH} \cdot 1000000_{USDC} = 1e9 = k$$

And now run the same swap depositing $\$1000_{USDC}$, we would expect to get closer to $dy \approx 1_{ETH}$:

$$dy = 1e3_{ETH} - \frac{1e9}{1e6_{USDC} + 997_{USDC}} = 0.99600698104_{ETH} \approx 1_{ETH} \quad \square$$

2.4.1 Shares, Fees & LP Profits

One might wonder why there is a fixed fee for every swap. This is to compensate LPs for the tokens they provide. In order to account for the profit generated with fees by LPs, a share mechanism is used. What the invariant really represents is $x \cdot y$ **at the end of the previous trade**.

When LPs provide liquidity to the pool shares are minted (created). When LPs withdraw liquidity they burn (destroy) shares and receive their assets back. The clever design here is that, as fees are subtracted from dx (respectively dy) before calculations (or added after price calculations in the pool as done in the contracts) and no shares are minted for the liquidity provided by the fee, more liquidity is now redeemable by LPs when they burn their shares (*share price has virtually increased*), therefore resulting in a net profit distributed in a *pro rata* basis among all LPs in the pool.

If no fees were set, then the invariant would actually remain constant for all swaps, having a detrimental effect on liquidity as pool prices would no be updated and the next trader would get a better rate. Let's use the prior example to see this updating k accounting for the fees and without:

$$\begin{cases} x = 1000_{ETH} \\ y = 1e6_{USDC} \\ k = 1e9 \\ dy^+ = \$1000 \\ dx^- = \end{cases} \tag{10}$$

TODO - COMPLETE EXAMPLE!

3 LP Math

3.1 LP Position Valuation

3.1.1 Full Range

Given a price $P := \frac{y}{x}$, the value of a LP position is:

$$V(P) := xP + y \tag{11}$$

The pool's liquidity L is defined by

$$L^2 := k \tag{12}$$

With some basic algebra we can see that

$$\begin{cases} x = \dfrac{y}{P} = \dfrac{L}{\sqrt{P}} \\[2ex] y = L\sqrt{P} \\[2ex] \dfrac{y^2}{P} = L^2 \end{cases} \tag{13}$$

Then it follows from 11 (using 13) that:

$$\begin{aligned} V(P) &= xP + y \\[1ex] &= 2 \cdot y \\[1ex] &= 2 \cdot \frac{k}{x} \\[1ex] &= 2 \cdot \frac{L^2}{x} \\[1ex] &= 2 \cdot \frac{L^2}{\frac{y}{P}} \\[1ex] &= 2 \cdot \frac{L^2 \cdot P}{y} \\[1ex] &= 2 \cdot \frac{L^2 \cdot P}{L \cdot \sqrt{P}} \\[1ex] &= 2L\sqrt{P} \end{aligned} \tag{14}$$

Using L and $\sqrt{P}$ is convenient because only one of them changes at a time.

If the price P changes, then $V(P)$ also changes. If we denote P_T the price at moment T, with the associated value V_T and the ratio of prices:

$$\alpha := \frac{P_T}{P_0} \tag{15}$$

Then

$$V_T = V_0\sqrt{\alpha} \tag{16}$$

This is, **the value function of a LP position is a square root function**.

We can easily verify this is true following from the above definition (considering only positive roots):

$$V_T = V_0\sqrt{\alpha}$$
$$V(P_T) = V(P_0)\sqrt{\alpha}$$
$$2L\sqrt{P_T} = 2L\sqrt{P_0}\sqrt{\alpha}$$
$$\sqrt{\frac{P_T}{P_0}} = \sqrt{\alpha}$$
$$\alpha = \frac{P_T}{P_0} \quad \square$$

To make sure this intuitively makes sense, imagine the following scenario:

ETH price at t_0 is $P_0 = \$1000$. We have a Uniswap position with `50:50` liquidity deployed. We deposit 1 `ETH` and $1000 USDC into the `ETH/USDC` Pool. Then it is trivial to see that:

$$V_0 = x \cdot P_0 + y = 1_{ETH} \cdot \$1000 + \$1000_{USDC} = \$2000$$

ETH price at t_1 increases to $P_1 = \$2000$. Our new position value following 16 will be:

$$V_1 = V_0\sqrt{\alpha} = \$2000\sqrt{2} \approx \$2828.43; \quad \alpha = \frac{2000}{1000} = 2$$

Now let's add an extra step. We will add one more observation before price reaches $2000. `ETH` prices will be $t_0 = \$1000, t_1 = \$1500, t_2 = \$2000$. We will measure (without doing any changes to our position) how its value changes over time. As we are not modifying anything we would expect the final value V_2 to be the same as above.

$$\begin{cases} V_0 = \$2000 \\ V_1 = V_0\sqrt{\alpha_1} \approx \$2449.49 \\ V_2 = V_1\sqrt{\alpha_2} = (V_0\sqrt{\alpha_1})\sqrt{\alpha_2} \approx \$2828.43 \quad \square \end{cases}$$

Where $\alpha_1 = \frac{1500}{1000} = 1.5$ and $\alpha_2 = \frac{2000}{1500} = \frac{4}{3}$.

3.1.2 Concentrated Liquidity

A custom LP range can be though of as providing liquidity with leverage.

With P a price, p_a, p_b the lower and upper limits of a concentrated liquidity position and given that $P \in [p_a, p_b]$, based on 10.1 the reserves for a concentrated position are defined by the curve:

$$\left(x + \frac{L}{\sqrt{p_b}}\right)(y + L\sqrt{p_a}) = L^2 \tag{17}$$

Notice that this is an expansion of 1 with virtual liquidity (more optimized liquidity):

$$x_{virtual} \cdot y_{virtual} = L^2$$

Define:

$$\begin{cases} x = x_{virtual} - \frac{L}{p_b} \overset{13}{=} L\left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{p_b}}\right) \\[2mm] y = y_{virtual} - L\sqrt{P_a} \overset{13}{=} L(\sqrt{P} - \sqrt{p_a}) \end{cases} \tag{18}$$

Then,

$$V_0 = y + xP$$

$$= L(\sqrt{P} - \sqrt{p_a}) + L\left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{p_b}}\right) \cdot P$$

$$= L\sqrt{P} - L\sqrt{p_a}) + L\sqrt{P} - \frac{L \cdot P}{\sqrt{p_b}}$$

$$= 2L\sqrt{P} - L\left(\sqrt{p_a} + \frac{P}{\sqrt{p_b}}\right)$$

$$\boxed{V_0 = 2L\sqrt{P} - L\left(\sqrt{p_a} + \frac{P}{\sqrt{p_b}}\right)} \tag{19}$$

Using that for V_{LP} on time t, $P_t = P_0 \cdot \alpha$:

$$\boxed{V_{LP} = 2L\sqrt{P\alpha} - L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)} \tag{20}$$

And finally if assets are just on a `50:50 HODL`, at time t:

$$V_{HODL} = y + xP\alpha$$

$$= L(\sqrt{P} - \sqrt{p_a}) + LP\alpha\left(\frac{1}{\sqrt{P}} - \frac{1}{\sqrt{p_b}}\right)$$

$$= L\sqrt{P} - L\sqrt{p_a} + L\alpha\sqrt{P} - \frac{LP\alpha}{\sqrt{p_b}}$$

$$\boxed{V_{HODL} = L\sqrt{P}(1 + \alpha) - L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)} \tag{21}$$

3.2 Liquidity Provider's Greeks

3.2.1 Full Range

Following the *Black-Scholes-Merton* model define Δ (delta) to be the price-dependence of the value function and Γ (gamma) the price-dependence of delta. These are the first and second order derivatives of the LP position's value function with respect to price:

$$\Delta_{LP} = V'(P) = \frac{\partial V(P)}{\partial P}$$

$$\Gamma_{LP} = V''(P) = \frac{\partial^2 V(P)}{\partial P^2} \tag{22}$$

Given the value function 16 as above we can find its derivatives:

$$\begin{cases} V_{LP} & = V_0\sqrt{\alpha} \\ \Delta_{LP} & = \frac{V_0}{2\sqrt{\alpha}} \\ \Gamma_{LP} & = -\frac{V_0}{4\alpha\sqrt{\alpha}} \end{cases} \tag{23}$$

Applying some basic calculus and some abuse of notation to facilitate comprehension:

$$\Delta_{LP} := \frac{\partial V_T}{\partial P} = \frac{\partial(V_0\sqrt{\alpha})}{\partial P} \implies V_0\frac{\partial\sqrt{\alpha}}{\partial P} = \frac{V_0}{2\sqrt{\alpha}}$$

$$\Gamma_{LP} := \frac{\partial\Delta_{LP}}{\partial P} = \frac{\partial}{\partial P}\frac{V_0}{2\sqrt{\alpha}} \implies \frac{V_0}{2}\cdot\frac{\partial}{\partial P}\left(\frac{1}{\sqrt{\alpha}}\right) = -\frac{V_0}{4\alpha\sqrt{\alpha}} \quad \square$$

A more in-depth study on how to hedge a full-range position can be found at *Section 7 - Hedging Strategies*.

Gamma term is a non-linear, always negative function that describes the impermanent loss of a LP position. In contrast, `50:50 HODL` and 100% `asset` holdings have **zero gamma, therefore no impermanent loss**. Normally the `50:50 HODL` is used a benchmark to compare the `LP` position, however, this is not always the best one to use.

$$\begin{cases} V_{\texttt{HODL}} = \frac{V_0 + V_0\cdot\alpha}{P} = \boxed{L\sqrt{P}(1+\alpha)} \\ \Delta_{\texttt{HODL}} = \frac{V_0}{2} = const \\ \Gamma_{\texttt{HODL}} = 0 \end{cases} \qquad \begin{cases} V_{\texttt{asset}} = V_0\cdot\alpha \\ \Delta_{\texttt{asset}} = V_0 = const \\ \Gamma_{\texttt{asset}} = 0 \end{cases} \tag{24}$$

To see why the second expression of $V_{\texttt{HOLD}}$ is true, we can start from the definition of value (11)

$$V_{\texttt{HODL}} = y + xP_t = y + xP\alpha$$

And now using the first and second formulas from 13 we obtain

$$V_{\texttt{HODL}} = y + xP\alpha = L\sqrt{P} + \frac{L}{\sqrt{P}}P\alpha = L\sqrt{P} + L\sqrt{P}\alpha = L\sqrt{P}\cdot(1+\alpha) \tag{25}$$

We can visualize this graphically:
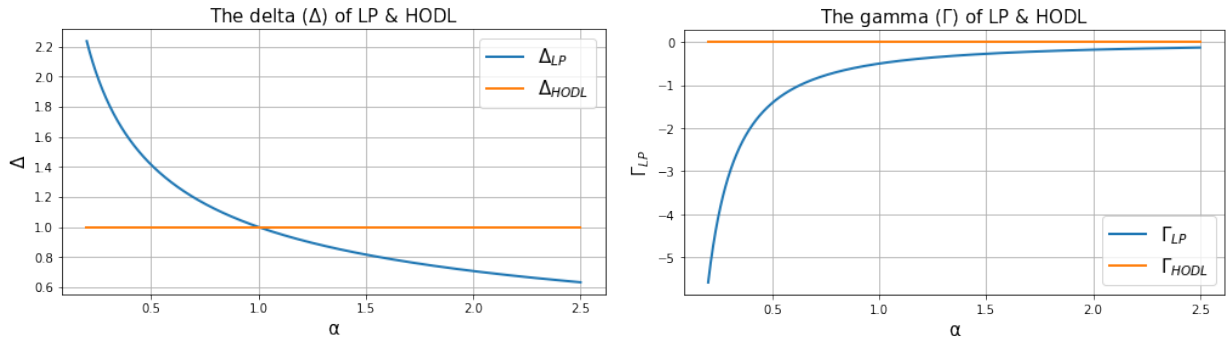
An alternative way to think about the LP position:

- If asset price increases, LP's delta is lower than the delta of a `50:50 HODL` position;

- If asset price decreases, LP's delta is higher.

The LP position is underexposed to price increases, and overexposed to price decreases. When price goes up, the gain of the LP is reduced; when the price goes down, their loses increased.

The primary intuition you should have about concentrated liquidity is that it makes all things more extreme. Fee income is amplified while the position is in range, but impermanent loss is amplified as well. Another important ideas is that CL introduces new risks (going out of range) as well as opportunities (range orders, limit orders).
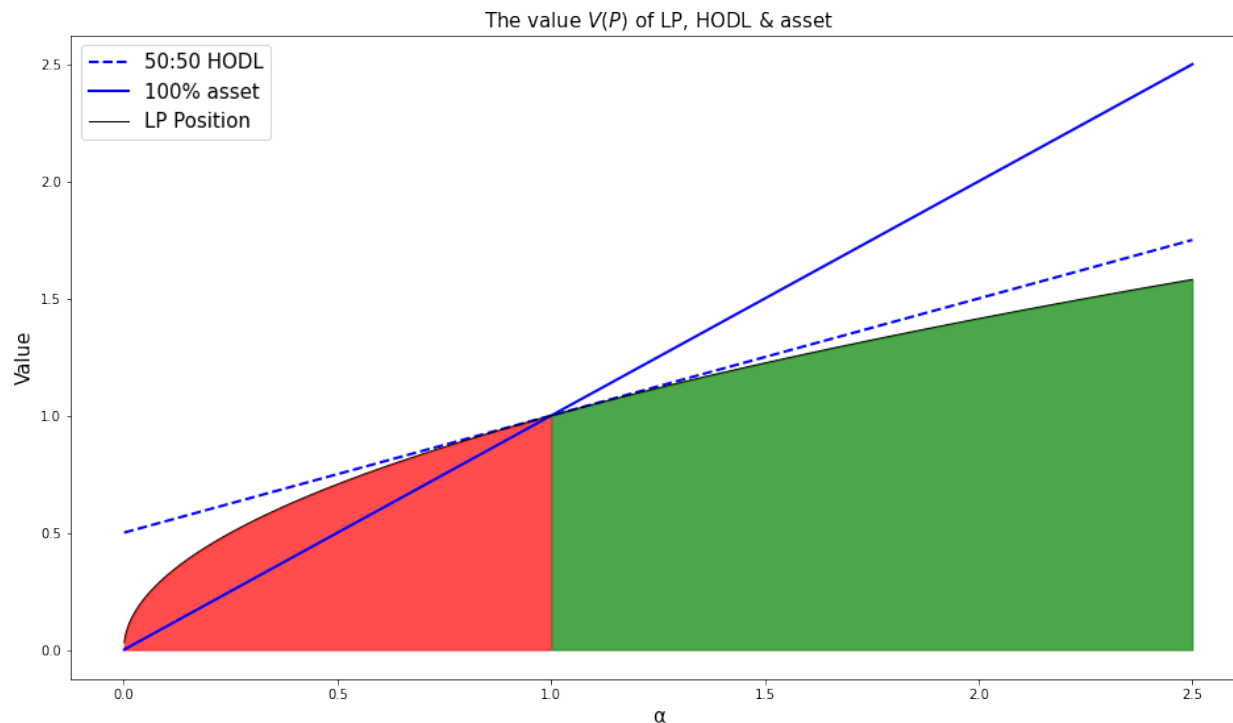
Full-range LP position's value:
Red area shows loss relative to initial value, the green area: gain or neutral.

The figure below shows the profit the LP has relative to a `50:50 HODL`:

The value $V(P)$ of LP vs. HODL with $\mu = 0\%$
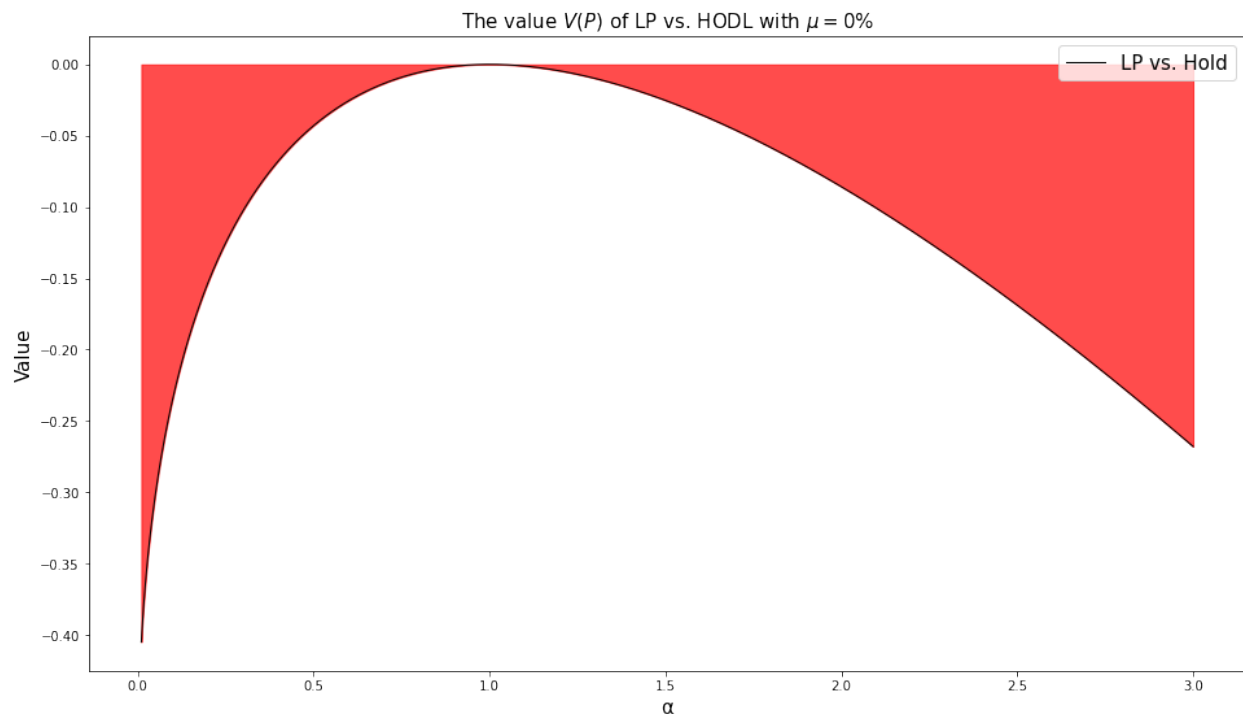
One can notice that the curve is always bellow 0. This means that if no fees are distributed Liquidity Providing is always a losing strategy. That is, fees have to be sufficiently high to offset the IL generated by the change in price α. Nevertheless, this is also incomplete: we are assuming the `50:50 HODL` strategy holds the tokens still; however, this is unrealistic as *risk free* strategies exist in DeFi. We can assume the *DeFi risk-free rate* to be the one of lending at Aave, Compound, etc. or staking funds at Lido even low-risk yield farming strategies at Yearn, Convex, ... and similar strategies.

