CS-E5740 Complex Networks, Answers to exercise set 2

Sara Cabodi, Student number: 784287

September 30, 2019

Compile with `pdflatex ex_template.tex`

Problem 1

a) With $N = 3$ we can obtain an ensamble of 8 different graphs.

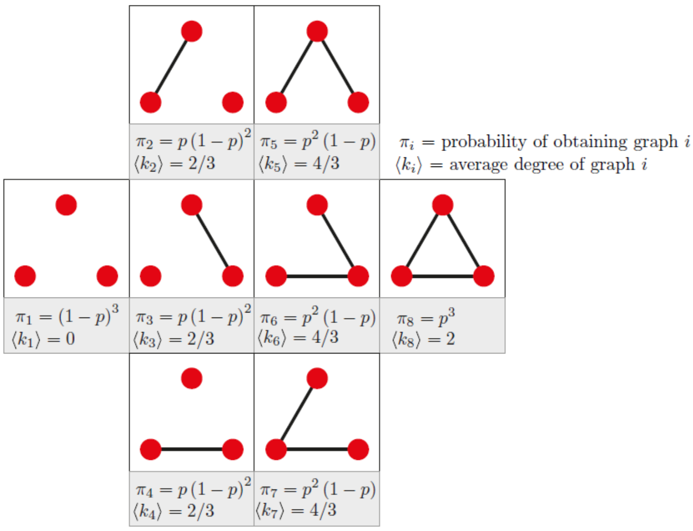

Figure 1: Graphs for N=3 for exercise 1.

The probability π_i of obtaining graph i is:

$$\pi_i = p^m(1-p)^{N-m}$$

with m number of links and N number of nodes in the graph.

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
0.296	0.148	0.148	0.148	0.074	0.074	0.074	0.037

The average quantities required are:

- Average degree for the ensemble for $i = \{1, ..., 8\}$ is

$$\langle k \rangle = \sum_i \pi_i k(G_i) = 0.666$$

with $k(G_i)$ values as follows

$k(G_1)$	$k(G_2)$	$k(G_3)$	$k(G_4)$	$k(G_5)$	$k(G_6)$	$k(G_7)$	$k(G_8)$
0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	2

- Average clustering coefficient for the ensemble is

$$\langle c \rangle = \sum_i \pi_i c(G_i) = 0.037$$

$c(G_1)$	$c(G_2)$	$c(G_3)$	$c(G_4)$	$c(G_5)$	$c(G_6)$	$c(G_7)$	$c(G_8)$
0	0	0	0	0	0	0	1

- Average diameter for the ensemble is

$$\langle d^* \rangle = \sum_i \pi_i d^*(G_i) = 0.925$$

$d^*(G_1)$	$d^*(G_2)$	$d^*(G_3)$	$d^*(G_4)$	$d^*(G_5)$	$d^*(G_6)$	$d^*(G_7)$	$d^*(G_8)$
0	1	1	1	2	2	2	1

b) The average formulas are:

$$\begin{aligned}
\langle k \rangle &= \sum_i \pi_i k(G_i) = \\
&= (1-p)^3 * 0 + 3 * p(1-p)^2 * \frac{2}{3} + 3 * p^2(1-p) * \frac{4}{3} + p^3 * 2 = \\
&= 0 + 2p(1-p)^2 + 4p^2(1-p) + 2p^3 = \\
&= 2p((1-p)^2 + 2p(1-p) + p^2) = \\
&= 2p(1 - 2p + p^2 + 2p - 2p^2 + p^2) = \\
&= 2p
\end{aligned}$$

$$\begin{aligned}
\langle c \rangle &= \sum_i \pi_i c(G_i) = \\
&= (1-p)^3 * 0 + 3 * p(1-p)^2 * 0 + 3 * p^2(1-p) * 0 + p^3 * 1 = \\
&= p^3
\end{aligned}$$

$$
\begin{aligned}
\langle d^* \rangle &= \sum_i \pi_i d^*(G_i) = \\
&= (1-p)^3 * 0 + 3 * p(1-p)^2 * 1 + 3 * p^2(1-p) * 2 + p^3 * 1 = \\
&= 0 + 3p(1-p)^2 + 6p^2(1-p) + p^3 = \\
&= p(3(1-p)^2 + 6p(1-p) + p^2) = \\
&= 2p(3 - 6p + 3p^2 + 6p - 6p^2 + p^2) = \\
&= p(3 - 2p^2) = 3p - 2p^3
\end{aligned}
$$

Problem 2

a) Starting from the formula of the degree distribution $P(k)$ of ER networks

$$P(k) = \binom{(N-1)}{k} p^k (1-p)^{(N-1)-k}$$

There are N-1 possible edges for each node. This is the number of trials, while k represents the exact number of edges taken into consideration, so P(k) represents the probability that one random node has degree k. Let's analyse each term in detail:

