CS-E5740 Complex Networks, Answers to exercise set 3

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Compile with `pdflatex ex_template.tex`

Problem 1

a) The degree of the node with the highest degree is 24 and the total number of links is 200 (3 in the initial clique and 197 generated).

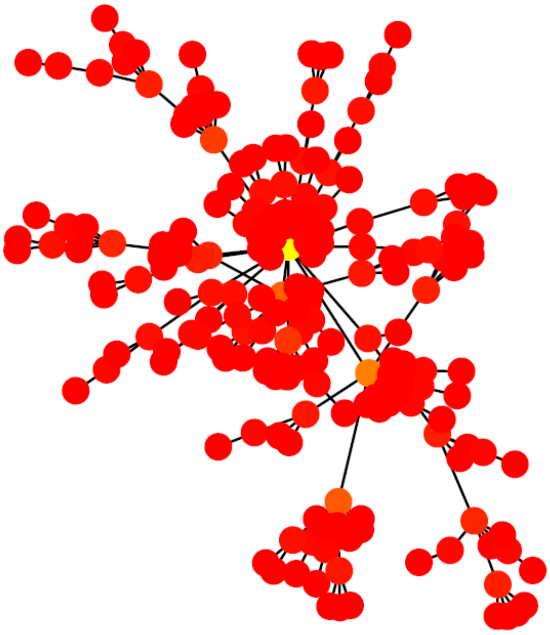

Figure 1: Graph for N=200 and m=1.

b) The following shows the logarithmically binned probability density function for degree, $P(k)$, for a network with $N = 10^4$ and $m = 2$ against the theoretical prediction.

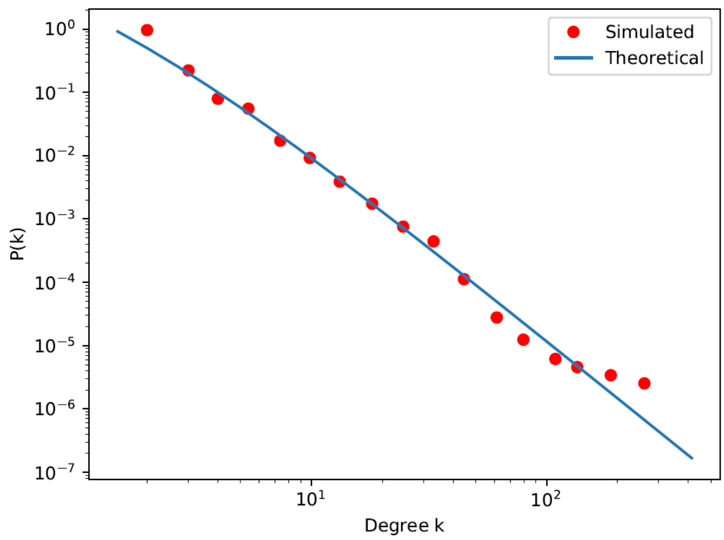

Figure 2: P(k) for a BA network with N=10000 and m=2

Problem 2

a) The initial network has N_o nodes, with $N_o > m$. Let us start from the probability Π_i that a new edge attaches to a particular vertex of degree k_i equals:

$$\Pi_i = \frac{k_i}{\sum_{j=1}^{N} k_i}$$

Knowing that $n_{k,N} = Np_{k,N}$ is the number of nodes with degree k, we can define the probability $\Pi(k)$ that a new edge attaches to any vertex of degree k in a network of N vertices as

$$\Pi(k) = Np_{k,N} * \Pi_i$$

The total sum of the degrees for all nodes is two times the total number of edges (NE) as follows:

$$NE(N) = \frac{N_0(N_0 - 1)}{2} + (N - N_0)m$$

Considering $\lim_{N\to\infty} NE(N)$ then we can express the total number of edges (edges in the intial clique + generated ones) as

$$NE(N) = N * m$$

Replacing the total number of edges in the summation of the degrees we obtain:

$$\sum_{j=1}^{N} k_i = 2 * NE = 2 * N * m$$

This allows us to express the probability Π_i as

$$\Pi_i = \frac{k_i}{2Nm}$$

Now we replace it in the $\Pi(k)$ formula obtaining the wanted probability

$$\Pi(k) = Np_{k,N} * \frac{k}{2Nm} = \frac{kp_{k,N}}{2m}$$

b) In the BA model, each node of the graph has at least m edges. Considering $k = m$, there aren't nodes with $k = m-1$ because of the construction of the graph. Therefore, the set of nodes with $k = m$ doesn't inherit any vertex from the set with $k = m - 1$. This is why the number n_k^+ is equal to 1, which represents the single node just added in the graph, with $k = m$.

The following equation represents the net change of the number of vertices of degree k as the network grows in size from N to N+1. Notice that $n_k^+ = n_{k-1}^-$ (with $k > m$). I'll analyse two cases: $k > m$ and $k = m$.

$\mathbf{k > m}$

$$(N+1)p_{k,N+1} - Np_{k,N} = n_k^+ - n_k^-$$
$$(N+1)p_{k,N+1} - Np_{k,N} = \frac{1}{2}(k-1)p_{k-1,N} - \frac{1}{2}kp_{k,N}$$

$\mathbf{k = m}$

$$(N+1)p_{m,N+1} - Np_{m,N} = n_m^+ - n_m^-$$
$$(N+1)p_{m,N+1} - Np_{m,N} = 1 - \frac{1}{2}mp_{m,N}$$

c) Letting the network grow towards the infinite and considering stationary solution, we can say that
$p_{k,N+1} = p_{k,N} = p_k$
$p_{k-1,N} = p_{k-1}$
$p_{m,N+1} = p_{m,N} = p_m$.
Replacing and simplifying the above equations, we obtain:

k > m

$$
\begin{aligned}
(N+1)p_k - Np_k &= \frac{1}{2}(k-1)p_{k-1} - \frac{1}{2}kp_k \\
(N+1-N+\frac{1}{2}k)p_k &= \frac{1}{2}(k-1)p_{k-1} \\
(1+\frac{1}{2}k)p_k &= \frac{1}{2}(k-1)p_{k-1} \\
(\frac{2+k}{2})p_k &= \frac{1}{2}(k-1)p_{k-1} \\
p_k &= \frac{k-1}{k+2}p_{k-1}
\end{aligned}
$$

k = m

$$
\begin{aligned}
(N+1)p_m - Np_m &= 1 - \frac{1}{2}mp_{m,N} \\
(N+1-N+\frac{1}{2}m)p_m &= 1 \\
(1+\frac{1}{2}m)p_m &= 1 \\
(\frac{2+m}{2})p_m &= 1 \\
p_m &= \frac{2}{2+m}
\end{aligned}
$$

d) With a recursive approach, we replace k with m+i, where i=1,2,3,4 and then we derive the p_k formula.

k = m + 1

$$p_{m+1} = (\frac{m}{3+m})p_m = (\frac{m}{3+m})(\frac{2}{2+m})$$

k = m + 2

$$p_{m+2} = (\frac{m+1}{4+m})(\frac{m}{3+m})(\frac{2}{2+m})$$

k = m + 3

$$p_{m+3} = (\frac{m+2}{5+m})(\frac{m+1}{4+m})(\frac{m}{3+m})(\frac{2}{2+m}) = (\frac{1}{5+m})(\frac{m+1}{4+m})(\frac{m}{3+m})2$$

k = m + 4

$$p_{m+4} = (\frac{3+m}{6+m})(\frac{1}{5+m})(\frac{m+1}{4+m})(\frac{m}{3+m})2 = (\frac{1}{6+m})(\frac{1}{5+m})(\frac{m+1}{4+m})2m$$

At each step only the denominator changes: one term is simplified out and one term is appended. So we can derive the general formula:

$$p_{m+i} = 2 * \frac{m(m+1)}{(m+i+2)(m+i+1)(m+i)}$$

And since, we considered $k = m + i$, the initial formula is derived:

$$p_k = 2 * \frac{m(m+1)}{(k+2)(k+1)(k)}$$