Therefore, fees do not only have to outweight IL but also the *DeFi risk-free rate*. This rarely happens. Now let's assume the LP position collects 5% of its initial value in fees: a simplification that is not fully realistic, but gives something to start with. Even under this assumption of the relatively high 5% fees, the LP suffers losses if the price changes significantly enough.
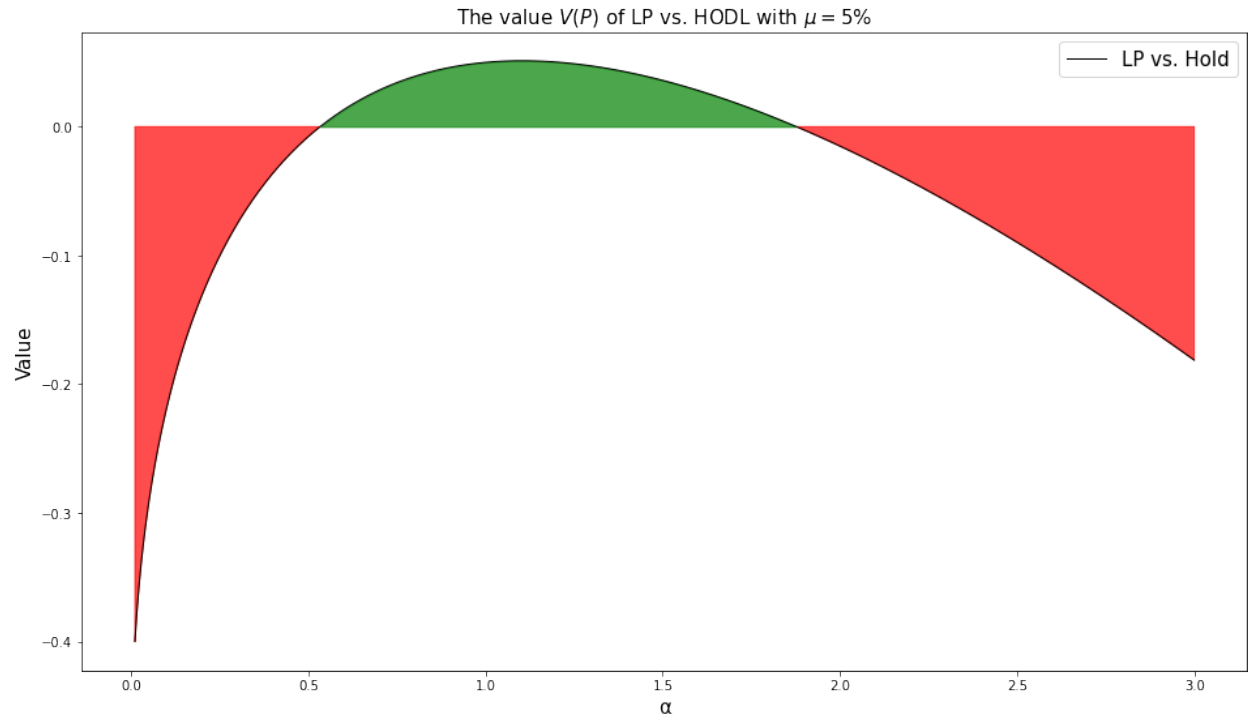
Full-range LP position with 5% fees, LP profit vs. HODL

3.3 Concentrated Liquidity

4 Impermanent Loss

4.1 IL on v1 & v2: $(0, +\infty)$ - Full range

With the information we have already covered we have sufficient knowledge to formalise the `Impermanent Loss` (also referred to as Divergence Loss or Loss-Versus-Holding, LVH for a stable-volatile pair. Impermanent loss is typically calculated with respect to one asset relative to another in a liquidity pool. When both assets are volatile (non-stable), the calculation becomes more complex.

Starting from 1 and using 13 and the value definition derived at 14 we define Impermanent Loss by computing the *percentage loss* of having V_{LP} compared to V_{HODL} (24 & 25).

$$IL(\alpha) := \frac{V_{LP} - V_{\text{HODL}}}{V_{\text{HODL}}} = \frac{V_{LP}}{V_{\text{HODL}}} - 1 = \frac{V_0\sqrt{\alpha}}{L\sqrt{P}(1+\alpha)} - 1 = \frac{2L\sqrt{P}\sqrt{\alpha}}{L\sqrt{P}(1+\alpha)} - 1 = \frac{2\sqrt{\alpha}}{(1+\alpha)} - 1$$

$$\boxed{IL(\alpha) = DL(\alpha) = \frac{2\sqrt{\alpha}}{1+\alpha} - 1} \tag{26}$$

We can use a plot to gain some intuition on how IL might behave depending on α:
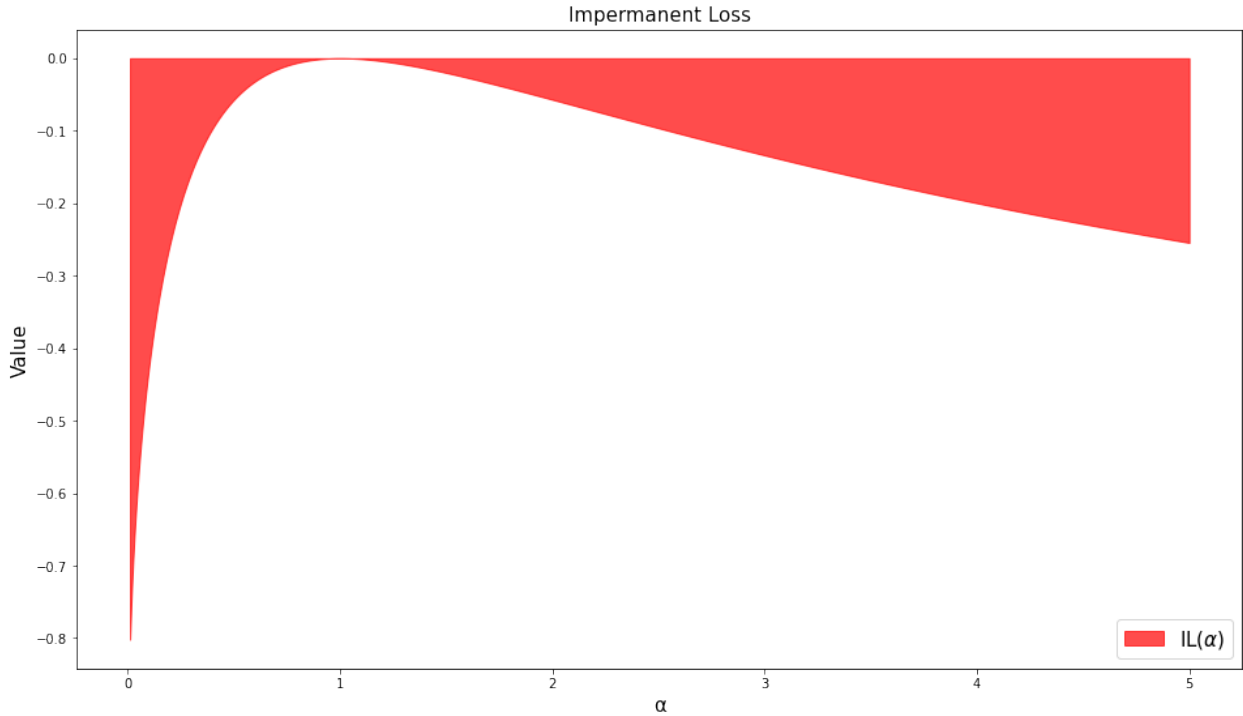
Notice value is always negative except at $\alpha = 1$ where the price has reverted back to the original price, hence the "*impermanence*". Many researchers prefer not to use this term as every loss is *impermanent* (or unrealised) until you close the trade, but that does not mean that price is going to return even near the initial price. Moreover, Impermanent Loss is not a loss in itself but rather an *opportunity cost* of having provided the assets as LP.

Example: We have provided liquidity to the `ETH/USDC` pool `50:50`. If the price of `ETH` rises $\alpha = 2$ (+200%)

and we obtained no fees ($\mu = 0$) our DL will be:

$$DL(\alpha = 2) = \frac{2\sqrt{2}}{3} - 1 \approx -5.72\%$$

4.1.1 The effect of fees on IL

With a small addition to the equation above (see [5]) we can visualize how different fee returns ($\mu \in [0\%, 20\%]$) affect IL:

$$IL_{\mu,t}(\alpha) = \frac{2\sqrt{\alpha}}{1 + \alpha} \cdot e^{\mu \cdot t} - 1 \tag{27}$$

Let $t = 1$, then:
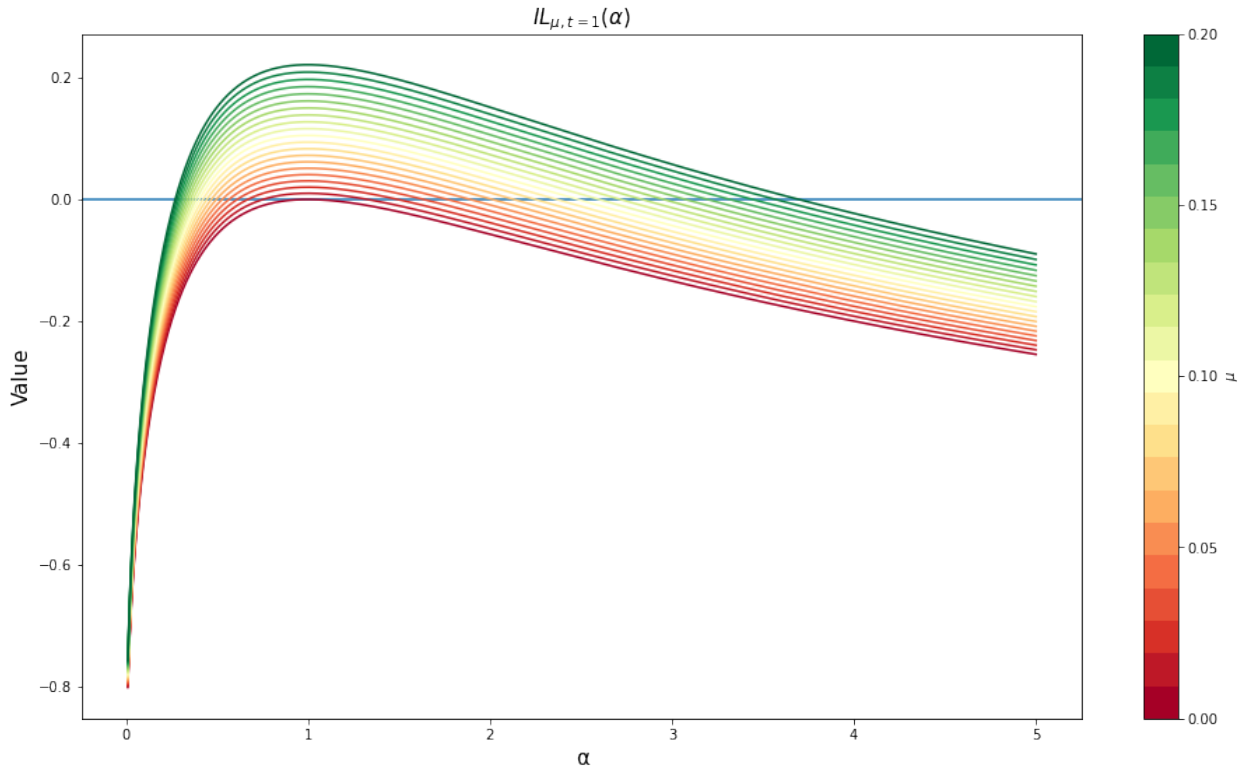
Due to the return's square root relationship with price, the effect of a high μ decreases as price deviates from the original price at which liquidity was provided.

4.1.2 Concentrated Liquidity

Using 20 and 21 we can compute IL based on α, p_a and p_b. We will compute some extra steps to obtain $IL_{a,b}(\alpha)$ as a function of $IL(\alpha)$. This is, we want to find $IL_{a,b}(\alpha) = IL(\alpha) \cdot C$:

$$IL_{a,b}(\alpha) = \frac{V_{LP} - V_{HODL}}{V_{HODL}}$$

$$= \frac{2L\sqrt{P\alpha} - L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right) - L\sqrt{P}(1+\alpha) + L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)}{L\sqrt{P}(1+\alpha) - L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)}$$

$$= \frac{2L\sqrt{P\alpha} - L\sqrt{P}(1+\alpha)}{L\sqrt{P}(1+\alpha) - L\left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)}$$

$$= \frac{\cancel{L} \cdot (2\sqrt{P\alpha} - \sqrt{P}(1+\alpha))}{\cancel{L} \cdot \left(\sqrt{P}(1+\alpha) - \left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)\right)}$$

$$= \frac{\sqrt{P} \cdot (2\sqrt{\alpha} - 1 - \alpha)}{\sqrt{P}(1+\alpha) - \left(\sqrt{p_a} + \frac{P\alpha}{\sqrt{p_b}}\right)} \cdot \underbrace{\left(\frac{\frac{1}{\sqrt{P}}}{\frac{1}{\sqrt{P}}}\right)}_{1}$$

$$= \frac{2\sqrt{\alpha} - 1 - \alpha}{(1+\alpha) - \sqrt{\frac{p_a}{P}} - \alpha\sqrt{\frac{P}{p_b}}} \cdot \underbrace{\left(\frac{1+\alpha}{1+\alpha}\right)}_{1}$$

$$= \frac{2\sqrt{\alpha} - 1 - \alpha}{1+\alpha} \cdot \left(\frac{1+\alpha}{1+\alpha - \sqrt{\frac{p_a}{P}} - \alpha\sqrt{\frac{P}{p_b}}}\right)$$

$$= \left(\frac{2\sqrt{\alpha}}{1+\alpha} - \cancel{\frac{1+\alpha}{1+\alpha}}^{1}\right) \cdot \left(\frac{1+\alpha}{1+\alpha - \sqrt{\frac{p_a}{P}} - \alpha\sqrt{\frac{P}{p_b}}}\right)$$

$$= \underbrace{\left(\frac{2\sqrt{\alpha}}{1+\alpha} - 1\right)}_{IL(\alpha)} \cdot \left(\frac{1+\alpha}{1+\alpha - \sqrt{\frac{p_a}{P}} - \alpha\sqrt{\frac{P}{p_b}}}\right)$$

$$= IL(\alpha) \cdot \left(\frac{1}{\frac{1+\alpha - \sqrt{\frac{p_a}{P}} - \alpha\sqrt{\frac{P}{p_b}}}{1+\alpha}}\right)$$

$$= IL(\alpha) \cdot \left(\frac{1}{\underbrace{\cancel{\frac{1+\alpha}{1+\alpha}}^{1} - \frac{\sqrt{\frac{p_a}{P}} + \alpha\sqrt{\frac{P}{p_b}}}{1+\alpha}}_{C}}\right)$$

$$\boxed{IL_{a,b}(\alpha) = IL(\alpha) \cdot \left(\frac{1}{1 - \frac{\sqrt{\frac{p_a}{P}} + \alpha\sqrt{\frac{P}{p_b}}}{1+\alpha}}\right)} \tag{28}$$

Note that this equation only applies when $P \in [p_a, p_b]$. Outside of this price range asset holding stop changing. Normally p_a and p_b are set as a percentage (fixed distance) of the current price. For example, if an LP wants to create a $(-r, r)$ range around the current price P:

$$\begin{cases} p_b = (1 + r) \cdot P \\ p_a = \frac{P}{1+r} \end{cases} \tag{29}$$

Intuitively, if $P = 100$ and $r = 0.1$ (10%),

$$\begin{cases} p_b = (1 + 0.1) \cdot 100 = 110 \\ p_a = \frac{100}{1+0.1} \approx 90.91 \end{cases}$$

We can visualize how much $IL_{a,b}(\alpha)$ increases as the range narrows (r gets smaller):

$IL_r(\alpha)$
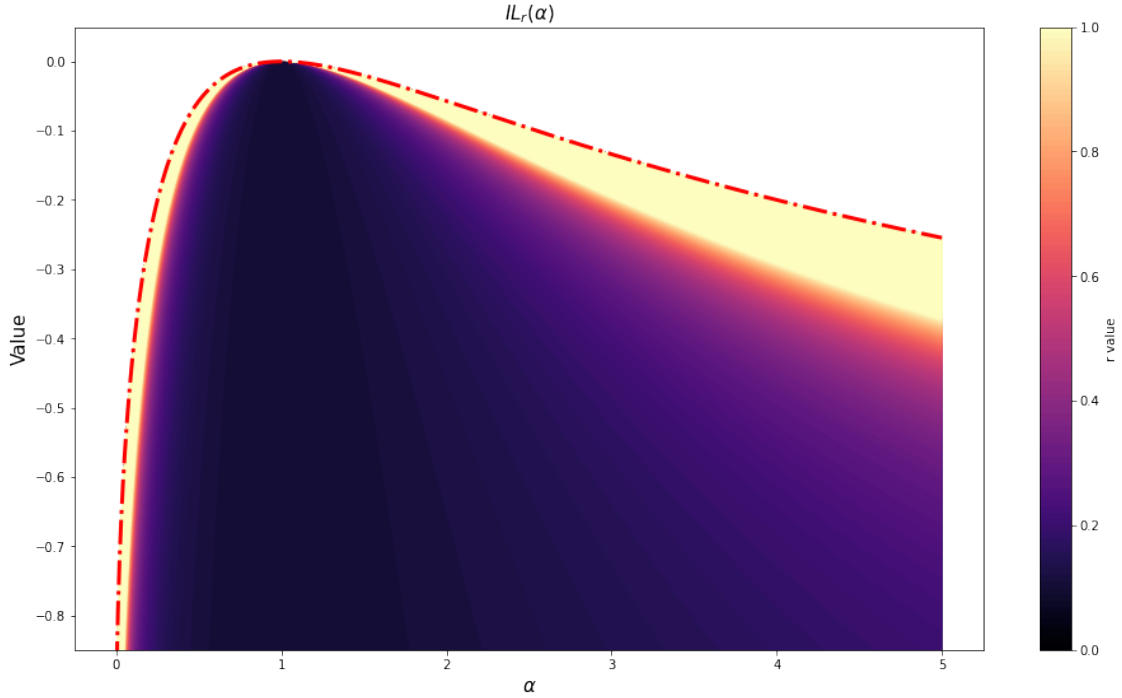
We see that, the narrower the range ($r \to 0$), the steeper IL becomes. On the other hand, the bigger r becomes ($r \to 1$), the more the IL curve resembles that one of v2 (full range). This can easily be seen as when $p_b \to +\infty, p_a \to 0$ we have a full range liquidity position and as $C \to 1 \implies IL_{a,b}(\alpha) \to IL(\alpha)$. More formally:

