Quantum Field Theory

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1 Листок

Упражнение 1.1. *Доказать, что* $\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}dt = \frac{1}{4n}\left|B_{2n}\right|$, *где* B_n *- числа Бернулли.*

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

Решение.

$$\int_0^\infty \frac{t^{2n-1}}{e^{2nt}-1}dt = \int_0^\infty dt \frac{t^{2n-1}}{e^{2\pi t}}\left(1-e^{-2\pi t}\right)^{-1} = \int_0^\infty dt\, t^{2n-1}\sum_{k=0}^{\infty} e^{-2\pi k t} = \sum_{k=0}^{\infty}\int_0^\infty dt\, \frac{t^{2n-1}}{e^{2\pi t}} e^{-2\pi k t}$$

$$\sum_{k=0}^{\infty}\int_0^\infty dt\, t^{2n-1}e^{-2\pi(k+1)t} = \left| \begin{array}{l} x = 2\pi(k+1)t \\ dx = 2\pi(k+1)dt \end{array} \right| = \sum_{k=0}^{\infty}\int_0^\infty \frac{dx}{2\pi(k+1)}\frac{x^{2n-1}e^{-x}}{(2\pi(k+1))^{2n-1}}dx$$

$$\sum_{k=0}^{\infty}\frac{1}{(2\pi(k+1))^{2n}}\int_0^\infty dx\, x^{2n-1}e^{-x} = \frac{1}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\,\Gamma(2n) = \frac{1}{(2\pi)^{2n}}\zeta(2n)\cdot\Gamma(2n) =$$

$$\left|\Gamma(2n) = (2n-1)! \qquad \zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}\right|$$

$$= (-1)^{n+1}\frac{B_{2n}}{4n}$$

Задача 1.

Решение.

$$\omega_{n,p_2,\ldots,p_d} = \left[\left(\frac{\pi n}{L}\right)^2 + p_2^2 + \ldots + p_d^2\right]^{1/2} = \left[\left(\frac{\pi n}{L}\right)^2 + \vec{p}_{\|}^{\,2}\right]^{1/2}, \quad n = 1,2,3,\ldots$$

$$\frac{E(L)}{A} = \sum_{n=1}^{\infty}\int \frac{d^{d-1}p_{\|}}{(2\pi)^{d-1}}\frac{1}{2}\left[\left(\frac{\pi n}{L}\right)^2 + \vec{p}_{\|}^{\,2}\right]^{1/2} =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\int_0^\infty p_{\|}^{d-2}dp_{\|}\left[\left(\frac{\pi n}{L}\right)^2 + \vec{p}_{\|}^{\,2}\right]^{1/2}$$

$$\Omega_{d-1} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$$

$$p_{\parallel} = \left(\frac{\pi n}{L}\right) t$$

$$\frac{E(L)}{A} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\pi n}{L}\right)^{d} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^\infty dt t^{d-2}\sqrt{1+t^2} = \frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\frac{\pi^d}{L^d}\zeta(-d)\int_0^\infty dt t^{d-2}\sqrt{1+t^2}$$

$$\int_0^\infty dt t^{d-2}\sqrt{1+t^2} = \frac{1}{2}\int_1^\infty du u^{1/2}(u-1)^{\frac{d-3}{2}} \overset{v=\frac{1}{u}}{=} \int_0^1 dv v^{-\frac{d}{2}-1}(1-v)^{\frac{d}{2}-\frac{3}{2}} =$$

$$= \frac{1}{2}B\left(-\frac{d}{2}, \frac{d-1}{2}\right) = \frac{\Gamma(-d/2)\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma(-1/2)}$$

$$\frac{E(L)}{A} = -\frac{1}{2^{d+1}}\zeta(-d)\frac{\pi^{d/2}}{L^d}\Gamma\left(-\frac{d}{2}\right)$$

$$\frac{F}{A} = -\frac{\partial}{\partial L}\left(\frac{E(L)}{A} + \frac{E(D-L)}{A}\right) \overset{D\to\infty}{=} \left(-\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\frac{\zeta(-d)\pi^{d/2}}{2^d L^{d+1}} = \frac{\pi^{d/2}\Gamma(1-d/2)\zeta(-d)}{2^d L^{d+1}}$$