- $\binom{(N-1)}{k}$: there are N-1 number of trials with k cases of success and the combination represents how many groups of k edges can be found having N-1 trials (possible edges)
- p^k : expresses the probability p of success for the number of cases of success. In other words, it represent the probability that the node has exactly degree k
- $(1-p)^{(N-1)-k}$: expresses the probability of failure for the number of cases of failure, so the total number of cases (N-1) minus the success ones (k). In other words, it represents the probability that the node has not degree k.

b) In general the average clustering coefficient $\langle c \rangle$ represents the average density between all subset of the graph, each one composed by the neighbors of each node. The probability p represents the likeliness that two nodes are connected. In ER networks, since the links are indipendent, the expected value of the average clustering coefficient $\langle c \rangle$ equals p, because both quantities represent the probability of two nodes to be connected.

c) Starting from the formula $\langle k \rangle = p(N-1)$, if N $\rightarrow \infty$ and $\langle k \rangle$, then $p \rightarrow 0$. Since $p = \langle c \rangle$, then also $\langle c \rangle \rightarrow 0$.

d) In the following, I present the graphics obtained running my pyhton code.

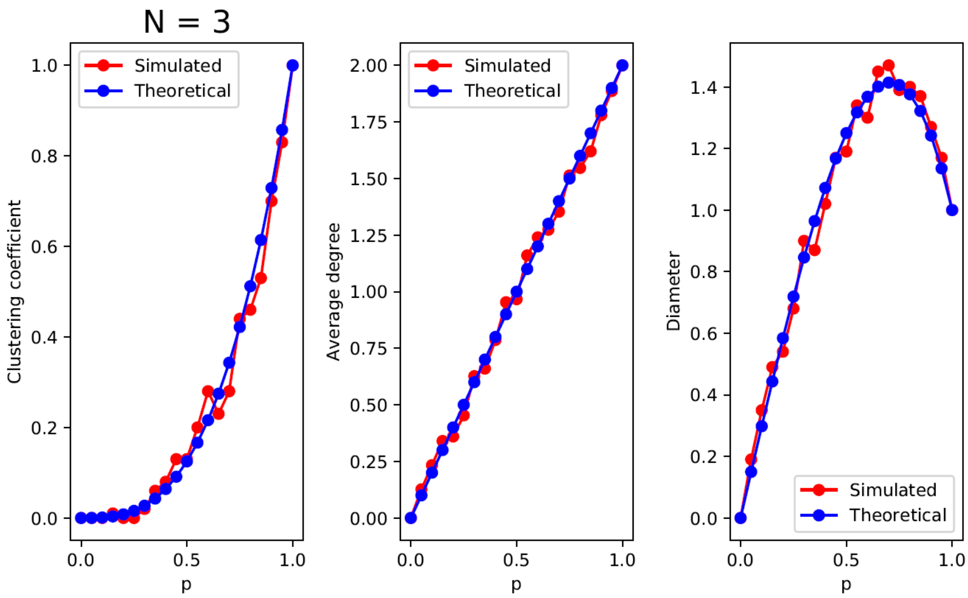

Figure 2: Values for N is 3

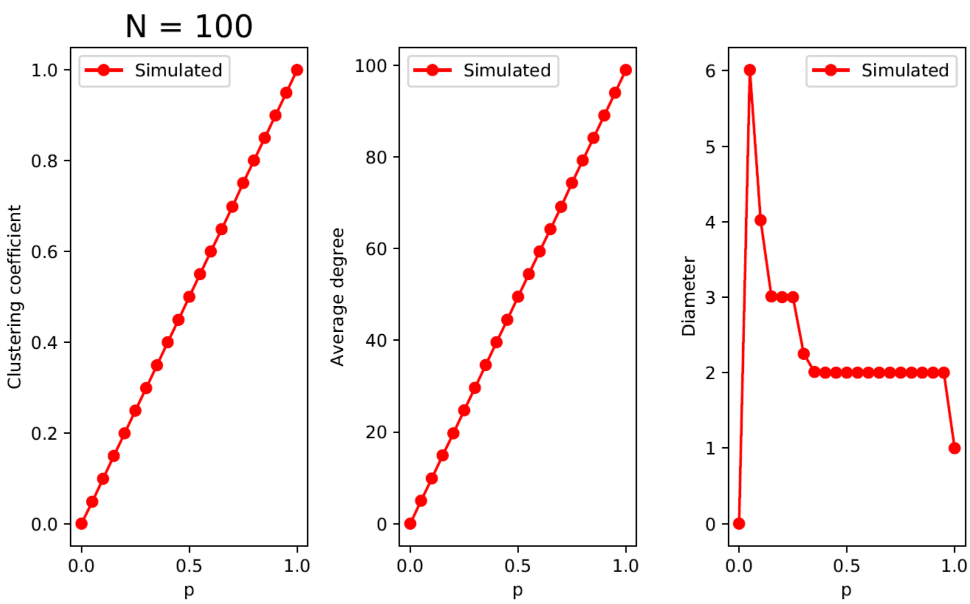

Figure 3: Values for N is 100

Problem 3

a) Total number of edges for N=15, m=2 and p=0.1 is 30. Total number of rewired edges is 4.