$$\lim_{\substack{p_a \to 0 \\ p_b \to +\infty}} IL_{a,b}(\alpha) = IL(\alpha) \because C \to 1 \tag{30}$$
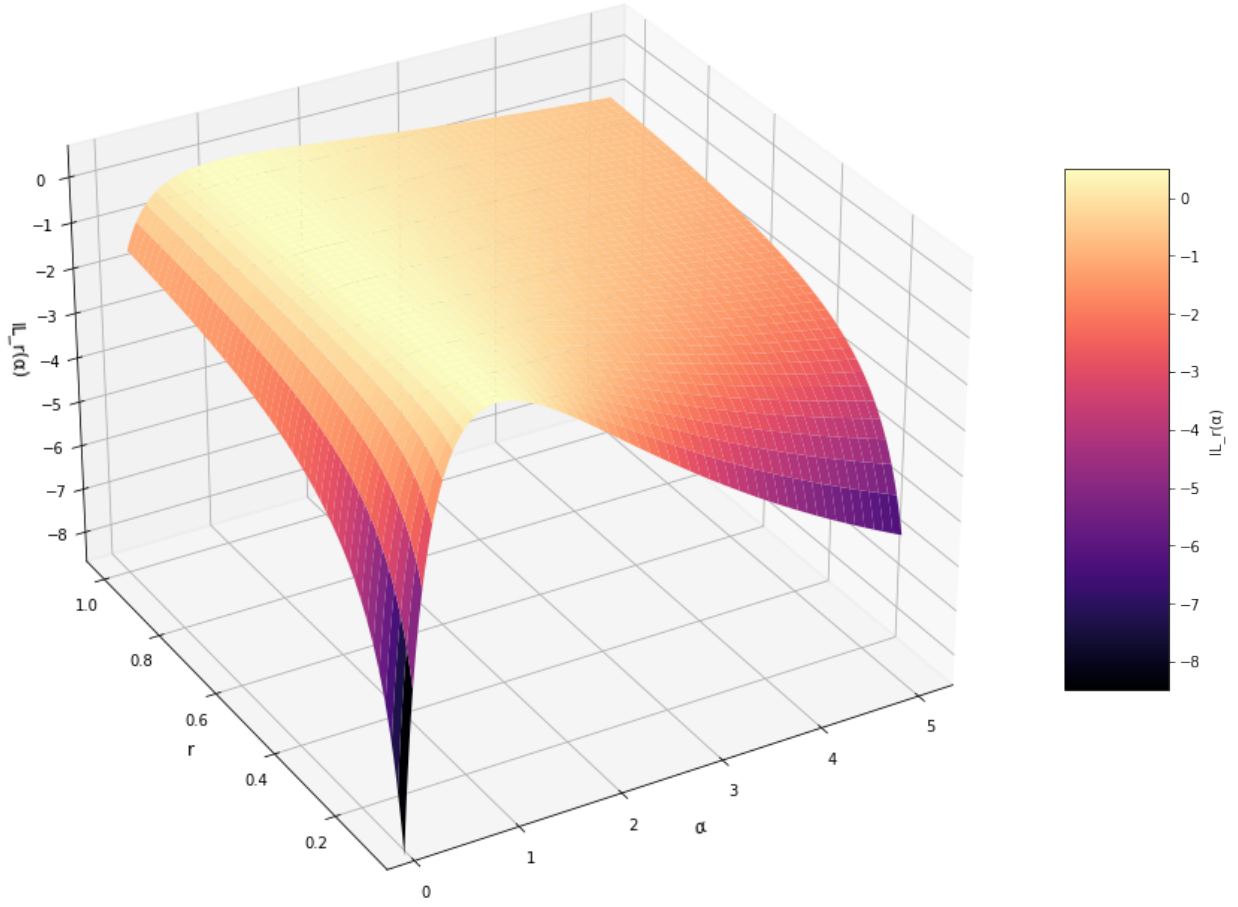
The following three sections can safely be skipped as they are intended for the most advanced and curious readers. This is to add completeness to the document.

4.2 Polynomial Approximation of IL

As it will become evident in future sections, having a polynomial approximation of the IL will come in handy when trying to hedge it. To do this, let's recall the general definition of the Taylor polynomial.

Theorem. *Let $D \subset \mathbb{R}^n$ be an open set, and $f : D \to \mathbb{R}$ an m-differentiable function at a point $a \in D$. Then, $\exists! p \in \mathbb{R}[x_1, \ldots, x_n]_m$ such that:*

$$\lim_{x \to a} \frac{f(x) - p(x)}{\|x - a\|^m} = 0$$

This unique polynomial is of the form:

$$T_{f,a,m}(x) := \sum_{k=0}^{m} D^k f(a) \frac{(x - a, \overset{k}{\cdots}, x - a)}{k!} \tag{31}$$

Note that $D^0 f(a) = f(a)$ and $(x - a)^0 = 1$.

In less technical terms, which some readers might find more familiar and helpful:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots$$

Therefore, we can apply this to the $IL(\alpha)$ formula we described above (26). First, we will need to compute its first and second order derivatives. We will ignore any higher order derivatives, we are interested in hedging delta (first-order derivative) and gamma (second-order derivative), as their marginal impact is negligible.

$$\frac{\partial IL(\alpha)}{\partial \alpha} = 2 \cdot \left(\frac{\frac{1}{2\sqrt{\alpha}}(1 + \alpha) - (\sqrt{\alpha} \cdot 1)}{(1 + \alpha)^2} \right)$$

$$= 2 \cdot \left(\frac{\frac{1+\alpha}{2\sqrt{\alpha}} - \sqrt{\alpha}}{(1 + \alpha)^2} \right)$$

$$= 2 \cdot \left(\frac{\frac{1+\alpha-2\alpha}{2\sqrt{\alpha}}}{(1 + \alpha)^2} \right)$$

$$= \cancel{2} \cdot \frac{1 - \alpha}{\cancel{2}\sqrt{\alpha} \cdot (1 + \alpha)^2}$$

$$\boxed{\frac{\partial IL(\alpha)}{\partial \alpha} = \frac{1 - \alpha}{\sqrt{\alpha}(1 + \alpha)^2}} \tag{32}$$

Now deriving that expression again with respect to α:

$$\frac{\partial^2 IL(\alpha)}{\partial \alpha^2} = \frac{-[\sqrt{\alpha}(1+\alpha)^2] - [(1+\alpha)^2 + 4\alpha(1+\alpha)]}{[\sqrt{\alpha}(1+\alpha)^2]^2}$$

$$= \frac{-\sqrt{\alpha}(1+\alpha)^2 - \left((1-\alpha) \cdot \frac{5\alpha^2 + 6\alpha + 1}{2\sqrt{\alpha}}\right)}{\left(\sqrt{\alpha}(1+\alpha)^2\right)^2}$$

$$= \frac{-\sqrt{\alpha}(1+\alpha)^2 - \left(\frac{5\alpha^2 + 6\alpha(1-\alpha) + (1-\alpha)}{2\sqrt{\alpha}}\right)}{\left(\sqrt{\alpha}(\alpha^2 + 2\alpha + 1)\right)^2}$$

$$= \frac{\frac{-2\alpha(1+\alpha)^2 - 5\alpha^2 - 6\alpha(1-\alpha) - (1-\alpha)}{2\sqrt{\alpha}}}{(\alpha^2\sqrt{\alpha} + 2\alpha\sqrt{\alpha} + \sqrt{\alpha})^2}$$

$$= \frac{3\alpha^3 - 3\alpha^2 - 7\alpha - 1}{2\sqrt{\alpha} \cdot (\alpha^5 + 4\alpha^4 + 6\alpha^3 + 4\alpha^2 + \alpha)}$$

$$= \frac{(\alpha+1)(3\alpha^2 - 6\alpha - 1)}{2\sqrt{\alpha} \cdot \alpha(\alpha+1)^4}$$

$$= \frac{\cancel{(\alpha+1)}(3\alpha^2 - 6\alpha - 1)}{2\sqrt{\alpha} \cdot \alpha(\alpha+1)^{\cancel{4}^{\,3}}}$$

$$\boxed{\frac{\partial^2 IL(\alpha)}{\partial \alpha^2} = \frac{3\alpha^2 - 6\alpha - 1}{2\alpha\sqrt{\alpha}(1+x)^3}} \tag{33}$$

Now we can formulate $T_{IL,a,2}$:

$$T_{IL,a,2} = \frac{2\sqrt{a}}{1+a} - 1 + \frac{1-a}{\sqrt{a}(1+a)^2} \cdot (\alpha - a) + \frac{3a^2 - 6a - 1}{2a\sqrt{a}(1+a)^3} \cdot \frac{(\alpha - a)^2}{2} \tag{34}$$

We are interested in approximating $IL(\alpha)$ at $a = 1$:

$$T_{IL,a=1,m=2} = \frac{2}{2} - 1 + \frac{1-1}{\sqrt{1}(1+1)^2} \cdot (\alpha - 1) + \frac{3 - 6 - 1}{2 \cdot (2)^3} \cdot \frac{(\alpha - a)^2}{2} = 0 + 0 + \frac{-4 \cdot (\alpha - 2)^2}{32} = -\frac{(\alpha - 1)^2}{8}$$

$$\boxed{T_{IL,1,2} = -\frac{1}{8}(\alpha - 1)^2} \tag{35}$$

For approximately $\alpha \in (-30\%, +30\%)$ it is a reasonable approximation and indeed it locally matches the negative convexity (concavity) as desired:

25

$T_{IL,1,2}$
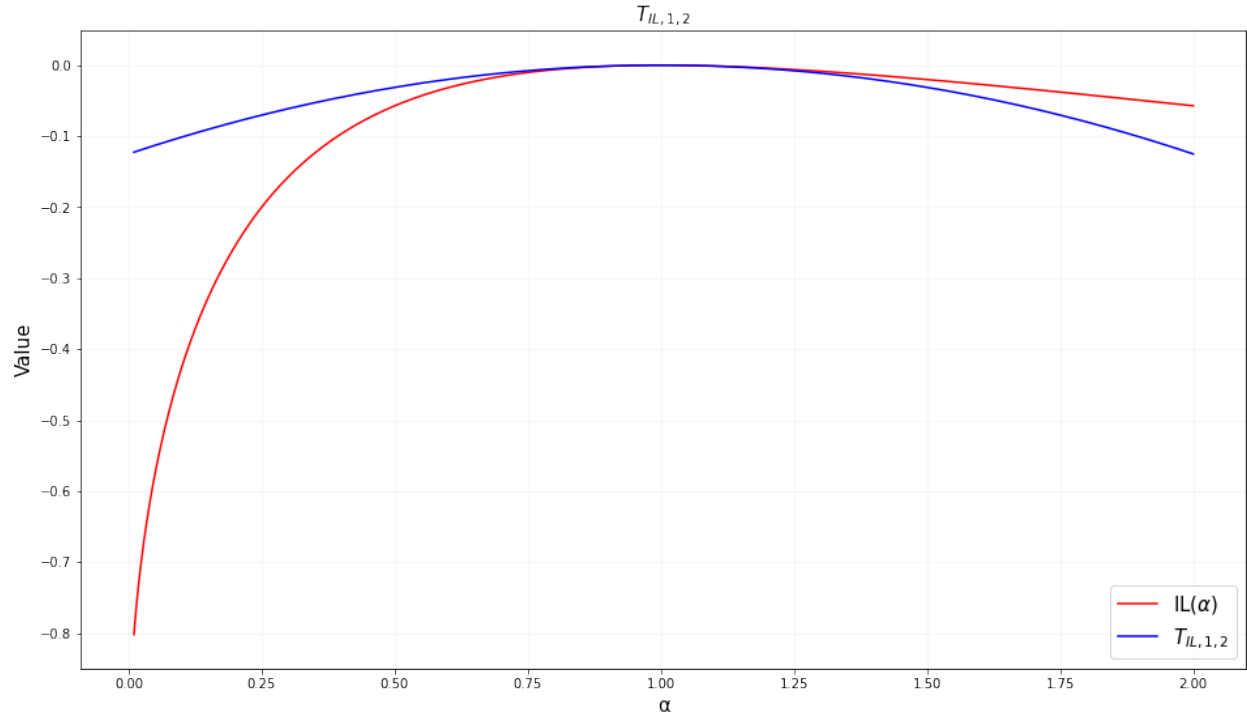
We will not be needing higher order derivatives however it can be interesting to visualise the Taylor expansion for higher degrees. In order to keep this document short, higher order derivatives are left as an exercise to the reader.

$$
\begin{cases}
IL(\alpha) = & \frac{2\sqrt{\alpha}}{(1+\alpha)} - 1 \\[2mm]
\frac{\partial IL(\alpha)}{\partial \alpha} = & \frac{1-\alpha}{\sqrt{\alpha}(1+\alpha)^2} \\[2mm]
\frac{\partial^2 IL(\alpha)}{\partial \alpha^2} = & \frac{1}{2} \cdot \frac{3\alpha^2 - 6\alpha - 1}{\alpha\sqrt{\alpha}(1+x)^3} \\[2mm]
\frac{\partial^3 IL(\alpha)}{\partial \alpha^3} = & -\frac{3}{4} \cdot \frac{5\alpha^3 - 15\alpha^2 - 5\alpha - 1}{\alpha^2\sqrt{\alpha}(1+\alpha)^4} \\[2mm]
\frac{\partial^4 IL(\alpha)}{\partial \alpha^4} = & \frac{3}{8} \cdot \frac{35\alpha^4 - 140\alpha^3 - 70\alpha^2 - 28\alpha - 5}{\alpha^3\sqrt{\alpha}(1+\alpha)^5} \\[2mm]
\frac{\partial^5 IL(\alpha)}{\partial \alpha^5} = & \frac{15}{16} \cdot \frac{-63\alpha^5 + 315\alpha^4 + 210\alpha^3 + 126\alpha^2 + 45\alpha + 7}{x^4\sqrt{\alpha}(1+\alpha)^6} \\[2mm]
\vdots & \vdots
\end{cases}
\tag{36}
$$

We will first evaluate each term at $a = 1$:

$$
\begin{cases}
IL(1) = & 0 \\[2mm]
\dfrac{\partial IL(1)}{\partial \alpha} = & 0 \\[2mm]
\dfrac{\partial^2 IL(1)}{\partial \alpha^2} = & -\dfrac{1}{4} \\[2mm]
\dfrac{\partial^3 IL(1)}{\partial \alpha^3} = & \dfrac{3}{4} \\[2mm]
\dfrac{\partial^4 IL(1)}{\partial \alpha^4} = & -\dfrac{39}{16} \\[2mm]
\dfrac{\partial^5 IL(1)}{\partial \alpha^5} = & \dfrac{75}{8} \\[2mm]
\vdots &
\end{cases}
$$

And now add it to our Taylor expansion:

$$
T_{IL,1,5} = -\frac{1}{4}\frac{(x-1)^2}{2} + \frac{3}{4}\frac{(x-1)^3}{6} - \frac{39}{16}\frac{(x-1)^4}{24} + \frac{75}{8}\frac{(x-1)^5}{120}
$$

$$
= -\frac{1}{8}(x-1)^2 + \frac{3}{24}(x-1)^3 - \frac{13}{128}(x-1)^4 + \frac{5}{64}(x-1)^5
$$

$T_{IL, 1, \{0, 1, 2, 3, 4, 5\}}$
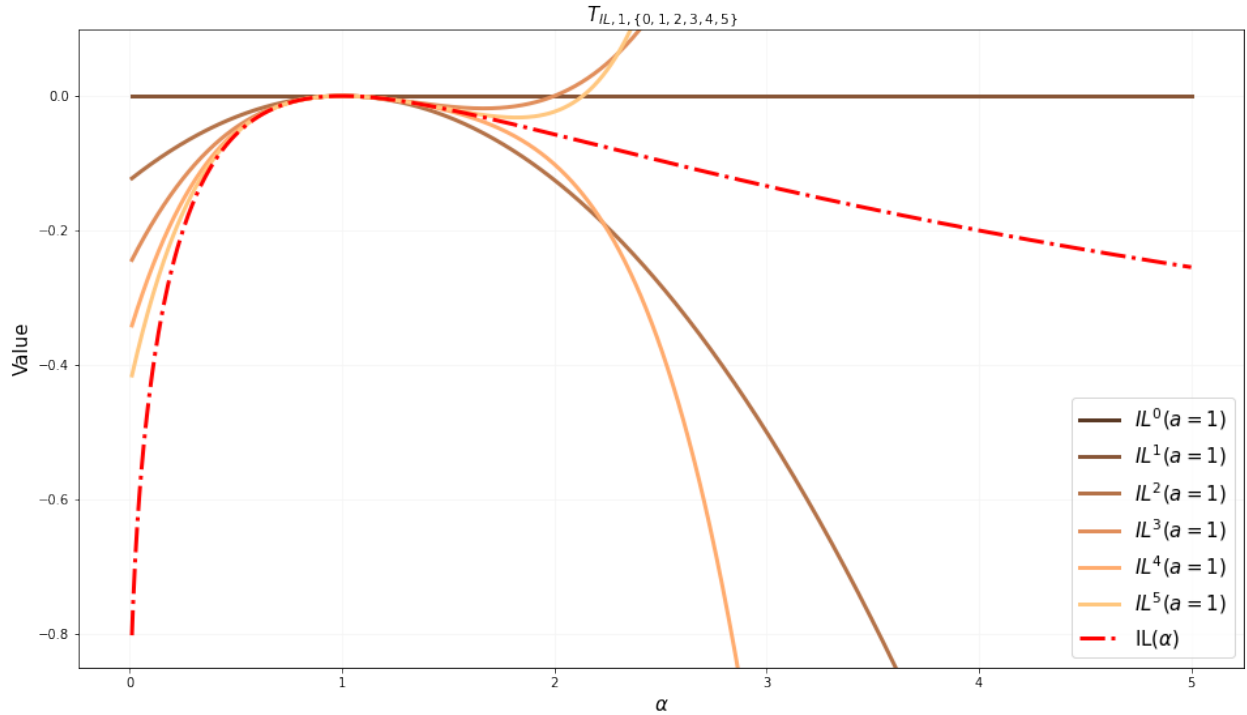
It is now apparent that the contribution of higher-order terms $(m > 2)$ is negligible. $\quad \square$

The construction of the general formula for the nth term is left as an exercise to the **author** (in some not-so-distant future).

4.3 General DL Derivation for Weighted Pools

The formula above for DL is actually a concrete example of a more general formula with only two assets [ETH, USDC] with respective prices and weights:

$$\mathbf{p} = \begin{bmatrix} 1000, \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix};$$

Other AMM's like Balancer do not follow a 50:50 distribution of tokens. Therefore, a more general definition of DL as a function of asset prices and weigths can be formulated as:

$$DL(\mathbf{p}, \mathbf{w}) = \frac{\prod_i (\Delta p_{USD}^i)^{w_i}}{\sum_i (\Delta p_{USD}^i \cdot w_i)} - 1 \tag{37}$$

Additionally, k can be computed as:

$$k = \prod_i x_i^{w_i}; \quad i \in \mathbb{N}^+$$

While it may look scary, the formula is actually quite simple: subtract one from the quotient of the product of all token USD price changes in each token to the power of their weight and the sum of all token USD price changes multiplied by their weight.

