Quantum Field Theory

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1 Листок

Упражнение 1.1. *Доказать, что* $\int_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}dt = \frac{1}{4n}\left|B_{2n}\right|$ *, где* B_n *- числа Бернулли.*

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

Решение.

$$\int_0^\infty \frac{t^{2n-1}}{e^{2nt}-1}dt = \int_0^\infty dt \frac{t^{2n-1}}{e^{2\pi t}} \left(1-e^{-2\pi t}\right)^{-1} = \int_0^\infty dt\, t^{2n-1} \sum_{k=0}^{\infty} e^{-2\pi k t} = \sum_{k=0}^{\infty} \int_0^\infty dt\, \frac{t^{2n-1}}{e^{2\pi t}} e^{-2\pi k t}$$

$$\sum_{k=0}^{\infty} \int_0^\infty dt\, t^{2n-1} e^{-2\pi(k+1)t} = \left| \begin{array}{l} x = 2\pi(k+1)t \\ dx = 2\pi(k+1)dt \end{array} \right| = \sum_{k=0}^{\infty} \int_0^\infty \frac{dx}{2\pi(k+1)} \frac{x^{2n-1}e^{-x}}{(2\pi(k+1))^{2n-1}} dx$$

$$\sum_{k=0}^{\infty} \frac{1}{(2\pi(k+1))^{2n}} \int_0^\infty dx\, x^{2n-1} e^{-x} = \frac{1}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}}\, \Gamma(2n) = \frac{1}{(2\pi)^{2n}} \zeta(2n) \cdot \Gamma(2n) =$$

$$\left| \Gamma(2n) = (2n-1)! \qquad \zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \right|$$

$$= (-1)^{n+1} \frac{B_{2n}}{4n}$$