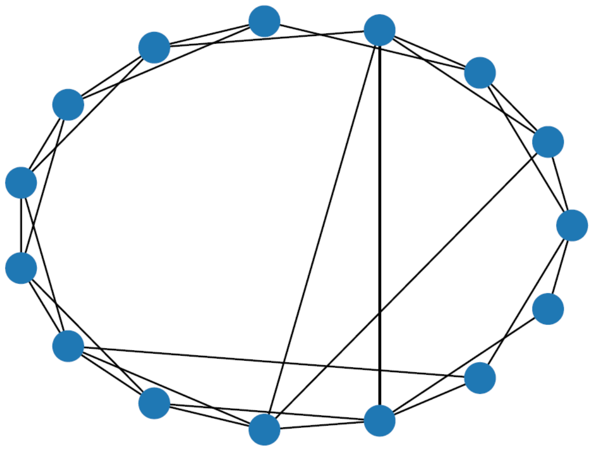

Figure 4: Small world ring

Total number of edges for N=100, m=2 and p=0.5 is 200. Total number of rewired edges is 96.

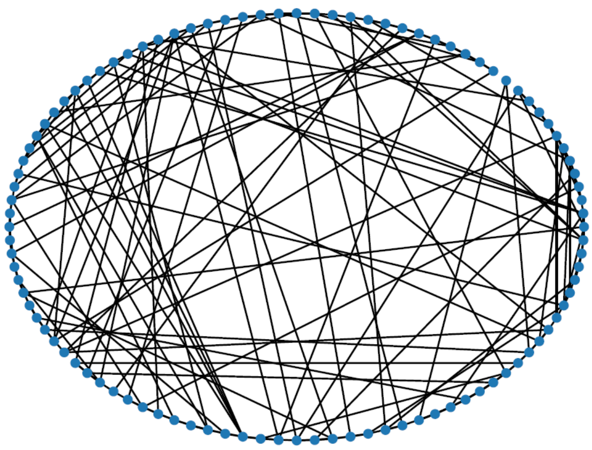

Figure 5: Small world rewired

b) The following graph shows the values of the relative average clustering coefficient $c(p)/c(p=0)$ and the average shortest path length $l(p)/l(p=0)$.

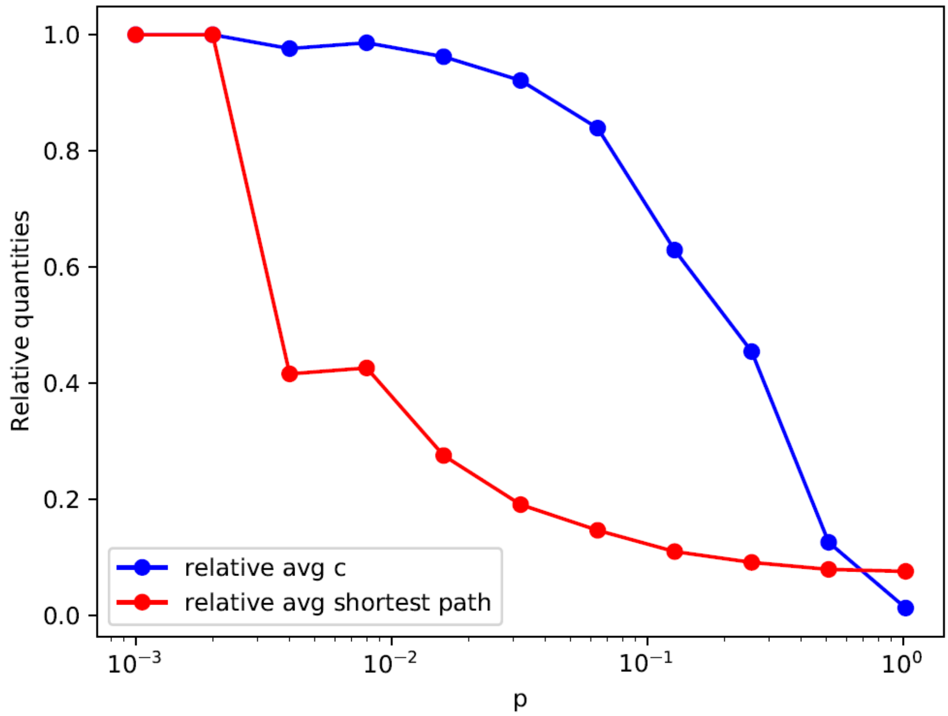

Figure 6: Relative quantities (c and l)