For example, imagine a similar pool for ETH:USDC pool where ETH trades for \$100 USDC. However instead of 50:50 this pool is 20:80. We will not be needing it but

$$k = 100^{0.2} + 1^{0.8} \approx 3.5119$$

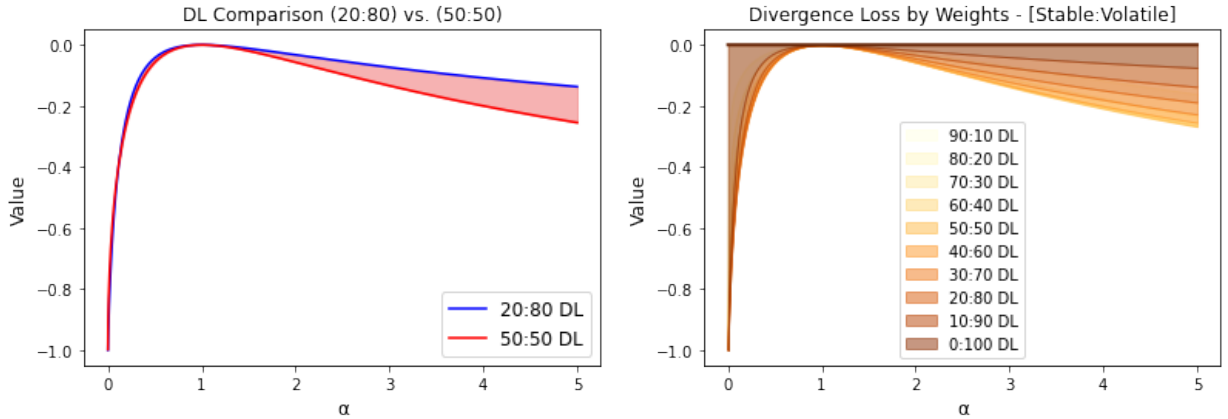Then ETH price increases to \$200. Our $DL(\alpha = 2)$ can be computed as:

$$DL_{20:80}(\alpha = 2) = \frac{2^{0.8} \cdot 1^{0.2}}{(2 \cdot 0.8) + (1 \cdot 0.2)} - 1 \approx -3.27\%$$

And we can clearly see that, as the weight of ETH is lower, so it is the Divergence Loss:

$$DL_{50:50}(2) > DL_{20:80}(2)$$

In fact we can see that the general inequality for the loss: $DL_{50:50} >= DL_{20:80}$ holds.

One might be tempted to think, well, if down-weighting the volatile asset we reduce DL, let's minimize that so that DL is also minimal. However, this is not so easy as it impacts the fees received as well (see 7.1).

4.4 Expected Impermanent Loss: $\mathbb{E}(\text{IL})$

Assume that asset prices follow a Geometric Brownian Motion (GBM) with log-normally distributed returns. This is a reasonable assumption typically made in traditional asset pricing models such as *Black-Scholes-Merton* model for options.

Define S_0 the price of the asset at time 0, S_t the price of the asset at time t, μ the expected log return of the asset, σ the volatility of the log returns of the asset and W_t a Wiener process that follows a normal distribution with mean 0 and variance t.

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}t + \sigma W_t\right)\right), \qquad W_t \sim N(0,t) \tag{38}$$

Given the values of the LP (24) and IL derived earlier (26):

$$\begin{cases} V = 2L\sqrt{P_0\alpha} \\[2mm] V_h = L\sqrt{P_0}(1+\alpha) \\[2mm] IL = \frac{2\sqrt{\alpha}}{1+\alpha} - 1 \end{cases}$$

Under the assumption of GBM we can model the price of the portfolio held outside of Uniswap V2 and the one with the assets provided as liquidity:

$$\begin{cases} V = 2L\sqrt{P_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)} \\[3mm] V_h = L\sqrt{P_0}\left(1 + \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)\right) \end{cases} \tag{39}$$

We can now compute the *Expected Value* of the portfolio held **outside** the AMM (`HODL`), which results in the same function but instead of the exponential we use the expected value of the GBM:

$$\mathbb{E}[V_h] = \mathbb{E}\left[L\sqrt{P_0}\left(1 + \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)\right)\right]$$

$$= L\sqrt{P_0}\left(1 + \mathbb{E}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}\right]\right)$$

$$\boxed{\mathbb{E}[V_h] = L\sqrt{P_0}(1 + e^{\mu t})} \tag{40}$$

We will now do the same for the *Expected Value* of the portfolio held inside the AMM:

$$\mathbb{E}[V] = \mathbb{E}\left[2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot e^{\frac{\sigma W_t}{2}}\right]$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \mathbb{E}\left[e^{\frac{\sigma W_t}{2}}\right]$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \int_{-\infty}^{+\infty} e^{\frac{\sigma W_t}{2}} \cdot \frac{e^{-\frac{W_t^2}{2t}}}{\sqrt{2\pi t}} dW_t$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}\left(W_t^2 - \sigma t W_t\right)}}{\sqrt{2\pi t}} dW_t$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}\left(W_t^2 - \sigma t W_t + \frac{1}{4}\sigma^2 t^2 - \frac{1}{4}\sigma^2 t^2\right)}}{\sqrt{2\pi t}} dW_t$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}\left[\left(W_t - \frac{\sigma}{2}\right)^2 - \frac{\sigma^2 t^2}{4}\right]}}{\sqrt{2\pi t}} dW_t$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}\left(W_t - \frac{\sigma}{2}\right)^2} \cdot e^{\frac{\sigma^2 t}{8}}}{\sqrt{2\pi t}} dW_t$$

$$= 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{4}\right)t} \cdot e^{\frac{\sigma^2 t}{8}} \underbrace{\int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{2t}\left(W_t - \frac{\sigma}{2}\right)^2}}{\sqrt{2\pi t}} dW_t}_{1 \ \because it \ is \ a \ PDF}$$

$$\boxed{\mathbb{E}[V] = 2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{8}\right)t}} \tag{41}$$

And then for the expected value of the IL:

$$\mathbb{E}[IL] = \frac{\mathbb{E}[V]}{\mathbb{E}[V_h]} - 1$$

$$= \frac{2L\sqrt{P_0} \cdot e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{8}\right)t}}{L\sqrt{P_0}\left(1 + e^{\mu t}\right)} - 1$$

$$= \frac{2e^{\left(\frac{\mu}{2} - \frac{\sigma^2}{8}\right)t}}{\left(1 + e^{\mu t}\right)} - 1$$

Using the *cosh* identity

$$cosh(x) = \frac{1 + e^{2x}}{2e^x} \tag{42}$$

And setting

$$x = \frac{\mu t}{2}$$

We can simplify the formula for the expected value of the impermanent loss to:

$$\boxed{\mathbb{E}[IL] = \frac{e^{-\frac{\sigma^2 t}{8}}}{cosh\left(\frac{\mu t}{2}\right)} - 1} \tag{43}$$

We can visualize the *expected impermanent loss surface* as a function of μ and σ:
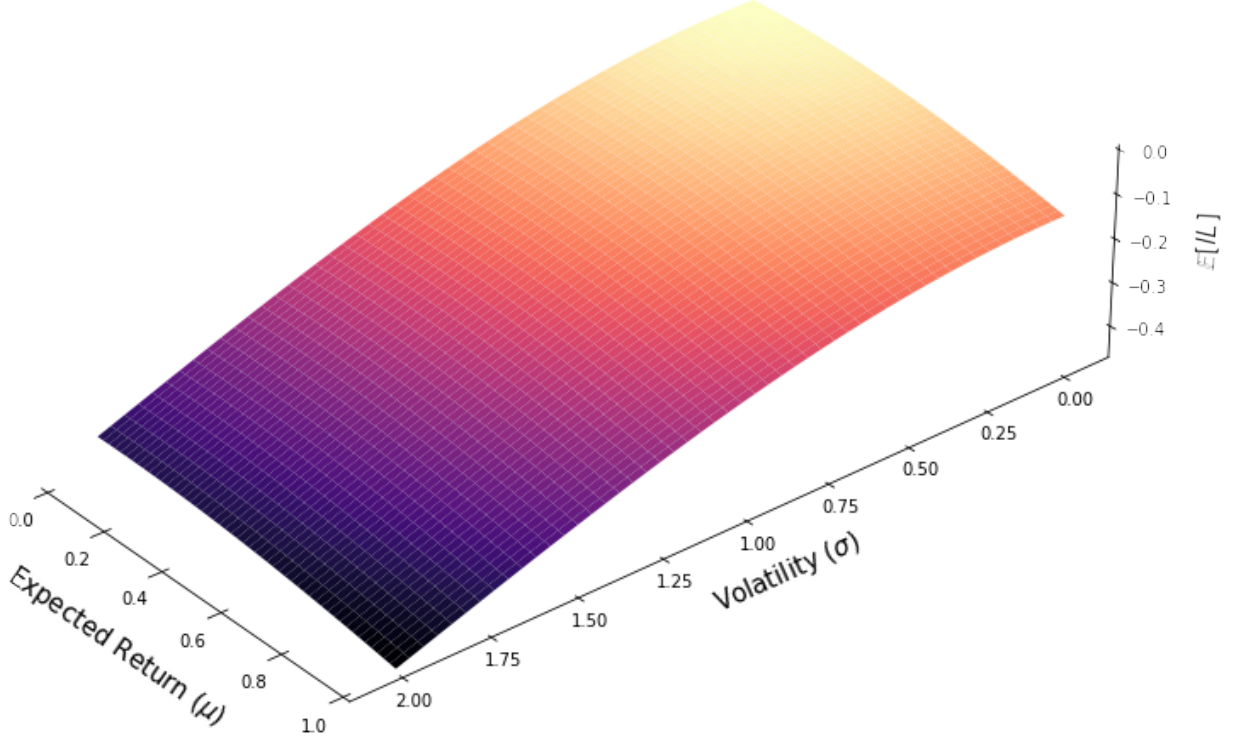
With the formulas to compute IL on Uniswap v3 as seen on [11]. Define the range factor $r = \sqrt{\frac{p_b}{p_a}}$, then:

$$IL_3 := \begin{cases} \frac{\sqrt{r}\cdot\alpha - 1}{\alpha+1} & \alpha < \frac{1}{r} \\ \frac{\sqrt{r}}{\sqrt{r}-1}\cdot\left(\frac{2\sqrt{\alpha}}{\alpha+1}-1\right) & \frac{1}{r} < \alpha < r \\ \frac{\sqrt{r}-\alpha}{\alpha+1} & \alpha > r \end{cases} \tag{44}$$

Additionally, we know that:

$$\lim_{r\to+\infty} IL_3 = IL_2 \tag{45}$$

If a non-trivial range is used, the top and bottom formulas on (44) become relevant. Due to the assumption of GBM, it would be necessary to integrate the normal distribution curve over all three separate price ranges. This would result in a very complex equation, as none of the integrals resolves to 1.

For this reason, computational methods are used to find the expected value of impermanent loss for Uniswap v3 pool positions that use a non-trivial price range.

Using a modification of the code provided by the author in [7] I have numerically recreated an $\mathbb{E}[IL_3]$ *Surface*

for $r = 0.5$ ($\pm 50\%$) as a function of μ and σ:
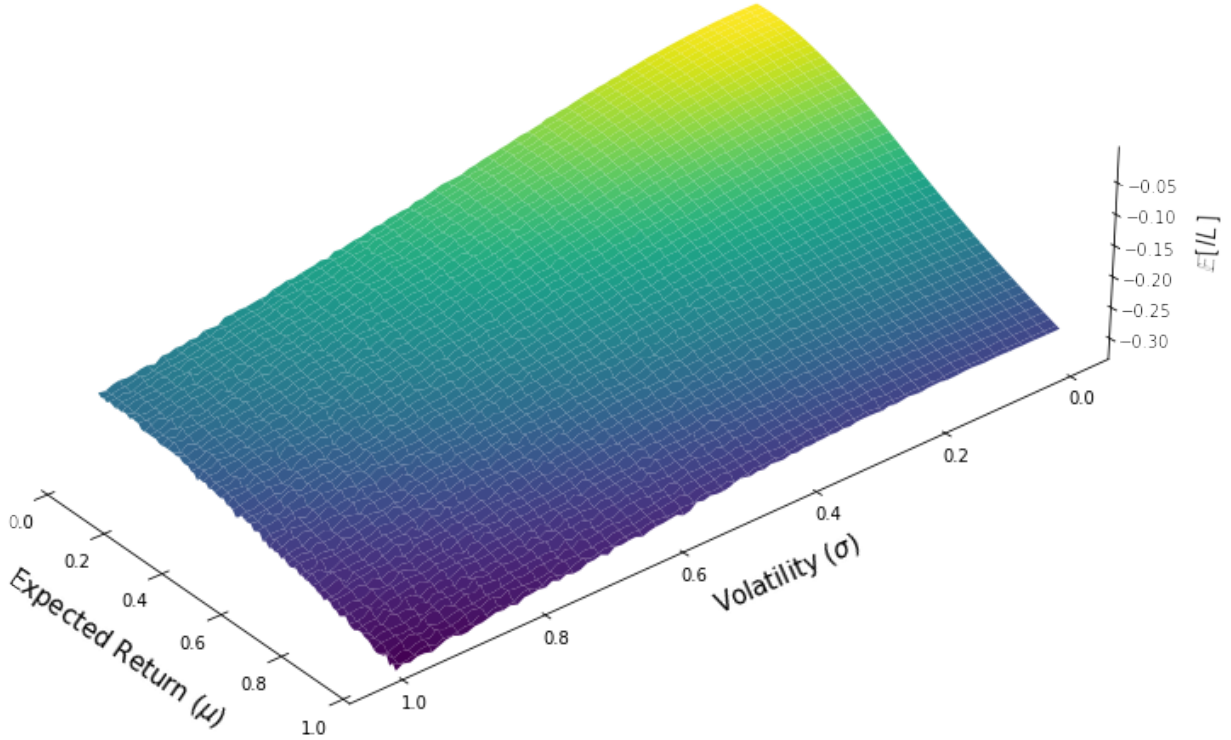
4.5 Implied Volatility

This section is mainly extracted from [5].

Under the *no-arbitrage principle*, there is no strategy between becoming an LP or just holding assets that inherently has positive EV. If the yields from a pool were much higher than the impermanent loss, providing liquidity would have a positive expected return, therefore arbitrageurs would take advantage of a statistical arbitrage and bring back EV to 0. If the reverse was true, the expected impermanent loss was higher than the yields from trading fees, the impermanent loss normally faced by LPs would become that borrower's impermanent gain.

Therefore, if we are to assume that there is no arbitrage opportunity in Uniswap, then the volatility the Uniswap trading fee yields imply is the expected return of the contract, and in this case that the return from trading fees and impermanent loss is equivalent to zero. At this level of volatility there is no strategy, providing liquidity or borrowing liquidity, that can generate an expected positive return.

Then, in order to find the *Implied Volatility* (σ) we assume the expected impermanent loss (43) with trading fees of the liquidity pool to be equal to zero and solve for σ:

$$0 = \frac{e^{\left(\alpha - \frac{\mu_{pool}^2}{8}\right)t}}{cosh\left(\frac{\mu t}{2}\right)} - 1$$

$$\boxed{\sigma = \sqrt{\frac{8}{t}\left[\mu_{pool} \cdot t - ln\left(cosh\left(\frac{\mu t}{2}\right)\right)\right]}} \tag{46}$$

For very short periods of time, where μ can assumed to be 0:

$$\sigma = \sqrt{8\mu_{pool}}$$

With a reasonably good assumption of μ, the expected price movement of a pair over a certain period of time, we can estimate what the `Uniswap v2` pool expected volatility for a given pair is based on that pair's current yield from trading fees.
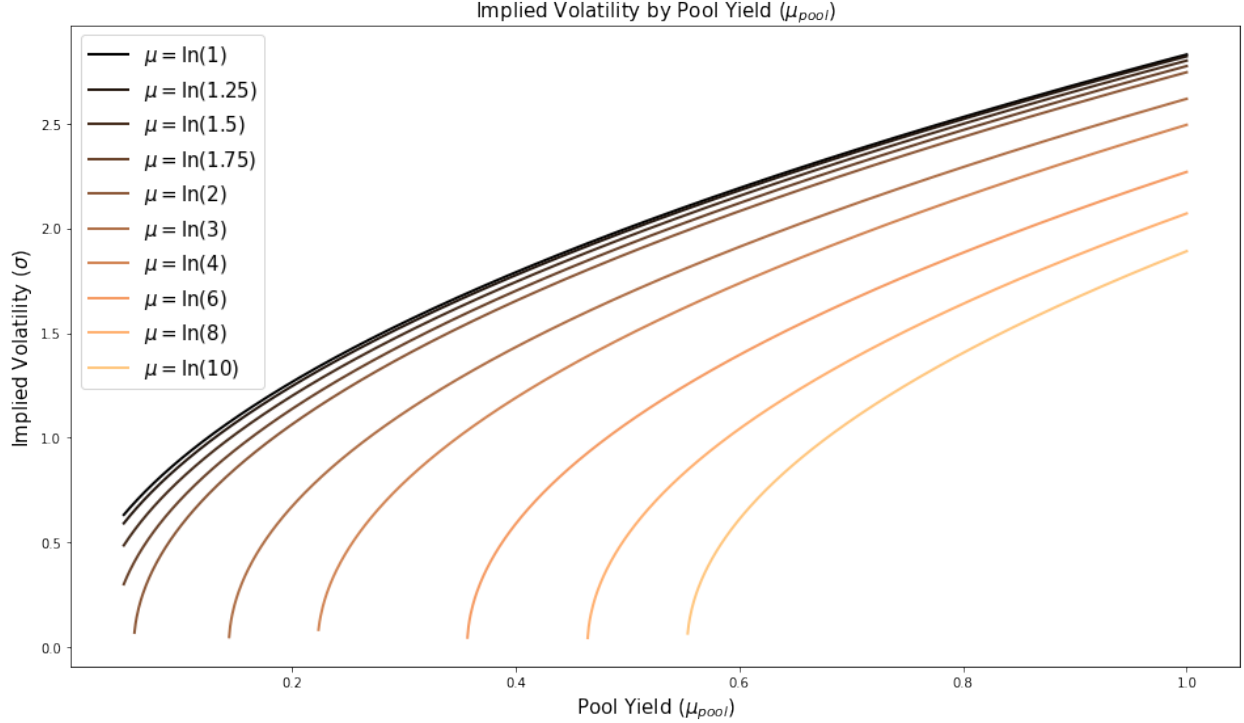
For concentrated liquidity ranges, the issue is the same as in the previous section, analytical solutions can get complicated, therefore is preferred to use numerical analysis to find the solutions to some of this equations.

An example follows with the *Implied Volatility* for a range $r = 0.5$ as a function of μ and α. A small modification of the author code's has been made: `k_=midSigma`, `i_=hiSigma`, `j_=loSigma` in order for the complete provided code to run.

IV(α) - μ = 20%, r = 5%

- calcIV(*range*, μ, r)
- Linear regression (y = 0.0042x + 1.5327)

Linear: 0.0663
Logarithmic: 0.2787
Square Root: 0.0776

IV (σ)

α (%)
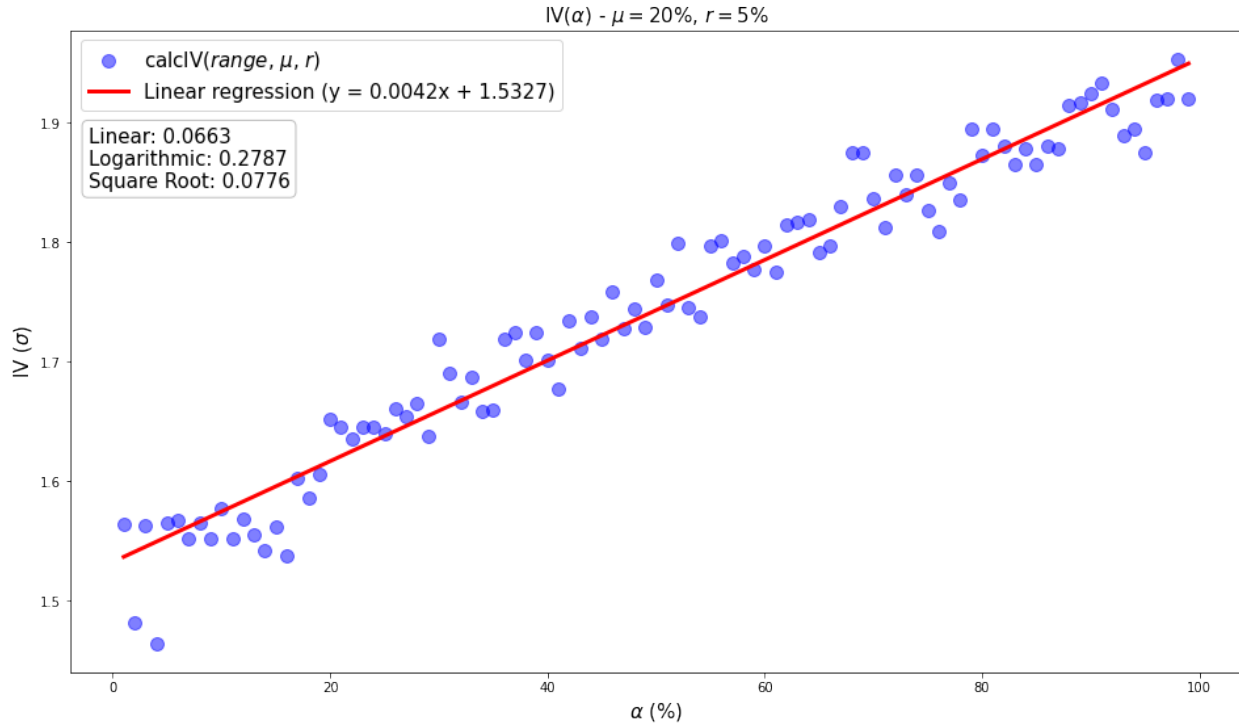
4.6   Volatility Risk Premium: Are LPs underpaid?

> Whenever **implied volatility is higher than historical** or expected volatility, **providing liquidity** to a Uniswap pool provides **positive expected return**.
>
> If the **implied volatility is lower than the historical** or expected volatility, the **return becomes negative**.

Therefore, implied volatility relative to whatever metric of expected volatility we use is a better determinate of EV than pool yield alone. In most cases, **Uniswap does not pay LPs an adequate risk adjusted return** [9].

UniV2 USDC-ETH IV vs Deribit IV vs Historical Volatility
Assuming ETH/USD Historical Avg Annual Return of 200%
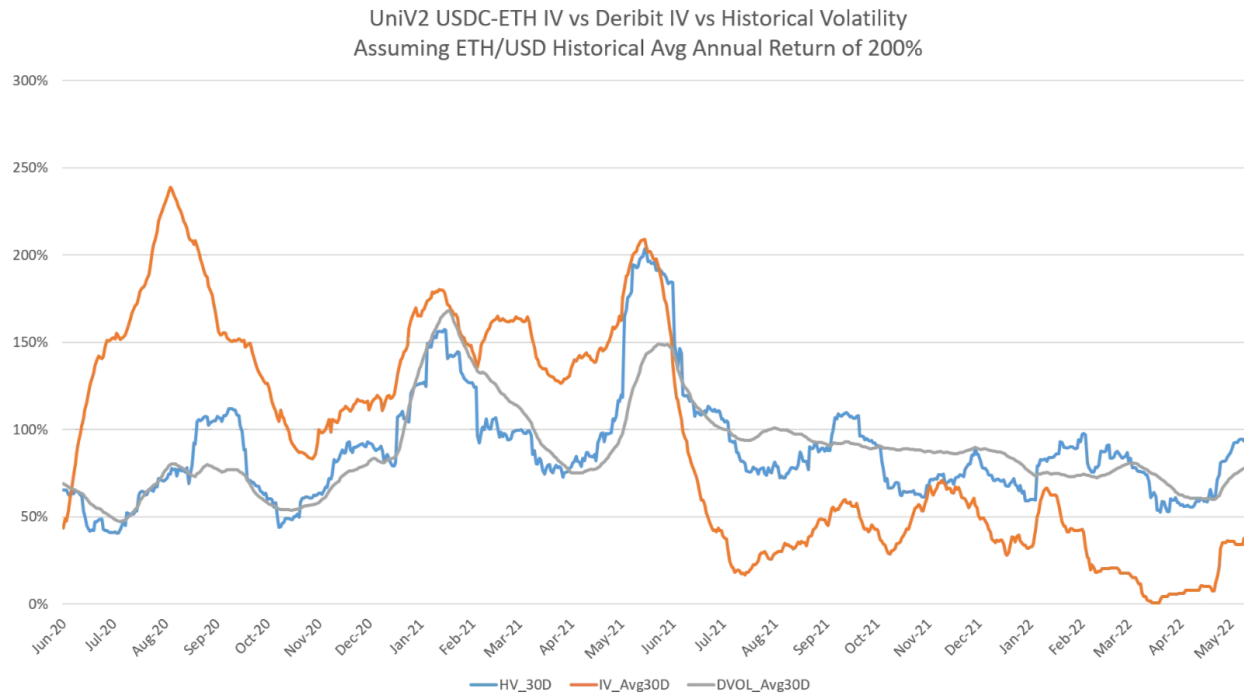
Figure extracted from [9]

The yield was worth the risk from June 2020 to the first half of 2021 but not after, assuming the annual return for this whole period of time is about 200%. If it is no longer positive EV to hold tokens in Uniswap, or at the very least less profitable than alternatives, then **why are LPs still providing liquidity?**

Liquidity providers (LPs) may continue to provide liquidity on Uniswap despite it no longer being as profitable due to revised market expectations, particularly surrounding ethereum's price appreciation against the US dollar. As ethereum's network grows, its volatility and risk diminish, leading to lower expected annual returns, now around 50% instead of 100%. Although historical performance, especially during the bull market of 2020-2021, showed exaggerated returns, LPs might believe that the market for volatility is fairly priced and expect lower, but still profitable, returns going forward. Some volatility discrepancies after mid-2021 may further reflect market adjustments (or LPs could just be uninformed - I doubt it).

Returns from Liquidity Provision (Fees - Impermanent Loss) on
UniswapV2 USDC-ETH Pool
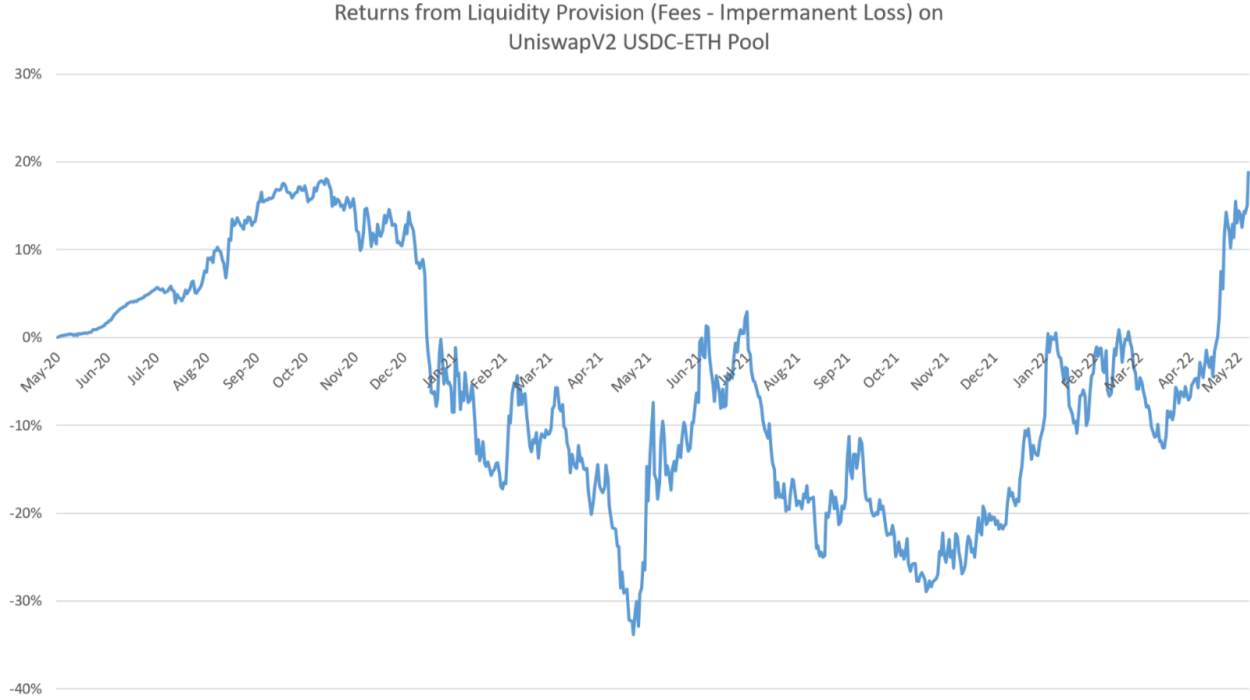
Figure extracted from [9]

Performance Statistic	Value
Mean Annual Return	8.54%
Std of Annual Return	35.81%
Total Return	18.82%
Sharpe Ratio	0.24
Sortino Ratio	0.36

Statistics of LP portfolio:
May 2020 to May 2022

Given the large drawdown the liquidity provider (LP) experienced, the risk adjusted returns are relatively low. The fact that the LP is still profitable is not statistically significant from random luck.

There was a liquidity mining program to issue `UNI` (Uniswap governance token) tokens, that ran from *Sep. 2020* to *Nov. 2020*, to LPs in some Uniswap V2 pools. `USDC-ETH` and `WBTC-ETH` pools were part of it but not the `WBTC-USDC` pool. Perhaps that's the reason why the `USDC-ETH` and `WBTC-ETH` pools were able to perform so well around that time period.

Unlike Deribit where a low price of volatility will attract buyers of volatility, Uniswap and other AMMs are a one sided volatility marketplace only enabling users to sell volatility. Therefore the volatility market in AMMs cannot function properly and adjust itself rapidly enough when volatility is cheap, unlike Deribit, without first causing losses to liquidity providers.

CFMM liquidity can be shorted by borrowing liquidity from the CFMM and withdrawing the reserve tokens it represents and holding them, such that the impermanent loss becomes an *impermanent gain*. The *impermanent gain* is the ability of the borrower to purchase more LP tokens with their reserve tokens.

Since the impermanent loss to LPs becomes the impermanent gain of the borrower, the borrower will have

achieved a long gamma position whose return can be characterized by the following function:

$$R = 1 - \frac{2\sqrt{\alpha} \cdot e^{(\mu_{pool}+g_b-r)t}}{1+\alpha} \tag{47}$$

In this example, g_b represents the interest rate charged by `GammaSwap` to liquidity borrowers. `GammaSwap` is the protocol whose founder inspired this section and author of the original article it is based on.
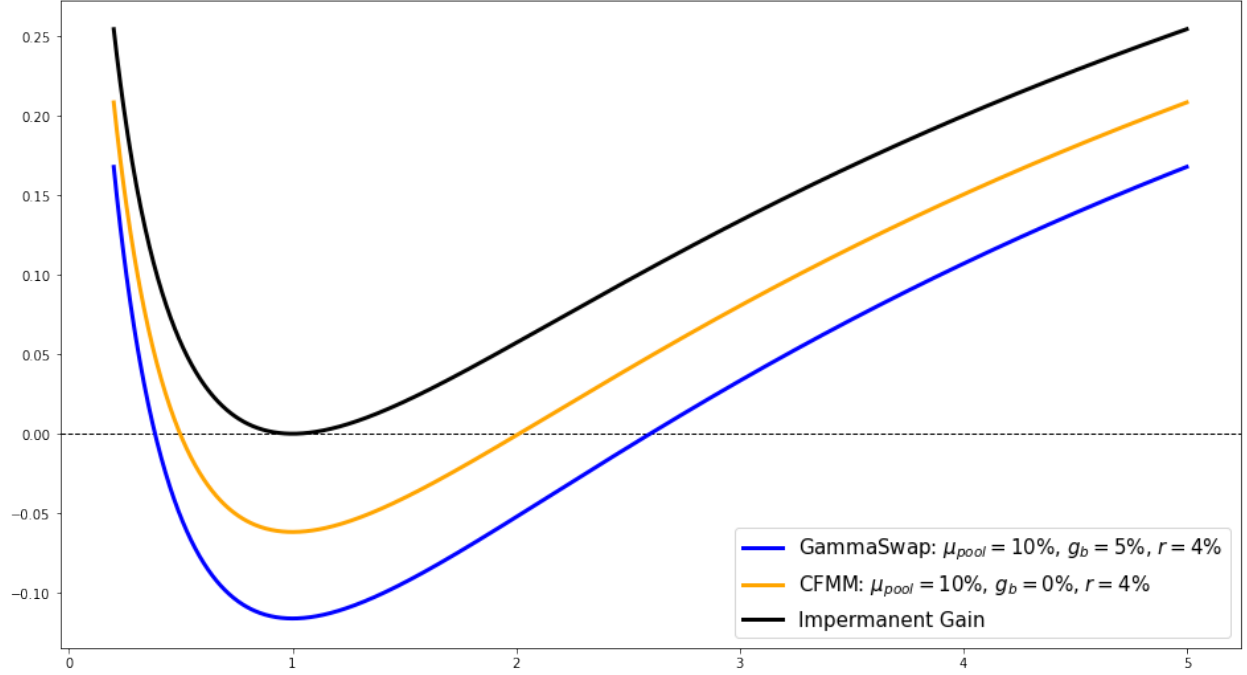
4.7   IL as a Stochastic Process of a G3M

The text below is entirely extracted from [19].

Given a portfolio P of fixed weights $(w_1, \ldots, w_n)$, which is of dynamic:

- $P_0 = 1$

- $\frac{dP}{dt} = \langle w | \frac{dS}{S} \rangle, \quad P_t = \exp\left(\int_0^t (\langle w|\mu\rangle - \frac{1}{2}\langle w|\Sigma w\rangle)dt + \int_0^t \langle w|\sigma \odot dW_t\rangle \right)$

- $V := \prod_i S_i^{w_i} = \exp\left(\int_0^t (\langle w|\mu\rangle - \frac{1}{2}\langle w|\sigma^2\rangle)dt + \int_0^t \langle w|\sigma \odot dW_t\rangle \right)$

And as such obtain:

$$V = P \cdot \exp\left(\frac{1}{2} \int_0^t (\langle w|\Sigma w\rangle) - \langle w|\sigma^2\rangle)dt \right)$$

As $w \in \Omega := \{w \in \mathbb{R}^n | w \geq 0, \langle w|1\rangle = 1\}$ and $f : w \mapsto \langle w|\Sigma w\rangle - \langle w|\sigma^2\rangle$ is a convex functional which is equal to zero on $G := \{w \in \mathbb{R}^n | \exists j \in [[1,n]], w_i = \delta_{i,j} \forall i\}$, we recall that Ω is the convex closure of G and obtain by direct application of the *Jensen Theorem* that $f \leq 0$ on Ω, i.e. that V is doomed to underperform P.

Define the Impermanent Loss (IL) process of the G3M as:

$$\mathcal{I}_t := \exp\left(\frac{1}{2}\int_0^t (\langle w|\Sigma w\rangle - \langle w|\sigma^2\rangle)dt\right)$$

Its negative trend is essentially proportional to Σ and this suggests that the G3M strategy is essentially '*selling gamma*' in option-trading language against the prospect of accruing transaction fees.

Actually the underperformance of V relative to the constant-mix portfolio P, and referring to the *Stochastic Portfolio Theory* developed by *Fernholz*, we further notice that f is the direct opposite of the excess growth rate provided by diversification in *Fernholz'* framework. In other words, investing in a G3M negates the boost provided by diversification in terms of portfolio growth rate. Indeed, one advantage of diversification is that the portfolio's growth rate is superior to the combination of its component stock growth: diversification in itself boosts portfolio returns. Alas, this is not the case for G3M strategies.

4.8   IL on v3 & v4: Concentrated Liquidity in $[a, b] \subseteq (0, +\infty)$ ranges

When providing liquidity in a custom fixed range one can optimize its capital efficiency up to x4000. But as one should know already, with no free lunch laying around on the market, that comes at a cost. This cost is materialised through an increase in the IL.

4.9   Loss Versus Rebalancing (LVR)

4.10 $\mathbb{E}[\mathbf{IL}] == \mathbb{E}[\mathbf{LVR}]$

5 Shares Math

5.1 Minting: Providing Liquidity

5.2 Burning: Withdrawing Liquidity

6 On LP Profitability

6.1 JIT, MEV and General LP PnL

Study on short/mid/long term profitability of LPs vs JIT bots. (Stefan et al., replicate from scratch with indexer to have up to date data)

6.2 Dynamic Fees Hooks

As it has been shown LPs rely on fees to offset IL.

Main debate seems to be doing it fully on-chain, oracle-free and saving gas costs. Main ideas extracted from [4]:

- **Volatility-based approach:** tie the swap fee to the historical volatility of the asset pair. The volatility can be tracked either by the pool itself, or by an external oracle. (Used by Trader Joe Liquidity Book, Algebra, Hypersea and many others.)

- **Lookback approach:** It requires a number of external pool with the same asset pair, but different fee tiers. The pool should be highly liquid, for example, they can be the mainnet Uniswap v3 pools. The idea is to set the fee same as the fee of the external pool most profitable in the lookback period. (Used by Ambient Finance.)

- **Price oracle approach:** use an external price oracle to determine the correct price of the asset. A higher fee is asked if the trade moves the pool price closer to this external price. (Used by Swaap.)

- **Price momentum approach:** look at the price history in the most recent block(s) and asymmetrically change the fee depending on the trade direction. The aim is to infer the dynamics of the external price without using an oracle. (Implemented as a Uniswap v4 hook.)

- **Asset composition approach:** lower fee for trades that make the pool more balanced, higher for trades that imbalance it. (Used in some Curve pools.)

- **Transaction-source based approach:** provide lower fee for transactions that are routed through aggregators, or are otherwise unlikely to be arbitrage-traded. (Example: Balancer has lower fees for trades coming from CowSwap solvers.)

Other ideas, such as using the swap size, the source address of the transaction, the transaction history of the source address, etc.

7 Hedging Strategies

To safeguard some of LPs capital, hedging strategies can be used. For simplicity sake we will be working with stable-volatile pairs such as `ETH/USDC`.

Hedging can fully be done on-chain via borrowing, however due to undercolateralisation this becomes more capital intensive than using margin. Additionally, borrowing on DeFi usually comes with a positive APR (%) that needs to be payed in order to hold the borrow. Meanwhile, because of funding fees, margin selling can have (this is not always true) a negative APR, therefore adding some extra yield on top of the fees. Liquidation risks are shared among strategies.

7.1 Linear and Non-Linear Payoffs

The main idea behind the hedge is to understand how the value of the position will behave with respect to our derivative of choice for the hedge.
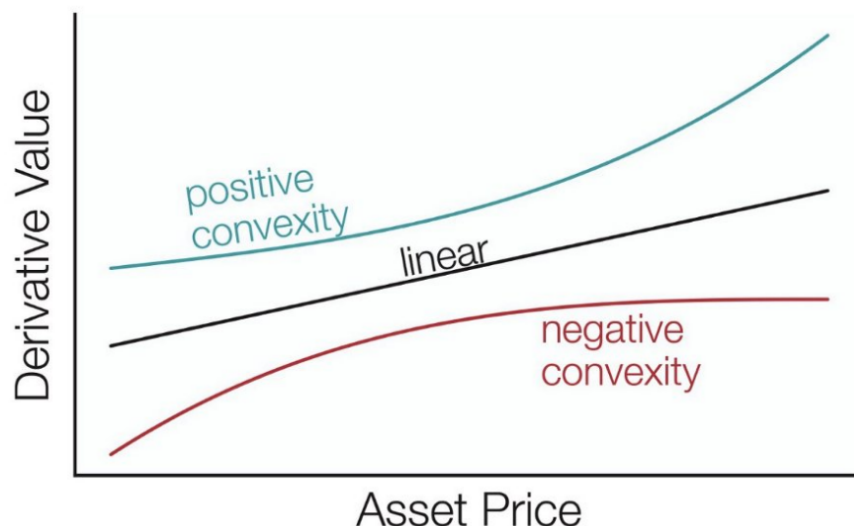
The payoff of a HODL position follows a linear pattern. In a literal sense, the value function resembles a straight line. This remains true for all HODL portfolios:

- The payoff of holding only the stable coins is a horizontal line,

- The payoff of `50:50 HODL` portfolio is the $y = \frac{x}{2} + c$; $c\ constant$ line,

- The payoff of 100% volatile asset is the $y = 1 \cdot x$ line

Longing the volatile asset through 10x leverage trading has y=10·x payoff and shorting it with 10x leverage has y= - 10·x payoff (up to the point of liquidation, and assuming a zero funding rate for simplicity).

In contrast, the payoff of an LP position is nonlinear:

- The value of a full range LP position, Uniswap v2 style, grows with the square root of the price.

- Uniswap v3 position values and payoffs are more tricky to describe mathematically, but the basic intuition is that concentrated liquidity makes all things more extreme, including the non-linearity of the LP value and payoff functions.

Position Payoff
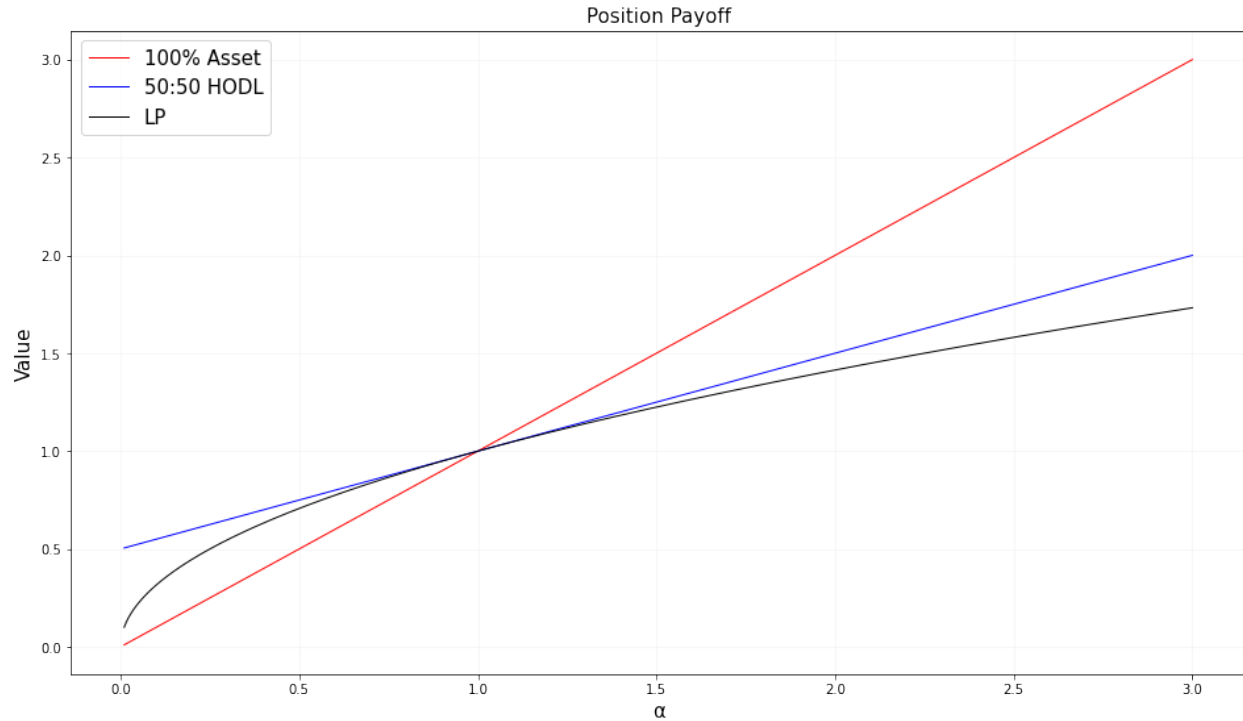
When a linear function is added to a nonlinear one, the result is always another nonlinear function, similar to how multiplying a negative number by a positive one always yields a negative number.

It means that LP positions cannot be perfectly hedged by just longing or shorting the volatile asset once.

It's the special kind of non-linearity called **concavity** or **negative convexity**, also characterized with **negative gamma**, that is **bad for LPs**.

This kind of non-linearity is present in all reasonable AMMs (as mathematically demonstrated in [17]), not just the Uniswap's xy=k type of AMM. Therefore it's usually fine to use the mental shortcuts "linear = beneficial for LPs" and "nonlinear = detrimental for LPs in this context.

There are AMMs with bonding curves that are close to a line for some values of the price. Curve is the most well known of these. However, while Curve offers nice properties for stable swaps, it has higher slippage and risks for LPs at the extremes of the bonding curve, when the price deviate s too much from the stable region. Essentially, Curve reduces the likelihood of losses, but increases the severity of the losses if they do occur.
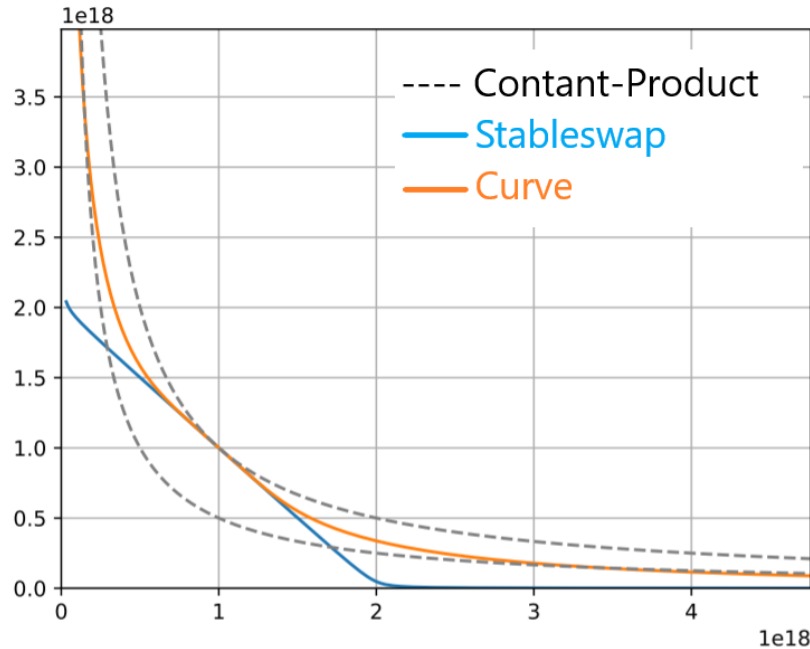
7.2 What can LPs do?

To counteract the negative gamma exposure that LPs face, several methods can be employed.

1. **Maximize fee income**. LPs are profitable relative to HODL when fee income is greater than the divergence loss. Select better pools, better fee tier and explore IV or Sharpe Ratio metrics for the pool.

2. Explore **less volatile pairs** as expected value of the impermanent loss is function of volatility squared.

3. **Hedge** the position by buying financial instruments that have **positive gamma**, such as buying options, power perpetuals or protocols like *Smilee Finance* or *GammaSwap*.

4. **Dynamically re-hedge** the position when ever price changes (the main goal of our hook is to facilitate this step).

7.3 Dynamic Re-Hedging with Delta One products

No exotic financial instruments are required to hedge LP positions with an **almost** perfect accuracy.

The key concept here is that, for relatively small price movements, gamma (the change in delta) is small and therefore convexity does not play a big role in our loss and a linear approximation with a Delta one product might suffice. Of course, for large price movements the hedge would be completely out-of-sync with the assets in the pool and therefore turn useless.

The main problem using this approach is that, of course, to re-hedge some trading action is required, either on-chain or off-chain and that carries some costs like fees, slippage, gas or borrowing APR's (although they are sometimes off-set by lended collateral ones). On the other hand, using this method *Divergence Loss* can be made as small as desired. The more experience reader might find this to be an optimization problem, with a trade-off between rebalance frequency (as a function of trading costs) and the negative convexity loss.

7.4 On-Chain Borrowing

7.4.1 Borrowing Basics

> **DeFi Concepts - Borrowing:**
>
> The **Loan to Value (LTV)** ratio defines the maximum amount of assets that can be borrowed with a specific collateral. It is expressed as a percentage (e.g., at LTV=75%, for every 1 ETH worth of collateral, borrowers will be able to borrow 0.75 ETH worth of the corresponding currency). Once a borrow occurs, the LTV evolves with market conditions.
>
> The **Liquidation Threshold** is the percentage at which a position is defined as undercollateralised. For example, a Liquidation threshold of 80% means that if the value rises above 80% of the collateral, the position is undercollateralised and could be liquidated. The difference between the LTV and the Liquidation Threshold is a safety mechanism in place for borrowers.
>
> When the **Health Factor** $H_f < 1$ the position may be liquidated to maintain solvency.

`Bob` deposits `120 USDC` of collateral at P_0=\$100/ETH and borrows `0.825 ETH` worth \$82.5 from a pool with a *liquidation threshold* of 86%. This means that if the value of his *collateral* (V^c) is less than $0.825 \cdot V^b$ (with V^b the value of his *borrowed assets*) he will get liquidated, equivalently, his $H_f < 1$.
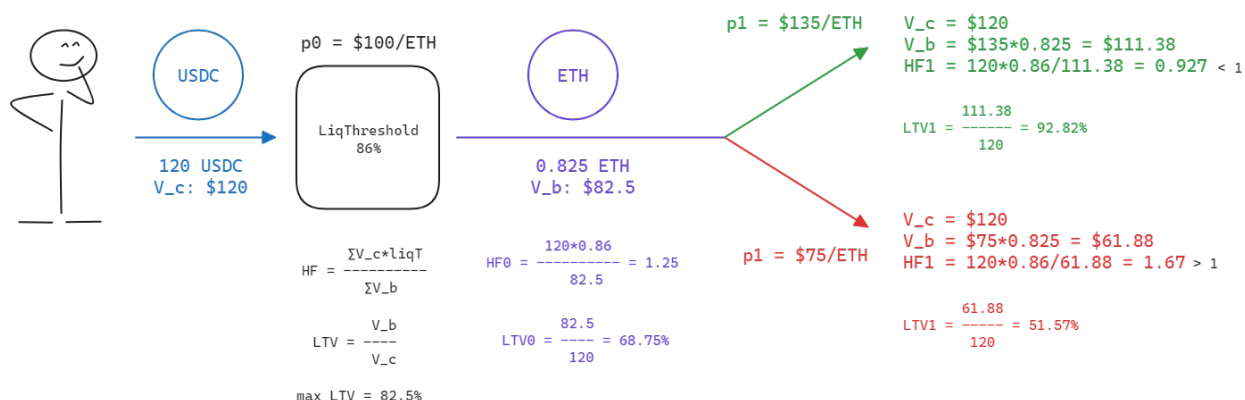
We see that, if `ETH` price increases, he will either have to provide more collateral, repay a part of his debt or get liquidated. If the price decreases however, the value of his collateral with respect to his borrowings has increased. Nevertheless, this does not imply he has made a profit. He still owes `0.825 ETH` to the pool. So, if at t_0 he did not sell the borrowed assets, at t_1 he has no profit in terms of `USDC`.

This is why some people do not consider borrowing an equivalent to shorting and somewhat closer to a neutral strategy depending on the actions taken by the borrower.

7.4.2 Example: Volatile/Stable Pair (ETH/USDC)

`Alice` has \$5000 USDC and wants to deploy them to Uniswap v2 (or v3 in full range) hedging with a borrowing strategy in Aave (we will ignore gas fees and suppose borrowing APR is compensated by lending APR which is not an unreasonable assumption).

This is what she will do:

1. Lend (deposit) $4000 USDC into Aave.

2. Borrow 1 ETH ($1000) from Aave.

3. Provide 50:50 liquidity full-range (1 ETH & $1000 USDC).
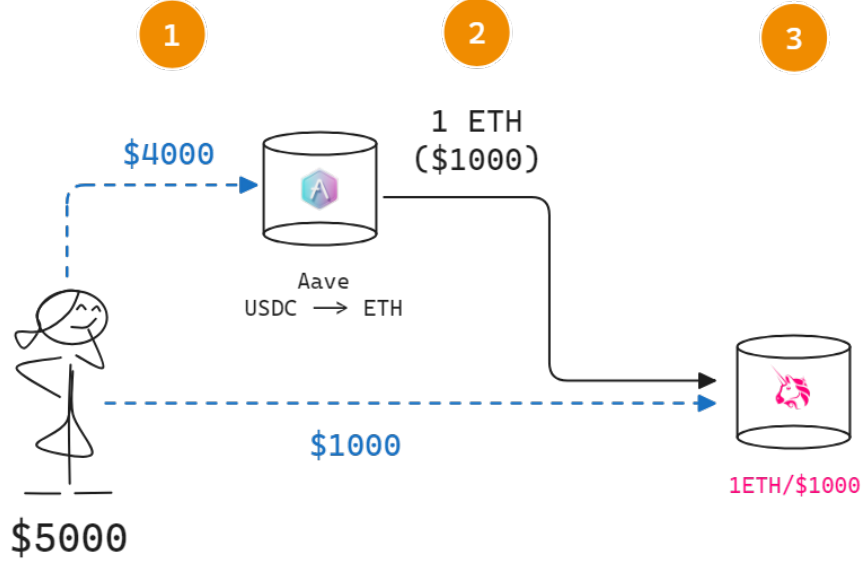
The value of her capital can be computed as:

$$
\begin{cases}
V_{collateral} = \$4000 \\
V_{debt} = \$1000 \\
V_{pool} = \$2000 \\
V_{hedge} = \$4000 - \$1000 = \$3000 \\
V_{capital} = V_{pool} + V_{hedge} = \$2000 + \$3000 = \$5000
\end{cases}
\tag{48}
$$

As expected. Then imagine `ETH` price doubles (1 `ETH` = $2000 USDC). We can now compute the new value of her capital using 16 to calculate V_{pool}:

$$
\begin{cases}
V_{collateral} = \$4000 \\
V_{debt} = \$2000 \\
V_{pool} = \$2000 \cdot \sqrt{2} \\
V_{hedge} = \$4000 - \$2000 = \$2000 \\
V_{capital} = V_{pool} + V_{hedge} = \$2000 \cdot \sqrt{2} + \$2000 = \$2000(\sqrt{2} + 1) \approx \$4828.43
\end{cases}
$$

Due to the increase in `ETH` price Alice has suffered a $\frac{4828.43}{5000} - 1 = -3.43\%$ loss.

Now meet Bob. Bob has the same initial capital and strategy than Alice. However, when he sees the price increase to $1500 he decides to rebalance the hedge. He observes that by AMM mechanics, he now holds less ETH than before (< 1 `ETH`). Therefore, as he had borrowed 1 `ETH`, he decides to repay some of the borrowed ETH. To do it, he removes some collateral `USDC` from *Aave*, swaps it for `ETH` (assuming 0 cost) and then repays the `ETH` to ensure that $borrowed_{ETH} = LP_{ETH}$.

Now when `ETH` price reaches \$2000 Bob has still a loss but smaller than Alice's.

More formally, if P_0 denotes the initial price, P the current price, α the ratio between them as defined in 15, V_0 the initial value of the LP or `HODL` position. For different `HODL` portfolios we have that (24):

$$
\begin{cases}
V_H^{100\%} = V_0 \cdot \alpha \\[2mm]
V_H^{50\%} = \frac{V_0(1+\alpha)}{2} \\[2mm]
V_H^{0\%} = V_0
\end{cases}
\tag{49}
$$

Full range LP position value is (16):

$$
V_{LP}(\alpha) = V_0 \cdot \sqrt{\alpha}
$$

By using a hedge as seen on 48, the LP position value becomes:

$$
V_{portfolio} = V_{LP}(\alpha) + V_{hedge}(\alpha)
\tag{50}
$$

We know Divergence (Impermanent) Loss is then defined by 26

$$
IL(\alpha) = DL(\alpha) = \frac{2\sqrt{\alpha}}{(1+\alpha)} - 1
$$

We can now decompose the value of the LP position using the `HODL` position's value and the divergence loss term:

$$
V_{LP}(\alpha) = V_H^{50\%}(\alpha) + V_H^{50\%}(\alpha) \cdot DL(\alpha)
\tag{51}
$$

This can be checked substituting the corresponding values from 49, 26 and 16:

$$
\begin{aligned}
V_{LP}(\alpha) &= V_H^{50\%}(\alpha) + V_H^{50\%}(\alpha) \cdot DL(\alpha) \\[2mm]
&= V_H^{50\%}(\alpha) \cdot (1 + DL(\alpha)) \\[2mm]
&= \frac{V_0 \cdot (1+\alpha)}{2} \cdot \left(\cancel{1} + \frac{2\sqrt{\alpha}}{1+\alpha} - \cancel{1} \right) \\[2mm]
&= \frac{V_0 \cdot \cancel{(1+\alpha)}}{2} \cdot \left(\frac{2\sqrt{\alpha}}{\cancel{1+\alpha}} \right) \\[2mm]
&= \frac{V_0}{\cancel{2}} \cdot (\cancel{2}\sqrt{\alpha}) \\[2mm]
&= V_0\sqrt{\alpha} \quad \square
\end{aligned}
$$

If we wished to obtain a delta-neutral portfolio then our hedge should be constructed so that it has inverse payoff to the `50:50 HODL` porfolio:

$$
V_{hedge}(\alpha_h) := V_0 - V_H^{50\%}(\alpha_h)
\tag{52}
$$

Where α_h denotes the price ratio at which the hedge portfolio was constructed. If $\alpha = \alpha_h$ then this gives a truly delta-neutral combined portfolio. This is, small variations in the volatile asset's price do not have a significant impact on the value of the portfolio.

$$V_{portfolio}(\alpha) = V_{LP}(\alpha) + V_{hedge}(\alpha) - H_c$$

$$= V_H^{50\%}(\alpha) + V_H^{50\%}(\alpha) \cdot DL(\alpha) + V_0 - V_H^{50\%}(\alpha) - H_c$$

$$= \cancel{V_H^{50\%}(\alpha)} + V_H^{50\%}(\alpha) \cdot DL(\alpha) + V_0 - \cancel{V_H^{50\%}(\alpha)} - H_c \qquad (53)$$

$$= V_0 + V_H^{50\%}(\alpha) \cdot DL(\alpha) - H_c$$

Where H_c are the *hedging costs* involved in performing the hedge (fees, slippage, gas, etc.). Additionally, notice that the term $V_H^{50\%}(\alpha) \cdot DL(\alpha) \leq 0$ describes the *loss due to price change*. If $\alpha = 1$ then this term is 0 (good). When the price change is small, this term is close to 0 as it's a non-linear function of the price change.

The main idea is to rebalance the hedge every time α grows or decreases above some fixed threshold called *rebalancing step*. The more frequent the rebalances the higher the *hedging costs* but the loss term can be made as small as possible.

7.4.3 Full range: Returns & Achieving Delta-Neutrality

This section is inspired by [6], although some notation changes have been made to keep formulas consistent with this document and new plots and comparissons have been added to better illustrate some points.

Given the value for an lp position from 14 and adding the return μ from fees (see [6]):

$$V_{LP} = 2L\sqrt{P\alpha} \cdot e^{\mu t}$$

This function is determined by the price P. Therefore, to hedge it we must take the opposite position, we must short asset X. We can achieve this by borrowing asset X using asset Y as collateral. The hedge portfolio becomes:

$$V_{LP} = 2L\sqrt{P\alpha} \cdot e^{\mu t} - xP\alpha \cdot e^{rt}\theta + y \cdot \theta$$

With θ a number that represents the percentage of the liquidity position that is being shorted. Usually $\theta \in [0,1)$, however it can be greater than 1 if we seek to obtain a net short position.

r being the interest rate on the loan per units of time t. Many times this cost will be negligible as it (approximately) cancels out with the APY from lending the other asset provided as collateral.

The sign next to x is negative as we are borrowing (*short*, as we will be providing it as liquidity) that asset and selling it for y (*long*). We are short x in terms of y. However, as we have just seen (7.4.1) we need to overcollateralise some of the funds when borrowing as margin.

We denote the LTV by $\varphi = \frac{V_b}{V_c}$; $\varphi \in (0,1)$. Additionally, let ω represent the percentage of liquidity provided in the pool due to the LTV. That is, if we need to overcollateralise the short position, we have less capital available to provide as liquidity.

ω is not chosen by the LP but rather dependent on θ and φ.

$$V_{LP} = \omega 2L\sqrt{P\alpha} \cdot e^{\mu t} - xP\alpha \cdot e^{rt}\theta + y\frac{\theta}{\varphi} \qquad (54)$$

$$\omega = 1 - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right)$$

Normally $\omega \leq 1$. This is because as a general rule $\theta \in [0, 1)$, furthermore, θ is divided by two as we can only create a short up until half our entire capital. $\varphi \in (0, 1)$ and therefore $\frac{1}{\varphi} - 1$ (the *collateral to debt ratio* of the lending platform) will always be > 0.

In the example above, at t_0 `Bob` had an LTV $\varphi = 66.67\%$. This implies that the percentage that his collateral represents over his debt is:

$$\frac{V_c - V_b}{V_b} = \frac{V_c}{V_b} - 1 = \frac{1}{\varphi} - 1 = \frac{1}{0.6667} - 1 = 0.5$$

which is precisely the *collateral to debt ratio* term.

Therefore, if we are not hedging our liquidity position at all $\omega = 1$, otherwise $\omega < 1$. Now to conclude, using 13 we can define the hedge portfolio as:

$$V_{LP} = \omega 2L\sqrt{P\alpha} \cdot e^{\mu t} - xP\alpha \cdot e^{rt}\theta + y\frac{\theta}{\varphi}$$

$$= \omega 2L\sqrt{P\alpha} \cdot e^{\mu t} - \frac{L}{\sqrt{P}}P\alpha \cdot e^{rt}\theta + L\sqrt{P}\frac{\theta}{\varphi}$$

$$\boxed{V_{LP} = \omega 2L\sqrt{P\alpha} \cdot e^{\mu t} - \theta L\sqrt{P}\left(\alpha e^{rt} - \frac{1}{\varphi}\right)} \tag{55}$$

Now we can compute the return function of the `hedged LP` as:

$$R_{LP} = \frac{V_t - V_0}{V_0}$$

$$= \frac{V_t}{V_0} - 1$$

$$= \frac{\omega e^{\mu t} \cdot 2L\sqrt{P} \cdot \sqrt{\alpha} - \theta L\sqrt{P}(\alpha e^{rt} - \frac{1}{\varphi})}{2L\sqrt{P}} - 1$$

$$= \omega\sqrt{\alpha} \cdot e^{\mu t} - \frac{\theta}{2}(\alpha e^{rt} - \frac{1}{\varphi}) - 1$$

$$\boxed{R_{LP} = \left(1 - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right)\right)\sqrt{\alpha} \cdot e^{\mu t} - \frac{\theta}{2}\left(\alpha \cdot e^{rt} - \frac{1}{\varphi}\right) - 1} \tag{56}$$

If we set for example $\varphi = 0.85$, $\mu = 20\%$, $r = 5\%$, $t = 1$, then we can see the evolution of the portfolio as a function of θ and α:

LP Hedged Returns ($\phi = 0.85, \mu = 20\%, r = 5\%, t = 1$)
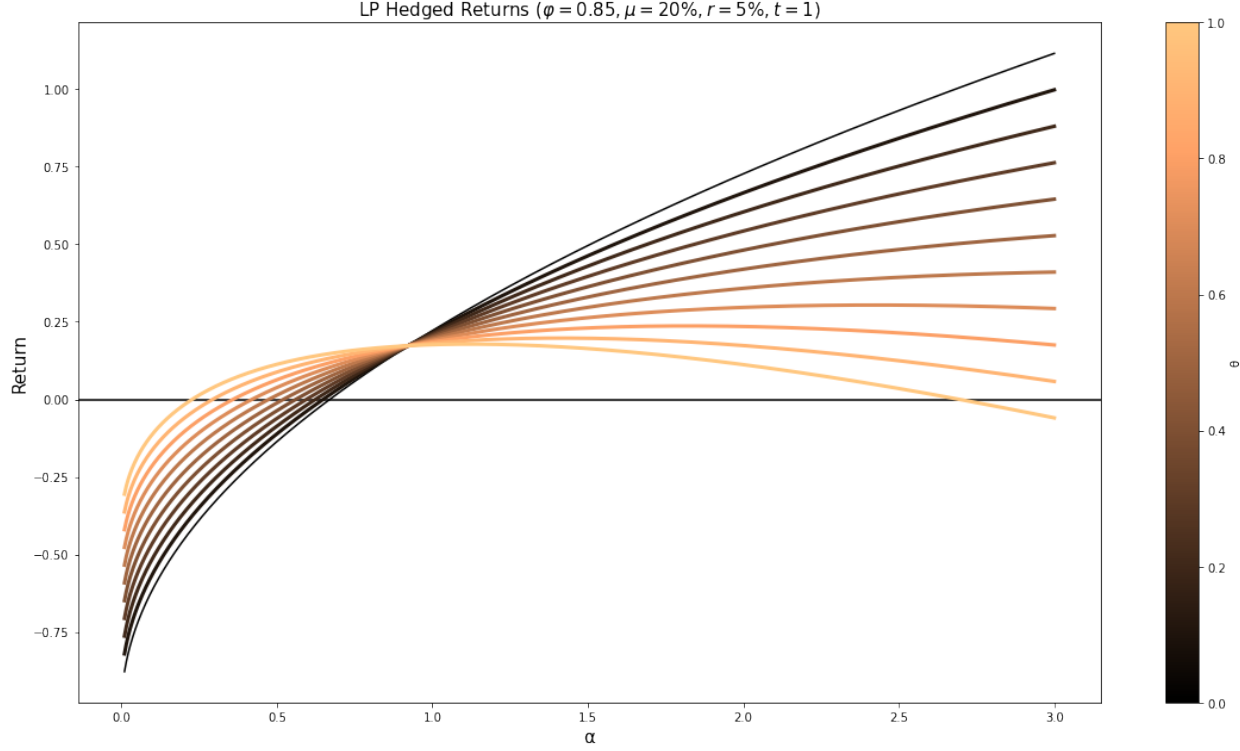
Notice that for, $\varphi = \theta = 1$, $\mu = r = 0$, $t = 1$, this formula exactly matches *The value V(P) of LP vs. HODL with $\mu = 0\%$ (on 3.2.1)*:
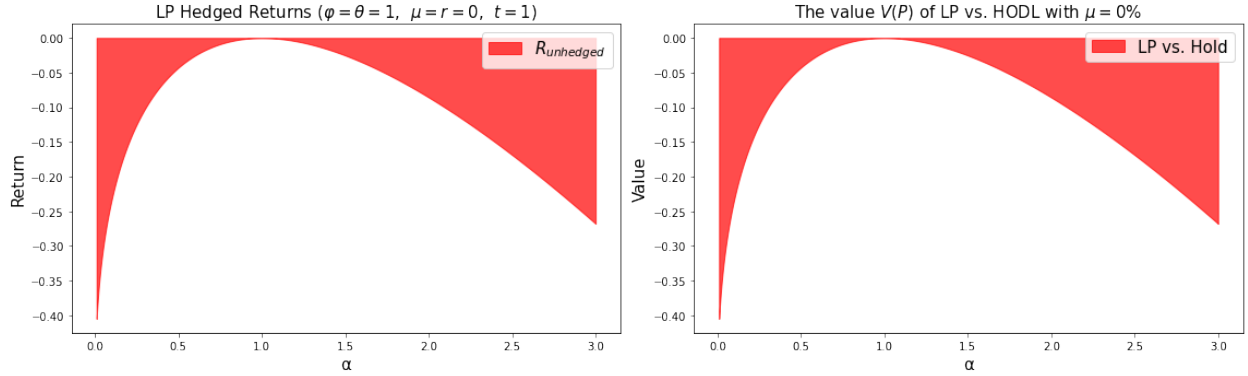
If we now study the variability our returns have against price movements, for the **unhedged** position (23), accounting for returns and setting $V_0 = 1$ we obtain:

$$\frac{\partial R_{LP}}{\partial \alpha} = \frac{e^{\mu t}}{2\sqrt{\alpha}} \tag{57}$$

If we do the same for a **hedged** position now:

$$\frac{\partial R_{LP}^h}{\partial \alpha} = \frac{e^{\mu t}}{2\sqrt{\alpha}} \cdot \omega - \frac{\theta}{2} e^{rt}$$

$$= \left(1 - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right)\right) \cdot \frac{e^{\mu t}}{2\sqrt{\alpha}} - \frac{\theta}{2} e^{rt} \tag{58}$$

48

In order to achieve Δ-neutrality, we simply equal that expression to 0, considering φ a constant given by the lending protocol, and solving for θ:

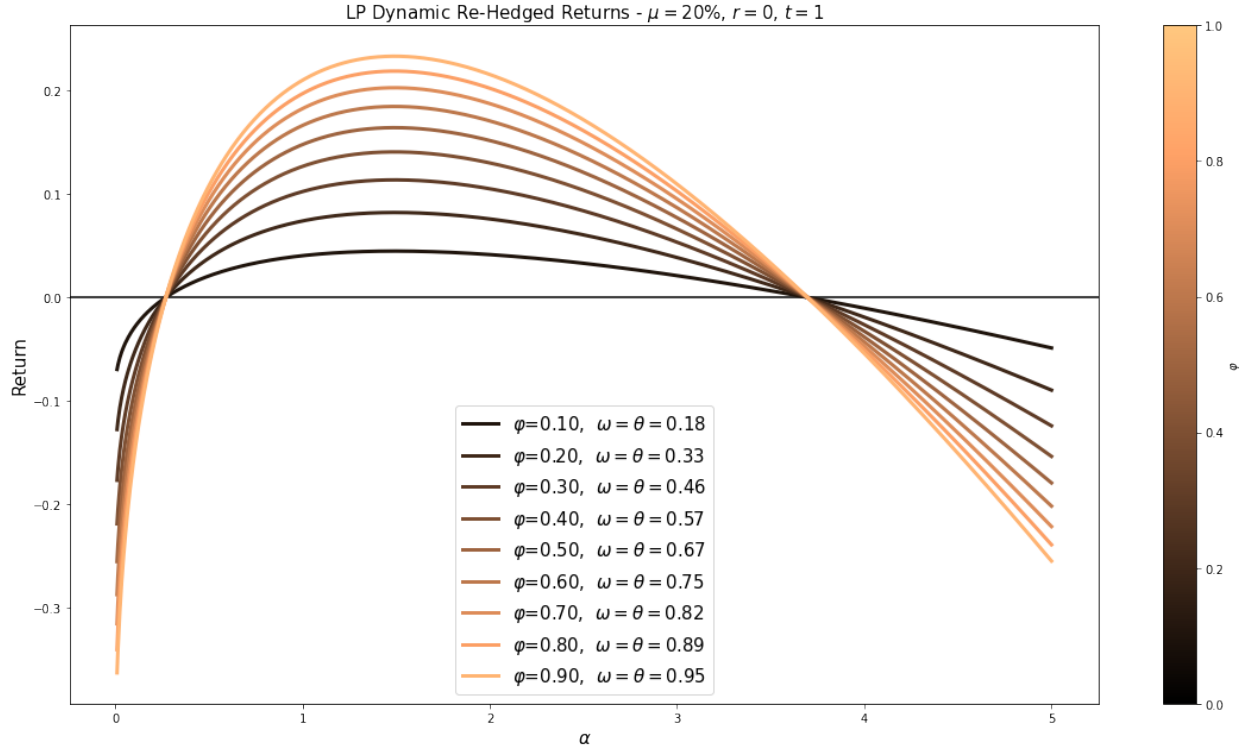$$0 = \frac{\partial R_{LP}^h}{\partial \alpha}$$

$$0 = \left(1 - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right)\right) \cdot \frac{e^{\mu t}}{2\sqrt{\alpha}} - \frac{\theta}{2}e^{rt}$$

$$0 = \frac{e^{\mu t}}{2\sqrt{\alpha}} - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right) \cdot \frac{e^{\mu t}}{2\sqrt{\alpha}} - \frac{\theta}{2} \cdot e^{rt}$$

$$\frac{\theta}{\cancel{2}}\left(\left(\frac{1}{\varphi} - 1\right) \cdot \frac{e^{\mu t}}{2\sqrt{\alpha}} + e^{rt}\right) = \frac{e^{\mu t}}{\cancel{2}\sqrt{\alpha}}$$

$$\theta = \frac{e^{\mu t}}{\left(\frac{1}{\varphi} - 1\right) \cdot \frac{1}{2}e^{\mu t} + \sqrt{\alpha}e^{rt}} \cdot \underbrace{\left(\frac{\frac{1}{e^{\mu t}}}{\frac{1}{e^{\mu t}}}\right)}_{1}$$

$$\theta = \frac{1}{\frac{1}{2}\left(\frac{1}{\varphi} - 1\right) + e^{(r-\mu)t}\sqrt{\alpha}}$$

Since we assume delta-neutrality is measured when we open the position we can consider $\alpha = 0$ and $t = 1$, therefore to maintain a delta-neutral portfolio, at any given time, our *theta* must be:

$$\theta = \frac{1}{\frac{1}{2}\left(\frac{1}{\varphi} - 1\right) + 1}$$

$$= \frac{1}{\frac{1}{2}\left(\frac{1}{\varphi} + 1\right)} \tag{59}$$

Then for example, if $\varphi = 0.8$, θ must always be $\frac{8}{9}$. Then $\omega = \frac{8}{9}$. Meaning, we should be providing $\frac{8}{9}$ of our capital as liquidity and hedging with the remaining $\frac{1}{9}$ ($\frac{\theta}{2}$). If the LTV was lower: $\varphi = 0.5$, $\theta = \omega = \frac{2}{3}$, therefore we would be providing $\frac{2}{3}$ of our capital as liquidity on Uniswap and $\frac{1}{3}$ as the capital for the hedge.

LP Dynamic Re-Hedged Returns - $\mu = 20\%$, $r = 0$, $t = 1$

Legend:
- $\varphi = 0.10$, $\omega = \theta = 0.18$
- $\varphi = 0.20$, $\omega = \theta = 0.33$
- $\varphi = 0.30$, $\omega = \theta = 0.46$
- $\varphi = 0.40$, $\omega = \theta = 0.57$
- $\varphi = 0.50$, $\omega = \theta = 0.67$
- $\varphi = 0.60$, $\omega = \theta = 0.75$
- $\varphi = 0.70$, $\omega = \theta = 0.82$
- $\varphi = 0.80$, $\omega = \theta = 0.89$
- $\varphi = 0.90$, $\omega = \theta = 0.95$

We can see that there are also some heavy losses on the right side of the chart. However, this **losses cannot occur** as the lending position would either get liquidated or need to be repayed before that as the borrow value would be too great against the collateral. If we constantly rebalance the portfolio, the hedge will always be readjusting so that returns continue to increase with increases in price. The same can be done when prices decrease to avoid negative P&L. To do this one must increase the hedge as price decreases, although the effect of price declines are less significant.

If we compare it against an unhedged position:

LP Returns ($\mu = r = 0$)
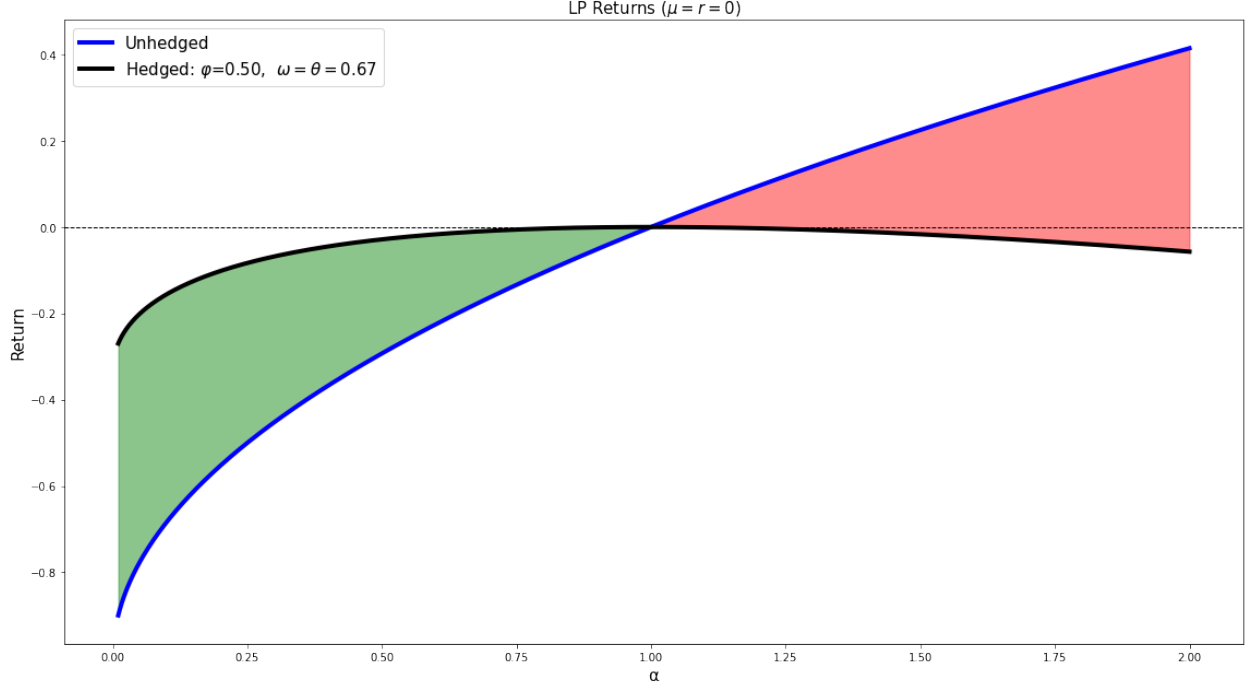
Change (%)	Hedged Ret. (%)	Unhedged Ret. (%)
-100	-33.33	-100.00
-80	-10.19	-55.28
-60	-4.50	-36.75
-40	-1.69	-22.54
-20	**-0.37**	-10.56
0.00	**0.00**	**0.00**
20	**-0.30**	9.54
40	-1.12	18.32
60	-2.34	26.49
80	-3.89	34.16
100	-5.72	41.42
120	-7.78	48.32
140	-10.05	54.92
160	-12.50	61.25

Table 1: *Comparison of Hedged and Unhedged Returns,*
$\mu = r = 0, \;\; \varphi = 0.5, \;\; \omega = \theta = \frac{2}{3}$

Then we see that if the LP rehedges for prices movements around $\pm 20\%$ the exposure to price movements will be minimal. Otherwise, if LP wanted to minimize hedging costs they could rebalance when prices move $\pm 40\%$.

The *Sharpe Ratio* is a metric that allow us to understand how much excess return above the *risk free rate* $(R_p - R_f)$ we are getting per unit of risk (volatility) in our portfolio (σ_p). Therefore it can be thought as a metric to define *risk-adjusted returns*.

$$SR = \frac{\mathbb{E}(R_p - R_f)}{\sigma_p} \tag{60}$$

A portfolio with a higher *Sharpe Ratio* than another one, even if it has lower returns, is preferable. That is because the higher Sharpe ratio portfolio can be leveraged to increase the risk to match that of the lower Sharpe ratio portfolio and thus increase the return to higher levels than those of the lower Sharpe ratio portfolio. For example, although delta neutral liquidity provision has lower returns than unhedged liquidity provision, the lower variability of the returns lead to a higher Sharpe Ratio, despite the lower absolute returns. Therefore, a leveraged delta neutral liquidity provision portfolio can have substantially higher returns than an unhedged liquidity provision portfolio.

The following explanation and table are both directly extracted from [6]:

Below are the Sharpe ratios of a delta neutral liquidity provision portfolio and a non hedged liquidity provision portfolio after a Monte Carlo simulation of 10,000 runs for a liquidity pool with different levels of growth (μ), different volatilities and returns for the underlying pair of the liquidity pool, and different rebalancing strategies to achieve delta neutrality or restart the position in the unhedged portfolio (rebalancing every week and every month). The instances where the Sharpe ratio of the delta neutral liquidity provision portfolio (hedging) was higher is highlighted in yellow. The risk free rate of return was assumed to be zero for the sake of simplicity. We also assumed a φ of 0.5 and therefore a θ of 1/3 to achieve delta neutrality.

		Monthly Rebalance					Weekly Rebalance				
$\mu = 20\%$, $r = 5\%$		**Annual Underlying Volatility**									
Annual Underlying Return		**25%**	**50%**	**100%**	**150%**	**200%**	**25%**	**50%**	**100%**	**150%**	**200%**
0%	non-hedging	0.48	0.23	0.13	0.10	0.06	0.24	0.11	0.08	0.05	0.02
	hedging	15.57	3.87	0.52	-0.09	-0.27	15.40	3.47	0.41	-0.17	-0.36
50%	non-hedging	1.08	0.54	0.29	0.19	0.16	0.50	0.23	0.13	0.08	0.06
	hedging	14.36	3.55	0.41	-0.12	-0.29	15.88	3.50	0.38	-0.18	-0.36
100%	non-hedging	1.64	0.82	0.45	0.30	0.23	0.77	0.40	0.21	0.16	0.11
	hedging	9.54	2.64	0.34	-0.15	-0.30	13.11	3.30	0.37	-0.18	-0.36
200%	non-hedging	2.80	1.43	0.72	0.48	0.37	1.31	0.68	0.34	0.26	0.17
	hedging	4.19	1.41	0.08	-0.21	-0.33	8.55	2.57	0.29	-0.21	-0.36
300%	non-hedging	3.88	1.97	1.00	0.69	0.51	1.92	0.93	0.48	0.32	0.25
	hedging	1.91	0.60	-0.09	-0.29	-0.37	5.85	1.95	0.19	-0.23	-0.38
400%	non-hedging	4.92	2.53	1.28	0.89	0.63	2.45	1.21	0.61	0.41	0.32
	hedging	0.62	0.08	-0.23	-0.36	-0.41	3.96	1.41	0.10	-0.26	-0.40

μ is the pool growth rate from trading fees, set at 20% APY, r is the lending rate in a decentralized exchange to achieve the hedge, set at 5% APY, the risk free rate of return was assumed to be 0.

As you can see in the table above, the portfolio's *Sharpe ratio* improves as *mu* and the rebalancing period increase and worsens as volatility and returns in the underlying increase. For additional tables with varying values of μ refer to the original article. I have decided not to include them as it is unlikely to get such high returns from fees over a long period of time. However, the reader is highly encouraged to review these elements, as they provide a clearer perspective on how each variable impacts the *SR*.

7.4.4 Leveraged Delta-Neutral Borrowing

Due to the higher *Sharpe Ratio* that the delta-neutral strategy has, one might be tempted to leverage it to further increase capital efficiency. Nevertheless, the increase in leverage implies a higher interest payment in loans to finance the higher leverage. In order to account for the leverage and its associated cost we only need to increase the return by the *leverage ratio* and subtract the additional *interest* accrued on the borrowed sums.

Let f be the *equity to total assets rate* (i.e. $f = 20\%$ means 20% is equity, 80% is borrowed sums: a *leverage ratio* of $x5$).

$$R_{LP} = \frac{1}{f}\left(\frac{V_1 - V_0}{V_0}\right) - \left(\frac{1-f}{f}\right)e^{rt}$$

Therefore, substituting the corresponding values we obtain:

$$R_{leveraged} = \frac{1}{f}\left(\left[1 - \frac{\theta}{2}\left(\frac{1}{\varphi} - 1\right)\right]e^{\mu t}\sqrt{\alpha} - \frac{\theta}{2}\left(\alpha e^{rt} - \frac{1}{\varphi}\right) - (1-f)e^{rt} - 1\right) \tag{61}$$

And we have that the delta of such portfolio is exactly:

$$\frac{\partial R_{leveraged}}{\partial \alpha} = \frac{1}{f}\left(\frac{\partial R^h_{LP}}{\partial \alpha}\right) \tag{62}$$

Therefore, achieving delta-neutrality (make the first derivative with respect to price zero) is the same process as for the non leveraged portfolio.

7.5 Leveraged Delta One Hedging

When hedging with Delta One products like futures (or perpetuals), one can utilise leverage to improve its capital efficiency. For example, using a 2x leverage one could hedge with 33.33% of their funds at a CEX the other 66.67% that are deployed as liquidity. By doing so, instead of splitting the capital 50/50 (x1 leverage), with only the 50% of capital earning fees, one can be earning fees with 66% while (approximately) neutral, gaining a +33.33% in capital efficiency.

This strategy could not be replicated on-chain via borrowing protocols as they always require an overcollateralisation.

7.6 Power Perpetuals

Power perps go brr

7.7 Gamma Hedging

8 Appendix A

8.1 Uniswap Whitepapers Takeaways

8.1.1 V1

`Uniswap v1` allows for an exchange rate between `ETH` and an `ERC20` token based on the relative sizes of their liquidity pools within the contract. This is done by maintaining the relationship:

$$eth\_pool * token\_pool = invariant$$

This invariant is held constant during trades and only changes when liquidity is added or removed from the market.

When doing a swap the fee (commission) is added after price calculation therefore the invariant increases slightly with every trade, making the system profitable for liquidity providers. Swapping fees are immediately deposited into liquidity reserves. Since total reserves are increased without adding any additional share tokens, this increases that value of all share tokens equally. This functions as a payout to liquidity providers that can be collected by burning shares. Since fees are added to liquidity pools, the invariant increases at the end of every trade. Within a single transaction, the `invariant` represents `eth_pool * token_pool` at the end of the previous transaction. Fees taken during trades are added to total liquidity pools without minting new liquidity tokens. Because of this, `ethWithdrawn` and `tokensWithdrawn` include a proportional share of all fees collected since the liquidity was first added.

Purchases that are large relative to the total size of the liquidity pools will cause price slippage. In an active market, aribitrage will ensure that the price will not shift too far from that of other exchanges.

The first liquidity provider to join a pool sets the initial exchange rate by depositing what they believe to be an equivalent value of ETH and ERC20 tokens. If this ratio is off, arbitrage traders will bring the prices to equilibrium at the expense of the initial liquidity provider.

Liquidity tokens are minted to track the relative proportion of total reserves that each liquidity provider has contributed.

ETH and ERC20 tokens are withdrawn at the current exchange rate (reserve ratio), not the ratio of their originial investment. This means some value can be lost from market fluctuations and arbitrage.

8.1.2 V2

`Uniswap v2` introduces some novel upgrades to the protocol like being able to swap arbitrary `ERC-20` tokens without routing through `ETH`, as well as re-writing the entire *Vyper* code base in *Solidity* and diminishing the surface of attack in the core contracts. Major updates are:

- ERC-20 Pairs
- Price Oracle (TWAP)
- Precision
- Flash Swaps
- Protocol Fee

- Meta Transactions
- Solidity & Contract Re-architecture
- Wrapping `ETH` (removal of native `ETH`)
- Deterministic Pair Addresses
- Maximum Token Balance

8.1.3 V3

`Uniswap v3` provides increased capital efficiency and fine-tuned control to liquidity providers, improves the accuracy and convenience of the price oracle, and has a more flexible fee structure.

- Concentrated Liquidity
- Flexible Fees
- Protocol Fee Governance
- Improved Price Oracle

- Liquidity Oracle
- Non-Fungible Liquidity
- Geometric Mean `TWAP` (sum of log prices)

The main change for LPs is the possibility to provide concentrated liquidity.
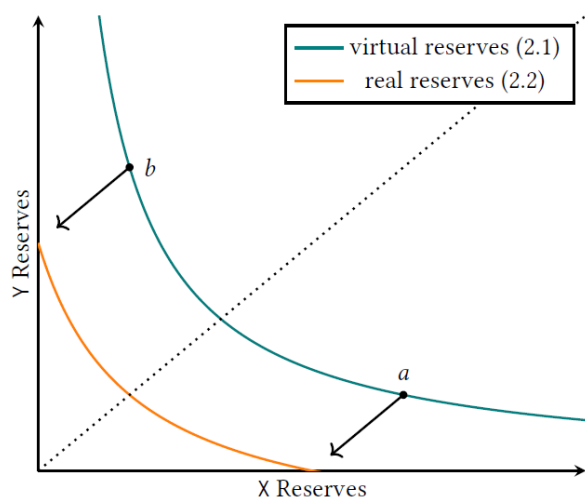
$$\left(x + \frac{L}{\sqrt{p_b}}\right)\left(y + L\sqrt{p_a}\right) = L^2$$

8.1.4 V4

9 Disclaimer

This article is for general information purposes only and does not constitute investment advice. The author does not guarantee the accuracy of the information provided in this article. This article was written in the author's free time and is not related to his professional activity. Furthermore, it reflects the current opinions of the author, and does not necessarily reflect the opinions of his employer.

10 References

10.1 Whitepapers

[W1]. Hayden A., (2018). Uniswap Whitepaper

[W2]. Hayden A., et al., (2020). Uniswap V2 Whitepaper

[W3]. Hayden A., et al., (2021). Uniswap V3 Whitepaper

[W4]. Hayden A., et al., (2024). Uniswap V4 Whitepaper

10.2 External References

1. Atis E., (2021). Liquidity Math In Uniswap v3

2. Atis E., (2023). Liquidity Provider Strategies for Uniswap v3 [Series]

3. Atis E., (2023). Uniswap LP Articles Code

4. Atis E., (2024). Dynamic Fees for Automated Market Makers: Liquidity, Volatility, and Collected Fees

5. Daniel A., (2021). Total Returns and Impermanent Loss in Uniswap V2

6. Daniel A., (2021). Optimizing Liquidity Provision: Hedging Market Risk in Uniswap V2

7. Daniel A., (2022). Expected Impermanent Loss in Uniswap V2 & V3

8. Daniel A., (2022). Calculating Implied Volatility Uniswap V2 & V3

9. Daniel A., (2022). Uniswap Liquidity Provision: Is the Yield Worth the Risk?

10. Guillaume L., (2021). Medium Articles

11. Guillaume L., (). An Analysis of the Expected Value of the Impermanent Loss in Uniswap (V3)

12. Joseph Clark, (2021). Hedging Uniswap v3 Positions with SQUEETH

13. Dave W., et al., (2021). Power Perpetuals

14. Joseph Clark, (2023). Spanning with Power Perpetuals

15. Joseph Clark, (2021).The replicating portfolio of a constant product market with bounded liquidity

16. Niccolò B., & Alessandro N. (2023). Liquidity Providers Greeks and Impermanent Gain

17. Jason M., et al. (2024). Automated Market Making and Loss-Versus-Rebalancing

18. Stefan L., et al. (2021). Impermanent Loss in Uniswap v3

19. Nassib B. (2021). G3M Impermanent Loss Dynamics

20. Compass Labs. (2024).Tracking Impermanent Loss on UniswapV3 using Dojo

21. Cyfrin Updraft. (2024). Uniswap v2 Developer Course

22. Peteris E. (2021). Impermanent Loss in Uniswap v3

23. Peteris E. (2021). How to calculate Impermanent Loss: full derivation

24. Doncho K. (2021). Impermanent Loss Explained With Examples & Math

25. Sia M., (2024). Convexity and Concavity in Investing, Life and Decision Making

26. Saleh H., Mohsen P., (2022). Concentrated Liquidity Analysis in Uniswap V3

27. Alex E., (2023). Navigating Uniswap V3: A Comprehensive Guide to APR Estimation and Pool Risk Analysis

28. Yi Z., et al., (2018). Formal Specification of Constant Product (xy = k) Market Maker Model and Implementation

29. Vitalik B., (2018). Improving front running resistance of x*y=k market makers

30. Vitalik B., (2016). Let's run on-chain decentralized exchanges the way we run prediction markets

31. Hayden A., (2018). *uniswap_exchange.vy* commit *4610c59*

32. RareSkills, (2023). Uniswap V2 Book

33. Chaisomsri, (2024). [DeFi Math] Uniswap V3 Concentrated Liquidity

34. Jeiwan, (2022). Uniswap V3 Development Book

35. Guillermo A., et al., (2019). An analysis of Uniswap markets

36. The-CTra1n, (2023). LVR-minimization in Uniswap V4

37. llllvvuu, (2021). Raw Moment Derivatives

38. mookid, (2014). Derivatives Pricing with Squared and Cubed Stock Prices

39. makemake, (2023). sneed, Uniswap V4 Hook to Hedge Against Impermanent Loss

40. Adam K., Xi Z., (2022). Delta Hedging Liquidity Positions on Automated Market Makers

41. Amy K., (2024). From CEX to DEX: A New Formulation For Implied Volatility - Part 2

42. Xue D., Chen Y., Yutian Z., (2024). Liquidity Pool Design on Automated Market Makers

43. CrocSwap, (2022). Benchmarking the Performance of Automated Liquidity Vault Strategies

44. rekt, (2021). Uniswap V3 LP Rekt

45. CowSwap, (2024). What is Loss-Versus-Rebalancing (LVR)?

46. DelphiDigital, (2024). Loss Versus Rebalancing (LVR)

11   Hook: VRP Discount Factor

Let σ_i denote the *Implied Volatility* and ζ_r the *Realized or Historical Volatility*. Then, the discount factor to reduce the Volatility Risk Premium (*VRP*) is defined by:

$$d(\sigma_i, \zeta_r) = \min\left(\frac{e^{\left(\sigma_i \cdot \frac{ln(\zeta_r)}{\zeta_r}\right)}}{\sigma_i} \cdot \left(e^{\zeta_r \cdot \frac{ln(2)}{\sigma_i}} - 1\right), 2 \right) \tag{63}$$